Completely Regular Codes in Johnson and Grassmann Graphs with Small Covering Radii

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Abstract

Let $\mathcal{L}$ be a Desarguesian 2-spread in the Grassmann graph $J_q(n, 2)$. We prove that the collection of the 4-subspaces, which do not contain subspaces from $\mathcal{L}$ is a completely regular code in $J_q(n, 4)$. Similarly, we construct a completely regular code in the Johnson graph $J(n, 6)$ from the Steiner quadruple system of the extended Hamming code. We obtain several new completely regular codes with covering radius 1 in the Grassmann graph $J_2(6, 3)$ using binary linear programming.

Mathematics Subject Classifications: 05B25, 05B30

1 Introduction

The notion of a completely regular code was introduced by Delsarte in [14] as a generalization of a perfect code. It is known that all perfect codes in Grassmann graphs [10] are trivial and their nonexistence in Johnson graphs is proven for a large number of cases [15], [19]. Therefore the completely regular codes in these graphs are of interest as they are related to designs and some geometrical objects.

During the years, researchers used several different names for the completely regular codes with covering radius 1. These objects could be equivalently defined in terms of: equitable 2-partitions [30], [32], perfect 2-colorings [1], [5], [13], [17],r [28], intriguing sets [12] and others.

We refer to Ph.D. thesis of Martin [26] for an introduction on completely regular codes in Johnson graphs and a survey of Borges, Rifa and Zinoviev [6] on a recent progress in the study of completely regular codes in Hamming and Johnson graphs.

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In [14] Delsarte noted interrelations of the completely regular codes in Hamming and Johnson graphs with orthogonal arrays and t-designs respectively. In order to emphasize this connection Martin [24] suggested the term "a completely regular design of strength t". He studied several well-known classes of designs from the point of view of complete regularity and showed that $(k-1)$-(n,k,λ)-designs are completely regular. This also holds for q-ary designs, therefore 2-spreads and 2-ary Steiner triple system [7] are completely regular. In [1] it was shown that if D is a $(k-1)$-$(n,k,1)$-design, then the code of the $(k+1)$-subsets that does not contain any block of D is completely regular in the Johnson graph $J(n,k+1)$. The integer necessary conditions for the existence of t-designs were exploited for showing the nonexistence of constant weight perfect codes [15], [19] and completely regular codes with covering radius 1 [28].

The completely regular designs of strength 0 in Hamming and Johnson graphs were characterized by Meyerowitz in [27] and have a rather simple structure. The completely regular codes of zero strength in the strongly regular Grassmann graphs are known as the Cameron-Liebler line classes [9]. Contrary to the ordinary designs of strength 0 [27], the complete classification of these objects is still open. Several approaches for studying completely regular codes in Johnson graphs were applied to Cameron-Liebler line classes in [18].

The completely regular codes of strength 0 with covering radius 1 in Grassmann graph $J_q(n,k)$ for $k \geqslant 3$ could be considered as a generalization of Cameron-Liebler line classes. Somewhat trivial examples of these codes are the subspaces containing a point; the subspaces contained in a hyperplane; the subspaces contained in a hyperplane or containing a point for a non-incident point-hyperplane pair. There are not known any other examples of such codes in $J_q(n,k)$ for $k \geqslant 3$. In [16] Filmus and Ihringer suggested a reductive concept for classification of completely regular codes of strength 0 and covering radius 1 if once has the classification of these codes for $k=2$. In particular, for $q=2,3,4,5$ they showed that such codes in $J_q(n,k)$ are only those described above.

The completely regular codes in Johnson graphs of strength 1 were characterized in the following cases: codes with the minimum distance at least 3 by Martin in [25] and covering radius 1 in $J(n,w)$ for $w \geqslant 4$ recently by Vorob'ev [32]. De Winter and Metsch in [13] considered completely regular codes with covering radius 1 and strength 1 in Grassmann graphs of diameter 3. They found two new series of examples of such codes: for a given 2-spread the 3-subspaces that do not contain a space of the 2-spread and the code arising from symplectic polar space. The first series is similar to a construction from [1] for completely regular codes in Johnson graphs $J(n,4)$ and $J(n,5)$ from Steiner triple and quadruples systems.

In Section 2 we give basic definitions and review the theory developed by Delsarte and Martin in the q-analog case. In Section 3 we consider the completely regular codes with covering radii 2 in regular graphs. We provide a sufficient condition for existence of such codes in terms of eigenvectors of these graphs. In Section 4 we discuss a spectral property of the inclusion matrix of t-subspaces vs k-subspaces. We show that the code of $(k+1)$-subspaces not containing subspaces of $(k-1)$-$(n,k,1)_q$-design is completely regular in the Grassmann graph of $(k+1)$-subspaces. This generalizes a series of completely regular codes from [1] and [13]. In Sections 5 and 6 we go further by applying this idea to the most symmetric cases, i.e. when a $(k-1)$-$(n,k,1)_q$-design is the Desarguesian 2-spread or the Steiner quadruple system of extended Hamming code. For these designs, the code of $(k+2)$-subspaces (subsets), which do not contain subspaces (subsets) of $(k-1)$-$(n,k,1)_q$-design is completely regular in the Grassmann (Johnson) graph.

The linear programming approach is a popular method for constructing and classifying codes. Binary versions of linear programming earlier showed promising results for settling the nonexistence of completely regular codes [4] as well as finding these objects [22]. In Section 7 we obtain several new completely regular designs of strength 1 and covering radius 1 in the Grassmann graph $J_2(6,3)$ by binary integer programming with a prescribed subgroup of its automorphism group. We outline the known results on the completely regular codes with $\rho = 1$ in this graph in a parameter table.

