On Nilpotent Orientably-Regular Maps of Nilpotency Class 4

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Abstract

By a *nilpotent map* we mean an orientably regular map whose orientation preserving automorphism group is nilpotent. The nilpotent maps are concluded to the maps whose automorphism group is a 2-group and a complete classification of nilpotent maps of (nilpotency) class 2 is given by Malnič *et al.* in [European J. Combin. 33 (2012), 1974–1986]. It is proved by Conder *et al.* in [J. Algebraic Combin. 44 (2016), 863–874] that given the class, there are finitely many simple nilpotent maps. However, for the nilpotent maps with multiple edges and given class, since its automorphism group may be infinitely big, it is impossible to list it by a computer. Therefore, to classify the nilpotent maps with small class c is necessary and interesting. In this paper, the nilpotent maps of class 4 will be determined.

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1 Introduction

A (topological) *map* is a cellular decomposition of a closed surface. A common way to describe maps is to consider them as cellular embeddings of graphs into closed surfaces. By an *automorphism* of a map we mean an automorphism of the underlying graph which extends to a self-homeomorphism of the surface, preserving incidence of vertices and edges and faces (open 2-cells) of the map. These automorphisms form a subgroup $\mathrm{Aut}\,(\mathcal{M}) \leqslant \mathrm{Aut}\,(\mathcal{G})$ of the automorphism group of the underlying graph $\mathcal{G}$. If $\mathrm{Aut}\,(\mathcal{M})$ acts regularly on the flags (vertex-edge-face incident triples), we call the map $\mathcal{M}$ is *regular*. For an orientable map $\mathcal{M}$, if its orientation persevering automorphism group $\mathrm{Aut}^{\,+}(\mathcal{M})$ acts regularly the arcs, then we call the map as well as the corresponding embedding of the underlying graph *orientably regular*.

In this paper we study the problem of the classification of regular maps with a given automorphism group. The research results on this problem are relatively less compared with classifying regular maps on a given surface, or with a given underlying graph. Now, complete classification results about regular maps with given families of solvable groups are very few, except for a folklore result classifying regular maps whose automorphism group is abelian, most of them are related to simple groups, such as $\mathrm{PSL}(2, q)$ [1, 7, 10, 11, 12, 17], Hurwitz groups or non-Hurwitz groups, including symmetric and alternating groups [4, 22], Suzuki groups [15], Ree groups [13, 20], and various sporadic simple groups, see [5] for a survey.

Here we focus on orientably regular maps whose orientation preserving automorphism group is nilpotent. We call these *nilpotent maps*. The study of such maps was initiated in [18] by Malnič *et al.*, where it is showed that every nilpotent map can be uniquely decomposed into a direct product of two regular maps: the automorphism group of one is a 2-group and the other map is a single vertex and an odd number of semiedges, see [18, Theorem 3.2]. They also gave a complete classification of nilpotent regular maps of nilpotency class 2. It is proved in [6] that given the class, there are finitely many simple nilpotent maps. However, for the nilpotent maps with multiple edges and given class, since its automorphism group may be infinitely big, it is impossible to list it by a computer. This kind of maps are very rich, and the classification of those of higher classes seems to be difficult, where one may feel the difficulties from the classification of nilpotent regular embeddings of the complete bipartite graphs $K_{n,n}$ and the n-dimensional hypercube Q_n, when n is a power of 2, see [8, 9, 16]. To determine the nilpotent maps, it has to be involved in complicate and difficult computation in 2-group theory. Since it is infeasible to classify all the nilpotent maps for any class c, it woluld be necessary and interesting to fix the cases for small class c. The nilpotent maps of class 3 have been determined by Ban *et al.* (unpublished). In the present paper, the nilpotent maps of class 4 will be determined.

Necessarily, we are having to introduce the following concepts and notations: by $|G|$ and $|g|$, we denote the order of a group G and an element $g \in G$, respectively. Set $[x, y] = x^{-1}y^{-1}xy$ for $x, y \in G$ and $[H, K] = \langle [h, k] \mid h \in H, k \in K\rangle$. Moreover, we use G', $\exp(G)$ and $\Phi(G)$ to denote the derived subgroup of G, the exponent of G and the

Frattini subgroup of G, respectively.

Next, for any group G, set $G_1 = G$ and then define $G_{n+1} = [G, G_n]$, for all $n \geqslant 1$, so that $G = G_1 \geqslant G_2 \geqslant G_3 \geqslant \cdots \geqslant G_n \cdots$ is the *lower central series* of G, with each G_n characteristic in G, where $G' = G_2$.

The group G is said to be *nilpotent* if G_{c+1} is trivial for some integer c, and then the smallest c for which this happens is called the *nilpotency class* of G. Note that $G_c \leqslant Z(G)$.

As usual, an orientably regular map will be presented by a triple $(G; a, b)$ for a group G generated by an element a and an involution b. To state the main theorem, let $G = \langle a, b\rangle$ be a two-generated 2-group of nilpotency class 4 where $b^2 = 1$. Since $c(G) = 4$, we have $G_4 \leqslant Z(G)$. Now, G is an extension of $\langle G', b\rangle$ by a, and $\langle G', b\rangle$ is an extension of G' by b, we may set

$$G = \langle a, b \mid R, T, a^{2^n} = x\rangle, \tag{1}$$

where

$$\begin{aligned} R = \ & \{b^2 = 1, [a, b] = c, [c, a] = d, [c, b] = e, [d, a] = u, [d, b] = v, [e, a] = w, \\ & [e, b] = z, \{u, v, w, z\} \text{ commute with both } a \text{ and } b\} \end{aligned}$$

and T is the set of defining relations of $G' = \langle c, d, e, u, v, w, z\rangle$ and $x \in \langle G', b\rangle$. Note that $G_3 = \langle d, e, u, v, w, z\rangle$ and $G_4 = \langle u, v, w, z\rangle \leqslant Z(G)$.

Theorem 1. *Let $\mathcal{M}$ be an orientably regular map whose automorphism group G is of nilpotency class 4. Then G is defined by Eq(1). Moreover, we have*

(1) *$G' = \langle c, d, e, u, v, w, z \mid T\rangle = \langle c, d, u\rangle$ is an abelian group, where $8 \leqslant |G'| \leqslant 64$ and T contains a subset of relations*

$$T_1 := \{[c, d] = [c, u] = [d, u] = 1, c^8 = d^4 = u^2 = 1, e = c^{-2}, w = v = d^2, z = c^4\}$$

so that $T = T_1 \cup T_2$ for a subset T_2 of relations. Moreover, $x \in G'$.

