

An Equivariant Quantum Pieri Rule for the Grassmannian on Cylindric Shapes

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Abstract

The quantum cohomology ring of the Grassmannian is determined by the quantum Pieri rule for multiplying by Schubert classes indexed by row or column-shaped partitions. We provide a direct equivariant generalization of Postnikov's quantum Pieri rule for the Grassmannian in terms of cylindric shapes, complementing related work of Gorbounov and Korff in quantum integrable systems. The equivariant terms in our Graham-positive rule simply encode the positions of all possible addable boxes within one cylindric skew diagram. As such, unlike the earlier equivariant quantum Pieri rule of Huang and Li and known equivariant quantum Littlewood-Richardson rules, our formula does not require any calculations in a different Grassmannian or two-step flag variety.

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1 Introduction

Equivariant cohomology is a powerful tool for studying the geometry and topology of algebraic varieties which admit an action by a reductive algebraic group; see the sur-

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vey [And]. With key ingredients appearing in Cartan’s complex of equivariant differential forms [Car], the Borel construction for equivariant singular cohomology dates back to the 1950’s [Bor]. The prevailing philosophy in the equivariant setting is to exploit the symmetries arising from the group action by replacing topological questions with a limited amount of algebraic information. For example, the theory of localization following [GKM] permits the equivariant cohomology of nice topological spaces, such as flag varieties and Grassmannians, to be represented using a graph determined by the fixed points and one-dimensional orbits of the group action; see [GZ] for an overview. Kim’s construction of the equivariant Gromov-Witten invariants [Kim] extends these principles to the (small) equivariant quantum cohomology of flag varieties, and thus to modern equivariant quantum Schubert calculus via various equivariant quantum-to-classical phenomena [BM, BMT].

1.1 Equivariant quantum Schubert calculus

The equivariant quantum cohomology ring for the Grassmannian $\text{Gr}(\mathbf{m}, \mathbf{n})$ has a basis of *Schubert classes*, indexed by partition shapes which fit inside an $\mathbf{m} \times (\mathbf{n} - \mathbf{m})$ rectangle. The fundamental question in *equivariant quantum Schubert calculus* is to find combinatorial formulas for the expansion of the product of two Schubert classes in terms of this Schubert basis. A Graham-positive *equivariant quantum Littlewood-Richardson rule* for $\text{QH}_T^*(\text{Gr}(\mathbf{m}, \mathbf{n}))$ is now available, by combining Buch’s generalization of Knutson and Tao’s equivariant puzzle rule [KT] to two-step flag varieties [Buc2], together with Buch and Mihalcea’s quantum-to-classical principle equating the equivariant quantum Littlewood-Richardson coefficients for $\text{Gr}(\mathbf{m}, \mathbf{n})$ to their classical counterparts in a related two-step flag variety [BM].

Other formulas for the equivariant quantum Littlewood-Richardson coefficients exist, the first of which was Mihalcea’s *equivariant quantum Pieri-Chevalley rule* for multiplying by the Schubert class corresponding to a single box [Mih2], which recursively determines the ring structure of $\text{QH}_T^*(\text{Gr}(\mathbf{m}, \mathbf{n}))$. Any of the numerous equivariant Littlewood-Richardson rules for $\text{H}_T^*(\text{Gr}(\mathbf{m}, \mathbf{n}))$ can be combined with the *equivariant rim hook rule* of the first, third, and fourth authors [BMT] to produce a signed method for computing products in $\text{QH}_T^*(\text{Gr}(\mathbf{m}, \mathbf{n}))$. Gorbounov and Korff take an integrable systems approach to provide an explicit determinantal formula for the equivariant quantum Littlewood-Richardson coefficients [GK2, GK3].

Without performing calculations in a related two-step flag variety, the most general Graham-positive combinatorial formula for the equivariant quantum Littlewood-Richardson coefficients is the *equivariant quantum Pieri rule* for multiplying by a Schubert class indexed by a row or column-shaped partition. Huang and Li proved the first equivariant quantum Pieri rule for any type A partial flag variety in Theorem 3.10 of [HL], though in the special case of the Grassmannian, this formula can also be recovered from Robinson’s earlier full flag Pieri rule [Rob], for which Li, Ravikumar, Sottile, and Yang later provided an alternate geometric proof [LRSY]. We remark that Li and Ravikumar have also generalized the underlying equivariant Pieri rule for the Grassmannian to other classical Lie types [LR]. The critical component in each of the aforementioned Pieri rules is the interpretation of certain equivariant quantum Littlewood-Richardson coefficients as *localizations* at a given T-fixed point of a Schubert variety in a smaller

Grassmannian. Gorbounov and Korff take a different approach in [GK2, GK3], providing an equivariant quantum Pieri rule in terms of an action by vicious and osculating walker transfer matrices in the Yang-Baxter algebra.

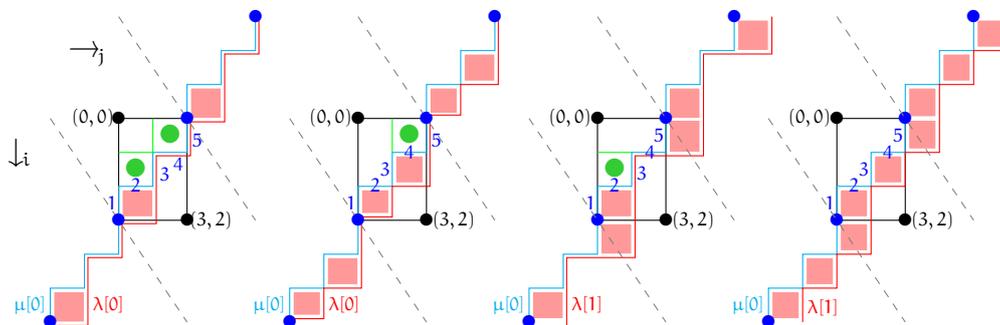


Figure 1: Equivariant cylindric shapes to calculate $\sigma_{(1)^3} \star \sigma_{(2,1)} \in \text{QH}_T^*(\text{Gr}(3, 5))$.

The goal of this paper is to provide a closed combinatorial Graham-positive equivariant quantum Pieri rule for $\text{Gr}(\mathbf{m}, \mathbf{n})$, formulated in terms of an equivariant generalization of *cylindric shapes*; see Figure 1 for an illustration which is fully explained in Example 16. Introduced by Gessel and Krattenthaler [GK1], cylindric shapes have since appeared in many mathematical contexts. In representation theory, cylindric diagrams enter Suzuki and Vazirani’s work on representations of the double affine Hecke algebra [SV]. In mathematical physics, Korff defines cylindric versions of specializations of skew Macdonald polynomials [Kor], which arise as partition functions of vertex models obtained by solving the Yang-Baxter equation. Arising from conjectures of McNamara [McN] recently settled by Lee [Lee], the cylindric (skew) Schur functions are a family of symmetric functions indexed by cylindric shapes, and they play an important role in algebraic combinatorics. Central in Lee’s work is the result of Lam that all cylindric Schur functions are affine Schur functions [Lam], which provides a connection to positroid varieties as studied by Knutson, Lam, and Speyer [KLS]. Cylindric shapes also arise in previous works on the quantum cohomology of the Grassmannian, implicitly in the development of quantum Kostka numbers by Bertram, Ciocan-Fontanine, and Fulton [BCFF], as well as explicitly in the quantum Pieri rule of Postnikov [Pos]; see Theorem 4 below. Gorbounov and Korff’s version of the equivariant quantum Pieri rule translates the graphical calculus of non-intersecting lattice paths into the language of toric skew shapes [GK2, GK3].

1.2 Statement of the main theorem

Postnikov’s version of the quantum Pieri rule provides an affine approach to $\text{QH}^*(\text{Gr}(\mathbf{m}, \mathbf{n}))$, and several additional quantum-to-affine connections from [Pos] are solidified in Knutson, Lam, and Speyer’s related work on positroid varieties [KLS]. A key motivation for our search for particular equivariant generalizations of these quantum-to-affine phenomena is the parabolic Peterson isomorphism [Pet, LS1], which implies that the equivariant homology of the affine Grassmannian surjects onto $\text{QH}_T^*(\text{Gr}(\mathbf{m}, \mathbf{n}))$, up to localization of the quantum parameter, in a manner which preserves the relevant Schubert calculus.

Inspired by these affine connections, we present an equivariant generalization of the quantum Pieri rule from [Pos], or equivalently closed combinatorial formulas for the equivariant quantum Pieri expansions provided in [GK2, GK3], as our main result. In this context, the equivariant terms are generated by *addable boxes* on the cylindric skew shape, which are depicted by the boxes with \bullet in Figure 1 and whose contributions are denoted by α in Theorem 1 below. For the reader who is familiar with similar combinatorial formulas, the following is our main theorem statement. All terminology is made precise in Section 2, but here we describe the key components in the product as they relate to Figure 1, with forward references to the relevant definitions in Section 2.

Theorem 1 (Equivariant Quantum Pieri Rule on Cylindric Shapes). *For any integers $1 \leq p \leq m$ and $1 \leq k \leq n - m$ and any partition $\mu \in P_{mn}$, we have*

$$\sigma_{(1)^p} \star \sigma_\mu = \sum_{0 \leq r \leq p} \sum_{\substack{\lambda/d/\mu = v^r \\ v^r \rightarrow v^p}} q^d \prod_{\alpha \in v^p \setminus v^r} \text{wt}^v(\alpha) \sigma_\lambda \quad \text{and}$$

$$\sigma_{(k)} \star \sigma_\mu = \sum_{0 \leq r \leq k} \sum_{\substack{\lambda/d/\mu = h^r \\ h^r \rightarrow h^k}} q^d \prod_{\alpha \in h^k \setminus h^r} \text{wt}^h(\alpha) \sigma_\lambda$$

in $\text{QH}_\mp^*(\text{Gr}(m, n))$. Since $\lambda/d/\mu$ is a vertical (resp. horizontal) strip, then $d \in \{0, 1\}$.

The inner sum in Theorem 1 ranges over cylindric skew diagrams $\lambda/d/\mu$ which are vertical (resp. horizontal) strips, as depicted by the red shaded boxes in Figure 1. These vertical (resp. horizontal) r -strips are then extended to full p - (resp. k)-strips by adding boxes which preserve the nature of the original vertical (resp. horizontal) strip, as do the boxes containing \bullet in Figure 1; see Definition 7 for details. The weight of each addable box is then easily calculated from several statistics which record the location of the box \bullet within the portion of the skew shape contained in the ambient $m \times (n - m)$ rectangle, outlined in black in Figure 1. We make the weight of an addable box precise in Definition 11, and we explicitly record the weights corresponding to each of the four cylindric shapes appearing in Figure 1 in Example 16. It is immediate from Definition 11 that both formulas in Theorem 1 are manifestly Graham-positive.

In comparison to other known equivariant quantum Pieri and/or Littlewood-Richardson rules, Theorem 1 more directly captures the quantum-to-affine phenomenon which governs the ring structure of $\text{QH}_\mp^*(\text{Gr}(m, n))$ via the parabolic Peterson isomorphism, as explicated in [CM] in the non-equivariant case. In addition, each of the existing closed combinatorial formulas for these products typically requires doing calculations in many different related two-step flags or smaller Grassmannians in order to calculate a single Pieri product; compare [HL, BM, Buc2]. We emphasize, by contrast, that Theorem 1 permits an entire equivariant quantum Pieri product to be carried out directly on the skew shapes $\lambda/d/\mu$ as in Figure 1, which fully illustrates the product $\sigma_{[pcY]} \star \sigma_{[pcY]} \in \text{QH}_\mp^*(\text{Gr}(3, 5))$; see Example 16. As such, even the classical case of Theorem 1 is quite distinct from existing formulations of the equivariant Pieri and Littlewood-Richardson rules for $H_\mp^*(\text{Gr}(m, n))$. Moreover, as we explain in the next subsection, Theorem 1 provides an ideal starting point for establishing equivariant generalizations of many related phenomena in combinatorics, representation theory, and mathematical physics, rather than serving exclusively as an illuminating and convenient means to an equivariant quantum Schubert calculus end.

The proof of Theorem 1 proceeds by comparing the classical terms indexed by cylindric shapes with the equivariant Pieri rule from [HL] reviewed here as Theorem 20. More specifically, we prove in Proposition 28 that the weight determined by a configuration of addable boxes is the same whether calculated on the cylindric skew shape λ/μ or on the partition λ_μ defined in [HL] by a join-and-cut algorithm which passes to a smaller Grassmannian. In Lemma 32, we then equate the equivariant Littlewood-Richardson coefficients appearing in Theorem 1 with certain specializations of elementary factorial Schur polynomials. These specialized factorial Schur polynomials agree with the classical terms appearing in Huang and Li's equivariant Pieri formula via the localization formula due to [KT, LRS, IN] reviewed as Theorem 31. Our equivariant quantum Pieri rule on cylindric shapes then follows by applying the equivariant rim hook rule of the first, third, and fourth authors [BMT] reviewed as Theorem 5.

1.3 Discussion of related work

We now highlight some applications of Theorem 1 suggested by related literature in the non-equivariant setting. Although the appropriate equivariant analog is often a direct consequence of the combinatorics developed here, an in-depth pursuit of any of the generalizations discussed in this subsection is beyond the scope of the present paper.