2 Definitions and Basic theory

2.1 The eigenspaces of Johnson and Grassmann graphs

In what follows we abbreviate k-element subset and k-dimensional subspace to k-subset and k-subspace respectively. The vertices of the Johnson graph $J(n,k)$ are k-subsets of the set $\{1,\dots,n\}$ and the edges are pairs of subsets meeting in a $(k-1)$-subset. The vertices of the Grassmann graph $J_q(n,k)$ are k-subspaces of $\mathbb{F}_q^n$ and the edges are pairs of subspaces meeting in a $(k-1)$-subspace. These graphs are well-known series of distance-regular graphs. We also denote the Johnson graph $J(n,k)$ by $J_1(n,k)$ to emphasize that a result holds for Johnson and Grassmann graphs simultaneously. Below we consider $k \leqslant n/2$ as the graphs $J_q(n,k)$ and $J_q(n,n-k)$ are isomorphic for all $q \geqslant 1$. We use the notations $\begin{bmatrix} n \\ k \end{bmatrix}_q$ for q-binomial coefficient and $\begin{bmatrix} n \\ k \end{bmatrix}_1$ for its limit value, i.e. the ordinary binomial coefficient.

A vector v is called an *e*igenvector of a graph Γ with eigenvalue θ if it is an eigenvector of the adjacency matrix of Γ with eigenvalue θ. The following representation for the eigenspaces of Johnson and Grassmann graphs can be found in [23], see also [14, Section 4.2]. We arrange the eigenvalues of $J_q(n,k)$, $q \geqslant 1$ in descending order starting from zeroth and denote them by $\theta_{i,q}(n,k)$, $i \in \{0,\dots,k\}$. For i-subspace (i-subset if $q=1$) X let us consider the characteristic vectors of all k-subspaces of $\mathbb{F}_q^n$ (k-subsets) containing X. Let $T_i(k)$ be the linear span of these vectors over real field where X runs through all i-subspaces (i-subsets) of $\mathbb{F}_q^n$ (the set $\{1,\dots,n\}$). Let $T^\perp$ denote the orthogonal completement of a subspace T.

Theorem 1. *[23, Theorem 4.16] For any $i, 0 \leqslant i \leqslant k$, $U_i = T_i(k) \cap T_{i-1}^\perp(k)$ is the eigenspace of $J_q(n,k)$, $q \geqslant 1$, with eigenvalue*

$$\theta_{i,q}(n,k) = q^{i+1}\begin{bmatrix} n-k-i \\ 1 \end{bmatrix}_q \begin{bmatrix} k-i \\ 1 \end{bmatrix}_q - \begin{bmatrix} i \\ 1 \end{bmatrix}_q.$$

Below we omit the index q in $\theta_{i,q}(n,k)$ when we speak of the eigenvalues of Johnson graphs.

2.2 Completely regular codes and q-ary designs

Let C be a code in a regular graph Γ. A vertex x is in C_i if the minimum of the distances between x and the vertices of C is i. The maximum of these distances is called *the covering radius* of C and is denoted by ρ. The *distance partition* of the vertices of Γ with respect to $C_0 = C$ is $\{C_i : i \in \{0, \ldots, \rho\}\}$.

A code C is called *completely regular* [31] if there are numbers $\alpha_0, \ldots, \alpha_\rho$, $\beta_0, \ldots, \beta_{\rho-1}$, $\gamma_1, \ldots, \gamma_\rho$ such that any vertex of C_i is adjacent to exactly α_i, β_i and γ_i vertices of C_{i-1}, C_i and C_{i+1} respectively. Note that $\alpha_0, \ldots, \alpha_\rho$ can be found from the remaining numbers and the valency of the graph. The set $\{\beta_0, \ldots, \beta_{\rho-1}; \gamma_1, \ldots, \gamma_\rho\}$ is called the *intersection array* of the completely regular code C.

For a completely regular code C consider the $(\rho+1) \times (\rho+1)$ *matrix* A such that $A_{i,j}$ equals the number of vertices of C_j adjacent to a fixed vertex of C_i. The eigenvalues of the matrix A are called the *eigenvalues* of the completely regular code C.

Theorem 2. *[11, Theorem 4.5] (Lloyd's theorem) The eigenvalues of a completely regular code in a graph Γ are eigenvalues of Γ.*

The eigenvalues of a completely regular code with covering radius 1 are easy to find, see e.g. [30, Proposition 1].

Proposition 3. *Let C be a completely regular code in a m-regular graph Γ with $\rho = 1$ and intersection array $\{\beta_0; \gamma_1\}$. Then the size of C is $|V(\Gamma)|\gamma_1/(\gamma_1 + \beta_0)$ and the eigenvalues of C are m and $m - \beta_0 - \gamma_1$.*

In the rest of the section we consider codes in Johnson or Grassmann graphs. A collection D of k-subspaces (k-subsets when $q = 1$) of $\mathbb{F}_q^n$ (of $\{1, \ldots, n\}$) is a t-$(n, k, \lambda)_q$-*design*, if any t-subspace of $\mathbb{F}_q^n$ (t-subset of $\{1, \ldots, n\}$) is contained in exactly λ elements of D. In throughout of what follows we consider only designs without repeated blocks. The *strength* of D is the maximum t such that D is a t-design. When $q \geqslant 2$, a 1-$(n, k, 1)_q$-design is called a k-*spread*. It is well-known that k-spreads exist if and only if k divides n.

Let χ_C be the characteristic vector of a code C in the graph $J_q(n, k)$. Consider the decomposition of χ_C over the eigenspaces $U_0, \ldots, U_k$ of $J_q(n, k)$:

$$\chi_C = u_0 + u_{i_1} + \cdots + u_{i_s}, \tag{1}$$

where $u_0 \in U_0$ and $u_{i_j} \in U_{i_j}$, $j = 1, \ldots, s$. The number s in the decomposition (1) is called the *dual degree* of the code C [14].

We make use of several results that were stated for the completely regular codes in Johnson graphs by Delsarte [14] and Martin [24]. The arguments of the proofs for the q-ary generalization of these results could be obtained by replacing "subset" with "subspace" and follow from the description of the eigenspaces of $J_q(n, k)$ in Section 2.1.