(2) *For the above R and T_1, the group G is uniquely determined by T_2 and $x \in G'$, which will be listed as follows: For every such group G, there exist at most two nonisomorphic maps $\mathcal{M}(G; ab^i, b)$, where either $i = 0$ or $i \in \{0, 1\}$, see below. Moreover, these maps are uniquely determined by given parameters.*

(I) $G' \cong \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$:
$G_1(n, \ell, s, t) : T_2 = \varnothing$; $x = c^{4\ell}d^{2s}u^t$, *where* $(\ell, s, t) = \mathbb{Z}_2^3 \setminus \{(1, 0, 0), (1, 1, 1)\}$ *for* $n = 3$, *and* $(\ell, s, t) = \mathbb{Z}_2^3 \setminus \{(1, 1, 0), (1, 0, 1)\}$ *for* $n \geqslant 4$; $i = 0, 1$ *for* $G_1(3, 0, 0, 0)$, $G_1(3, 0, 1, 1)$, $G_1(n, 0, 1, 0)$ $(n \geqslant 4)$, $G_1(n, 0, 0, 1)$ $(n \geqslant 4)$, *and* $i = 0$ *for otherwise.*

(II) $G' \cong \mathbb{Z}_8 \times \mathbb{Z}_4$:
(1) $G_{21}(n, \ell, s) : T_2 = \{u = 1\}$; $x = c^{4\ell}d^{2s}$, *where* $n \geqslant 3$, $\ell, s \in \{0, 1\}$; $i = 0, 1$.

(2) $G_{22}(n,\ell,s): T_2=\{u=d^2\}$; $x=c^{-2\ell}d^s$, *where* $(\ell,s)=(-1,1),(-1,-1)$, $(1,-1)$ *for* $n=2$, $(2,0),(2,2),(0,2)$ *for* $n=3$, *and* $(0,0)$, $(2,0),(0,2)$ *for* $n\geqslant 4$; $i=0,1$ *for* $G_{22}\cong G_{22}(2,-1,1)$, $G_{22}(3,2,0)$, $G_{22}(n,0,2)(n\geqslant 4)$, *and* $i=0$ *for otherwise.*

(3) $G_{23}(n,\ell,s): T_2=\{u=d^2c^4\}$; $x=u^\ell d^{2s}$, *where* $(\ell,s)=(0,0)$,$(0,1)$, $(1,0)$ *for* $n=3$, *and* $(0,0),(0,1),(1,1)$ *for* $n\geqslant 4$; $i=0,1$ *for* $G_{23}(3,0,0)$, $G_{23}(n,0,1)(n\geqslant 4)$, *and* $i=0$ *for otherwise.*

(III) $G'\cong \mathbb{Z}_8\times\mathbb{Z}_2\times\mathbb{Z}_2$:
$G_3(n,s,t): T_2=\{d^2c^4=1\}$; $x=d^{2s}u^t$, *where* $n\geqslant 2$ *and* $s,t\in\{0,1\}$; $i=0,1$.

(IV) $G'\cong \mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_2$:
$G_4(n,s,t): T_2=\{c^4=1\}$; $x=d^{2s}u^t$, *where* $n\geqslant 3$ *and* $s,t\in\{0,1\}$; $i=0$.

(V) $G'\cong \mathbb{Z}_8\times\mathbb{Z}_2$:

(1) $G_{51}(n,\ell,t): T_2=\{uc^4=d^2=1\}$; $x=c^{-2\ell}u^t$, *where* $(\ell,t)=(-1,0),(1,0)$ *for* $n=2$, *and* $(0,0),(0,1)$ *for* $n\geqslant 3$; $i=0,1$.

(2) $G_{52}(n,\ell,s): T_2=\{u=d^2=1\}$; $x=c^{-2\ell}d^s$, *where* $(\ell,s)=(-1,0),(-1,1)$, $(1,0)$, $(1,1)$ *for* $n=2$, *and* $(0,0),(0,1),(2,0),(2,1)$ *for* $n\geqslant 3$; $i=0,1$.

(VI) $G'\cong \mathbb{Z}_4\times\mathbb{Z}_4$:

(1) $G_{61}\cong G_{61}(n,s): T_2=\{u=c^4=1\}$; $x=d^s$, *where* $s=1$ *for* $n=2$, *and* $s=0,2$ *for* $n\geqslant 3$; $i=0,1$ *for* $G_{61}(2,1)$, *and* $i=0$ *for otherwise,*

(2) $G_{62}\cong G_{62}(n,\ell,s): T_2=\{ud^2=c^4=1\}$; $x=c^{-2\ell}d^s$, *where* $(\ell,s)=(-1,1),(-1,-1)$ *for* $n=2$, *and* $(\ell,s)=(0,0),(0,2)$ *for* $n\geqslant 3$; $i=0$.

(VII) $G'\cong \mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_2$:
$G_7(n,\ell,t): T_2=\{d^2=c^4=1\}$; $x=c^{-2\ell}u^t$, *where* $(\ell,t)=(0,0),(0,1)$ *for* $n=2$, *and* $(\ell,t)=(0,0),(0,1),(-1,0),(-1,1)$ *for* $n\geqslant 3$; $i=0,1$ *for* $G_7(2,0,0),G_7(2,0,1)$, *and* $i=0$ *for otherwise.*

(VIII) $G'\cong \mathbb{Z}_8$:

(1) $G_{81}(n,\ell): T_2=\{u=d=1\}; x=c^\ell$, *where* $\ell=-1,3$ *for* $n=1$, *and* $\ell=2,-2$ *for* $n=2$, *and* $\ell=0,4$, *for* $n\geqslant 3$; $i=0,1$.

(2) $G_{82}(n,\ell): T_2=\{u=1,c^4=d\}$; $x=c^{-2\ell}$, *where* $\ell=1,-1$ *for* $n=2$, *and* $\ell=0,2$ *for* $n\geqslant 3$; $i=0,1$.

(IX) $G'\cong \mathbb{Z}_4\times\mathbb{Z}_2$:
$G_9(n,t): T_2=\{d^2=c^4=1,u=c^2\}$; $x=u^t$, *where* $n\geqslant 2$ *and* $t=0,1$; $i=0$.

