Dense Eulerian Graphs are $(1,3)$-Choosable

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Abstract

A graph G is total weight (k,k')-choosable if for any total list assignment L which assigns to each vertex v a set $L(v)$ of k real numbers, and each edge e a set $L(e)$ of k' real numbers, there is a proper total L-weighting, i.e., a mapping $f: V(G) \cup E(G) \to \mathbb{R}$ such that for each $z \in V(G) \cup E(G)$, $f(z) \in L(z)$, and for each edge uv of G, $\sum_{e \in E(u)} f(e) + f(u) \neq \sum_{e \in E(v)} f(e) + f(v)$. This paper proves that if G decomposes into complete graphs of odd order, then G is total weight $(1,3)$-choosable. As a consequence, every Eulerian graph G of large order and with minimum degree at least $0.91|V(G)|$ is total weight $(1,3)$-choosable. We also prove that any graph G with minimum degree at least $0.999|V(G)|$ and sufficiently large order is total weight $(1,4)$-choosable.

Mathematics Subject Classifications: 05C15, 05C72

1 Introduction

Assume $G=(V,E)$ is a graph with vertex set $V=\{1,2,\ldots,n\}$. Each edge $e \in E$ of G is a 2-subset $e=\{i,j\}$ of V. For $i \in V$, we denote by $E(i)$ the set of edges incident to i. A *total weighting* of G is a mapping ϕ: $V \cup E \to \mathbb{R}$. A total weighting ϕ is *proper* if for any edge $\{i,j\} \in E$,

$$\sum_{e \in E(i)} \phi(e) + \phi(i) \neq \sum_{e \in E(j)} \phi(e) + \phi(j).$$

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A proper total weighting ϕ with $\phi(i) = 0$ for all vertices i is also called a *vertex coloring edge weighting*. A vertex coloring edge weighting of G using weights $\{1, 2, \dots, k\}$ is called a *vertex coloring k-edge weighting*. Note that if G has an isolated edge, then G does not admit a vertex coloring edge weighting. We say a graph is *nice* if it does not contain any isolated edge.

Karoński, Łuczak and Thomason [11] conjectured that every nice graph has a vertex coloring 3-edge weighting. This conjecture received considerable attention [1, 2, 9, 10, 14, 15, 19], and it is known as the 1-2-3 conjecture. The best result on 1-2-3 conjecture so far was obtained by Kalkowski, Karoński and Pfender [10], who proved that every nice graph has a vertex coloring 5-edge weighting.

The list version of edge weighting of graphs was introduced by Bartnicki, Grytczuk and Niwczyk [5]. The list version of total weighting of graphs was introduced independently by Przybyło and Woźniak in [12] and by Wong and Zhu in [17]. Let $\psi : V \cup E \to \mathbb{N}^+$. A ψ-list assignment of G is a mapping L which assigns to $z \in V \cup E$ a set $L(z)$ of $\psi(z)$ real numbers. Given a total list assignment L, a *proper L-total weighting* is a proper total weighting ϕ with $\phi(z) \in L(z)$ for all $z \in V \cup E$. We say G is *total weight ψ-choosable* (ψ-choosable for short) if for any ψ-list assignment L, there is a proper L-total weighting of G. We say G is *total weight (k, k')-choosable* ((k, k')-choosable for short) if G is ψ-total weight choosable, where $\psi(i) = k$ for $i \in V(G)$ and $\psi(e) = k'$ for $e \in E(G)$.

The list version of edge weighting also received a lot of attention [5, 6, 7, 8, 13, 14, 16, 17, 18, 20]. As strengthenings of the 1-2-3 conjecture, it was conjectured in [17] that every nice graph is $(1, 3)$-choosable. A weaker conjecture was also proposed in [17], which asserts that there is a constant k such that every nice graph is $(1, k)$-choosable. This weaker conjecture was recently confirmed by Cao [6], who proved that every nice graph is $(1, 17)$-choosable. This result was improved in [20], where it was shown that every nice graph is $(1, 5)$-choosable.