Theorem 4. *Let C be a code in $J_q(n, k)$, $q \geqslant 1$ such that the decomposition (1) holds. Then we have the following:*

1. [14, Theorem 4.2] The strength of C as q-ary design is equal to $min\{i_j : j \in \{1,\dots,s\}\} - 1$.

2. [14, Theorem 5.13] If the minimum distance of C is greater or equal to $2s-1$ then C is completely regular.

3. [14, Theorem 5.10] The covering radius of C is not greater than s.

4. [24, Corollary 3.4] If C is completely regular then its strength is equal to

$$min\{i \geqslant 0 : \theta_{i,q}(n,k) \text{ is an eigenvalue of } C\} - 1.$$

The following statement for Johnson graphs could be found in [28, Corollary 1].

Corollary 5. *Let C be a completely regular code in $J_q(n,k)$, $q \geqslant 1$ with covering radius 1 and intersection array $\{\beta_0;\gamma_1\}$. Then we have the following:*

1. The code C is of size $\begin{bmatrix} n \\ k \end{bmatrix}_q \gamma_1/(\gamma_1+\beta_0)$ and its eigenvalues are numbers $\begin{bmatrix} n-k \\ 1 \end{bmatrix}_q \begin{bmatrix} k \\ 1 \end{bmatrix}_q$ and $\begin{bmatrix} n-k \\ 1 \end{bmatrix}_q \begin{bmatrix} k \\ 1 \end{bmatrix}_q - \gamma_1 - \beta_0$.

2. The strength of C is t where

$$\theta_{t+1,q}(n,k) = \begin{bmatrix} n-k \\ 1 \end{bmatrix}_q \begin{bmatrix} k \\ 1 \end{bmatrix}_q - \gamma_1 - \beta_0.$$

Moreover, the numbers $\left[\begin{smallmatrix} n-i \\ k-i \end{smallmatrix}\right]_q \gamma_1/(\gamma_1+\beta_0)$ are integers for any $i \in \{0,\dots,t\}$.

Proof. The eigenvalues of C and the expression $|C| = \begin{bmatrix} n \\ k \end{bmatrix}_q \gamma_1/(\gamma_1+\beta_0)$ follow from Proposition 3 and the strength of C is by the fourth Statement of Theorem 4. The integer necessary conditions for t-design imply that for any $i \in \{0,\dots,t\}$ the number of the subspaces in C containing an i-subspace is $\begin{bmatrix} n-i \\ k-i \end{bmatrix}_q \gamma_1/(\gamma_1+\beta_0)$. □

Corollary 6. *1. [24, Corollary 3.5], [3, Corollary 8] For $q \geqslant 1$ any $(k-1)$-$(n,k,\lambda)_q$-design is a completely regular code in $J_q(n,k)$ with eigenvalue $\theta_{k,q}(n,k)$ and covering radius 1.*

2. Any 3-spread is a completely regular code in the Grassmann graph $J_q(n,3)$ with covering radius 2.

Proof. We follow the considerations from [24, Corollary 3.5]. By Statement 1 of Theorem 4 we see that the dual degrees of a $(k-1)$-$(n,k,\lambda)_q$-design and a 3-spread are 1 and s respectively, where $s \leqslant 2$. Since the covering radius of 3-spread is 2, its dual degree is also 2 by Statement 3 of Theorem 4. Note that the minimum distance of $(k-1)$-$(n,k,\lambda)_q$-design in $J(n,k)$ is 2 while that of 3-spread in $J(n,3)$ is 3. The result follows from Statement 2 of Theorem 4. □

3 Auxilary statements

Let the positions of a vector u be indexed by the vertices of a graph Γ. If the set of the pairwise different values of u is $\{a_0,\dots,a_r\}$, denote by C^i the set of the vertices of Γ such that $u_x = a_i$, $i \in \{0,\dots,r\}$. The partition $\{C^0,\dots,C^r\}$ is called the *p*artition associated to the vector u. We see that the eigenvectors taking two or three values are tightly related with the completely regular codes with covering radii 1 and 2.

Theorem 7. *1. A code C is completely regular in a m-regular graph Γ with covering radius 1 and eigenvalue θ, $\theta \neq m$ if and only if $\{C, \overline{C}\}$ is the partition associated to an eigenvector of Γ with eigenvalue θ.*

2. Let Γ be a m-regular graph Γ, $\{C^0, C^1, C^2\}$ be the partition associated to an eigenvector u of Γ. If there are no edges between C^0 and C^2 and any vertex of C^1 is adjacent to exactly β_1 vertices of C^2 then C^0 is a completely regular code in Γ with covering radius 2.

Proof. 1. The result could be found in [12, Proposition 3.2] or [1, Proposition 1].

2. Let a_0, a_1 and a_2 be the values of u on C^0, C^1 and C^2 respectively. Let a vertex x of C^0 be adjacent to $\alpha_0(x)$ vertices of C^0. Since there are no edges between the vertices C^0 and C^2, the vertex x is adjacent to $\beta_0(x) = m - \alpha_0(x)$ vertices of C^1. Consider the sum of the values of u on the neighbors of x. Since u is an eigenvector with eigenvalue θ we have the following:

$$\theta a_0 = (m - \beta_0(x))a_0 + \beta_0(x)a_1. \tag{2}$$

This implies that $\alpha_0(x)$ and $\beta_0(x)$ do not depend on x. The same argument on C^1 and C^2 implies that $x \in C^2$ has exactly γ_2 and α_2 neighbors in C^1 and C^2 respectively. This follows from $\alpha_2 + \gamma_2 = m$ and the equation

$$\theta a_2 = (m - \alpha_2)a_1 + \alpha_2 a_2. \tag{3}$$

Let a vertex x of C^1 be adjacent to $\gamma_1(x)$, $\alpha_1(x)$ and β_1 vertices of C^0, C^1 and C^2 respectively. Note that β_1 does not depend on x by the condition of the theorem. Taking into account that $\alpha_1(x) + \beta_1 + \gamma_1(x) = m$, the sum of the values of u on the neighbors of x is

$$\theta a_1 = \gamma_1(x)a_0 + (m - \gamma_1(x) - \beta_1)a_1 + \beta_1 a_2, \tag{4}$$