(X) $G'\cong \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$:
$G_{10}(n,t): T_2=\{d^2=c^2=1\}$; $x=u^t$, *where* $t=1$ *for* $n=2$, *and* $t=0,1$ *for* $n\geqslant 3$; $i=0,1$ *for* $G_{10}(2,1)$, *and* $i=0$ *for otherwise.*

Remark 2. Remind that all the maps in here are quotients of that in Case (I). Moreover, we may pick up all simple maps (with no multiple edges) of class 4 from Theorem 1, which coincide with that in [6]:

(1) $|G| = 32 : \mathcal{M}(G_{81}(1,-1); ab, b))$;

(2) $|G| = 64 : \mathcal{M}(G_{82}(2,1); ab, b), \mathcal{M}(G_9(2,0); a, b), \mathcal{M}(G_{10}(2,1); ab, b)$;

(3) $|G| = 128 : \mathcal{M}(G_{51}(2,-1,0); ab, b), \mathcal{M}(G_{52}(2,1,0); ab, b),\ \mathcal{M}(G_7(2,0,0); a, b)$, $\mathcal{M}(G_{81}(3,0); a, b),\quad \mathcal{M}(G_{82}(3,0); a, b)$;

(4) $|G| = 256 : \mathcal{M}(G_3(2,0,0); a, b), \mathcal{M}(G_{51}(3,0,0); a, b), \mathcal{M}(G_{52}(3,0,0); a, b)$, $\mathcal{M}(G_{61}(3,0); a, b),\ \mathcal{M}(G_{62}(3,0,0); a, b)$;

(5) $|G| = 512 : \mathcal{M}(G_{21}(3,0,0); a, b), \mathcal{M}(G_{21}(3,1,0); ab, b), \mathcal{M}(G_{22}(3,2,0); ab, b)$, $\mathcal{M}(G_{23}(3,0,0); a, b),\ \mathcal{M}(G_3(3,1,0); ab, b), \mathcal{M}(G_4(3,0,0); a, b)$;

(6) $|G| = 1024 : \mathcal{M}(G_1(3,0,0,0); a, b)$.

2 Algebraic maps

An orientable map is a triple $\mathcal{M} = (D; R, L)$ where D is a finite nonempty set of darts, R and L are permutations of D such that $L^2 = \mathrm{id}$, and the permutation group $\langle R, L\rangle$ acts transitively on D. The permutations R and L are called the *rotation* and the *dart-reversing* involution of $\mathcal{M}$, respectively, and the group $\langle R, L\rangle = \mathrm{Mon}(\mathcal{M})$ is the *monodromy group* of $\mathcal{M}$. Each map $\mathcal{M}$ has its underlying graph $K_{\mathcal{M}}$ whose vertices are the orbits of R and whose edges are the orbits of L, with incidence between vertices and edges defined by nonempty intersection. Since $\mathrm{Mon}(\mathcal{M})$ is transitive on D, the graph $K_{\mathcal{M}}$ is connected.

Given two maps $\mathcal{M} = (D; R, L)$ and $\mathcal{M}' = (D'; R', L')$. An orientation preserving homomorphism $\phi : \mathcal{M} \to \mathcal{M}'$ is a mapping $\phi : D \to D'$ such that $\phi R = R'\phi$ and $\phi L = L'\phi$. Since the monodromy groups of both maps are transitive on darts, it follows that every map homomorphism is surjective. As usual, an isomorphism between two maps is a bijective homomorphism, and an automorphism is a self-isomorphism of a map. All automorphisms of a map form the automorphism group $\mathrm{Aut}^+(\mathcal{M})$. By definition, it is easy to see that $\mathrm{Aut}^+(\mathcal{M}) \leqslant C_{S_D}(\mathrm{Mon}(\mathcal{M}))$. As $\mathrm{Mon}(\mathcal{M})$ acts transitively D, we get $\mathrm{Aut}^+(\mathcal{M})$ acts semi-regularly on D. If the action is regular, then the map $\mathcal{M}$ is called *orientably regular*. As a consequence of some well-known results in a permutation group theory (see [3]), we infer that for an orientably regular map $\mathcal{M}$, $\mathrm{Mon}(\mathcal{M}) \cong \mathrm{Aut}^+(\mathcal{M})$ although their action on dart-set are different.

Given a group $G = \langle r, \ell\rangle$, where $\ell^2 = 1$, from the above arguments, we may deduce an algebraic map$\mathcal{M}(G; r, \ell)$ as follows: Set $D = G$ and consider the left multiplication action $L(G)$ of G on D. The vertices, edges and faces are just cosets of $\langle r\rangle$, $\langle \ell\rangle$ and $\langle r\ell\rangle$, respectively, with the natural intersection relation. Moreover, $\mathrm{Mon}(\mathcal{M}) = L(G)$ and $\mathrm{Aut}^+(\mathcal{M}) = R(G)$.

It is a matter of routine to check that every regular embedding of a graph can be described by an algebraic map (see [14, 19]), and that two such algebraic maps $\mathcal{M}(G; r_1, \ell_1)$ and $\mathcal{M}'(G; r_2, \ell_2)$ are isomorphic if and only if there exists an automorphism $\sigma \in \mathrm{Aut}\,(G)$ such that $r_1^\sigma = r_2$ and $\ell_1^\sigma = \ell_2$.

The following proposition gives some formulas about meta-abelian groups (G' is abelian) (see [23]).

Proposition 3. *Let G be a meta-abelian group. For any $a, b, c \in G$, the following facts hold:*

(1) $[a, b^{-1}, c]^b = [b, a, c]$;

(2) $[a, b, c][b, c, a][c, a, b] = 1$;

(3) *if* $b \in G'$, *then* $[ab, c] = [a, c][b, c]$ *and* $[c, ab] = [c, a][c, b]$;

(4) *if* $c \in G'$ *and* $n \in \mathbb{Z}$, *then* $[c^n, a] = [c, a]^n$ *and* $[c, a, b] = [c, b, a]$;

3 Properties of the group G

In this section, we derive some properties and commutator formulas for G, and in particular, we prove Theorem 1(1).

Lemma 4 (Theorem 1(1))**.** *For the group G of* Eq(1), *we have*

$$G' = \langle c, d, e, u, v, w, z \mid T \rangle = \langle c, d, u \rangle,$$

which is an abelian group such that $8 \leqslant |G'| \leqslant 64$ and T contains a subset

$$T_1 := \{[c, d] = [c, u] = [d, u] = 1, c^8 = d^4 = u^2 = 1, e = c^{-2}, w = v = d^2, z = c^4\}.$$

Moreover, $x \in G'$.