Both [Pos] and [BCFF] define quantum Kostka numbers by iterating the quantum Pieri rule. These quantum Kostka numbers are the coefficients in the monomial expansion of the cylindric Schur functions, a specialization of which give the toric Schur polynomials whose Schur expansions encode all quantum Littlewood-Richardson coefficients; see [Pos, Theorem 5.3]. Iterating the equivariant quantum Pieri rule in Theorem 1 thus yields an immediate equivariant generalization for each of these phenomena, which all carry important representation-theoretic meaning in the non-equivariant setting. For example, quantum Kostka numbers coincide with Wess-Zumino-Novikov-Witten (WZNW) fusion coefficients; compare the generalization [Kor, Conjecture 1.1] concerning cylindric skew Macdonald functions, special cases of which connect to Kostka-Foulkes polynomials, graded characters of Demazure modules of the current algebra, and Feigin-Loktev fusion products of Kirillov-Reshetikhin modules. To date, the only explicit reference to equivariant quantum Kostka numbers in the literature is the unpublished work of Gorbounov and Korff [GK2, Corollary 6.27], in which they are defined by iterating an equivariant quantum Pieri rule phrased in terms of vicious and osculating walkers. As such, Theorem 1 also provides a simple combinatorial model for performing such calculations in quantum integrable systems.

The cylindric Schur functions of [Pos] are examples of affine Schur functions [Lam], also known as the dual k -Schur functions which represent Schubert classes in the cohomology of the affine Grassmannian, as one might expect by applying the parabolic Peterson isomorphism [Pet, LS1]. Analogously, the equivariant version of the cylindric Schur functions obtained by iterating Theorem 1 should give rise to a family of affine double Schur functions. By [LS3, Theorem 27], specializing one set of variables in an affine double Schur function recovers the localizations of the corresponding affine Schubert class at a T -fixed point, deepening the geometric meaning of this family of symmetric functions, and simultaneously illuminating the critical role of localization in the proof of

Theorem 1. The equivariant k -Kostka numbers are defined in [LS3] by expanding the affine double Schur functions in terms of the double monomial symmetric functions, or equivalently by iterating the affine equivariant Pieri rule of [LS2], which offers an affine approach to studying equivariant quantum Kostka numbers. The authors also expect the cyclic factorial Schur functions defined in [BMT] to configure into this same framework of affine symmetric functions.

Many additional applications of Theorem 1 are implicit in Postnikov's wide-ranging work [Pos]. There are natural interpretations of the correspondence between cylindric shapes and non-vanishing monomials in the affine nil-Temperley-Lieb algebra, which would give rise to an equivariant generalization of [Pos, Proposition 8.5], as well as its physical interpretation in [KS, Theorem 10.11]. This perspective also yields equivariant analogs of the pairwise commuting Dunkl elements in the quadratic algebra which Fomin and Kirillov use to describe the quantum cohomology of the complete flag variety [FK]. Postnikov defines a representation of the symmetric group by its action on the toric Specht module, and Yoo develops a cylindric version of jeu de taquin to study certain families of these toric Specht modules [Yoo]. Since the quantum Littlewood-Richardson coefficients conjecturally determine the decomposition of toric Specht modules into irreducible components, an algebraic explanation of the Graham-positivity of the equivariant Littlewood-Richardson coefficients results from this representation-theoretic approach. See [Pos, Section 9] for further discussion of open problems, each of which now has a natural equivariant analogue.

1.4 Organization of the paper

We begin with some background on the equivariant quantum Schubert calculus of the Grassmannian in Section 2. In particular, we formally review Postnikov's quantum Pieri rule on cylindric shapes from [Pos] as Theorem 4, and we make all additional terminology appearing in Theorem 1, our equivariant generalization of Postnikov's quantum Pieri rule, precise in Section 2.4. The localizations occurring in the equivariant Pieri rule of [HL] are developed in Section 3, which permits a comparison to the equivariant Littlewood-Richardson coefficients from Theorem 1 by recognizing both as certain specializations of factorial Schur polynomials in Section 4. The proof of Theorem 1 then follows in Section 4.3 by applying the equivariant rim hook rule of [BMT].

2 Quantum Pieri rules on cylindric shapes

The purpose of this section is to briefly develop the Schubert calculus required to formally state our main theorem. We review the combinatorics of the quantum cohomology of the Grassmannian in Section 2.1, and we formulate Postnikov's quantum Pieri rule on cylindric shapes in Section 2.2. We pass to the equivariant setting in Section 2.3, and we present our equivariant quantum Pieri rule on cylindric shapes as Theorem 1 in Section 2.4. Throughout the paper, we fix a pair of integers $m, n \in \mathbb{N}$.

2.1 Quantum cohomology of the Grassmannian

Let $G = \mathrm{SL}_n$, fix a Borel subgroup B and a split maximal torus T , and denote the associated Weyl group by $W = S_n$. Fix a Cartan subalgebra \mathfrak{h} of $\mathfrak{g} = \mathrm{Lie}(G)$. The choice

of B determines a set of positive roots R^+ , in the sense that $\mathfrak{b} = \text{Lie}(B) = \mathfrak{h} \oplus (\bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha)$. Denote by $\Delta = \{\alpha_i\}_{i=1}^{n-1}$ a basis of simple roots, satisfying that any root in $\bigoplus_{\mathbb{Z}_{\geq 0}} \alpha_i$ is positive with respect to B . In this paper, we choose B to be the subgroup of lower-triangular matrices and T the diagonal matrices, in which case $\alpha_i = e_{i+1} - e_i$ where e_i denotes the standard coordinate vector for the lattice \mathbb{Z}^n . There is a bijection between the simple roots $\alpha_i \in \Delta$ and the generators $s_i \in S$ for the Coxeter group (W, S) . The standard parabolic subgroups $P \supseteq B$ are in bijection with subsets $\Delta_P \subseteq \Delta$, and the associated Weyl group W_P is generated by those simple reflections corresponding to the elements of Δ_P . The set of minimal length coset representatives in the quotient W/W_P is denoted by W^P , using the standard length function on (W, S) .

The homogeneous variety G/P admits a Bruhat decomposition into *opposite Schubert cells*, given by $G/P = \bigsqcup_{w \in W^P} B^- wP/P$. Here, B^- denotes the *opposite Borel subgroup*, defined by $B \cap B^- = T$. Following the conventions in this paper, B^- is the subgroup of upper-triangular matrices. For any $w \in W^P$, we denote the corresponding *Schubert variety* by $X_w := \overline{B^- wP/P}$. The cohomology of G/P admits an additive \mathbb{Z} -basis of *Schubert classes* σ_w indexed by $w \in W^P$, where σ_w denotes the Poincaré dual of the fundamental class $[X_w]$. Throughout this paper, we focus exclusively on the *maximal* parabolic subgroups P , equivalently $\Delta_P = \Delta \setminus \{\alpha_m\}$ for a fixed $m \in [n-1]$.

When P is maximal, the complex algebraic variety G/P is the *Grassmannian* $\text{Gr}(m, n)$, whose points consist of all m -dimensional subspaces of \mathbb{C}^n . The cohomology of $\text{Gr}(m, n)$ in this case is determined by a Schubert basis which is indexed by the set P_{mn} of *partitions* $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}_{\geq 0}^m$ such that $n - m \geq \lambda_1 \geq \dots \geq \lambda_m$. Equivalently, $\lambda \in P_{mn}$ determines a *Young diagram* in English notation by drawing λ_i left-justified boxes in row i , placing row 1 at the top. Thus, $\lambda \in P_{mn}$ if and only if its Young diagram fits inside an $m \times (n - m)$ rectangle. The number of boxes in $\lambda \in P_{mn}$ is denoted by $|\lambda| := \sum \lambda_i$.

When the homogeneous space G/P is identified with the Grassmannian $\text{Gr}(m, n)$, the Schubert variety X_λ associated to the partition $\lambda \in P_{mn}$ is determined by the rank conditions

$$X_\lambda := \{V \in \text{Gr}(m, n) \mid \dim(V \cap F_{r_i}) \geq i, \text{ for } i \in [m]\},$$

where $r_i := \lambda_{m-i+1} + i$ and $F_{r_i} := \text{Span}(e_1, \dots, e_{r_i})$ denotes the subspace spanned by the first r_i column vectors of the standard basis for \mathbb{C}^n . The Schubert class σ_λ is Poincaré dual to the fundamental class $[X_\lambda]$. These Schubert classes form an additive \mathbb{Z} -basis for the ring $H^*(\text{Gr}(m, n))$, and the product of two Schubert classes $\sigma_\lambda \cdot \sigma_\mu = \sum c_{\lambda, \mu}^\nu \sigma_\nu$ expands non-negatively in this basis. The *Littlewood-Richardson coefficients* $c_{\lambda, \mu}^\nu \in \mathbb{Z}_{\geq 0}$ are well understood, with many available combinatorial interpretations; see [Ful] for an overview.

If we introduce a parameter q of degree n , the *quantum cohomology of the Grassmannian* is defined as $\text{QH}^*(\text{Gr}(m, n)) := \mathbb{Z}[q] \otimes H^*(\text{Gr}(m, n))$, and thus admits a $\mathbb{Z}[q]$ -basis of Schubert classes indexed by $\lambda \in P_{mn}$, which we again denote by σ_λ . The *quantum Littlewood-Richardson coefficients* are determined by the product of two quantum Schubert classes

$$\sigma_\lambda * \sigma_\mu = \sum_{\nu, d} c_{\lambda, \mu}^{\nu, d} q^d \sigma_\nu,$$

where the sum now additionally ranges over $d \in \mathbb{Z}_{\geq 0}$. The quantum Littlewood-Richardson coefficients are also non-negative integers, and may be computed by various rules, including several variations on the quantum-to-classical principle; see [BCFF, BKT]. The

classical Littlewood-Richardson coefficients are recorded as the degree $\mathbf{d} = \mathbf{0}$ terms in the associated quantum product.

2.2 A quantum Pieri rule on cylindric shapes

The product in $\mathrm{QH}^*(\mathrm{Gr}(\mathbf{m}, \mathbf{n}))$ is completely determined by the *quantum Pieri rule*, a closed combinatorial formula for multiplying by the quantum Schubert classes $\sigma_{(1)^k}$ or $\sigma_{(k)}$; that is, classes indexed by column or row-shaped partitions. The first quantum Pieri rule for the Grassmannian was obtained by Bertram using Grothendieck's quot scheme [Ber], and Buch then gave a linear-algebraic proof [Buc1]. The quantum Pieri rule can also be easily recovered from its classical counterpart using the rim hook algorithm of Bertram, Ciocan-Fontanine, and Fulton [BCFF].

An alternative quantum Pieri rule was formulated by Postnikov in terms of cylindric shapes [Pos]. Postnikov's rule more directly captures the quantum-to-affine phenomenon which governs the ring structure of $\mathrm{QH}^*(\mathrm{Gr}(\mathbf{m}, \mathbf{n}))$ via the parabolic Peterson isomorphism, as explicated by the third author in joint work with Cookmeyer [CM]. The purpose of this subsection is to review the version of the quantum Pieri rule from [Pos], which we record as Theorem 4 below.

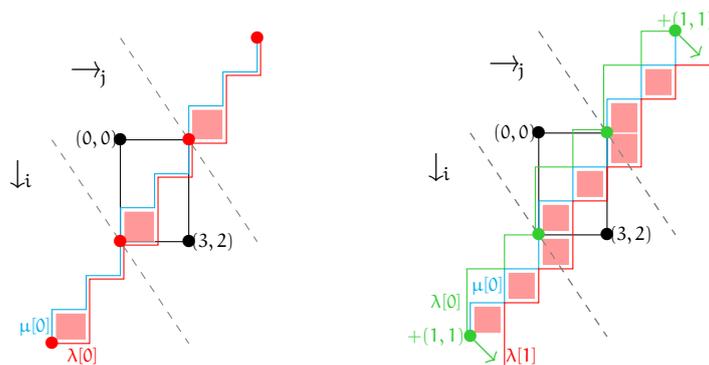


Figure 2: The cylindric loop $\mu[0]$ for $\mu = (2, 1)$ on the cylinder \mathcal{C}_{35} in blue. A cylindric diagram $\lambda/\mathbf{d}/\mu$ in red, with $\lambda = (2, 1, 1)$ and $\mathbf{d} = \mathbf{0}$ on the left, and $\lambda = (1)$ and $\mathbf{d} = \mathbf{1}$ on the right. The skew shape on the left is both a vertical and horizontal strip, whereas the skew shape on the right is only a vertical strip.