The above implies that $\alpha_1(x)$ and $\gamma_1(x)$ do not depend on x and C^0 is a completely regular code by the definition. □

4 Inducing map and completely regular codes

For $q \geqslant 1$ and $l \geqslant k$ consider the matrix $I_{l,k}$ whose rows and columns are indexed by the vertices of $J_q(n,l)$ and those of $J_q(n,k)$ respectively, $I_{l,k}(x,y)$ is 1 if y is contained in x and 0 otherwise. Kantor in [20] proved that $I_{l,k}$ is a full rank matrix. Moreover, it is clear that the left multiplication by $I_{l,k}$ maps the column-vectors of the subspace $T_i(k)$ to those of $T_i(l)$ (see Section 2.1 for their definitions). Therefore, $I_{l,k}$ establishes isomorphism from the eigenspace $U_{i,q}(n,k)$ to $U_{i,q}(n,l)$ for any $i \in \{0,\dots,k\}$.

Theorem 8. *Let u be an eigenvector of the graph $J_q(n,k)$ with eigenvalue $\theta_{i,q}(n,k)$, $q \geqslant 1$. Then for any $l \geqslant k$ the vector $I_{l,k}u$ is an eigenvector of the graph $J_q(n,l)$ with eigenvalue $\theta_{i,q}(n,l)$.*

We make use of the theorem above for obtaining a series of completely regular designs.

Theorem 9. *1. [1, Proposition 4] Let D be a $(k-1)$-$(n,k,1)$-design. Then*

$$\{U : U \subset \{1,\dots,n\}, |U| = k+1, |\{V \in D : V \subset U\}| = 0\}$$

is a completely regular code in the Johnson graph $J(n,k+1)$ with covering radius 1.

2. Let D be a $(k-1)$-$(n,k,1)_q$-design. Then

$$\{U : U < \mathbb{F}_q^n, dim(U) = k+1, |\{V \in D : V < U\}| = 0\}$$

is a completely regular code in the Grassmann graph $J_q(n,k+1)$ with covering radius 1.

Proof. 2. We use the approach of work [1] based on the inclusion matrix. The code D is completely regular with covering radius 1 by Corollary 6. By the first statement of Theorem 7 there is an eigenvector u of $J_q(n,k)$ with associated partition $\{D,\overline{D}\}$. A $(k+1)$-subspace of $\mathbb{F}_q^n$ contains either 0 or 1 subspaces of D. These two facts combined imply that $I_{k+1,k}u$ takes only two values. Moreover, the partition associated to the vector $I_{k+1,k}u$ is the partition into the codes

$$\{U : U < \mathbb{F}_q^n, dim(U) = k+1, |\{V \in D : V < U\}| = 0\} \text{ and}$$

$$\{U : U < \mathbb{F}_q^n, dim(U) = k+1, |\{V \in D : V < U\}| = 1\}.$$

By the first statement of Theorem 8 we see that $I_{k+1,k}u$ is an eigenvector of $J_q(n,k+1)$. By the first statement of Theorem 7 we obtain the required. □

Remark 10. A combinatorial proof for the statement above in case when k is 2 (i.e. 2-spreads) for the Grassmann graphs could be found in [13, Lemma 12]. Apart from 2-spreads the only known example of $(k-1)$-$(n,k,1)_q$-design, $q \geqslant 2$ is the 2-ary Steiner triple system constructed in [7]. This implies the existence of a completely regular code in $J_2(13,4)$ by Theorem 9.

In the sections below we proceed further with the idea described in Theorem 9 and show that the Steiner quadruple systems of the extended Hamming code and the Desarguesian 2-spreads produce completely regular codes in the Johnson graph $J(n,6)$ and the Grassmann graph $J_q(n,4)$ respectively.

5 Completely regular code in the Johnson graph $J(n,6)$ from the SQS of the extended Hamming code

We use the traditional point-block terms throughout the section. A 3-$(n,4,1)$-design is called a *S*teiner quadruple system of order n. By $B\Delta B'$ we denote the symmetric difference of subsets B and B'.

If $\mathcal{Q}$ is any Steiner quadruple system of order n then $\mathcal{Q}$ in $J(n,4)$ and $\{x : x \subset \{1,\dots,n\}, |x| = 5, |\{B \in \mathcal{Q}, B \subset x\}| = 0\}$ in $J(n,5)$ are completely regular codes respectively by Corollary 6 and Theorem 9. We now describe a case where Steiner quadruple system yields a completely regular code in $J(n,6)$.

Theorem 11. *Let $\mathcal{Q}$ be a Steiner quadruple system of order n such that for any distinct B, $B' \in \mathcal{Q}$: $|B \cap B'| = 2$ we have $B \Delta B' \in \mathcal{Q}$. The code $\{x : x \subset \{1, \dots, n\}, |x| = 6, |\{B \in \mathcal{Q}, B \subset x\}| = 0\}$ is completely regular in $J(n, 6)$ with covering radius 2.*

Proof. Let B and B' be two blocks of $\mathcal{Q}$, $|B \cap B'| = 2$. By condition of the theorem, the symmetric difference $B\Delta B'$ is also a block of $\mathcal{Q}$. Therefore any 6-subset of $\{1, \dots, n\}$ contains 0, 1 or 3 blocks of $\mathcal{Q}$. We consider the following codes:

$$C^0 = \{x : x \subset \{1, \dots, n\}, |x| = 6, |\{B \in \mathcal{Q} : B \subset x\}| = 0\},$$

$$C^1 = \{x : x \subset \{1, \dots, n\}, |x| = 6, |\{B \in \mathcal{Q} : B \subset x\}| = 1\},$$
$$C^2 = \{x : x \subset \{1, \dots, n\}, |x| = 6, |\{B \in \mathcal{Q} : B \subset x\}| = 3\}$$

that partition the vertices of $J(n, 6)$.