Proof. Recall that $G = \langle a, b \mid R, T, a^{2^n} = x \rangle$, as shown in Eq(1), where

$$[a, b] = c, [c, a] = d, [c, b] = e, [d, a] = u, [d, b] = v, [e, a] = w, [e, b] = z$$

and $G' = \langle c, d, e, u, v, w, z \rangle$. Since $G' = \langle c, G_3 \rangle$ and $[G_2, G_3] \leqslant G_5 = 1$, it follows that G' is abelian, that is, G is meta-abelian. Then by Proposition 3 we have

$$[a, b, c][b, c, a][c, a, b] = 1.$$

Noting $[a, b, c] = 1$, we get $[b, c, a][c, a, b] = 1$, that is $v = w$, as desired. From

$$1 = [a, b^2] = [a, b][a, b]^b = cc^b = c^2[c, b] = c^2 e,$$

we get $c^2 = e^{-1}$. Similarly, from $[c, b^2] = [d, b^2] = [e, b^2] = 1$, one may deduce $e^2 = z^{-1}$ and $v^2 = z^2 = 1$. Again, by $[G_2, G_3] \leqslant G_5 = 1$, we have

$$w = [e, a] = [c^{-2}, a] = [c^{-1}, a]^{c^{-1}}[c^{-1}, a] = [c^{-1}, a]^2 = d^{-2}.$$

Thus, $d^2 = w^{-1} = w = v$, which in further concludes $1 = [w, a] = [d^2, a] = [d, a]^2 = u^2$. Finally, $c^8 = e^{-4} = z^2 = 1$ and $d^4 = w^2 = 1$. Therefore, T contains T_1. Clearly, $G' = \langle c, d, u \rangle$, whose order is at most 64.

Note that $G = \langle b, G' \rangle \langle a \rangle$. Now, suppose that $a^{2^n} = bx$, for some integer n and $x \in G'$. Then $bG' \in \langle aG' \rangle$ and so $G/G' = \langle aG', bG' \rangle = \langle aG' \rangle$, a contradiction, as G (and so $G/\Phi(G)$) is a 2-generated group. Hence, $a^{2^n} = x \in G'$. □

By computation, one may get the following two lemmas:

Lemma 5. *Let G be the group defined in* Eq(1). *For any integers h and ℓ, we have*

$$\begin{array}{ll} [g_1^h, g_2] = [g_1, g_2]^h,\ \forall g_1 \in G', g_2 \in G, & \\ [c^\ell, a^h] = d^{h\ell} u^{\frac{h\ell(h-1)}{2}}, & [a^h, b] = c^h d^{\frac{h(h-1)}{2}} u^{\frac{h(h-1)(h-2)}{6}}, \\ [d, a^h] = u^h, & [e, a^h] = v^h. \end{array}$$

Lemma 6. *Let G be the group defined in* Eq(1). *For any integers h, ℓ, m and j, we have*

(1) $(a^h c^\ell d^m u^j)^2 = a^{2h} c^{2\ell} d^{h\ell+2m} u^{\frac{\ell h(h-1)}{2}+hm}$,

(2) $(a^h b c^\ell d^m u^j)^2 = a^{2h} c^h d^{3h\ell+\frac{h(h-1)}{2}} u^{mh+\frac{\ell h(h-1)}{2}+\frac{h(h-1)(h-2)}{6}}$,

(3) $(a^h c^\ell d^m u^j)^4 = a^{4h} c^{4\ell} d^{2h\ell}$,

(4) $(a^h b c^\ell d^m u^j)^4 = a^{4h} c^{2h} u^{h^2} d^{3h^2-h+2h\ell}$,

(5) $(a^h c^\ell d^m u^j)^8 = a^{8h}$,

(6) $(a^h b c^\ell d^m u^j)^8 = a^{8h} c^{4h} d^{2h(h-1)}$,

(7) $(a^h b^i c^\ell d^m u^j)^{16} = a^{16h}$ *for $i = 0$ or 1.*

The following lemma determines the derived group G'.

Lemma 7. *Let $G = \langle a, b\rangle$ be a two-generated 2-group of nilpotency class 4 where $b^2 = 1$, as defined in* Eq(1). *Then G' is one of the following ten groups:*

(I) $G' = \langle c\rangle \times \langle d\rangle \times \langle u\rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$ *with* $\langle u, v, z\rangle \cong \mathbb{Z}_2^3$;

(II) $G' = \langle c\rangle \times \langle d\rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_4$ *with* $\langle u, v, z\rangle \cong \mathbb{Z}_2^2$, *where* $u = 1$; $u = v$; $u = z$; *or* $uvz = 1$;

(III) $G' = \langle c, d\rangle \times \langle u\rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *with* $\langle u, v, z\rangle \cong \mathbb{Z}_2^2$, *either* $v = 1$ *or* $v = z$;

(IV) $G' = \langle c\rangle \times \langle d\rangle \times \langle u\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$ *with* $\langle u, v, z\rangle \cong \mathbb{Z}_2^2$, *where* $z = 1$;

(V) $G' = \langle c, d\rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_2$ *with* $\langle u, v, z\rangle \cong \mathbb{Z}_2$, *where* $uz = v = 1$; $vz = u = 1$; $u = v = 1$; *or* $u = v = z$;

(VI) $G' = \langle c\rangle \times \langle d\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4$ *with* $\langle u, v, z\rangle \cong \mathbb{Z}_2$, *either* $u = z = 1$ *or* $uv = z = 1$;

(VII) $G' = \langle c\rangle \times \langle d\rangle \times \langle u\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *with* $\langle u, v, z\rangle \cong \mathbb{Z}_2$, *where* $v = z = 1$;

(VIII) $G' = \langle c\rangle \cong \mathbb{Z}_8$ *with* $\langle u, v, z\rangle \cong \mathbb{Z}_2$, *where* $u = v = d = 1$; $u = v = 1$, $d = z$; $u = v = z$, $d = e^{-1}$; *or* $u = v = z$, $d = e$;

(IX) $G' = \langle c\rangle \times \langle d\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ *with* $\langle u, v, z\rangle \cong \mathbb{Z}_2$, *where* $v = z = eu = 1$;

(X) $G' = \langle c\rangle \times \langle d\rangle \times \langle u\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *with* $\langle u, v, z\rangle \cong \mathbb{Z}_2$, *where* $v = z = e = 1$.

Proof. By Lemma 4, we know that $G' = \langle c, d, u\rangle$ is an abelian group and $w = v$ so that $G_4 = \langle u, v, z\rangle \cong \mathbb{Z}_2^3$, $\mathbb{Z}_2^2$ or $\mathbb{Z}_2$. In what follows, we deal with them, separately.

Case 1: $\langle u, v, z\rangle \cong \mathbb{Z}_2^3$. In this case, $G' = \langle c\rangle \times \langle d\rangle \times \langle u\rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$, that is the case (I) in the lemma.