Postnikov associates each partition in $\mathcal{P}_{\mathbf{m}\mathbf{n}}$ to a loop on the *cylinder*

$$\mathcal{C}_{\mathbf{m}\mathbf{n}} := \mathbb{Z}^2 / (-\mathbf{m}, \mathbf{n} - \mathbf{m})\mathbb{Z},$$

where the coordinates $(i, j) \in \mathbb{Z}^2$ follow the conventions for indexing matrix entries, meaning that the i^{th} coordinate increases going down, and the j^{th} coordinate increases going right; see Figure 2 for an illustration of the cylinder \mathcal{C}_{35} . When referring to a *box* of $\mathcal{C}_{\mathbf{m}\mathbf{n}}$, we use the cylinder coordinate $(i, j) \in \mathcal{C}_{\mathbf{m}\mathbf{n}}$ to refer to the box to its northwest, located between coordinates (i, j) and $(i - 1, j - 1)$ in \mathbb{Z}^2 . *Row* k of the cylinder consists of the boxes indexed by $(k, j) \in \mathcal{C}_{\mathbf{m}\mathbf{n}}$, and *column* k of the cylinder consists of the boxes indexed by $(i, k) \in \mathcal{C}_{\mathbf{m}\mathbf{n}}$; note that $\mathcal{C}_{\mathbf{m}\mathbf{n}}$ has exactly \mathbf{m} rows and $\mathbf{n} - \mathbf{m}$ columns.

Given a partition $\lambda = (\lambda_1, \dots, \lambda_m) \in P_{mn}$, define a doubly infinite integer sequence $\lambda[0] := (\dots, \ell_{-1}, \ell_0, \ell_1, \dots)$ by setting $\ell_k := \lambda_k$ for $k \in [m]$, and extending to \mathbb{Z} by defining $\ell_{i+m} := \ell_i - (n - m)$ for all $i \in \mathbb{Z}$. The sequence $\lambda[0]$ determines a *closed loop* on \mathcal{C}_{mn} by plotting $(k, \ell_k) \in \mathbb{Z}^2$ for all $k \in \mathbb{Z}$, and then following the path $(k, \ell_k) \rightarrow (k - 1, \ell_k) \rightarrow (k - 1, \ell_{k-1})$. The constructed path traces out the boundary of λ in the $m \times (n - m)$ rectangle whose northwest corner is the origin. The closed loop $\lambda[0]$ outlines the boundary of the Young diagram for λ periodically in \mathbb{Z}^2 , in a manner which is invariant under translation by the vector $(-m, n - m)$; see the left-hand illustration in Figure 2 for two examples of closed loops. Given any integer $d \in \mathbb{Z}_{\geq 0}$, the *cylindric loop* $\lambda[d]$ is obtained by shifting the image of $\lambda[0]$ in \mathbb{Z}^2 by d steps in the southeast direction, equivalently by adding the vector (d, d) to every point $(i, \ell_i) \in \mathbb{Z}^2$. For any $j \in \mathbb{Z}$, the j^{th} entry of the integer sequence corresponding to the cylindric loop $\lambda[d]$ is denoted by $\lambda[d]_j = \ell_{j-d} + d$. The right-hand illustration in Figure 2 depicts $\lambda[0]$ in green and its shifted image $\lambda[1]$ in red.

Definition 2. Given any $\lambda, \mu \in P_{mn}$ and any $d \in \mathbb{Z}_{\geq 0}$ such that $\lambda[d]_i \geq \mu_i$ for all $i \in [m]$, the *cylindric diagram* $\lambda/d/\mu$ is represented by those boxes of \mathcal{C}_{mn} which are located between the two cylindric loops $\lambda[d]$ and $\mu[0]$.

In Figure 2, the red shaded boxes depict two different cylindric diagrams on \mathcal{C}_{35} .

A (weak) *composition* $\eta = (\eta_1, \dots, \eta_m) \in \mathbb{Z}_{\geq 0}^m$ satisfies $n - m \geq \eta_k \geq 0$ for all $k \in [m]$, and has an associated diagram given by drawing η_k boxes in row k of the $m \times (n - m)$ rectangle. Given a partition $\lambda \in P_{m,n}$, the composition $\eta \subseteq \lambda$ if and only if $\eta_k \leq \lambda_k$ for all $k \in [m]$. If $\eta \subseteq \lambda$, the *skew diagram* λ/η is the set-theoretic difference of the diagrams for λ and η . As such, all skew Young diagrams also correspond to cylindric diagrams of the form $\lambda/0/\eta$. We thus typically write λ/η rather than $\lambda/0/\eta$, and use the term *skew shape* to simultaneously represent the two equivalent perspectives. The left-hand diagram in Figure 2 illustrates this equivalence of skew shapes in both P_{35} and \mathcal{C}_{35} , though for convenience we frequently illustrate only the portion of the skew shape contained in the $m \times (n - m)$ rectangle.

A skew shape is *connected* if for each pair of consecutive rows in the shape, there is also at least one pair of boxes in each row sharing a common edge; neither skew shape depicted in Figure 2 is connected. A *rim hook* is a connected skew shape which does not contain any 2×2 squares, and a k -rim hook for any $k \in \mathbb{N}$ is a rim hook which contains k distinct boxes on \mathcal{C}_{mn} . Importantly, note that the cylindric loop $\lambda[1]$ is equivalently constructed from the Young diagram obtained by adjoining an n -rim hook along the boundary of $\lambda \in P_{mn}$ such that the lower leftmost box is located in position $(m + 1, 1)$; compare the cylindric loops $\lambda[0]$ in green and $\lambda[1]$ in red in the right-hand diagram of Figure 2. Removing all possible n -rim hooks from a Young diagram of any size, in any order, results in the *n-core* partition.

Definition 3. Let $\lambda \in P_{mn}$, and let η be a composition such that $\eta \subseteq \lambda$. The skew shape λ/η is a *vertical r-strip* (resp. *horizontal r-strip*), denoted $\lambda/\eta = v^r$ (resp. $\lambda/\eta = h^r$), if its diagram contains exactly r distinct boxes on \mathcal{C}_{mn} , no two in the same row (resp. column) of the cylinder \mathcal{C}_{mn} .

In Figure 2, the left-hand diagram is both a vertical and horizontal 1-strip, and the right-hand diagram is a vertical 3-strip. However, the right-hand diagram contains two boxes in the first column of the cylinder \mathcal{C}_{35} , and is thus not a horizontal 3-strip.

We are now able to formally state the quantum Pieri rule from [Pos, Prop. 4.1], for which we provide an equivariant generalization in Theorem 1 as our main result.

Theorem 4 (Quantum Pieri Rule on Cylindric Shapes [Pos]). *For any integers $1 \leq p \leq m$ and $1 \leq k \leq n - m$ and any partition $\mu \in P_{mn}$, we have*

$$\sigma_{(1)^p} * \sigma_\mu = \sum_{\lambda/d/\mu=v^p} q^d \sigma_\lambda \quad \text{and} \quad \sigma_{(k)} * \sigma_\mu = \sum_{\lambda/d/\mu=h^k} q^d \sigma_\lambda$$

in $QH^*(Gr(m, n))$. Since $\lambda/d/\mu$ is a vertical (resp. horizontal) strip, then $d \in \{0, 1\}$.

2.3 Equivariant quantum cohomology of the Grassmannian

There is always a natural left action of the torus T on the homogenous space G/P . In the case of maximal P , the Schubert varieties X_λ are T -invariant with codimension $|\lambda|$, and thus also determine a basis of Schubert classes σ_λ in the T -equivariant cohomology of the Grassmannian $H_T^*(Gr(m, n))$. In fact, these Schubert classes also form a basis for $H_T^*(Gr(m, n))$ as a module over $H_T^*(pt)$. Furthermore, since $G = SL_n$, then the equivariant cohomology of a point is isomorphic to a polynomial ring $H_T^*(pt) \cong \mathbb{Z}[t_1, \dots, t_n]$. As such, $H_T^*(Gr(m, n))$ admits a $\mathbb{Z}[t_1, \dots, t_n]$ -basis of Schubert classes σ_λ indexed by $\lambda \in P_{mn}$.

The *equivariant Littlewood-Richardson coefficients* are defined by the product of two equivariant Schubert classes

$$\sigma_\lambda \circ \sigma_\mu = \sum c_{\lambda, \mu}^\nu \sigma_\nu$$

in $H_T^*(Gr(m, n))$, where here $c_{\lambda, \mu}^\nu \in \mathbb{Z}[t_1, \dots, t_n]$. The equivariant Littlewood-Richardson coefficients satisfy the *Graham-positivity* property [Gra], which means that $c_{\lambda, \mu}^\nu \in \mathbb{Z}_{\geq 0}[\alpha_1, \dots, \alpha_{n-1}]$. Having chosen B to be lower-triangular in Section 2.1, then $c_{\lambda, \mu}^\nu \in \mathbb{Z}_{\geq 0}[t_2 - t_1, \dots, t_n - t_{n-1}]$. In this context, $\mathbb{Z}_{\geq 0}$ -linear combinations of the $(t_{i+1} - t_i)$ are referred to as (torus) *weights*. Several different combinatorial formulas for the equivariant Littlewood-Richardson coefficients exist, such as the puzzle rule of Knutson and Tao [KT], reinterpreted by Zinn-Justin through quantum integrability of tilings [ZJ], a barred skew tableaux formula obtained independently by Molev [Mol] and Kreiman [Kre], or Thomas and Yong's equivariant adaptation of jeu de taquin [TY]. Since we shall work in the equivariant setting for the remainder of the paper, there should be no confusion from adopting the same notation here as in our brief review of the non-equivariant context.

Using the same quantum parameter q as in the non-equivariant setting, the *equivariant quantum cohomology of the Grassmannian* is defined as

$$QH_T^*(Gr(m, n)) := \mathbb{Z}[q] \otimes H_T^*(Gr(m, n)),$$

and is simultaneously a deformation of both $H_T^*(Gr(m, n))$ and $QH^*(Gr(m, n))$. The ring $QH_T^*(Gr(m, n))$ has an additive $\mathbb{Z}[t_1, \dots, t_n][q]$ -basis of Schubert classes σ_λ for $\lambda \in P_{mn}$. The *equivariant quantum Littlewood-Richardson coefficients* are determined by the product of two Schubert classes

$$\sigma_\lambda \star \sigma_\mu = \sum_{\nu, d} c_{\lambda, \mu}^{\nu, d} q^d \sigma_\nu$$

in $\mathrm{QH}_T^*(\mathrm{Gr}(\mathbf{m}, \mathbf{n}))$. The equivariant Littlewood-Richardson coefficients are encoded as the degree $\mathbf{d} = 0$ terms in the same product, and the corresponding quantum Littlewood-Richardson coefficients can be obtained by setting all $\mathbf{t}_i = 0$. The equivariant quantum Littlewood-Richardson coefficients also display Graham-positivity, as proved by Mihalcea [Mih1], in the sense that $c_{\lambda, \mu}^{\gamma, \mathbf{d}} \in \mathbb{Z}_{\geq 0}[\mathbf{t}_2 - \mathbf{t}_1, \dots, \mathbf{t}_n - \mathbf{t}_{n-1}]$ using the conventions of this paper. A positive combinatorial formula for the equivariant Littlewood-Richardson coefficients is obtained by combining Buch and Mihalcea's equivariant quantum-to-classical principle [BM] with Buch's equivariant puzzle rule for two-step flag varieties [Buc2].

A simple, signed formula for the equivariant quantum Littlewood-Richardson coefficients is given by combining the equivariant rim hook rule of the first, third, and fourth authors with any available equivariant Littlewood-Richardson rule. We now briefly review [BMT, Theorem 2.6], in order to apply it in the proof of our main theorem in Section 4.

Theorem 5 (Equivariant Rim Hook Rule [BMT]). *Let $\lambda, \mu \in \mathcal{P}_{mn}$, and consider the product of the corresponding classes $\sigma_\lambda \circ \sigma_\mu = \sum c_{\lambda, \mu}^\gamma \sigma_\gamma \in \mathrm{H}_T^*(\mathrm{Gr}(\mathbf{m}, 2\mathbf{n} - 1))$. Then in $\mathrm{QH}_T^*(\mathrm{Gr}(\mathbf{m}, \mathbf{n}))$, we have*

$$\sigma_\lambda \star \sigma_\mu = \sum \varphi(c_{\lambda, \mu}^\gamma) \varphi(\sigma_\gamma), \quad \text{where } \varphi : \begin{cases} \mathbf{t}_i \mapsto \mathbf{t}_{i \pmod{\mathbf{n}}} \\ \sigma_\gamma \mapsto \begin{cases} \prod_{i=1}^{\mathbf{d}} ((-1)^{\varepsilon_i - m}) \mathbf{q} \sigma_\nu & \text{if } \nu \in \mathcal{P}_{mn} \\ 0 & \text{if } \nu \notin \mathcal{P}_{mn}. \end{cases} \end{cases}$$

Here, ν denotes the \mathbf{n} -core of γ , the integer \mathbf{d} is the number of \mathbf{n} -rim hooks removed from γ to obtain ν , and ε_i is the height of the i^{th} rim hook removed.

Our conventions on Graham-positivity in this paper differ by a sign from those in much of the literature, in order to remain consistent with the torus weight conventions of [BMT].

The structure of the Grassmannian lends itself to many symmetries, which we exploit to relate the Pieri rule on row and column-shaped partitions in the proof of our main theorem. Given $\lambda \in \mathcal{P}_{mn}$, denote by $\lambda' \in \mathcal{P}_{n-m, n}$ the *transpose* partition which exchanges the rows and columns. The isomorphism which identifies an m -dimensional subspace of $V = \mathbb{C}^n$ with an $(n - m)$ -dimensional subspace of the dual space V^* exchanges the Schubert class σ_λ for the Schubert class $\sigma_{\lambda'}$ indexed by the transposed partition. The induced isomorphism on equivariant quantum cohomology $\mathrm{QH}_T^*(\mathrm{Gr}(\mathbf{m}, \mathbf{n})) \cong \mathrm{QH}_T^*(\mathrm{Gr}(\mathbf{n} - \mathbf{m}, \mathbf{n}))$ also involves an involution on the torus weights. Below we review this *level-rank duality* on equivariant quantum Littlewood-Richardson coefficients, as stated by Gorbounov and Korff in [GK3, Corollary 4.14]; see also [GK2, Corollary 6.8].