We are now to show that C^0 is a completely regular code and $\{C^0, C^1, C^2\}$ is the distance partition. In view of the second statement of Theorem 7 we prove that the vertices of C^0 and C^2 are disjoint and a vertex of C^1 is adjacent to exactly 6 vertices of C^2. We finish the proof by noting that $\{C^0, C^1, C^2\}$ is the partition associated to an eigenvector.

A 6-subset from C^2 contains three blocks of $\mathcal{Q}$. Moreover, by condition of the theorem, the symmetric difference of any two of these blocks is the third block. We see that the following holds:

$$\text{for any } x \in C^2 \text{ and any } i \in x, \text{ there is } B \in \mathcal{Q}, i \notin B, B \subset x. \tag{5}$$

From (5) we conclude that the vertices of C^0 and C^2 are nonadjacent in the Johnson graph $J(n, 6)$.

Let y be a vertex of C_1 and B be a unique block of $\mathcal{Q}$ such that $B \subset y$. Let $x \in C^2$ be a vertex that is adjacent to y in $J(n, 6)$, so $x = (y \setminus \{i\}) \cup \{j\}$ for some $i \in y, j \notin y$.

Suppose that i is in B. Then by property (5) there is a block B' of $\mathcal{Q}$ that is a subset of $x \setminus \{j\}$. We see that $B' \subset x \setminus \{j\} \subset y$. Moreover B' is not B, because $i \in B$, $i \notin x$ and $B' \subset x$. We have that distinct blocks B' and B from $\mathcal{Q}$ are subsets of y, which contradicts $y \in C^1$.

We have that $i \in y \setminus B$. Since $x \in C^2$, let the following blocks of $\mathcal{Q}$ be the subsets of x: B, $\{s, t, l, j\}$ and $B\Delta\{s, t, l, j\}$ for some $s, t \in B$, $\{l\} = x \setminus (B \cup \{j\})$. On the other hand, given the 2-subset $\{s, t\}$ of B and the point l from $y \setminus B$ the point j could be reconstructed. Indeed, the block $\{s, t, l, j\}$ is a unique block in 3-$(n, 4, 1)$-design $\mathcal{Q}$ containing $\{s, t, l\}$. Since $\{s, t, l, j\}$ and $B\Delta\{s, t, l, j\}$ are contained in the same x from C^2, we conclude that there are exactly ${\begin{bmatrix}4\\2\end{bmatrix}}_1 \cdot 2/2 = 6$ neighbors of y in C^2.

We show that the partition $\{C^0, C^1, C^2\}$ is the partition associated to an eigenvector of $J(n, 6)$. The Steiner quadruple system $\mathcal{Q}$ is a completely regular code with covering radius 1 and eigenvalue $\theta_4(n, 4)$ by Corollary 6. Then by the first statement of Theorem 7 there is an eigenvector v with eigenvalue $\theta_4(n, 4)$ such that $\{\mathcal{Q}, \overline{\mathcal{Q}}\}$ is the partition associated to v. By Theorem 8 the vector $I_{6,4}v$ is an eigenvector of $J(n, 6)$ with eigenvalue $\theta_4(n, 6)$. The definitions of the inclusion matrix $I_{6,4}$ and the codes C^0, C^1, C^2 imply that these codes form the partition associated to $I_{6,4}v$. By the second statement of Theorem 7 we conclude that C^0 is completely regular. $\square$

Remark 12. One might obtain the intersection array $\{60, 6; 4(n-15), 6(n-8)\}$ of the code from Theorem 11 by combinatorial arguments or following the proof of Theorem 7 from equations (2)-(4).

Consider the extended Hamming code of length n. It is well-known that the set of the supports of the codewords of weight 4 of this code form a Steiner quadruple system of order n. Since the extended Hamming code is linear, the symmetric difference of two blocks of its Steiner quadruple system meeting in exactly 2 points is also a block of the Steiner quadruple system. Thus we obtain the following.

Corollary 13. *Let $\mathcal{Q}$ be the Steiner quadruple system of the extended Hamming code of length n. Then the code $\{x : x \subset \{1,\dots,n\}, |x| = 6, |\{B \in \mathcal{Q}, B \subset x\}| = 0\}$ is completely regular in $J(n,6)$ with $\rho = 2$.*

Corollary 14. *The set of the codewords of weight 6 of the extended Hamming code of length 16 is a completely regular code in $J(16,6)$ with $\rho = 2$.*

Proof. Since the minimum distance of the extended Hamming code is 4, any of its codeword of weight 6 is at Hamming distance at least 4 from a codeword of weight 4. This implies that the support of any codeword of the extended Hamming code of weight 6 is always contained in $C = \{x : x \subset \{1,\dots,n\}, |x| = 6, |\{B \in \mathcal{Q}, B \subset x\}| = 0\}$. The definition of a completely regular code and the double counting of edges between C_i and C_{i+1} imply the following:

$$|C|\beta_0 = |C_1|\gamma_1,\ |C_1|\beta_0 = |C_2|\gamma_1,\ |C| + |C_1| + |C_2| = \begin{bmatrix} n \\ 6 \end{bmatrix}_1.$$

The intersection array of C was obtained in Remark 12, so for $n = 16$ we have

$$|C| + |C|\frac{60}{4} + |C|\frac{6}{48} = \begin{bmatrix} 16 \\ 6 \end{bmatrix}_1 = 8008$$

and therefore $|C|$ is 448. Note that there are exactly 448 codewords of weight 6 in extended Hamming code of length 16. We conclude that they coincide with C, which is a completely regular code in $J(16,6)$ by Theorem 11. $\square$

6 Completely regular code in $J_q(n,4)$ from Desarguesian 2-spread

Let $\mathbb{F}'$ be the subfield of the field $\mathbb{F}_{q^n}$ of order q^2. The elements of the multiplicative group of $\mathbb{F}_{q^n}$ are parted into the cosets of that of $\mathbb{F}'$. We treat $\mathbb{F}_{q^n}$ as the vector space $\mathbb{F}_q^n$ and any multiplicative coset of $\mathbb{F}'$corresponds to a 2-subspace of $\mathbb{F}_q^n$. The collection of such subspaces is a 2-spread, which is called *D*esarguesian. A subspace is called $\mathbb{F}'$-*c*losed if its vectors (threated as elements of $\mathbb{F}_{q^n}$) are closed under the multiplication by the elements of $\mathbb{F}'$. In particular, the subspaces of a Desarguesian 2-spread are $\mathbb{F}'$-closed. For a subset of $\mathbb{F}_q^n$ the minimal inclusion-wise $\mathbb{F}'$-closed subspace that contains the subset is called its $\mathbb{F}'$-*closure*.