Case 2: $\langle u, v, z\rangle \cong \mathbb{Z}_2^2$. In this case, there exists only one triple $(i, j, k) \neq (0, 0, 0)$ such that $u^i v^j z^k = 1$. So we have totally the following seven possibilities:

$$u = 1; u = v; u = z; uvz = 1; v = 1; v = z; z = 1.$$

Take into account,

$$e = c^{-2}, \quad w = v = d^2, \quad z = c^{-4}.$$

Suppose that $u = 1$; $u = v$; $u = z$ or $uvz = 1$. Then $u \in \langle v, z\rangle$ and so $\mathbb{Z}_2^2 \cong G_4 = \langle v, z\rangle \leqslant \langle c, d\rangle$. Thus $|c| = 8$, $|d| = 4$ and $\langle c\rangle \cap \langle d\rangle = 1$, otherwise, $z = c^{-4} = d^2 = v$. Therefore, $G' = \langle c\rangle \times \langle d\rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_4$, that is Case (II) in the lemma.

Suppose that $v = 1$ or $v = z$. Then $v \in \langle u, z\rangle$ and so $\mathbb{Z}_2^2 \cong G_4 = \langle u, z\rangle$. Now $|c| = 8$, $|u| = 2$, $d \neq 1$ (as $u = [d, a]$) and $\langle c\rangle \cap \langle u\rangle = 1$. Suppose that $v = 1$. Then $|d| = 2$. It is obvious that $d \in G_3 \setminus G_4$. Since $\Omega_1(\langle c\rangle \times \langle u\rangle) = \langle z, u\rangle = G_4$, we get $d \notin \langle c\rangle \times \langle u\rangle$, and so $G' = \langle c\rangle \times \langle d\rangle \times \langle u\rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, that is Case (III) in the lemma. Suppose that $v = z$. Then $|d| = 4$, $(dc^2)^2 = vz = 1$ and $dc^2 \notin \langle c\rangle \times \langle u\rangle$, which implies $G' = \langle c\rangle \times \langle dc^2\rangle \times \langle u\rangle$. In both cases, $G' \cong \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, that is Case (III) in the lemma.

Suppose that $z = 1$. Then $G_4 = \langle u, v\rangle$ and so $|c| = |d| = 4$, $|u| = 2$ and $\langle u\rangle \cap \langle d\rangle = 1$. Clearly $c \in G_2 \setminus G_3$. Therefore, $G' = \langle c\rangle \times \langle d\rangle \times \langle u\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$, which is Case (IV) in the lemma.

Case 3: $\langle u, v, z\rangle \cong \mathbb{Z}_2$. In this case, there exists only one subgroup of order 4 in $\mathbb{Z}_2^3$, each of whose nontrivial element (i, j, k) satisfies $u^i v^j z^k = 1$. So we have the following seven possibilities:

$$uz = v = 1; vz = u = 1; u = v = 1; uv = vz = 1; u = z = 1; uv = z = 1; v = z = 1.$$

Suppose $uz = v = 1$. Then $u = z = c^4 \neq 1$, $d^2 = 1$ ($d \neq 1$, as $u = [d, a]$)). Clearly, $c \in G_2 \setminus G_3$, which implies $d \notin \langle c\rangle$. Therefore, $G' = \langle c\rangle \times \langle d\rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_2$, that is Case (V) in the lemma.

Suppose $vz = u = 1$. Then $d^2 = v = z = c^{-4} \neq 1$, which implies $|d| = 4$, $|dc^2| = 2$, $|c| = 8$, $d \notin \langle c\rangle$. Therefore, $G' = \langle c\rangle \times \langle c^2 d\rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_2$, that is Case (V) in the lemma.

Suppose $u = v = 1$. Then $c^{-4} = z \neq 1$, which implies $|c| = 8$; $d^2 = v = 1$; and $G' = \langle c, d\rangle$. Now we have two cases: (i) If $d \in \langle c\rangle$, that is $d \in \{1, z\}$, then

$G' = \langle c\rangle \cong \mathbb{Z}_8$, that is the case (VIII) in the lemma; and (ii) if $d \notin \langle c\rangle$, then $G' = \langle c\rangle \times \langle d\rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_2$, that is the Case (V) in the lemma.

Suppose $uv = vz = 1$. Then $u = d^2 = e^2 = c^4$ and $G' = \langle c, d\rangle$, where $|c| = 8$, $d^2 \in \langle c\rangle$ and $|d| = 4$. Moreover, if $d \notin \langle c\rangle$ (that is $d \neq e^{\pm 1}$), then $G' = \langle c\rangle \times \langle c^2 d\rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_2$, that is the Case (V) of the lemma; and if $d = e^{\pm 1}$ then $G' = \langle c\rangle \cong \mathbb{Z}_8$, that is the Case (VIII) in the lemma.

Suppose $u = z = 1$ or $uv = z = 1$. Then $|d| = 4$ (as $v \neq 1$), $c^4 = z = 1$ and $G' = \langle c, d\rangle$. Moreover, we have $c^2 \neq 1$. Otherwise, $1 = [c^2, a] = [c, a]^2 = d^2$. Hence, $|c| = 4$. Furthermore, since $[d^2, a] = [d, a]^2 = u^2 = 1$ and $[c^2, a] = [e, a] = w = v \neq 1$, it follow that $d^2 \neq c^2$ and so $\langle c\rangle \cap \langle d\rangle = 1$. Therefore, $G' = \langle c\rangle \times \langle d\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4$, that is case (VI) in the lemma.

Finally, suppose $v = z = 1$. Then $u = [d, a] \neq 1$, which implied $|d| = 2$ and $d \neq u$; $c^4 = 1$ and $G' = \langle c, \langle d\rangle \times \langle u\rangle\rangle$. Since $c \notin G_3$, we have $c \notin \langle d\rangle \times \langle u\rangle$ and so we have three possibilities: $G' = \langle c\rangle \times \langle d\rangle \times \langle u\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2^3$, that is Cases (VII) and (X) in the lemma, respectively; and $c^2 = d^i u^j$ for some $i, j \in \mathbb{Z}_2$. For the third case, since $[d, a] = u \neq 1$, $[c^2, a] = [e, a] = w = v = 1$ but $[du^i, a] = u \neq 1$, we get $c^2 \neq d, du$ and so $c^2 = u$, which implies $G' = \langle c\rangle \times \langle d\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, that is Case (IX) in the lemma. The case $v = z = 1$ is also considered by Miss Liu (unpublished). □

The following lemma gives the sufficient and necessary condition of the existence of the group G.