Theorem 6 (Level-Rank Duality [GK2, GK3]). *Given $\lambda, \mu, \nu \in \mathcal{P}_{mn}$ and any $\mathbf{d} \in \mathbb{Z}_{\geq 0}$, then*

$$c_{\lambda, \mu}^{\gamma, \mathbf{d}}(\mathbf{t}) = c_{\lambda', \mu'}^{\gamma', \mathbf{d}}(-w_0 \mathbf{t}),$$

where the involution on torus weights is given by the substitution $-w_0 : \mathbf{t}_i \mapsto -\mathbf{t}_{n+1-i}$.

2.4 An equivariant quantum Pieri rule on cylindrical shapes

In the equivariant setting, Mihalcea showed in [Mih2] that the product in $\mathrm{QH}_T^*(\mathrm{Gr}(\mathbf{m}, \mathbf{n}))$ is completely determined by the *equivariant quantum Pieri-Chevalley*

rule, a closed combinatorial formula for multiplying by the equivariant quantum Schubert class $\sigma_{(1)}$ corresponding to a single box. The method for determining the algebra structure from this Pieri-Chevalley rule is both recursive and signed, and so more general Graham-positive product formulas have since emerged, including the full equivariant quantum Littlewood-Richardson rule given by combining [BM] and [Buc2].

Without appealing to some variation on the quantum-to-classical principle as in [BM] or [BMT], the most general combinatorial formula for products in $\text{QH}_T^*(\text{Gr}(\mathbf{m}, \mathbf{n}))$ is the *equivariant quantum Pieri rule* for multiplying by the equivariant quantum Schubert classes $\sigma_{(1)^k}$ or $\sigma_{(k)}$. Huang and Li proved the first equivariant quantum Pieri rule for any type A partial flag variety in Theorem 3.10 of [HL]; see Theorem 20 below, which reviews a special case. Also compare the approach of Gorbounov and Korff using the six-vertex model from statistical mechanics [GK2, GK3]. As in the non-equivariant context, the equivariant quantum Pieri rule can also be easily recovered from its classical counterpart using the equivariant rim hook rule of [BMT].

The goal of this paper is to present a closed combinatorial Graham-positive equivariant quantum Pieri rule in terms of cylindric shapes. In addition to more directly capturing the quantum-to-affine phenomenon which governs the ring structure of $\text{QH}_T^*(\text{Gr}(\mathbf{m}, \mathbf{n}))$ via the parabolic Peterson isomorphism of [Pet, LS1], our rule does not require doing calculations in any related two-step flags or smaller/larger Grassmannians. Theorem 1 below is a direct equivariant generalization of Theorem 4, in which the equivariant terms enter by recording statistics on boxes which may be “added back” to the skew shape, in a manner which we now formalize.

Definition 7. Let $\lambda, \mu \in \mathcal{P}_{mn}$ be such that $\mu \subseteq \lambda$, and let $\mathbf{d} \in \mathbb{Z}_{\geq 0}$. Suppose that the cylindric shape $\lambda/\mathbf{d}/\mu$ is a vertical r -strip \mathbf{v}^r (resp. horizontal r -strip \mathbf{h}^r) for some $0 \leq r \leq m$. Fix another integer $r \leq \mathbf{p} \leq m$.

1. \mathbf{v}^r (resp. \mathbf{h}^r) is *extendable* to a vertical (resp. horizontal) \mathbf{p} -strip if by adding $\mathbf{p} - r$ boxes from μ to $\lambda/\mathbf{d}/\mu$, each of which shares a vertical (resp. horizontal) edge with the boundary of $\lambda/\mathbf{d}/\mu$, we can form a vertical \mathbf{p} -strip $\mathbf{v}^{\mathbf{p}}$ (resp. horizontal \mathbf{p} -strip $\mathbf{h}^{\mathbf{p}}$). Note that the complement $\lambda \setminus \mathbf{v}^{\mathbf{p}}$ (resp. $\lambda \setminus \mathbf{h}^{\mathbf{p}}$) may no longer be a partition shape, but rather a composition.
2. The $\mathbf{p} - r$ boxes from μ which are added to $\lambda/\mathbf{d}/\mu$ in the process of extending \mathbf{v}^r to $\mathbf{v}^{\mathbf{p}}$ (resp. \mathbf{h}^r to $\mathbf{h}^{\mathbf{p}}$) are called *addable boxes*. We typically denote an individual addable box by $\alpha \in \mathbf{v}^{\mathbf{p}} \setminus \mathbf{v}^r$ (resp. $\alpha \in \mathbf{h}^{\mathbf{p}} \setminus \mathbf{h}^r$), where the original vertical (resp. horizontal) r -strip is always assumed to be of the form $\lambda/\mathbf{d}/\mu$.
3. We write $\mathbf{v}^r \rightarrow \mathbf{v}^{\mathbf{p}}$ (resp. $\mathbf{h}^r \rightarrow \mathbf{h}^{\mathbf{p}}$) to denote that the vertical r -strip \mathbf{v}^r (resp. horizontal r -strip \mathbf{h}^r) is being *extended* to a vertical \mathbf{p} -strip $\mathbf{v}^{\mathbf{p}}$ (resp. horizontal \mathbf{p} -strip $\mathbf{h}^{\mathbf{p}}$) via addable boxes, without specific reference to which boxes are being added. Equivalently, we say that $\mathbf{v}^{\mathbf{p}}$ (resp. $\mathbf{h}^{\mathbf{p}}$) is an *extension* of \mathbf{v}^r (resp. \mathbf{h}^r).

We illustrate the concept of extending vertical and horizontal strips via addable boxes in the following example, which we shall then continue to utilize throughout the paper.

Example 8. Consider $\mu = (6, 6, 6, 3, 2, 0, 0) \subset \lambda = (7, 6, 6, 4, 2, 1, 0) \in \mathcal{P}_{7,15}$. As Figure 3 illustrates in the 7×8 rectangle, the skew shape λ/μ consists of the three boxes containing

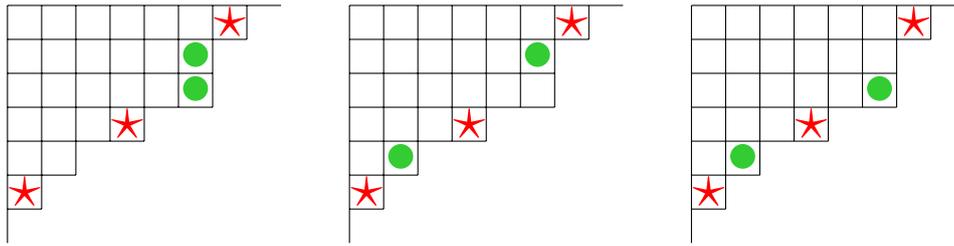


Figure 3: Extending a vertical 3-strip to a vertical 5-strip via addable boxes.

red stars. Since no two of the boxes containing a \star are in the same row or column, this skew shape is both a vertical 3-strip and a horizontal 3-strip on the cylinder $\mathcal{C}_{7,15}$.

If we now fix $p = 5$, then there are 3 different ways to extend ν^3 to a vertical 5-strip, shown in Figure 3. In each diagram, the $5 - 3 = 2$ boxes of μ containing green dots are addable, since each box with a \bullet shares a vertical edge with the boundary of the original skew shape λ/μ , and the union of the boxes with \star and \bullet forms a vertical 5-strip. For the 5-strip ν^5 depicted in the middle diagram, note that the complement $\lambda \setminus \nu^5$ is not a partition shape, whereas the other two are. Each of the three configurations in Figure 3 would contribute to a sum indexed by the extension $\nu^3 \rightarrow \nu^5$.

As noted above, the cylindric shape λ/μ is also a horizontal 3-strip h^3 . However, only the right-hand arrangement of addable boxes in Figure 3 represents a valid extension of h^3 to a horizontal 5-strip, since each addable box \bullet is required to share a horizontal edge with the boundary of λ/μ . The other 5 configurations which would contribute to a sum indexed by $h^3 \rightarrow h^5$ are not pictured.

We now define several natural statistics on an addable box, determined by its location within the extension. We first enumerate the edges of the partition $\mu \in \mathcal{P}_{mn}$, equivalently the closed loop $\mu[0]$ on \mathcal{C}_{mn} .

Definition 9. Given $\mu \in \mathcal{P}_{mn}$, starting in the lower left-hand corner of the $m \times (n - m)$ rectangle, number all the edges of the path that traces out the boundary of μ .

1. The *up-steps* of μ , denoted by $\mathbf{U}(\mu)$, are the numbers indexing the vertical edges of μ .
2. The *side-steps* of μ , denoted by $\mathbf{S}(\mu)$, are the numbers indexing the horizontal edges of μ .

Note that $\mathbf{U}(\mu) \sqcup \mathbf{S}(\mu) = [n]$.

The required statistics depend on whether an addable box is viewed as part of a vertical or horizontal strip, and so the following definition is presented as two corresponding sets of statistics.

Definition 10. Given a vertical strip $\lambda/d/\mu = \nu^r$, an extension $\nu^r \rightarrow \nu^p$, and an addable box $\alpha \in \nu^p \setminus \nu^r$, define:

- (1v) The *up-step* of α , denoted $\mathbf{u}(\alpha)$, is the index of the vertical edge of the box α from $\mathbf{U}(\mu)$.

- (2v) The *row number* of α , denoted $r(\alpha)$, enumerates the row containing α , where we count rows from the bottom of the $m \times (n - m)$ rectangle, equivalently from row m of \mathcal{C}_{mn} .
- (3v) The *number of boxes below* α , denoted $b(\alpha)$, is defined to be the total number of boxes in the extension v^p lying in rows strictly below α in the $m \times (n - m)$ rectangle, equivalently boxes in v^p with row index in \mathcal{C}_{mn} strictly greater than $r(\alpha)$.

Given a horizontal strip $\lambda/d/\mu = h^r$, an extension $h^r \rightarrow h^p$, and an addable box $\alpha \in h^p \setminus h^r$, define:

- (1h) The *side-step* of α , denoted $s(\alpha)$, is the index of the horizontal edge of the box α from $S(\mu)$.
- (2h) The *column number* of α , denoted $c(\alpha)$, enumerates the column containing α , where we count columns from the right of the $m \times (n - m)$ rectangle, equivalently from column $n - m$ of \mathcal{C}_{mn} .
- (3h) The *number of boxes to the right* of α , denoted $rt(\alpha)$, is defined to be the total number of boxes in the extension h^p lying in columns strictly to the right of α in the $m \times (n - m)$ rectangle, equivalently boxes in h^p with column index in \mathcal{C}_{mn} strictly greater than $c(\alpha)$.

We can now formally define the weight of an addable box.

Definition 11. Let $\lambda, \mu \in P_{mn}$ be such that $\mu \subseteq \lambda$, and let $d \in \mathbb{Z}_{\geq 0}$.

1. If the cylindric shape $\lambda/d/\mu$ is a vertical r -strip v^r , we define the *weight of an addable box* $\alpha \in v^p \setminus v^r$ to be

$$\text{wt}^v(\alpha) := t_{u(\alpha)} - t_{r(\alpha)-b(\alpha)}.$$

2. If the cylindric shape $\lambda/d/\mu$ is a horizontal r -strip h^r , we define the *weight of an addable box* $\alpha \in h^p \setminus h^r$ to be

$$\text{wt}^h(\alpha) := t_{n+1-(c(\alpha)-rt(\alpha))} - t_{s(\alpha)}.$$

When the original skew shape $\lambda/d/\mu = v^r$ (resp. $\lambda/d/\mu = h^r$) is not clear from context, we write $\text{wt}_\mu^v(\alpha)$ (resp. $\text{wt}_\mu^h(\alpha)$) to indicate that the box is being added from μ .

Remark 12. Since addable boxes belong to the interior diagram of the skew shape $\lambda/d/\mu$, each of the statistics in Definition 10 is equivalent whether we consider the partition $\mu \in P_{mn}$ in the rectangle or the closed loop $\mu[0] \subset \mathcal{C}_{mn}$ on the cylinder. The weight of an addable box is thus independent of whether we consider the cylindric shape $\lambda/d/\mu$ or the equivalent skew diagram which fits inside the $m \times (n - m)$ rectangle. We thus refer to the weight of an addable box in a skew shape to simultaneously invoke both meanings.

To illustrate this definition, we calculate the weight of several addable boxes from Figure 3.