Any 2-spread $\mathcal{L}$ and all 3-subspaces of $\mathbb{F}_q^n$ that do not contain any subspace from $\mathcal{L}$ are completely regular codes in $J_q(n,2)$ and $J_q(n,3)$ respectively by Corollary 6 and Theorem 9. In case when $\mathcal{L}$ is a Desarguesian spread we will show that the code $\{U : U < \mathbb{F}_q^n, dim(U) = 4, |\{X \in \mathcal{L} : X < U\}| = 0\}$ is a completely regular code in $J_q(n,4)$.

Consider a 4-subspace U of $\mathbb{F}_q^n$. It can contain 0, 1 or at least 2 subspaces from $\mathcal{L}$. Suppose X and X' are 2-subspaces from $\mathcal{L}$ that are contained in U. Because X and X' meet only in a zero vector, all vectors of U are linear combinations of the vectors of X and X'. Moreover, since X and X' are $\mathbb{F}'$-closed, so is U. In other words, the nonzero vectors of U are parted by nonzero vectors from q^2+1 subspaces from $\mathcal{L}$. We have the following partition of the vertices of $J_q(n,4)$:

$$C^0 = \{U : U < \mathbb{F}_q^n, dim(U) = 4, |\{V \in \mathcal{L} : V < U\}| = 0\},$$
$$C^1 = \{U : U < \mathbb{F}_q^n, dim(U) = 4, |\{V \in \mathcal{L} : V < U\}| = 1\},$$
$$C^2 = \{U : U < \mathbb{F}_q^n, dim(U) = 4, |\{V \in \mathcal{L} : V < U\}| = q^2+1\}.$$

We now show some structural properties of these codes.

Lemma 15. *Any* 3*-subspace of* U*,* $U \in C^2$ *contains exactly one subspace from* $\mathcal{L}$*. In particular, the subspaces from* C^0 *and* C^2 *are nonadjacent in* $J_q(n,4)$*.*

Proof. Let W be a 3-subspace of U that does not contain subspaces from $\mathcal{L}$. Since $\mathbb{F}_q < \mathbb{F}'$ we see that the $\mathbb{F}'$-closure of any of nonzero vectors of W meets W in exactly $q-1$ nonzero vectors. We see that the $\mathbb{F}'$-closure of W has at least $(q^3-1)(q+1)$ vectors, so its dimension is at least 5. We found a subspace W of $\mathbb{F}'$-closed 4-subspace U whose $\mathbb{F}'$-closure has dimension 5, a contradiction.

This contradicts that the subspace U is $\mathbb{F}'$-closed and that W is a subspace of U. □

In view of Lemma 15 one might consider C^2 to be the $\mathbb{F}'$-closure of all 3-subspaces that contain exactly one subspace from $\mathcal{L}$. Indeed any such 3-subspace is spanned by a subspace X (which is $\mathbb{F}'$-closed) from $\mathcal{L}$ and a 1-subspace and the $\mathbb{F}'$-closure of the latter one has dimension 2.

Lemma 16. *Any subspace* U*,* $U \in C^1$ *is adjacent to exactly* $q+1$ *subspaces from* C^2 *in* $J_q(n,4)$*.*

Proof. Let $X \in \mathcal{L}$ be that such that $X < U$. Let $V \in C^2$ be adjacent to U, i.e. $dim(U \cap V) = 3$. By Lemma 15 the subspace $U \cap V$ of U must contain X. There are exactly $q+1$ 3-subspaces of the 4-subspace U that contain the given 2-subspace X. Their $\mathbb{F}'$-closures are in C^2 and we obtain the required. □

Theorem 17. *Let* $\mathcal{L}$ *be a Desarguesian* 2*-spread. The code* $\{U : U < \mathbb{F}_q^n, dim(U) = 4, |\{V \in \mathcal{L} : V < U\}| = 0\}$ *is completely regular in* $J_q(n,4)$*.*

Proof. The vertex set of $J_q(n,4)$ is parted into the codes C^0, C^1 and C^2. By Lemmas 15 and 16 there are no edges between C^0 and C^2 and any vertex from C^1 is adjacent to exactly 6 subspaces from C^2.

Since $\mathcal{L}$ is a completely regular code with covering radius 1, we see that $\{\mathcal{L}, \overline{\mathcal{L}}\}$ is the partition associated to an eigenvector v of $J_q(n,2)$ with eigenvalue $\theta_{2,q}(n,2)$. By Theorem 8 the vector $I_{4,2}v$ is an eigenvector of $J_q(n,4)$ with eigenvalue $\theta_{2,q}(n,4)$. The definition of the inclusion matrix $I_{4,2}$ and the definition of the codes C^0, C^1, C^2 imply that these codes form the partition associated to $I_{6,4}v$. The result follows from Theorem 7. $\square$

We note that following the proof of Theorem 7 one can obtain that the code given in Theorem 17 has the intersection array

$$\{\begin{bmatrix}4\\1\end{bmatrix}_q\begin{bmatrix}3\\1\end{bmatrix}_q, q+1; q^5(q+1)\begin{bmatrix}n-6\\1\end{bmatrix}_q, \begin{bmatrix}4\\1\end{bmatrix}_q\begin{bmatrix}n-4\\1\end{bmatrix}_q q\}.$$

7 Completely regular codes with $\rho = 1$ in $J_2(6,3)$

We start this section with formulating the existence problem of completely regular codes with prescribed automorphism group as a binary linear programming problem. This approach showed good results for codes in Grassmann, halved cube and Star graphs [4], [22], [29].