Given a graph G and a family of graphs $\mathcal{H}$, we say that G has an $\mathcal{H}$-decomposition, if the edges of G can be partitioned into the edge sets of copies of graphs from $\mathcal{H}$. In particular, a triangle decomposition of G is a partition of $E(G)$ into triangles, and for a given graph H, an H-decomposition of G partitions $E(G)$ into subsets, each inducing a copy of H. The following is the main result of this paper.

Theorem 1. *If $E(G)$ can be decomposed into cliques of odd order, then G is $(1, 3)$-choosable.*

As a consequence of Theorem 1, we prove the following result.

Theorem 2. *If G is an n-vertex Eulerian graph with minimum degree at least $0.91n$ and n is sufficiently large, then G is $(1, 3)$-choosable.*

In [19], Zhong confirmed the 1-2-3 conjecture for graphs that can be edge-decomposed into cliques of order at least 3. As a consequence of this result, it was proved in [19] that the 1-2-3 conjecture holds for every n-vertex graph with minimum degree at least $0.99985n$, where n is sufficiently large.

Our result is the list version of Zhong's result, but with one degree restriction: $E(G)$ needs to be decomposed into complete graphs of odd order. Hence we can only show that dense Eulerian graphs are $(1,3)$-choosable. For general dense graphs, we prove the following result:

Theorem 3. *If G is an n-vertex graph with minimum degree at least $0.999n$ and n is sufficiently large, then G is $(1,4)$-choosable.*

2 Some preliminaries

The proofs of Theorems 1, 2 and 3 use tools that were introduced in [6] and were further developed in [20]. In this section, we introduce some definitions and present a result from [6] that will be used in this paper.

Given a graph $G=(V,E)$, let

$$\tilde{P}_G(\{x_z : z \in V \cup E\}) = \prod_{\{i,j\}\in E, i<j} \left(\left(\sum_{e\in E(i)} x_e + x_i \right) - \left(\sum_{e \in E(j)} x_e + x_j \right) \right).$$

Assign a real number $\phi(z)$ to each variable x_z, and view $\phi(z)$ as the weight of z. Let $\tilde{P}_G(\phi)$ be the evaluation of the polynomial at $x_z = \phi(z)$, $z \in V \cup E$. Then ϕ is a proper total weighting of G if and only if $\tilde{P}_G(\phi) \neq 0$. Thus the problem of finding a proper L-total weighting of G (for a given total list assignment L) is equivalent to finding a non-zero point of the polynomial $\tilde{P}_G(\{x_z : z \in V \cup E\})$ in the grid $\prod_{z\in V\cup E} L(z)$.

Combinatorial Nullstellensatz [3] gives a sufficient condition for the polynomial $\tilde{P}_G(\{x_z : z \in V \cup E\})$ has a non-zero point in the grid $\prod_{z\in V\cup E} L(z)$: If some non-vanishing (i.e., with non-zero coefficient) highest degree monomial $\prod_{z\in V\cup E} x_z^{K(z)}$ in the expansion of $\tilde{P}_G(\{x_z : z \in V \cup E\})$ satisifes $K(z) \leqslant |L(z)| - 1$ for $z \in V \cup E$, then $\tilde{P}_G(\{x_z : z \in V \cup E\})$ has a non-zero point in the grid $\prod_{z\in V\cup E} L(z)$.

We denote by $\mathbb{N}$ the set of non-negative integers. To prove a graph $G=(V,E)$ is $(1,k)$-choosable, it suffices to show that for some $K : V \cup E \to \mathbb{N}$ such that $K(v) = 0$ and $K(e) \leqslant k-1$, and the monomial $\prod_{z\in V\cup E} x_z^{K(z)}$ has non-zero coefficient in the expansion of $\tilde{P}_G(\{x_z : z \in V \cup E\})$.