Example 13. Recall from Figure 3 that $\lambda = (7, 6, 6, 4, 2, 1, 0) \in P_{7,15}$. Tracing the boundary of μ in the 7×8 rectangle, we obtain

$$U(\mu) = \{1, 2, 5, 7, 11, 12, 13\} \quad \text{and} \quad S(\mu) = \{3, 4, 6, 8, 9, 10, 14, 15\}.$$

1. We first focus on the vertical strip depicted in the left-hand diagram, for which we calculate the weight of each addable box containing a \bullet . Denote by α_T the top addable box, and by α_B the bottom addable box. Compute using Definition 10 that

$$\begin{aligned} u(\alpha_T) &= 12 & r(\alpha_T) &= 6 & b(\alpha_T) &= 3 \\ u(\alpha_B) &= 11 & r(\alpha_B) &= 5 & b(\alpha_B) &= 2. \end{aligned}$$

Using Definition 11, we then compute the weight of each of the two addable boxes occurring in the left-hand diagram from Figure 3 to be

$$\text{wt}^v(\alpha_T) = t_{12} - t_{6-3} = t_{12} - t_3 \quad \text{and} \quad \text{wt}^v(\alpha_B) = t_{11} - t_{5-2} = t_{11} - t_3.$$

2. For the purpose of illustrating these calculations in the case of a horizontal strip, we now choose to view the right-hand diagram in Figure 3 as a horizontal strip. Denote by α_R the right addable box, and by α_L the left addable box. Using Definition 10,

$$\begin{aligned} s(\alpha_R) &= 10 & c(\alpha_R) &= 3 & \text{rt}(\alpha_R) &= 1 \\ s(\alpha_L) &= 4 & c(\alpha_L) &= 7 & \text{rt}(\alpha_L) &= 3. \end{aligned}$$

By Definition 11, we then compute the weight of each of the two addable boxes occurring in the right-hand diagram from Figure 3 to be

$$\text{wt}^h(\alpha_R) = t_{16-(3-1)} - t_{10} = t_{14} - t_{10} \quad \text{and} \quad \text{wt}^h(\alpha_L) = t_{16-(7-3)} - t_4 = t_{12} - t_4.$$

We are now able to formally state the main theorem in this paper, which provides an equivariant generalization of the quantum Pieri rule on cylindric shapes from [Pos], as well as closed combinatorial formulas for the equivariant quantum Pieri expansions provided in [GK2, GK3]. For ease of reference, we repeat the theorem statement from the introduction here.

Theorem 1 (Equivariant Quantum Pieri Rule on Cylindric Shapes). *For any integers $1 \leq p \leq m$ and $1 \leq k \leq n - m$ and any partition $\mu \in P_{mn}$, we have*

$$\begin{aligned} \sigma_{(1)^p} \star \sigma_\mu &= \sum_{0 \leq r \leq p} \sum_{\substack{\lambda/d/\mu = v^r \\ v^r \rightarrow v^p}} q^d \prod_{\alpha \in v^p \setminus v^r} \text{wt}^v(\alpha) \sigma_\lambda \quad \text{and} \\ \sigma_{(k)} \star \sigma_\mu &= \sum_{0 \leq r \leq k} \sum_{\substack{\lambda/d/\mu = h^r \\ h^r \rightarrow h^k}} q^d \prod_{\alpha \in h^k \setminus h^r} \text{wt}^h(\alpha) \sigma_\lambda \end{aligned}$$

in $\text{QH}_T^*(\text{Gr}(m, n))$. Since $\lambda/d/\mu$ is a vertical (resp. horizontal) strip, then $d \in \{0, 1\}$.

Given an addable box α in a vertical strip, since $u(\alpha) > r(\alpha) > r(\alpha) - b(\alpha)$, then clearly $wt^v(\alpha) \in \mathbb{Z}_{\geq 0}[t_2 - t_1, \dots, t_n - t_{n-1}]$, and similarly for horizontal strips. In particular, this equivariant quantum Pieri rule is manifestly Graham-positive, using the conventions of this paper. Moreover, setting all $t_i = 0$ immediately recovers Postnikov's quantum Pieri rule from Theorem 4.

We now provide several examples illustrating how to use Theorem 1 to efficiently compute equivariant quantum Littlewood-Richardson coefficients, directly on the skew shape λ/μ or $\lambda/1/\mu$.

Example 15. Fix $p = 5$, and let $\mu = (6, 6, 6, 3, 2, 0, 0) \in P_{7,15}$. Since we have already seen that the cylindric shape λ/μ for the partition $\lambda = (7, 6, 6, 4, 2, 1, 0)$ is a vertical 3-strip \mathfrak{v}^3 , we may use Theorem 1 to calculate the equivariant quantum Littlewood-Richardson coefficient $c_{(1)^5, \mu}^{\lambda, 0}$. Figure 3 shows the three possible ways to extend $\mathfrak{v}^3 \rightarrow \mathfrak{v}^5$. By Theorem 1, the left-hand configuration of addable boxes depicted in Figure 3 contributes the product $(t_{12} - t_3)(t_{11} - t_3)$ of the weights calculated in Example 13. In like manner, by Theorem 1, in $QH_T^*(Gr(7, 15))$ we have

$$c_{(1)^5, \mu}^{\lambda, 0} = (t_{12} - t_3)(t_{11} - t_3) + (t_{12} - t_3)(t_5 - t_2) + (t_{11} - t_2)(t_5 - t_2),$$

recording the contributions from the remaining two configurations of addable boxes from left to right, where the weight of the top addable box is recorded on the left within each product.

For comparison, we now use the equivariant quantum Pieri rule on cylindric shapes to calculate a full product in a smaller Grassmannian. We highlight the fact that the entire calculation in Example 16 is performed quickly and directly on the four skew shapes in Figure 4, without passing to any other Grassmannian or two-step flag variety.

Example 16. We now use Theorem 1 to compute the product $\sigma_{[pcY]} \star \sigma_{[pcY]} \in QH_T^*(Gr(3, 5))$. Since $p = 3$, we first identify all partitions $\lambda \in P_{35}$ such that $\lambda/d/\mu$ is a vertical r -strip for some $0 \leq r \leq 3$ and $d \in \{0, 1\}$, and such that $\lambda/d/\mu$ can be extended to a vertical 3-strip. A quick inspection reveals that $\lambda \in \{[pcY], [pcY], \bullet, [pcY]\}$, and the 4 respective cylindric diagrams are depicted left to right by the red shaded boxes in Figure 4; refer to Figure 2 and the surrounding discussion for more details on constructing the relevant cylindric loops.

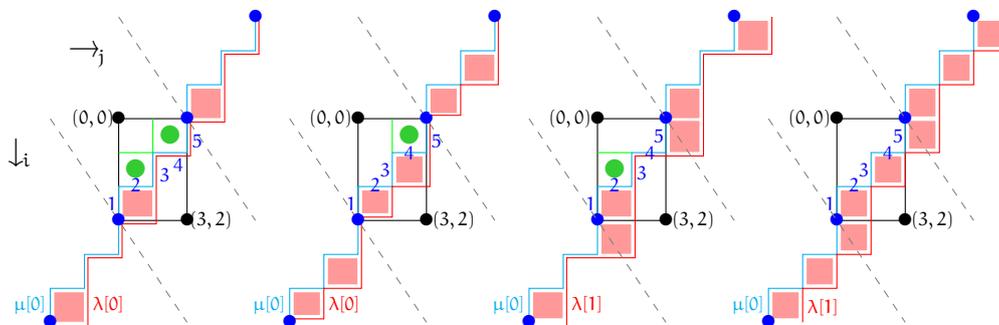


Figure 4: Using Theorem 1 to calculate $\sigma_{(1)^3} \star \sigma_{(2,1)} \in QH_T^*(Gr(3, 5))$.

In this example, for each $\lambda/\mathbf{d}/\mu = \mathbf{v}^r$, there is a unique extension $\mathbf{v}^r \rightarrow \mathbf{v}^3$, and the addable boxes in each extension are indicated by a green \bullet in Figure 4. Calculating the weight of each addable box using Definition 11, by Theorem 1 we then immediately obtain

$$\sigma_{[\text{pc}\gamma]} \star \sigma_{[\text{pc}\gamma]} = (\mathbf{t}_5 - \mathbf{t}_1)(\mathbf{t}_3 - \mathbf{t}_1)\sigma_{[\text{pc}\gamma]} + (\mathbf{t}_5 - \mathbf{t}_1)\sigma_{[\text{pc}\gamma]} + \mathbf{q}(\mathbf{t}_3 - \mathbf{t}_1)\sigma_{\bullet} + \mathbf{q}\sigma_{[\text{pc}\gamma]} \in \text{QH}_{\mathbb{T}}^*(\text{Gr}(3, 5)),$$

where we record the product in the same order as the corresponding diagram in Figure 4.

3 Equivariant Pieri rules via localization

The goal of this section is to relate the weights which arise in the equivariant quantum Pieri rule on cylindric shapes of Theorem 1 with those appearing as localizations in the corresponding formula of Huang and Li reviewed in Section 3.2 as Theorem 20. We provide the required background on the localization map in Section 3.1, and then relate the localizations appearing in Theorem 20 to the weights occurring in Theorem 1 in Section 3.3.

3.1 The localization map

In [KK], Kostant and Kumar define a family of functions which determines the ring structure for the equivariant cohomology of any Kac-Moody flag variety. In the special case of the Grassmannian, the Schubert class $[X_{\gamma}] \in H_{\mathbb{T}}^*(\text{Gr}(\mathbf{m}, \mathbf{n}))$ for any partition $\gamma \in \mathcal{P}_{\mathbf{m}\mathbf{n}}$ is identified with a function $\xi^{\gamma} : \mathcal{P}_{\mathbf{m}\mathbf{n}} \rightarrow \mathbb{Z}[\mathbf{t}_1, \dots, \mathbf{t}_n]$. When evaluated at an element $\eta \in \mathcal{P}_{\mathbf{m}\mathbf{n}}$, the polynomial $\xi^{\gamma}(\eta)$ is homogeneous of degree $|\gamma|$. Moreover, the polynomials $\xi^{\gamma}(\eta)$ satisfy the Graham-positivity property, meaning that the function values $\xi^{\gamma}(\eta) \in \mathbb{Z}_{\geq 0}[\mathbf{t}_2 - \mathbf{t}_1, \dots, \mathbf{t}_n - \mathbf{t}_{n-1}]$, using the conventions in this paper; refer to Sections 2.1 and 2.3 for more details.

We briefly review the precise relationship between the function ξ^{γ} and the Schubert class $[X_{\gamma}]$, referring the reader to [Kum, Section 11.3] for more details. The inclusion of the \mathbb{T} -fixed points in $\text{Gr}(\mathbf{m}, \mathbf{n})$, themselves also indexed by partitions $\eta \in \mathcal{P}_{\mathbf{m}\mathbf{n}}$, induces an injection

$$H_{\mathbb{T}}^*(\text{Gr}(\mathbf{m}, \mathbf{n})) \hookrightarrow H_{\mathbb{T}}^*(\text{Gr}(\mathbf{m}, \mathbf{n})^{\mathbb{T}}) \cong \bigoplus_{\eta \in \mathcal{P}_{\mathbf{m}\mathbf{n}}} \mathbb{Z}[\mathbf{t}_1, \dots, \mathbf{t}_n].$$

By definition, the induced map sends the Schubert class $[X_{\gamma}]$ to the collection of localizations $[X_{\gamma}]|_{\eta}$ at all \mathbb{T} -fixed points indexed by $\eta \in \mathcal{P}_{\mathbf{m}\mathbf{n}}$. In turn, under the above isomorphism, the localization $[X_{\gamma}]|_{\eta}$ is identified with the polynomial $\xi^{\gamma}(\eta)$. Altogether, this localization map identifies each \mathbb{T} -equivariant Schubert class with a polynomial-valued function via $[X_{\gamma}] \mapsto \xi^{\gamma}$. As such, the ring structure of $H_{\mathbb{T}}^*(\text{Gr}(\mathbf{m}, \mathbf{n}))$ with respect to the Schubert basis is completely determined by the pointwise product of the family of functions ξ^{γ} . Moreover, the image of $H_{\mathbb{T}}^*(\text{Gr}(\mathbf{m}, \mathbf{n}))$ in this polynomial ring has a simple characterization due to Goresky, Kottwitz, and MacPherson [GKM], which is the starting point for GKM theory; see the survey by Tymoczko [Tym2]. If the ambient Grassmannian is not clear from context (e.g. if it differs from $\text{Gr}(\mathbf{m}, \mathbf{n})$), we shall use a subscript $\xi_k^{\gamma}(\eta)$ to indicate that $\gamma, \eta \in \mathcal{P}_{k\mathbf{n}}$ instead.

The functions ξ^γ are uniquely determined by several straightforward properties, including the following support condition.

Lemma 17 (Corollary 11.1.12 [Kum]). *For any $\gamma, \eta \in \mathcal{P}_{mn}$, we have $\xi^\gamma(\eta) = 0$ if and only if $\gamma \not\subseteq \eta$.*

The polynomials $\xi^\gamma(\eta)$ themselves can be computed in many different ways, including by the original formula of Billey [Bil], formulated independently by Anderson, Jantzen, and Soergel [AJS]. Ikeda and Naruse [IN] interpret Billey’s formula for the Grassmannian using excited Young diagrams; see Example 21 for an illustration of this method. As we shall see, certain polynomials $\xi^\gamma(\eta)$ also coincide with special cases of equivariant Littlewood-Richardson coefficients.