Throughout this section by $t$he automorphism group of a code (subset of the vertices) in a graph we mean the setwise stabilizer of the code in the automorphism group of the graph. Let G be a subgroup of the automorphism group of a m-regular graph Γ. Let $O_1, \dots, O_r$ be the orbits of the action of G on the vertex set of Γ. Because $O_1, \dots, O_r$ are orbits we see that given any i, $j \in \{1, \dots, r\}$ any vertex x of O_i is adjacent to exactly A_{ij} vertices of O_j and A_{ij} does not depend on x.

Let A be the matrix $\{A_{ij}\}_{i,j\in\{1,\dots,r\}}$. Suppose the automorphism group of a code C has a subgroup G. We consider the characteristic vector $\chi_{C,G}$ of C in the orbits $O_1, \dots, O_r$, i.e. $(\chi_{C,G})_i = 1$ if and only if $O_i \subseteq C$ and zero otherwise. If $\mathbf{1}$ is the all-one vector then $\mathbf{1} - \chi_{C,G}$ is the characteristic vector of the complement of C. The notations above imply that

$$A\chi_{C,G} = (m - \beta_0)\chi_{C,G} + \gamma_1(\mathbf{1} - \chi_{C,G}) \tag{6}$$

holds if and only if C is a completely regular code in Γ with covering radius 1, intersection array $\{\beta_0; \gamma_1\}$ and G is a subgroup of its automorphism group.

As $\chi_{C,G}$ is a binary vector one might consider (6) to be a binary linear programming problem with the binary variable vector $\chi_{C,G}$. From this perspective with the help of computer we show the existence of completely regular codes in the Grassmann graph $J_2(6,3)$.

Let a be a primitive element of $\mathbb{F}_{2^6}$. We set G_{21} to be the group generated by the multiplication of the vectors of $\mathbb{F}_2^6$ (treated as the elements of $\mathbb{F}_{2^6}$) by a^{21}. The vertices of $J_2(6,3)$ are parted into 465 orbits of the group G_{21}.

Let C be a completely regular code in the graph $J_2(6,3)$ with intersection array $\{\beta_0; \gamma_1\}$. Due to Corollary 5 the eigenvalues of C are the valency of $J_2(6,3)$ which is

98 and

$$98 - \beta_0 - \gamma_1 = \theta_{i,2}(6,3), \tag{7}$$

where $i-1$ is the strength of C as a design.

Let the eigenvalue $98 - \beta_0 - \gamma_1$ be $\theta_{2,2}(6,3)$, i.e. C is a q-ary 1-design. In this case a binary linear programming solver found solutions of system (6) for 8 different values of γ_1. Two of the constructed codes (with $\gamma_1 = 9$ and 21) were previously obtained in [13]. With exception of these two codes, all other constructed codes have G_{21} as their full automorphism group. This fact makes a further generalization of these codes to other Grassmann graphs difficult as each code consist of at least 60 orbits, each of size 3.

Theorem 18. *There are completely regular codes in $J_2(6,3)$ with covering radius 1, intersection array $\{93-\gamma_1;\gamma_1\}$ for any $\gamma_1 \in \{12,15,18,24,27,30\}$ such that G_{21} is their automorphism group.*

From (7) we see that the intersection array $\{\beta_0;\gamma_1\}$ of any completely regular code could be found from the strength of the code and γ_1. We summarize the information on the intersection arrays of the completely regular codes in $J_2(6,3)$ with $\rho = 1$ in Table 1. By integer conditions in the table we mean the integer necessary existence conditions for designs imposed by the second statement of Corollary 5.

Eigen-value	Design strength	Integer conditions	Nonexis-, tence, γ_1	Existence, γ_1	Open cases, γ_1
35	0	$\gamma_1 \mod 7 = 0$	$21^F, 28^F$	$7^H, 14^{HP}$	
5	1	$\gamma_1 \mod 3 = 0$	$3^{M''}$	$9^M, 21^{M'}$, $3l, l = 4,5,6$ $8,9,10^A$	$3l$, $l \in \{2\} \cup$ $\{10,\dots,15\}$
-7	2	$\gamma_1 \mod 21 = 0$		$21^B, 42^B$	

Table 1: Completely regular codes in $J_2(6,3)$ with $\rho = 1$. The intersection array $\{\beta_0;\gamma_1\}$ of any completely regular code is obtained from its strength and γ_1 using (7).

^{F} completely regular codes with covering radius 1 and strength 0 in $J_2(n,k)$ were classified in [16];
^{H} subspaces belonging to a hyperplane;
^{HP} subspaces are in a hyperplane H or contain a vector v, where $v \notin H$;
^{A} exists by Theorem 18;
^{B} correspond to 2-$(6,3,3)_2$- and 2-$(6,3,6)_2$-designs, which exist by [8];
^{M} totally isotropic subspaces of a symplectic polarity [13, Example 6];
$^{M'}$ 3-subspaces that do not contain subspaces from a 2-spread;
[13, Example 5], see also Theorem 9;
$^{M''}$ nonexistence follows from [13, Lemma 21].

Remark 19. Apart from the integer necessary conditions there are other techniques for proving the nonexistence of a completely regular code with $\rho = 1$ given a putative intersection array. One of them is a method based on the finding weight distribution of the code [2, Theorem 1] (see also [21]). In the case of Johnson and Hamming graphs some completely regular codes have intersection arrays feasible by integer necessary conditions but infeasible by the weight distribution method [5], [21]. However, in the particular instance of the graph $J_2(6,3)$ this approach is not stronger than the integer necessary conditions. Another techniques could be also used, like counting argument in [13, Lemma 21] that implies the nonexistence of the completely regular codes with strength 1 and $\gamma_1 = 3$ in this graph.