Lemma 8. *The group G of* Eq(1) *exists if and only if*

$$d^\ell u^s = 1,\ d^{2^n} u^{2^{n-1}(2^n-1)} = 1,\ c^{2^n} d^{2^{n-1}} u^{\frac{2^{n-1}(2^n-1)(2^n-2)}{3}} = v^s e^\ell. \tag{2}$$

Proof. By the group extension theory (see [21, pp. 245-268]), the cyclic extension G of $\langle b, G'\rangle = \langle b, c, d, u\rangle$ by the element a can be determined by

$$a^{2^n} = c^\ell d^s u^t,\ (c^\ell d^s u^t)^a = c^\ell d^s u^t,\ b^{a^{2^n}} = b^{c^\ell d^s u^t},\ c^{a^{2^n}} = c,\ d^{a^{2^n}} = d. \tag{3}$$

Since

$$(c^\ell d^s u^t)^a = (cd)^\ell (du)^s u^t = c^\ell d^{\ell+s} u^{s+t},$$

the second relation of Eq(3) gives $d^\ell u^s = 1$. Using the formulas in Lemma 5, we get

$$b^{a^{2^n}} = b[b, a^{2^n}] = bc^{-2^n} d^{-2^{n-1}(2^n-1)} u^{\frac{2^{n-1}(2^n-1)(2^n-2)}{3}},$$

$$b^{c^\ell d^s u^t} = (b[b, c^\ell])^{d^s} = (be^{-\ell})^{d^s} = b[b, d^s]e^{-\ell} = bv^s e^{-\ell}.$$

The the third relation of Eq(3) implies $c^{2^n} d^{2^{n-1}} u^{\frac{2^{n-1}(2^n-1)(2^n-2)}{3}} = v^s e^\ell$. Moreover, from the forth relation, we have $1 = [c, a^{2^n}] = d^{2^n} u^{2^{n-1}(2^n-1)}$. The last relation of Eq(3) is obvious true. Hence, Eq(3) is equivalent to Eq(2). □

With a routine checking one may get the following result.

Lemma 9. *Let $G_1 = \langle a_1, b_1\rangle$ and $G_2 = \langle a_2, b_2\rangle$ are groups as* Eq(1). *Suppose that $G_1 \cong G_2$, and let*

$$\alpha : a_1 \longrightarrow a_2^h b_2^i c_2^\ell d_2^m u_2^j,\ b_1 \longrightarrow a_2^{h'} b_2 c_2^{\ell'} d_2^{m'} u_2^{j'}$$

be an isomorphism from G_1 to G_2, where h is odd, $i = 0$ or 1, and $h', \ell, \ell', m, m', j, j'$ are all integers. Then, we have

(1) $i = 0$.

$$\begin{aligned}
c_1^\alpha &= [a_1^\alpha,\ b_1^\alpha] = c_2^{h-2\ell} d_2^{\frac{h(h-1)}{2}-h\ell'+h'\ell} u_2^{\frac{\ell h'(h'-1)}{2}+mh'+m'h+\frac{\ell' h(h-1)}{2}+\frac{h(h-1)(h-2)}{6}} v_2^{m+h'\ell},\\
d_1^\alpha &= [c_1^\alpha, a_1^\alpha] = d_2^{h^2} u_2^{h'\ell+\ell'} v_2^{\ell},\\
e_1^\alpha &= [c_1^\alpha, b_1^\alpha] = c_2^{-2h} d_2^{hh'} u_2^{h'^2\ell+h'\ell'} v_2^{h'+\ell'+\frac{h(h-1)}{2}} z_2^{\ell},\\
u_1^\alpha &= [d_1^\alpha, a_1^\alpha] = u_2,\\
w_1^\alpha &= [e_1^\alpha, a_1^\alpha] = u_2^{h'} v_2,\\
z_1^\alpha &= [e_1^\alpha, b_1^\alpha] = u_2^{h'^2} z_2.
\end{aligned}$$

(2) $i = 1$.

$$\begin{aligned}
c_1^\alpha &= [a_1^\alpha,\ b_1^\alpha] = c_2^{h'-h+2(\ell'-\ell)} d_2^{\frac{h(h-1)}{2}-\frac{h'(h'-1)}{2}-h\ell'+h'\ell}\\
&\quad u_2^{hm'+h'm+\frac{h\ell'(h-1)}{2}+\frac{h'\ell(h'-1)}{2}+\frac{h(h-1)(h-2)}{6}+\frac{h'(h'-1)(h'-2)}{6}} v_2^{m'-h\ell'+h'\ell+m+\frac{h(h-1)}{2}+\frac{(h'-1)h'}{2}},\\
d_1^\alpha &= [c_1^\alpha, a_1^\alpha] = c_2^{2(h'-h)} d_2^{h(h-h')} u_2^{\ell'+h'\ell+\frac{hh'(h'-1)}{2}+\frac{h'h(h-1)}{2}} v_2^{h'\ell+\ell+\frac{h(h-1)}{2}+\frac{h'(h'-1)}{2}} z_2^{\ell-\ell'+h-h'},\\
e_1^\alpha &= [c_1^\alpha, b_1^\alpha] = c_2^{2(h'-h)} d_2^{h'(h-h')} u_2^{\frac{hh'(h'-1)}{2}+\frac{h'h(h-1)}{2}+h'\ell'+h'^2\ell} v_2^{\frac{h(h-1)}{2}+\frac{h'(h'-1)}{2}+h'\ell'+\ell'} z_2^{\ell-\ell'+h-h'},\\
u_1^\alpha &= [d_1^\alpha, a_1^\alpha] = u_2^{h-h'} z_2^{h-h'},\\
w_1^\alpha &= [e_1^\alpha, a_1^\alpha] = u_2^{h'(h-h')} v_2^{h^2-h'^2} z_2^{h-h'},\\
z_1^\alpha &= [e_1^\alpha, b_1^\alpha] = u_2^{h'^2(h-h')} z_2^{h-h'}.
\end{aligned}$$

4 Proof of Theorem 1

The first part of Theorem 1 has been proved in Lemma 4. To prove the second part, we need to discuss ten classes in Lemma 7. Clearly, any two groups in distinct classes are not mutually isomorphic, as they have nonisomorphic derived group. For every such class, some groups having different subset T_2 of relations and the element $x \in G'$ might be mutually isomorphic and so we need to determine isomorphism class of these groups. Finally, for every given group G, we shall show that there are at most two maps $\mathcal{M}(G; ab^i, b)$, where $i = 0$ or $i \in \{0.1\}$.

The discussion for all the cases are similar, for the sake of the length of the paper, we just give a proof in details for one subcase of Case (VI), where $G' \cong \mathbb{Z}_4 \times \mathbb{Z}_4$ and $u = c^4 = 1$.

Suppose $u = c^4 = z = 1$. By Lemma 4, we have

$$G = \langle a, b \mid R, T_1, u = c^4 = 1, a^{2^n} = c^\ell d^s\rangle, \tag{4}$$

where R and T_1 are simplified as

$$\begin{aligned} R = \ & \{b^2=1, [a,b]=c, [c,a]=d, [c,b]=e, [d,a]=1, [d,b]=[e,a]=v, \\ & [e,b]=1, [v,a]=[v,b]=1\}, \end{aligned}$$

$$T_1 := \{[c,d]=1, c^4=d^4=1, e=c^2, v=d^2\}.$$

Therefore, $G' = \langle c\rangle \times \langle d\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4$ and n, ℓ, s are some integers; and $G_4 = \langle v\rangle$.