3.2 An equivariant Pieri rule via localization

Huang and Li proved the first equivariant quantum Pieri rule for any type A partial flag variety in Theorem 3.10 of [HL], a special case of which we review below as Theorem 20. Their Pieri rule for the Grassmannian records the (classical) equivariant Littlewood-Richardson coefficient $c_{(1)^r, \mu}^\lambda$ by specifying a column-shape Schubert variety in a smaller Grassmannian, and localizing at a T -fixed point indexed by a partition obtained from λ and μ via a join-and-cut algorithm.

More precisely, given a pair of partitions $\mu \subseteq \lambda \in \mathcal{P}_{mn}$ such that the skew shape $\lambda/\mu = \mathbf{v}^r$, Huang and Li define a new diagram λ_μ in the shorter rectangle $\mathcal{P}_{m-r, n}$. The closed formula for λ_μ is provided in [HL, Definition 3.14], and interpreted through a join-and-cut process explained in [HL, Definition 3.15]. We now review the construction of λ_μ , using a slight reformulation of the join-and-cut algorithm from [HL]. Given any $\mathbf{k} \in [m] := \{1, \dots, m\}$, the *staircase* diagram with \mathbf{k} parts is defined as $\delta^{\mathbf{k}} := (\mathbf{k} - 1, \dots, 1, 0)$.

Definition 18. Let $\mu = (\mu_1, \dots, \mu_m), \lambda \subseteq (\lambda_1, \dots, \lambda_m) \in \mathcal{P}_{mn}$ be such that the skew shape $\lambda/\mu = \mathbf{v}^r$ for some $0 \leq r \leq m$. Construct a new partition by performing the following *join-and-cut algorithm*:

1. Add the staircase δ^m to the diagram λ , where parts are added coordinate-wise.
2. Delete from the diagram resulting from step (1) those r rows in which the original shapes λ and μ differ; i.e. remove every row from $\lambda + \delta^m$ for which $\lambda_i \neq \mu_i$.
3. Remove the staircase δ^{m-r} from the shape resulting from step (2), where parts are subtracted coordinate-wise.

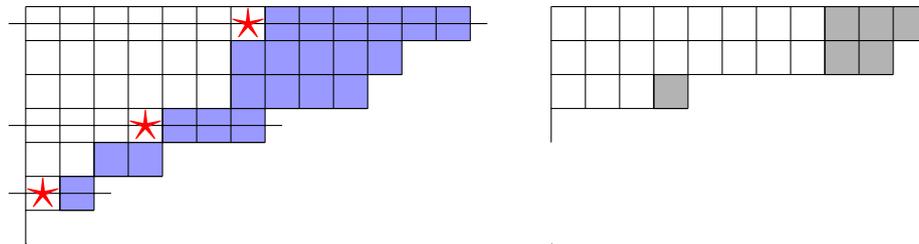
The partition resulting from this join-and-cut algorithm is denoted by $\lambda_\mu \in \mathcal{P}_{m-r, n}$. Equivalently, if we denote by $i_1 < \dots < i_{m-r}$ those parts such that $\lambda_{i_t} = \mu_{i_t}$, we have

$$\lambda_\mu = (\mu_{i_1} - i_1 + r + 1, \mu_{i_2} - i_2 + r + 2, \dots, \mu_{i_{m-r}} - i_{m-r} + m) \in \mathcal{P}_{m-r, n}. \quad (1)$$

We illustrate this join-and-cut algorithm in the following example.

Example 19. Consider $\mu = (6, 6, 6, 3, 2, 0, 0) \subset \lambda = (7, 6, 6, 4, 2, 1, 0) \in \mathcal{P}_{7, 15}$, and recall that the skew shape λ/μ in the 7×8 rectangle is the vertical 3-strip consisting of the boxes containing a \star in the figure below. We illustrate the join-and-cut algorithm, as

described in Definition 18. Step (1) adds the staircase δ^7 to λ coordinate-wise. In the left-hand figure, the boxes of δ^7 in the partition $\lambda + \delta^7$ are indicated with blue shading:



$$\lambda + \delta^7 = (13, 11, 10, 7, 4, 2, 0) \quad \mapsto \quad (11, 10, 4, 0) \in \mathcal{P}_{4,15}$$

Step (2) calls for removing the rows where $\lambda_i \neq \mu_i$; these 3 rows (each of which contains a \star) are indicated in the left-hand figure using strike-through. Finally, Step (3) requires us to remove the staircase δ^4 , indicated in the right-hand figure with gray shading, from which we obtain

$$\lambda_\mu = (11, 10, 4, 0) - \delta^4 = (8, 8, 3, 0).$$

For comparison, those parts such that $\lambda_{i_\ell} = \mu_{i_\ell}$ are given by $\{i_1, i_2, i_3, i_4\} = \{2, 3, 5, 7\}$, so that

$$\lambda_\mu = (6 - 2 + 4, 6 - 3 + 5, 2 - 5 + 6, 0 - 7 + 7) = (8, 8, 3, 0) \in \mathcal{P}_{4,15}$$

by the closed formula (1).

Together with any available formula for computing the localization $\xi^\gamma(\eta)$, the following (special case of a) theorem of Huang and Li provides a rule for computing products by Schubert classes indexed by column shapes in the equivariant cohomology of the Grassmannian.

Theorem 20 (Equivariant Pieri Rule, Theorem 3.17 [HL]). *For any integer $1 \leq p \leq m$ and any partition $\mu \in \mathcal{P}_{m,n}$, we have*

$$\sigma_{(1)^p} \circ \sigma_\mu = \sum_{0 \leq r \leq p} \sum_{\lambda/\mu = \nu^r} \xi_{m-r}^{(1)^{p-r}}(\lambda_\mu) \sigma_\lambda$$

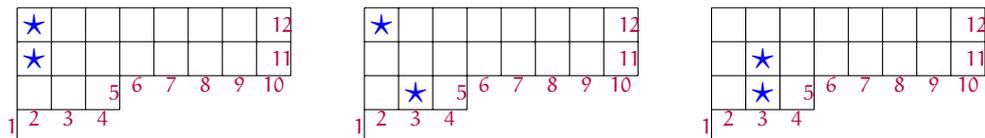
in $H_T^*(\text{Gr}(m, n+1))$, where $\lambda \in \mathcal{P}_{m, n+1}$.

The switch to $\text{Gr}(m, n+1)$ in Theorem 20, which persists throughout the remainder of the paper, is intentionally made to prepare for the application of the equivariant rim hook rule in the proof of Theorem 1, in order to obtain an equivariant quantum Pieri rule for $\text{Gr}(m, n)$. Note that Huang and Li also deliberately work in $\text{Gr}(m, n+1)$ throughout [HL], albeit for different reasons.

For the sake of completeness, we now illustrate Theorem 20 by calculating the same equivariant Littlewood-Richardson coefficient as in Example 15. See [Tym1] for a survey on the localizations occurring in Theorem 20, including for more details on the method of excited Young diagrams illustrated in the following example.

Example 21. Let $p = 5$, and recall that for $\mu = (6, 6, 6, 3, 2, 0, 0) \subset \lambda = (7, 6, 6, 4, 2, 1, 0) \in \mathcal{P}_{7,15}$ with $r = 3$, we have $\lambda_\mu = (8, 8, 3, 0) \in \mathcal{P}_{4,15}$ by Example 19.

Following [IN], we use the method of excited Young diagrams for calculating the localization $\xi_4^{(1)^2}(\lambda_\mu)$ required to determine the equivariant Littlewood-Richardson coefficient $c_{(1)^5, \mu}^\lambda$ using Theorem 20. In this case, there are 3 possible excited states corresponding to the column shape $(1)^2$ inside the diagram $\lambda_\mu \in \mathcal{P}_{4,15}$, as follows:



Recording the up-steps and side-steps for λ_μ along the boundary of the diagram in red, the weight of an excited box (indicated in the figure above with a blue star \star) is obtained directly from the corresponding up-step and side-step as $t_{u(\star)} - t_{s(\star)}$, using the conventions of this paper, which differ in sign from [IN]. For example, the weight of the top excited box in the left-hand figure is $t_{12} - t_2$. Altogether, we then have

$$c_{(1)^5, \mu}^\lambda = (t_{12} - t_2)(t_{11} - t_2) + (t_{12} - t_2)(t_5 - t_3) + (t_{11} - t_3)(t_5 - t_3)$$

in $H_T^*(\text{Gr}(7, 15))$, where the weights are summed over the 3 excited states from left to right, with the weight of the top excited box recorded on the left within each product. Most importantly for our purposes, we invite the reader to verify by regrouping terms that the above equivariant Littlewood-Richardson coefficient $c_{(1)^5, \mu}^\lambda = c_{(1)^5, \mu}^{\lambda, 0}$ from Example 15.

Remark 22. It is purely a coincidence that the number of excited states for the localization in Example 21 equals the number of configurations of addable boxes in Example 15. In general, there are many more extensions $\mathbf{v}^r \rightarrow \mathbf{v}^p$ of the skew shape λ/μ than excited states of $(1)^{p-r}$ in the diagram λ_μ , and equating individual equivariant Littlewood-Richardson coefficients via Theorems 1 and 20 is an exercise in clever polynomial algebra.

3.3 Addable boxes as localizations

In order to directly compare the equivariant Littlewood-Richardson coefficients in Theorem 1 to the corresponding localizations occurring in Theorem 20, we now explain how to calculate the weight of an addable box using the partition λ_μ resulting from the join-and-cut procedure, instead of the original skew shape λ/μ . In light of Remark 12, throughout the remainder of this section, we typically refer to addable boxes within a fixed rectangle, as opposed to on the cylinder.

Lemma 23. *Fix an integer $1 \leq p \leq m$, and let $\mu = (\mu_1, \dots, \mu_m) \in \mathcal{P}_{mn}$ be any partition. Suppose that the skew shape $\lambda/\mu = \mathbf{v}^r$ for some $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathcal{P}_{m, n+1}$ and $0 \leq r \leq p$. Denote by $i_1 < \dots < i_{m-r}$ those parts such that $\lambda_{i_\ell} = \mu_{i_\ell}$, indexed from the top of the rectangle. Given any extension $\mathbf{v}^r \rightarrow \mathbf{v}^p$, consider an addable box $\alpha \in \mathbf{v}^p \setminus \mathbf{v}^r$ in row i_j of μ for some $j \in [m - r]$.*

Viewing λ_μ/λ_μ as a vertical 0-strip \mathbf{v}^0 , then there is an addable box β in row j of λ_μ , indexing rows from the top of the rectangle. As such, the given extension $\mathbf{v}^r \rightarrow \mathbf{v}^p$ corresponds uniquely to an extension $\mathbf{v}^0 \rightarrow \mathbf{v}^{p-r}$ of addable boxes in λ_μ , identifying $\alpha \in \mathbf{v}^p \setminus \mathbf{v}^r$ in row i_j of μ with $\beta \in \mathbf{v}^{p-r} \setminus \mathbf{v}^0$ in row j of λ_μ . Moreover, under this correspondence,

$$\text{wt}_\mu^{\mathbf{v}}(\alpha) = \text{wt}_{\lambda_\mu}^{\mathbf{v}}(\beta). \tag{2}$$

Proof. We first make explicit the correspondence between extensions of vertical strips by addable boxes in μ and λ_μ . If for some $j \in [m-r]$, row i_j of μ contains an addable box α in the extension $\mathbf{v}^r \rightarrow \mathbf{v}^p$, we first claim that row j in λ_μ is also nonempty. Since the skew shape λ/μ supports an addable box in row i_j counting from the top of the rectangle, then by definition $\lambda_{i_j} = \mu_{i_j} \neq 0$. By (1), the number of boxes in row j of λ_μ , counting from the top of the rectangle, equals

$$(\lambda_\mu)_j = \mu_{i_j} - i_j + r + j. \quad (3)$$

The largest possible value for i_j in the $m \times (n - m + 1)$ rectangle occurs when the $m - r$ parts such that $\lambda_{i_j} = \mu_{i_j}$ coincide with the bottom $m - r$ rows of λ/μ , which would then be indexed by $i_1 = r + 1 < \dots < i_{r-m} = m$. In this scenario, $i_j = r + j$, and so in general we necessarily have $i_j \leq r + j$. Therefore, $-i_j + r + j \geq 0$, and since $\mu_{i_j} > 0$, then (3) implies that $(\lambda_\mu)_j > 0$ as well. In particular, whenever μ supports an addable box α in row i_j , then λ_μ supports a corresponding addable box β in row j , indexing the rows from the top of the corresponding rectangles. Identifying the addable box $\alpha \in \mu$ in row i_j with the corresponding addable box $\beta \in \lambda_\mu$ in row j , the $p - r$ addable boxes in the extension $\lambda/\mu = \mathbf{v}^r \rightarrow \mathbf{v}^p$ correspond uniquely to an extension $\mathbf{v}^0 \rightarrow \mathbf{v}^{p-r}$ of addable boxes in λ_μ .

We now proceed to calculate the weight of two corresponding addable boxes. By definition,

$$\text{wt}_\mu^v(\alpha) = \mathbf{t}_{\mathbf{u}(\alpha)} - \mathbf{t}_{\mathbf{r}(\alpha)-\mathbf{b}(\alpha)} \quad \text{and} \quad \text{wt}_{\lambda_\mu}^v(\beta) = \mathbf{t}_{\mathbf{u}(\beta)} - \mathbf{t}_{\mathbf{r}(\beta)-\mathbf{b}(\beta)}.$$

First, recall that the row numbers $\mathbf{r}(\alpha)$ and $\mathbf{r}(\beta)$ are calculated from the bottom of their respective rectangles, which means that we can immediately calculate $\mathbf{r}(\alpha) = m - i_j + 1$ and $\mathbf{r}(\beta) = m - r - j + 1$.