The only known completely regular codes in $J_2(6,3)$ with $\rho = 2$ are the code of the subspaces containing a fixed 2-space and 3-spread in $J_2(6,3)$ (Corollary 6).

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References

[1] S. V. Avgustinovich and I. Yu. Mogilnykh. Induced perfect colorings. *Sib. Electron. Math. Rep.*, 8:310–316, 2011.

[2] S. V. Avgustinovich and I. Yu. Mogilnykh. Perfect 2-colorings of Johnson graphs J(6, 3) and J(7, 3). In *Coding Theory and Applications (Second International Castle Meeting), ICMCTA 2008, Castillo de la Mota, Medina del Campo, Spain*, volume 5228 of *Lecture Notes in Comput. Sci.*, pages 11–19. Springer, 2008.

[3] S. V. Avgustinovich and V. N. Potapov. Combinatorial designs, difference sets, and bent functions as perfect colorings of graphs and multigraphs. *Siberian Mathematical Journal*, 61(5):867–877, 2020.

[4] J. Bamberg. There is no Cameron–Liebler line class of $PG(3,4)$ with parameter 6, https://symomega.wordpress.com/2012/04/01/there-is-no-cameron-liebler-line-class-of-pg34-with-parameter-6/.

[5] E. A. Bespalov, D. S. Krotov, A. A. Matiushev, A. A. Taranenko, and K. V. Vorob'ev. Perfect 2-colorings of Hamming graphs. *Journal of Combinatorial Designs*, 29(6):367–396, 2021.

[6] J. Borges, J. Rifá, and V.A. Zinoviev. On completely regular codes. *Problems of Information Transmission*, 55:1–45, 2019.

[7] M. Braun, T. Etzion, P. R. J. Ostergard, A. Vardy, and A. Wassermann. Existence of q-analogs of Steiner systems. *Forum of Mathematics, Pi*, 4, E7, 2016.

[8] M. Braun, A. Kerber, and R. Laue. Systematic Construction of q-Analogs of $t-(v,k,l)$-designs. *Designs, Codes and Cryptography*, 34:55–70, 2005.

[9] P. J. Cameron and R. A. Liebler. Tactical decompositions and orbits of projective groups. *Linear Algebra and Its Applications*, 46:91–102, 1982.

[10] L. Chihara. On the zeros of the Askey-Wilson polynomials, with applications to coding theory. *SIAM Journal on Mathematical Analysis*, 18(1):191–207, 1987.

[11] D. M. Cvetkovic, M. Doob, and H. Sachs. Spectra of Graphs. Academic Press, New York, London, 1980.

[12] B. De Bruyn and H. Suzuki. Intriguing sets of vertices of regular graphs. *Graphs and Combinatorics*, 26(5):629–646, 2010.

[13] S. De Winter and K. Metsch. Perfect 2-Colorings of the Grassmann Graph of Planes. *The Electronic Journal of Combinatorics*, 27(1):#P1.21, 2020.

[14] P. Delsarte. An algebraic approach to the association schemes of coding theory. Philips Res. Rep. Suppl., no. 10., 1973.

[15] T. Etzion and M. Schwarz. Perfect Constant-Weight Codes, *IEEE Transactions on Information Theory*, 50(9):2156–2165, 2004.

[16] Y. Filmus and F. Ihringer. Boolean degree 1 functions on some classic association schemes. *Journal of Combinatorial Theory, Series A*, 162:241–270, 2019.

[17] D. G. Fon-Der-Flaass. Perfect 2-colorings of a hypercube. *Siberian Mathematical Journal*, 48(4):740–745, 2007.

[18] A. L. Gavrilyuk and I. Y. Mogilnykh. Cameron-Liebler line classes in PG(n,4). *Designs, Codes, and Cryptography*, 73(3):969–982, 2014.

[19] M. D. Gordon. Perfect single error-correcting codes in the Johnson scheme. *IEEE Transactions on Information Theory*, 52(10):4670–4672, 2006.

[20] W. M. Kantor. On incidence matrices of finite projective and affine space. *Mathematische Zeitschrift*, 24:315–318, 1972.

[21] D. S. Krotov. On weight distributions of perfect colorings and completely regular codes. *Designs, Codes, and Cryptography*, 61(3):315–329, 2011.

[22] D. S. Krotov, I. Yu. Mogilnykh, and A. Yu. Vasil'eva. On completely regular codes of covering radius 1 in the halved hypercubes. arXiv:1812.03159.

[23] C. Maldonado and D. Penazzi. Lattices and Norton algebras of Johnson. Grassmann and Hamming graphs. arXiv:1204.1947.

[24] W. J. Martin. Completely regular designs. *Journal of Combinatorial Designs*, 4:261–273, 1998.

[25] W. J. Martin. Completely regular designs of strength one. *Journal of Algebraic Combinatorics*, 3(2):177–185, 1994.

[26] W. J. Martin. Completely regular subsets. Ph.D. Thesis, University of Waterloo, 1992.

[27] A. Meyerowitz. Cycle-balanced conditions for distance-regular graphs. *Discrete Mathematics*, 264(3):149–166, 2003.

[28] I. Yu. Mogilnykh. On the regularity of perfect 2-colorings of the Johnson graph. *Problems of information transmission*, 43(4):271–277, 2007.

[29] I. Mogilnykh. Codes in Star and Bubble-sort graphs. International Conference Automatics and Informatics (ICAI), Varna, Bulgaria: 230–233, 2021.

[30] I. Mogilnykh and A. Valyuzhenich. Equitable 2-partitions of the Hamming graphs with the second eigenvalue. *Discrete Mathematics*, 343(11):112039, 2020.

[31] A. Neumaier. Completely regular codes. *Discrete Mathematics*, 106/107(1):353–360, 1992.

[32] K. Vorob'ev. Equitable 2-partitions of Johnson graphs with the second eigenvalue. arXiv:2003.10956.