As $K(v) = 0$ for all $v \in V$, the monomials in concern are of the form $\prod_{e\in E} x_e^{K(e)}$. Such monomials have the same coefficient in the expansions of $\tilde{P}_G(\{x_z : z \in V \cup E\})$ and

$$P_G(\{x_e : e \in E\}) = \prod_{\{i,j\}\in E, i<j} \left(\sum_{e\in E(i)} x_e - \sum_{e\in E(j)} x_e \right).$$

We denote by $\mathbb{N}^E$ the set of mappings $K : E \to \mathbb{N}$. Let

$$\mathbb{N}^E_m = \{K \in \mathbb{N}^E : \sum_{e\in E} K(e) = m\},\ \mathbb{N}^E_{(b^-)} = \{K \in \mathbb{N}^E : K(e) \leqslant b, \forall e \in E\}.$$

For $K \in \mathbb{N}^E$, let

$$x^K = \prod_{e \in E} x_e^{K(e)},\ K! = \prod_{e \in E} K(e)!.$$

Denote the coefficient of the monomial x^K in the expansion of P_G by $\mathrm{coe}(x^K, P_G)$.

For a positive integer b, to prove that $G = (V, E)$ is $(1, b+1)$-choosable, it suffices to show that $\mathrm{coe}(x^K, P_G) \neq 0$ for some $K \in \mathbb{N}^E_{(b^-)}$. For this purpose, we use a formula given in [6] for the calculation of $\mathrm{coe}(x^K, P_G)$.

For $m, n \in \mathbb{N}$, let $\mathbb{C}[x_1, x_2, \ldots, x_n]_m$ be the vector space of homogeneous polynomials of degree m in variables $x_1, \ldots, x_n$ over the field $\mathbb{C}$ of complex numbers.

Assume $|E| = m$. Consider the vector space of homogeneous polynomials of degree m in $\mathbb{C}[x_e : e \in E]$. For $f, g \in \mathbb{C}[x_e : e \in E]$, we define the *inner product* of f and g as

$$\langle f, g \rangle = \sum_{K \in \mathbb{N}^n_m} K! \mathrm{coe}(x^K, f) \overline{\mathrm{coe}(x^K, g)}.$$

The following lemma was proved in [6].

Lemma 4. *Assume $G = (V, E)$, $|E| = m$ and $K \in \mathbb{N}^E_m$. Let*

$$Q_E = \prod_{\{i,j\} \in E, i<j} (x_i - x_j),\ H_E^K = \prod_{\{i,j\} \in E, i<j} (x_i + x_j)^{K(e)}.$$

Then

$$\mathrm{coe}(x^K, P_G) = \frac{1}{K!} \langle Q_E, H_E^K \rangle.$$

Definition 5. For $K \in \mathbb{N}^E$, let $W^K_{E,m}$ be the complex linear space spanned by

$$\{H_E^{K'} : K' \leqslant K, K' \in \mathbb{N}^E_m\}.$$

It is obvious that there exists $K' \in \mathbb{N}^E_m$ such that $K' \leqslant K$ and $\langle Q_E, H_E^{K'} \rangle \neq 0$ if and only if there exists $F \in W^K_{E,m}$ such that $\langle Q_E, F \rangle \neq 0$. Thus we have the following corollary.

Corollary 6. *If $K \in \mathbb{N}^E_{(b^-)}$ and there exists $F \in W^K_{E,m}$ such that $\langle Q_E, F \rangle \neq 0$, then G is $(1, b+1)$-choosable.*

3 Proofs of Theorems 1, 2, 3

The following lemma is an easy observation, but it is the key tool for proving the main results of this paper.

Lemma 7. *If $Q_E \in W^K_{E,m}$ for some $K \in \mathbb{N}^E_{(b^-)}$, then G is $(1, b+1)$-choosable.*

Proof. Assume $Q_E \in W^K_{E,m}$. As $Q_E \neq 0$, we have $\langle Q_E, Q_E \rangle > 0$. By Corollary 6, G is $(1, b+1)$-choosable. $\square$

As an example, consider a triangle T with vertex set $\{i,j,k\}$. By definition, $Q_E = (x_i - x_j)(x_j - x_k)(x_i - x_k)$. To prove that $Q_E \in W^K_{E,3}$, it suffices to express each of the three factors of Q_E, $(x_i - x_j)$, $(x_j - x_k)$ and $(x_i - x_k)$, as a linear combination of $(x_i + x_j), (x_j + x_k), (x_i + x_k)$, and for each edge e, say for $e = \{i,j\}$, the term $(x_i + x_j)$ occurs in at most $K(e)$ of such linear combinations. We can write Q_E as

$$Q_E = ((x_i + x_k) - (x_j + x_k))((x_i + x_j) - (x_i + x_k))((x_i + x_j) - (x_j + x_k)).$$

It is easy to check that for each edge, say for $e = \{i,j\}$, the term $(x_i + x_j)$ occurs in two of the linear combinations. Thus $Q_E \in W^K_{E,3}$, where $K(e) = 2$ for each edge e of T.