To determine the group G in Eq(4), we need to know the values s and l, depending on a given n. To do that, we need to employ the group extension theory, that is Eq(2).

Lemma 10. *With the above notations, we have that $l=0$ and either $n=2$ and $s=1$; or $n \geqslant 3$ and $s \in \{0,\ 2\}$. Moreover, G is uniquely determined by given parameters (n,s).*

Proof. By Lemma 8, we know that G exists if and only if Eq(2) holds.

Noting that $u=1$, the first relation of Eq(2) means $d^\ell = 1$, namely $\ell \equiv 0 (\mathrm{mod}\ 4)$. The second relation of Eq(2) implies $d^{2^n}=1$. Since $|d|=4$, we get $n \geqslant 2$. Combining $\ell \equiv 0(\mathrm{mod}\ 4)$ with the third relation of Eq(2), we get $c^{2^n} d^{2^{n-1}} = v^s$. So,

$$d^2 = v^s \text{ for } n=2; \text{ and } v^s = 1 \text{ for } n \geqslant 3,$$

which is equivalent to

$$s \equiv \pm 1 (\mathrm{mod}\ \ 4) \text{ if } n=2; \text{ and } s \equiv 0, 2 (\mathrm{mod}\ \ 4) \text{ if } n \geqslant 3.$$

Now our group G may be denoted by $G(n,s)$.

Since $|G/G'| = 2^{n+1}$, it follows $G(n_1,s_1) \not\cong G(n_2,s_2)$ for any two distinct n_1 and n_2.

Checking by Magma System [2], we get $G(2,1) \cong G(2,-1)$.

Finally, we show $G(n,0) \not\cong G(n,2)$ for $n \geqslant 3$. Note that each element of $G(n,0)$ is of the form $a^h b^i c^\ell d^m$, where $i=0$ or 1, and h, ℓ, m are some integers. As $n \geqslant 3$, if $s \equiv 0 (\mathrm{mod}\ \ 4)$, that is $a^{2^n}=1$, then we get from Lemma 6 that

$$(a^h c^\ell d^m)^{2^n} = ((a^h c^\ell d^m)^8)^{2^{n-3}} = (a^{8h})^{2^{n-3}} = a^{2^n h} = 1,$$

$$(a^h b c^\ell d^m)^{2^n} = ((a^h b c^\ell d^m)^8)^{2^{n-3}} = (a^{8h} d^{2h(h-1)})^{2^{n-3}} = a^{2^n h} w^{h(h-1)2^{n-3}} = 1.$$

Therefore, $\exp(G(n,0)) = 2^n$. With the similar arguments, one may get $\exp(G(n,2)) = 2^{n+1}$. Hence, $G(n,0) \not\cong G(n,2)$ for $n \geqslant 3$. $\square$

For a later use, we are determining the automorphism group of $G(n,s)$.

Lemma 11. *For the group $G(n,s)$, we have $\alpha \in \mathrm{Aut}\,(G(n,s))$ if and only if it can be expressed as follows:*

$$\alpha : a \mapsto a^h b^i c^\ell d^m,\ b \mapsto a^{h'} b^{i'} c^{\ell'} d^{m'},$$

where h is odd, $i'=1$, $\ell, \ell', m, m' \in \mathbb{Z}_4$, and moreover,

(1) $G(2,1)$*: $h \equiv 1, -1 (\mathrm{mod}\ 4)$, if $i=0$ and $i=1$, respectively, and $4|h'$;*

(2) $G(n,0)$, $n \geqslant 3$: $8|h'$ *if* $n=3$; *and* $2^{n-1}|h'$ *if* $n \geqslant 4$;

(3) $G(n,2)$, $n \geqslant 3$: $4|h'$ *if* $n=3$; *and* $2^{n}|h'$ *if* $n \geqslant 4$.

Proof. Set $G = G(n,s)$. A direct checking shows that the mapping α given in the lemma can be extended to an automorphism of G.

Conversely, assume that $\alpha \in \mathrm{Aut}\,(G)$. From

$$G/G' = G^{\alpha}/G' = \langle a^{\alpha}G'\rangle \times \langle b^{\alpha}G'\rangle = \langle a^{h}b^{i}G'\rangle \times \langle a^{h'}b^{i'}G'\rangle \cong \mathbb{Z}_{2^n} \times \mathbb{Z}_2,$$

one may see that h is odd, as $n \geqslant 2$. From $1 = (a^{h'}b^{i'}G')^2 = a^{2h'}G'$, we have $2^{n-1}|h'$. Now, suppose the contrary that $i' = 0$. Since $2^{n-1}|h'$, it should be $a^{h'}b^{i'}G' = a^{2^{n-1}}G'$. However, recalling that h is odd and $n \geqslant 2$, it means that $a^{2^{n-1}}G' \in \langle a^{h}b^{i}G'\rangle$, a contradiction. Thus, $i' = 1$. In what follows, we continue our proof according to the three cases of G.

Case 1: $(n,s) = (2,1)$. Since $b^2 = 1$ and $a^4 = d$, it follows that $(b^{\alpha})^2 = 1$ and $(a^{\alpha})^4 = d^{\alpha}$. By Lemma 6, we have

$$1 = (b^{\alpha})^2 = (a^{h'}bc^{\ell'}d^{m'})^2 = a^{2h'}c^{h'}d^{3h'\ell' + \frac{h'(h'-1)}{2}}. \tag{5}$$

Since $2|h'$ and $a^4 = d$, we have $a^{2h'} \in \langle d\rangle$. Thus, Eq(5) is equivalent to

$$c^{h'} = 1 \text{ and } a^{2h'}d^{3h'\ell' + \frac{h'(h'-1)}{2}} = 1. \tag{6}$$

The first equation of Eq(6) means that $4|h'$. Moreover, from the second equation of Eq(6), combining $4|h'$, we have $1 = a^{2h'}d^{3h'\ell' + \frac{h'(h'-1)}{2}} = d^{\frac{h'^2}{2}}$, which is obvious true as $4|h'$.

For $a^{\alpha} = a^{h}b^{i}c^{\ell}d^{m}$, we discuss the cases $i = 0$ and 1, separately.