The index $\mathbf{u}(\alpha)$ is the up-step for box α , which can also be decomposed as the sum of the count of up-steps of $\mathbf{U}(\mu)$ plus the count of side-steps of $\mathbf{S}(\mu)$, up to and including the vertical edge of α . Since α is in row i_j of $\mu \in \mathbf{P}_{mm}$ indexed from the top of the rectangle, the number of up-steps to this point is given by $m - i_j + 1$. The number of side-steps to this point equals μ_{i_j} , and so $\mathbf{u}(\alpha) = \mu_{i_j} + m - i_j + 1$. Since β is in row j of λ_μ counted from the top of the rectangle, which only has $m - r$ rows, we instead have $\mathbf{u}(\beta) = (\lambda_\mu)_j + m - r - j + 1$. Applying (3), we thus see that indeed

$$\mathbf{u}(\beta) = (\mu_{i_j} - i_j + r + j) + (m - r - j + 1) = \mathbf{u}(\alpha).$$

We now compare the total number of addable and skew boxes below α and β , denoted $\mathbf{b}(\alpha)$ and $\mathbf{b}(\beta)$ respectively. First note that skew boxes do not occur in λ_μ , since all rows in which λ and μ have unequal parts are deleted in the join-and-cut procedure which produces λ_μ . To carry out the calculation of $\mathbf{b}(\beta)$, we define an intermediate statistic $\mathbf{n}_k := \#\{i > k \mid \lambda_i \neq \mu_i\}$ which counts the number of rows below row k in which λ and μ have unequal parts, counting from the top of the rectangle. Since exactly r rows are deleted from μ in step (2) of Definition 18, and \mathbf{n}_{i_j} of these deleted rows lie below row i_j in μ , then $r - \mathbf{n}_{i_j}$ rows are deleted from μ above row i_j . Thus after the join-and-cut algorithm is carried out, the row number $\mathbf{r}(\beta)$ for β in λ_μ equals $m - i_j + 1 - \mathbf{n}_{i_j}$. Notice, though, that we already computed $\mathbf{r}(\beta) = m - r - j + 1$ above. Comparing these two calculations, we deduce that $\mathbf{n}_{i_j} = -i_j + r + j$.

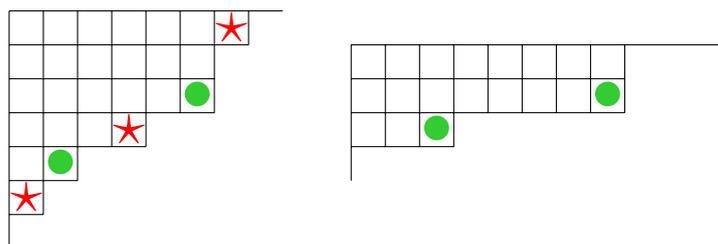
By construction, all of the boxes in the extension $\nu^r \rightarrow \nu^p$ that are below α remain below β in the corresponding extension $\nu^0 \rightarrow \nu^{p-r}$. Since the rows in which $\lambda_i \neq \mu_i$ are precisely those rows which are removed in the join-and-cut algorithm to produce λ_μ , and those unequal rows below i_j in the skew shape λ/μ each contribute one box to $\mathbf{b}(\alpha)$, then by definition $\mathbf{b}(\beta) = \mathbf{b}(\alpha) - \mathbf{n}_{i_j}$. Using our formulas for $r(\alpha)$, $r(\beta)$, and \mathbf{n}_{i_j} from above, we thus have

$$\begin{aligned} r(\beta) - \mathbf{b}(\beta) &= (m - r - j + 1) - (\mathbf{b}(\alpha) - \mathbf{n}_{i_j}) \\ &= (m - r - j + 1) - \mathbf{b}(\alpha) + (-i_j + r + j) \\ &= (m - i_j + 1) - \mathbf{b}(\alpha) \\ &= r(\alpha) - \mathbf{b}(\alpha), \end{aligned}$$

as required to conclude our verification of (2). □

We illustrate the correspondence defined by Lemma 23 as follows.

Example 24. Recall that for $\mu = (6, 6, 6, 3, 2, 0, 0) \subset \lambda = (7, 6, 6, 4, 2, 1, 0) \in P_{7,15}$, we have $\lambda/\mu = \nu^3$, and $\{i_1, i_2, i_3, i_4\} = \{2, 3, 5, 7\}$ denote those parts in which λ and μ agree. Consider the extension $\nu^3 \rightarrow \nu^5$ given by the pair of addable boxes in rows $i_2 = 3$ and $i_3 = 5$, counting from the top of the rectangle, depicted on the left in the figure below.



By Lemma 23, the left-hand configuration of addable boxes corresponds to the right-hand configuration of addable boxes in the partition $\lambda_\mu = (8, 8, 3, 0) \in P_{4,15}$ obtained by the join-and-cut algorithm in Example 19. In λ_μ , the corresponding addable boxes are instead in rows 2 and 3, counting from the top of the shorter rectangle. Moreover, recall from Example 15 that

$$\text{wt}_\mu^\nu(\alpha_{i_2}) = \mathbf{t}_{11} - \mathbf{t}_{(5-3)} = \mathbf{t}_{11} - \mathbf{t}_2 \quad \text{and} \quad \text{wt}_\mu^\nu(\alpha_{i_3}) = \mathbf{t}_5 - \mathbf{t}_{(3-1)} = \mathbf{t}_5 - \mathbf{t}_2,$$

where here we denote by α_{i_j} the addable box in row i_j of μ , counting from the top of the rectangle. Similarly, we may compute that

$$\begin{aligned} \text{wt}_{\lambda_\mu}^\nu(\beta_2) &= \mathbf{t}_{11} - \mathbf{t}_{(3-1)} = \mathbf{t}_{11} - \mathbf{t}_2 = \text{wt}_\mu^\nu(\alpha_{i_2}), \\ \text{wt}_{\lambda_\mu}^\nu(\beta_3) &= \mathbf{t}_5 - \mathbf{t}_{(2-0)} = \mathbf{t}_5 - \mathbf{t}_2 = \text{wt}_\mu^\nu(\alpha_{i_3}), \end{aligned}$$

illustrating Equation (2).

In light of Lemma 23, we aim next to repackage the weights on skew shapes in Theorem 1 in terms of weights on a partition shape, for which we require some auxiliary notation.

Recall that $1 \leq p \leq m$ is a fixed integer in the context of Theorem 1. For any fixed integer $0 \leq r \leq p$, throughout the rest of the paper, we write

$$m' := m - r \quad \text{and} \quad p' := p - r.$$

Define an indexing set

$$\mathcal{A}_{p'} := \{\vec{\tau} \in [m - p + 1]^{p'} \mid \tau_1 \leq \dots \leq \tau_{p'}\}.$$

The value $m - p + 1$ arises here as the largest attainable row number (counting from the bottom of the rectangle) indexing the bottom-most addable box for a partition in $\mathcal{P}_{m',n}$.

Remark 25. Given any vector $\vec{\tau} = (\tau_1, \dots, \tau_{p'}) \in \mathcal{A}_{p'}$, define

$$\vec{\tau}_\delta := \vec{\tau} + (0, 1, \dots, p' - 1),$$

which adds the reverse of the staircase $\delta^{p'}$ to $\vec{\tau}$. Equivalently, writing $\vec{\tau}_\delta = (\tau_1^\delta, \dots, \tau_{p'}^\delta)$, we see that

$$\tau_j^\delta := \tau_j + j - 1$$

for any $j \in [p']$. Since $\tau_j \in [m - p + 1]$, the entries of $\vec{\tau}_\delta$ are p' distinct integers from the set $[m']$. Conversely, any increasing choice $\vec{\tau}_\delta$ of p' distinct integers in $[m']$ uniquely corresponds to $\vec{\tau} = \vec{\tau}_\delta - (0, 1, \dots, p' - 1) \in \mathcal{A}_{p'}$. The correspondence $\vec{\tau} \longleftrightarrow \vec{\tau}_\delta$ is clearly a bijection between $\mathcal{A}_{p'}$ and the set $\mathcal{A}_{p'}^\delta$ of p' strictly increasing integers in $[m']$.

In the remainder of this section, we also fix a diagram $\eta \in \mathcal{P}_{m',n+1}$. Recall that $\mathbf{U}(\eta)$ denotes the set of up-steps of η , and enumerate these up-steps here as $\{\mathbf{u}_1, \dots, \mathbf{u}_{m'}\}$, recorded from the bottom of the rectangle. Given any $\vec{\tau} = (\tau_1, \dots, \tau_{p'}) \in \mathcal{A}_{p'}$ and any index $j \in [p']$, define the weight

$$\Psi_\eta(\vec{\tau}, j) := t_{\mathbf{u}_{\tau_j^\delta}} - t_{\tau_j}. \tag{4}$$

The following example illustrates how to calculate these weights.

Example 26. Recall $\lambda_\mu = (8, 8, 3, 0) \in \mathcal{P}_{4,15}$ from Example 19, which originated from a pair of partitions $\lambda, \mu \in \mathcal{P}_{7,15}$ such that the skew shape $\lambda/\mu = \mathbf{v}^3$. Recalling that $m = 7$ and fixing $p = 5$ as in Example 15, we have $m' = 7 - 3 = 4$ and $p' = 5 - 3 = 2$, and so $\mathcal{A}_2 = \{\vec{\tau} \in [3]^2 \mid \tau_1 \leq \tau_2\}$.

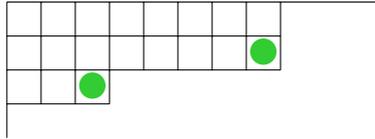
As an example, fix $\vec{\tau} = (2, 2) \in \mathcal{A}_2$. We demonstrate how to calculate the weights $\Psi_{\lambda_\mu}(\vec{\tau}, 1)$ and $\Psi_{\lambda_\mu}(\vec{\tau}, 2)$. First, the vector $\vec{\tau}_\delta = \vec{\tau} + (0, 1) = (2, 3) \in \mathcal{A}_2^\delta$ indexes two rows in the partition λ_μ , counting from the bottom of the rectangle. The up-steps of λ_μ are

$$\mathbf{U}(\lambda_\mu) = \{1, 5, 11, 12\},$$

and so the up-steps isolated by $\vec{\tau}_\delta = (2, 3)$ are $\mathbf{u}_2 = 5$ and $\mathbf{u}_3 = 11$. Therefore, by (4) we have

$$\Psi_{\lambda_\mu}(\vec{\tau}, 1) = t_5 - t_2 \quad \text{and} \quad \Psi_{\lambda_\mu}(\vec{\tau}, 2) = t_{11} - t_2.$$

To take this calculation one step further, the vector $\vec{\tau}_\delta = (2, 3)$ indexes two rows of λ_μ , each of which supports an addable box as follows:



Denote here by γ_j the addable box in row j of λ_μ , counting from the bottom of the rectangle. Then, comparing the addable boxes β_j from Example 24 which are indexed from the top down, we have

$$\begin{aligned} \text{wt}_{\lambda_\mu}^v(\gamma_2) &= \text{wt}_{\lambda_\mu}^v(\beta_3) = t_5 - t_2 = \Psi_{\lambda_\mu}(\vec{t}, 1), \\ \text{wt}_{\lambda_\mu}^v(\gamma_3) &= \text{wt}_{\lambda_\mu}^v(\beta_2) = t_{11} - t_2 = \Psi_{\lambda_\mu}(\vec{t}, 2). \end{aligned}$$

We formalize this additional observation in the next lemma.

As illustrated by Example 26, the weights $\Psi_\eta(\vec{t}, j)$ encode the weights of those addable boxes occurring in Lemma 23 precisely as follows.

Lemma 27. *Let $\eta \in \mathcal{P}_{m', n+1}$, and suppose that $\vec{t}_\delta = (t_1^\delta, \dots, t_{p'}^\delta) \in \mathcal{A}_p^\delta$, indexes the rows of η which contain addable boxes in an extension $\eta = \mathbf{v}^0 \rightarrow \mathbf{v}^{p'}$, counting rows from the bottom of the rectangle. Then for $\vec{t} = \vec{t}_\delta - (0, 1, \dots, p' - 1) \in \mathcal{A}_{p'}$, and any $j \in [p']$,*

$$\text{wt}_\eta^v(\beta_{t_j^\delta}) = \Psi_\eta(\vec{t}, j),$$

where $\beta_{t_j^\delta}$ is the addable box in row t_j^δ of η , counting rows from the bottom of the rectangle.