A path of length k in G connecting i and j is a sequence of distinct vertices $P = (i_0, i_1, \ldots, i_k)$ such that $i_0 = i$, $i_k = j$ and $\{i_l, i_{l+1}\} \in E$ for $l = 0, 1, \ldots, k-1$.

Definition 8. Assume $G = (V,E)$ is a graph. A *path covering family* of G is a family of paths

$$\mathcal{P} = \{P_e : e \in E\},$$

where for each edge $e = \{i,j\} \in E$, P_e is an even length path connecting i and j.

For a subgraph H of G, $K_H \in \mathbb{N}^E$ is the characteristic function of $E(H)$, i.e., $K_H(e) = 1$ if $e \in E(H)$ and $K_H(e) = 0$ otherwise. For a family $\mathcal{F}$ of subgraphs of G,

$$K_{\mathcal{F}} = \sum_{H \in \mathcal{F}} K_H.$$

Observe that if $F_i \in W^{K_i}_{E,m_i}$ for $i = 1, 2, \ldots, t$, then $\prod_{i=1}^t F_i \in W^{\sum_{i=1}^t K_i}_{E,\sum_{i=1}^t m_i}$.

Lemma 9. *If G has a path covering family $\mathcal{P}$ with $K_{\mathcal{P}}(e) \leqslant b$ for each edge e, then G is $(1, b+1)$-choosable.*

Proof. Assume $\mathcal{P}$ is a path covering family with $K_{\mathcal{P}}(e) \leqslant b$ for each edge e. For each edge $e = \{i,j\}$ of G, let $P_e = (i_0, i_1, \ldots, i_{2k_e})$ be the even length path in $\mathcal{P}$ connecting i and j, i.e., $i_0 = i$ and $i_{2k_e} = j$. Then

$$x_i - x_j = \sum_{l=0}^{2k_e - 1} (-1)^l (x_{i_l} + x_{i_{l+1}}) \in W^{K_{P_e}}_{E,1}.$$

Hence

$$Q_E = \prod_{\{i,j\} \in E} (x_i - x_j) \in W^{K_{\mathcal{P}}}_{E,m}.$$

Since $K_{\mathcal{P}}(e) \leqslant b$ for each edge e, we have $Q_E \in W^K_{E,m}$ for some $K \in \mathbb{N}^E_{(b^-)}$. By Lemma 7, G is $(1, b+1)$-choosable. □

The following lemma follows easily from the definitions and its proof is omitted.

Lemma 10. *If G decomposes into graphs $H_1, H_2, \dots, H_q$, and each H_i has a path covering family $\mathcal{P}_i$ with $F_{\mathcal{P}_i} \in W^{K_i}_{E(H_i),m_i}$ and $K_i \in \mathbb{N}^{E(H_i)}_{(b^-)}$, then $\mathcal{P} = \cup_{i=1}^q \mathcal{P}_i$ is a path covering family of G and $K_{\mathcal{P}} \in W^K_E$ for $K = \sum_{i=1}^q K_i \in \mathbb{N}^E_{(b^-)}$.* □

Proof of Theorem 1. By Lemmas 9 and 10, it suffices to show that each complete graph K_n of odd order has a path covering family $\mathcal{P}$ with $K_{\mathcal{P}} \in \mathbb{N}^E_{(2^-)}$. Assume K_n has vertex set $\{1, 2, \dots, n\}$.