First suppose $i = 0$. In view of Lemmas 6 and 9, combining h is odd, we have

$$(a^4)^{\alpha} = (a^{h}c^{\ell}d^{m})^4 = a^{4h}c^{4\ell}d^{2h\ell} = d^{h+2h\ell} = d^{h+2\ell} \text{ and } d^{\alpha} = d^{h^2}v^{\ell} = d^{1+2\ell}.$$

Thus, $(a^{\alpha})^4 = d^{\alpha}$ is equivalent to $d^{h-1} = 1$, that is $h \equiv 1 (\mathrm{mod}\ 4)$.

Now suppose $i = 1$. By Lemma 6 and 9, combining $4|h'$, we have

$$(a^4)^{\alpha} = (a^{h}bc^{\ell}d^{m})^4 = a^{4h}c^{2h}d^{3h^2-h+2h\ell} = c^{2h}d^{3h^2+2h\ell} \text{ and } d^{\alpha} = c^{-2h}d^{h^2}v^{\frac{h(h-1)}{2}+\ell}.$$

Thus, $(a^{\alpha})^4 = d^{\alpha}$ is equivalent to $v^{\frac{h(h+1)}{2}} = 1$, and so $4|h(h+1)$ which forces that $h \equiv -1 (\mathrm{mod}\ 4)$.

Case 2: $n \geqslant 3$ and $s = 0$. In this case, we have $a^{2^n} = 1$. As $2^{n-1}|h'$, by Lemma 6, it follows that

$$1 = (b^{\alpha})^2 = a^{2h'}c^{h'}d^{3h'\ell' + \frac{h'(h'-1)}{2}} = d^{\frac{h'(h'-1)}{2}},$$

which is equivalent to $8|h'$. Hence, we have $8|h'$ while $n = 3$, and $2^{n-1}|h'$ while $n \geqslant 4$.

Since $n \geqslant 3$ and $a^{2^n} = 1$, by Lemma 6, we have

$$1 = (a^{2^n})^\alpha = (a^h b^i c^\ell d^m)^{2^n} = (a^{8h} c^{4hi} d^{2h(h-1)i})^{2^{n-3}},$$

where $i = 0, 1$, which is obvious true.

Case 3: $n \geqslant 3$ and $s = 2$. In this case, we have $a^{2^n} = d^2$. As $2^{n-1}|h'$, by Lemma 6, it follows that

$$1 = (b^\alpha)^2 = a^{2h'} c^{h'} d^{3h'\ell' + \frac{h'(h'-1)}{2}} = a^{2h'} (d^2)^{\frac{h'(h'-1)}{4}} = a^{2h' + 2^{n-2}h'(h'-1)},$$

which is equivalent to $2^n|h'(1 + 2^{n-3}(h'-1))$. Hence, we have $4|h'$ while $n = 3$, and $2^n|h'$ while $n \geqslant 4$.

Since $a^{2^n} = d^2 = v$, $n \geqslant 3$ and h is odd, by Lemma 6 and 9, we have

$$(a^{2^n})^\alpha = (a^h b^i c^\ell d^m)^{2^n} = (a^{8h} d^{2h(h-1)i})^{2^{n-3}} = a^{2^n h} = v^h = v = v^\alpha,$$

where $i = 0, 1$. □

Finally, we determine all the maps with the automorphism group $G \cong G(n, s)$. Equivalently, we need to determine the representatives of the generating pairs (r, ℓ) of G under the action of $\mathrm{Aut}\,(G)$, where $|\ell| = 2$.

Lemma 12. *Let $\mathcal{M}$ be an orientably-regular map with the automorphism group $G \cong G(n, s)$. Then, $\mathcal{M}$ is isomorphic to $\mathcal{M}(G; ab^i, b)$, where*

(i) *If $G \cong G(2, 1)$, then $i = 0$ and 1;*

(ii) *If $G \cong G(n, s)$ where $n \geqslant 3$, then $i = 0$.*

Proof. From the proof of Lemma 11, $\mathrm{Aut}\,(G)$ acts transitively on involutions in $G \setminus \Phi(G)$ and so we set $\ell = b$. In what follows, we need to consider two cases, separately.

Case 1: $G \cong G(2, 1)$. Let $G = G(2, 1)$. Now our $r \in G \setminus \langle \Phi(G), b \rangle$ and this set can be divided into the following four subsets which are mutually disjoint:

$\Omega_1 := \langle a^h bc^\ell d^m \mid \mathrm{h} \equiv -1 (\mathrm{mod}\ \ 4), \ell, m \in \mathbb{Z}_4 \rangle$, $\Omega_2 := \langle a^h c^\ell d^m \mid \mathrm{h} \equiv 1 (\mathrm{mod}\ \ 4), \ell, m \in \mathbb{Z}_4 \rangle$,

$\Omega_3 := \langle a^h bc^\ell d^m \mid \mathrm{h} \equiv 1 (\mathrm{mod}\ \ 4), \ell, m \in \mathbb{Z}_4 \rangle$, $\Omega_4 := \langle a^h c^\ell d^m \mid \mathrm{h} \equiv -1 (\mathrm{mod}\ \ 4), \ell, m \in \mathbb{Z}_4 \rangle$.

By Lemma 11, we know that for each $r \in \Omega_1 \cup \Omega_2$, there exists $\alpha \in \mathrm{Aut}\,(G)$ such that $(a, b)^\alpha = (r, b)$.

Take an arbitrary $r = a^h bc^\ell d^m \in \Omega_3$, where $h \equiv 1 \pmod 4$. By Lemma 11, we know that there exists $\alpha \in \mathrm{Aut}\,(G)$ such that

$$a^\alpha = a^h c^{-\ell} d^{-m}, \ \ b^\alpha = b,$$

which follows that $(ab, b)^\alpha = (r, b)$. Similarly, for each $r \in \Omega_4$, there exists $\alpha \in \mathrm{Aut}\,(G)$ such that $(ab, b)^\alpha = (r, b)$. Hence, for each $r \in \Omega_3 \cup \Omega_4$, there exists $\alpha \in \mathrm{Aut}\,(G)$ such that $(ab, b)^\alpha = (r, b)$.

By observing the automorphisms of G from Lemma 11, one may conclude that (a, b) cannot be mapping to (ab, b) by any element of $\mathrm{Aut}\,(G)$. Consequently, up to isomorphism, there exist two maps $\mathcal{M}(G; a, b)$ and $\mathcal{M}(G; ab, b)$.

Case 2: $\boldsymbol{G \cong G(n, s)}$ where $\boldsymbol{n \geqslant 3}$. For the generating pairs (r, b) with $r = a^h b^i c^\ell d^m$, where h is odd, $i = 0$ or 1, and ℓ, m are some integers. By Lemma 11, we note that there exists $\alpha \in \mathrm{Aut}\,(G)$ such that $(a, b)^\alpha = (r, b)$. Consequently, we get, up to isomorphism, a unique regular map $\mathcal{M}(G; a, b)$. □

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