Put the n vertices $\{1, 2, \dots, n\}$ of K_n equally spaced along the perimeter of a circle C. For an edge $e = \{i, j\}$ of K_n, denote by $[i, j]$ the interval of C from i to j along the clockwise direction (containing both i and j). Since n is odd, exactly one of $[i, j]$ and $[j, i]$ contains an odd number of vertices of K_n. Let $t_{i,j}$ be the vertex that is in the center of the interval $[i, j]$ or $[j, i]$ that contains an odd number of vertices, and let $P_e = (i, t_{i,j}, j)$. Then $\mathcal{P} = \{P_e : e \in E(K_n)\}$ is a path covering family of K_n. For each edge $e = \{i, j\}$ of K_n, let $a_e = \{i, 2j - i\}$ and $b_e = \{j, 2i - j\}$ (where calculations are modulo n). It is easy to verify that e is contained in $P_{e'}$ if and only if $e' \in \{a_e, b_e\}$. So each edge of K_n is contained in two paths in $\mathcal{P}$, i.e., $K_{\mathcal{P}}(e) = 2$ for each edge e of K_n. This completes the proof of Theorem 1. □

For a graph G, let $\gcd(G)$ be the largest integer dividing the degree of every vertex of G. We say that G is F-divisible if $|E(G)|$ is divisible by $|E(F)|$ and $\gcd(G)$ is divisible by $\gcd(F)$.

The following result was proved in [4]:

Theorem 11. *For every $\epsilon > 0$, there is an integer n_0 such that if G is a triangle-divisible graph of order $n \geqslant n_0$ and minimum degree at least $(0.9 + \epsilon)n$, then G has a triangle decomposition.*

Proof of Theorem 2. Assume G is an n-vertex Eulerian graph of minimum degree $\delta(G) > (0.9 + \epsilon)n$ with large enough n. By Theorem 1, it suffices to show that G decomposes into complete graphs of odd order.

Assume $|E(G)| \equiv i \pmod 3$, where $i \in \{0, 1, 2\}$. Let $H_1, \dots, H_i$ be vertex disjoint 5-cliques in G. Then $G' = G - \cup_{j=1}^i E(H_j)$ is triangle divisible and $\delta(G') \geqslant \delta(G) - 4 \geqslant (0.9 + \epsilon')n$. By Theorem 11, G' is triangle decomposible. Hence G decomposes into complete graphs of odd order. This completes the proof of Theorem 2. □

Lemma 12. *Let $H = (V, E)$ be the graph shown in Figure 1. Then H has a path covering family $\mathcal{P}$ with $K_{\mathcal{P}} \in \mathbb{N}^E_{(3^-)}$.*

Proof. We denote by $T_1 = (1,2,4)$, $T_2 = (2,3,5)$ the two edge disjoint triangles in H. For each triangle T_i, let $\mathcal{P}_i$ be the path covering family with $K_{\mathcal{P}_i} \in \mathbb{N}^{E(T_i)}_{(2^-)}$. For the edge $e = \{1,3\}$ which is not contained in the two triangles, let $P_e = (1,2,3)$. Then

$$\mathcal{P} = \cup_{i=1}^{2} \mathcal{P}_i \cup \{P_e\}$$

is a path covering family of H with $K_{\mathcal{P}} \in \mathbb{N}^{E}_{(3^-)}$. This completes the proof of Lemma 12. □

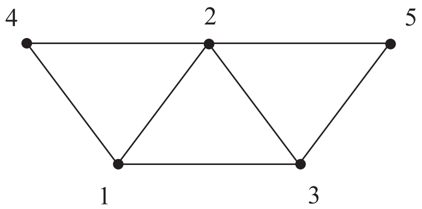

Figure 1: The graph H.

The following theorem was proved in [4]:

Theorem 13. *For every $\epsilon > 0$, there is an integer n_0 such that if G is an H-divisible graph of order $n \geqslant n_0$ and minimum degree at least $(1 - 1/t + \epsilon)n$, where $t = \max\{16\chi(H)^2(\chi(H)-1)^2, |E(H)|\}$, then G has an H-decomposition.*

Proof of Theorem 3. Assume G is a graph of sufficiently large order and with minimum degree $\delta(G) \geqslant 0.999|V(G)|$. If $|E(H)|$ divides $|E(G)|$, then G decomposes into copies of H and Theorem 3 follows from Lemma 9. Otherwise, the same argument as in the proof of Theorem 2 shows that G can be decomposed into at most 12 copies of triangles and copies of H, and hence again Theorem 3 follows from Lemma 9. □

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