Proof. For ease of notation, throughout this proof we denote by $\beta := \beta_{t_j^\delta}$. Since β is in row t_j^δ counting from the bottom of the rectangle, we immediately have $r(\beta) = t_j^\delta$. Having also enumerated $\mathbf{u}(\eta) = \{u_1, \dots, u_{m'}\}$ from the bottom of the rectangle, we automatically have $\mathbf{u}(\beta) = u_{t_j^\delta}$. By hypothesis, the entries of $\vec{t}_\delta = (t_1^\delta, \dots, t_{p'}^\delta)$ index the rows in η which contain addable boxes in an extension $\mathbf{v}^0 \rightarrow \mathbf{v}^{p'}$, counting from the bottom of the rectangle, and so $\mathbf{b}(\beta) = j - 1$. Therefore,

$$r(\beta) - \mathbf{b}(\beta) = t_j^\delta - (j - 1) = (t_j + j - 1) - (j - 1) = t_j,$$

and so $\text{wt}_\eta^v(\beta) = t_{\mathbf{u}(\beta)} - t_{r(\beta) - \mathbf{b}(\beta)} = t_{u_{t_j^\delta}} - t_{t_j} = \Psi_\eta(\vec{t}, j)$, as claimed. \square

Ultimately, the equivariant Pieri rule on cylindric shapes requires consideration of all possible extensions $\mathbf{v}^r \rightarrow \mathbf{v}^p$. We thus define the following sum

$$\Psi(\eta, p') := \sum_{\vec{t} \in \mathcal{A}_{p'}} \prod_{j \in [p']} \Psi_\eta(\vec{t}, j), \tag{5}$$

which yields the equivariant Littlewood-Richardson coefficients in Theorem 1 as follows.

Proposition 28. *Fix an integer $1 \leq p \leq m$ and a partition $\mu \in \mathcal{P}_{mn}$. For any integer $0 \leq r \leq p$ and any partition $\lambda \in \mathcal{P}_{m, n+1}$ such that the skew shape $\lambda/\mu = \mathbf{v}^r$, we have*

$$\Psi(\lambda_\mu, p') = \sum_{\mathbf{v}^r \rightarrow \mathbf{v}^p} \prod_{\alpha \in \mathbf{v}^p \setminus \mathbf{v}^r} \text{wt}_\mu^v(\alpha). \tag{6}$$

Proof. By Lemma 23, any extension $\mathbf{v}^r \rightarrow \mathbf{v}^p$ in μ corresponds to a unique extension $\mathbf{v}^0 \rightarrow \mathbf{v}^{p'}$ in λ_μ , and the weights of corresponding addable boxes are equal. Therefore, by Lemma 23, we have

$$\sum_{\mathbf{v}^r \rightarrow \mathbf{v}^p} \prod_{\alpha \in \mathbf{v}^p \setminus \mathbf{v}^r} \text{wt}_\mu^{\mathbf{v}}(\alpha) = \sum_{\mathbf{v}^0 \rightarrow \mathbf{v}^{p'}} \prod_{\beta \in \mathbf{v}^{p'} \setminus \mathbf{v}^0} \text{wt}_{\lambda_\mu}^{\mathbf{v}}(\beta). \quad (7)$$

We proceed to rephrase the right-hand expression in (7) in terms of the weights occurring in Lemma 27. By definition, any extension $\mathbf{v}^0 \rightarrow \mathbf{v}^{p'}$ consists of p' addable boxes in distinct, nonzero rows of $\lambda_\mu \in \mathbf{P}_{m', n+1}$. Recording these row numbers in increasing order, counted from the bottom of the rectangle, gives an element $\vec{t}_\delta \in \mathcal{A}_p^\delta$, by Remark 25. Identifying the extension $\mathbf{v}^0 \rightarrow \mathbf{v}^{p'}$ with the vector \vec{t}_δ , since row ι_j^δ of λ_μ supports an addable box, note that $(\lambda_\mu)_{\iota_j^\delta} \neq 0$ for all $j \in [p']$.

Note, that Lemma 27 only applies to those rows of λ_μ which are nonzero, and so our first reformulation of (7) will reflect this constraint. Recalling that the number of boxes in each row weakly increases going upward from the bottom of the rectangle, since the bottom-most row in the extension $\mathbf{v}^0 \rightarrow \mathbf{v}^{p'}$ is nonzero, so are all the rows above it. Rephrasing this observation in terms of the parts of λ_μ , we have $(\lambda_\mu)_{\iota_j^\delta} \neq 0$ for all $j \in [p']$ if and only if $(\lambda_\mu)_{\iota_1^\delta} \neq 0$. In addition, since $\iota_j^\delta = \iota_j + j - 1$, then for $j = 1$ we simply have $\iota_1^\delta = \iota_1$. Lemma 27 then permits us to convert the weights $\text{wt}_{\lambda_\mu}^{\mathbf{v}}(\beta)$ into polynomials of the form $\Psi_{\lambda_\mu}(\vec{t}, j)$, for those \vec{t} such that \vec{t}_δ index nonzero rows of λ_μ , as follows

$$\sum_{\mathbf{v}^0 \rightarrow \mathbf{v}^{p'}} \prod_{\beta \in \mathbf{v}^{p'} \setminus \mathbf{v}^0} \text{wt}_{\lambda_\mu}^{\mathbf{v}}(\beta) = \sum_{\substack{\vec{t} \in \mathcal{A}_{p'} \\ (\lambda_\mu)_{\iota_1} \neq 0}} \prod_{j \in [p']} \Psi_{\lambda_\mu}(\vec{t}, j). \quad (8)$$

We now consider those $\vec{t} \in \mathcal{A}_{p'}$ such that $(\lambda_\mu)_{\iota_1} = 0$, which are currently omitted from the sum in the right-hand expression in (8). In particular, we claim that each such \vec{t} contributes zero to this sum. To see this, recall that $\Psi_{\lambda_\mu}(\vec{t}, 1) = \mathbf{t}_{\mathbf{u}_{\iota_1}} - \mathbf{t}_{\iota_1}$. Since the up-steps $\mathbf{U}(\lambda_\mu)$ are indexed from the bottom of the rectangle, and since the number of parts weakly increases going upward while $(\lambda_\mu)_{\iota_1} = 0$, then the first up-steps $\mathbf{u}_1, \dots, \mathbf{u}_{\iota_1}$ all satisfy $\mathbf{u}_k = k$. In particular, $\mathbf{u}_{\iota_1} = \iota_1$, and so $\Psi_{\lambda_\mu}(\vec{t}, 1) = \mathbf{t}_{\mathbf{u}_{\iota_1}} - \mathbf{t}_{\iota_1} = \mathbf{t}_{\iota_1} - \mathbf{t}_{\iota_1} = 0$. Since we take the product over all $j \in [p']$, we have

$$\sum_{\substack{\vec{t} \in \mathcal{A}_{p'} \\ (\lambda_\mu)_{\iota_1} = 0}} \prod_{j \in [p']} \Psi_{\lambda_\mu}(\vec{t}, j) = 0. \quad (9)$$

We can thus add these terms back into the sum in (8) without affecting the value:

$$\begin{aligned} \sum_{\substack{\vec{t} \in \mathcal{A}_{p'} \\ (\lambda_\mu)_{\iota_1} \neq 0}} \prod_{j \in [p']} \Psi_{\lambda_\mu}(\vec{t}, j) &= \sum_{\substack{\vec{t} \in \mathcal{A}_{p'} \\ (\lambda_\mu)_{\iota_1} \neq 0}} \prod_{j \in [p']} \Psi_{\lambda_\mu}(\vec{t}, j) + \sum_{\substack{\vec{t} \in \mathcal{A}_{p'} \\ (\lambda_\mu)_{\iota_1} = 0}} \prod_{j \in [p']} \Psi_{\lambda_\mu}(\vec{t}, j) \\ &= \sum_{\vec{t} \in \mathcal{A}_{p'}} \prod_{j \in [p']} \Psi_{\lambda_\mu}(\vec{t}, j) \\ &= \Psi(\lambda_\mu, \mathbf{p}'). \end{aligned} \quad (10)$$

Equation (6) now follows by combining (7), (8), and (10). \square

We demonstrate Proposition 28 in the following example.

Example 29. Fix $p = 5$, and $\mu = (6, 6, 6, 3, 2, 0, 0) \in P_{7,15}$. Then $\lambda = (7, 6, 6, 4, 2, 1, 0) \in P_{7,15}$ satisfies $\lambda/\mu = v^3$. Recall from Example 15 that

$$\sum_{v^r \rightarrow v^p} \prod_{\alpha \in v^p \setminus v^r} \text{wt}_\mu^v(\alpha) = (t_{12} - t_3)(t_{11} - t_3) + (t_{12} - t_3)(t_5 - t_2) + (t_{11} - t_2)(t_5 - t_2),$$

by calculating the weights of the three possible extensions $v^3 \rightarrow v^5$ depicted in Figure 3.

On the other hand, we may apply Proposition 28 to calculate the same polynomial. Recall from Example 26 that $p' = 5 - 3 = 2$ and

$$\mathcal{A}_2 = \{\vec{t} \in [3]^2 \mid t_1 \leq t_2\} = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}.$$

By Example 19, the partition $\lambda_\mu = (8, 8, 3, 0) \in P_{4,15}$, which has up-steps $\mathbf{U}(\lambda_\mu) = \{1, 5, 11, 12\}$. We shall compute

$$\Psi(\lambda_\mu, 2) = \sum_{\vec{t} \in \mathcal{A}_2} \Psi_{\lambda_\mu}(\vec{t}, 1) \Psi_{\lambda_\mu}(\vec{t}, 2).$$

First, consider any of the vectors of the form $\vec{t} = (1, k) \in \mathcal{A}_2$. In each case, the weight $\Psi_{\lambda_\mu}(\vec{t}, 1) = t_1 - t_1 = 0$, and so the three vectors $(1, 1), (1, 2), (1, 3) \in \mathcal{A}_2$ contribute zero to the sum $\Psi(\lambda_\mu, 2)$, illustrating Equation (9) from the proof of Proposition 28.

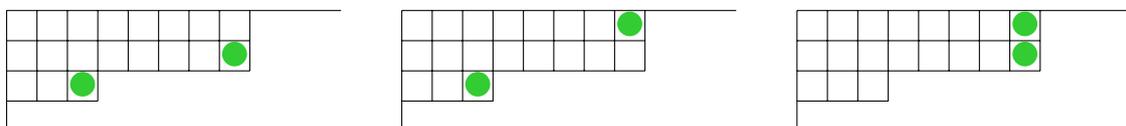
Each of the remaining three vectors in \mathcal{A}_2 contributes a nonzero term. Recall from Example 26 that we calculated the weights $\Psi_{\lambda_\mu}(\vec{t}, j)$ for $\vec{t} = (2, 2)$ and $j \in [2]$. In particular, we see that

$$\Psi_{\lambda_\mu}((2, 2), 1) \Psi_{\lambda_\mu}((2, 2), 2) = (t_5 - t_2)(t_{11} - t_2).$$

Similarly, applying Equation (4) directly, we have

$$\begin{aligned} \Psi_{\lambda_\mu}((2, 3), 1) \Psi_{\lambda_\mu}((2, 3), 2) &= (t_5 - t_2)(t_{12} - t_3), \\ \Psi_{\lambda_\mu}((3, 3), 1) \Psi_{\lambda_\mu}((3, 3), 2) &= (t_{11} - t_3)(t_{12} - t_3). \end{aligned}$$

For reference, the configurations of addable boxes corresponding to the vectors $(2, 2), (2, 3), (3, 3) \in \mathcal{A}_2$ via Lemma 27 are recorded left to right, respectively, in the following figure:



Taking the sum, we obtain

$$\Psi(\lambda_\mu, 2) = (t_5 - t_2)(t_{11} - t_2) + (t_5 - t_2)(t_{12} - t_3) + (t_{11} - t_3)(t_{12} - t_3), \quad (11)$$

which is the same polynomial reviewed above from Example 15, after reversing the order of the summands and multiplicands.

As we shall see in the next section, the polynomial $\Psi(\lambda_\mu, \mathbf{p}')$ is the one which most naturally relates to the localizations appearing in Theorem 20 of Huang and Li. In particular, to prove Theorem 1, we shall equate the polynomials occurring in Proposition 28 with Huang-Li's localization formula for equivariant Littlewood-Richardson coefficients from Theorem 20 via the following key proposition.

Proposition 30. *Fix an integer $1 \leq p \leq m$ and a partition $\mu \in \mathcal{P}_{mn}$. For any integer $0 \leq r \leq p$ and any partition $\lambda \in \mathcal{P}_{m,n+1}$ such that the skew shape $\lambda/\mu = \mathbf{v}^r$, we have*

$$\xi_{\mathbf{m}'}^{(1)^{p'}}(\lambda_\mu) = \Psi(\lambda_\mu, \mathbf{p}').$$

Combining Proposition 28 with Proposition 30 proves the equality of all classical Littlewood-Richardson coefficients in Theorems 1 and 20. Since the quantum terms are easily obtained from the classical ones by applying the equivariant rim hook rule from [BMT], the proof of Proposition 30 will be the focus of the rest of the paper.

4 Specializations of Factorial Schur Polynomials

The proof of Proposition 30 is the primary goal of this final section of the paper. This proof proceeds by realizing in Section 4.2 the localizations $\xi^Y(\eta)$ as specializations of factorial Schur polynomials, whose definition we review in Section 4.1. The proof of Theorem 1 then immediately follows in Section 4.3.

4.1 Factorial Schur polynomials

A *semi-standard Young tableaux (SSYT)* of shape $\lambda \in \mathcal{P}_{mn}$ is a filling of each box of the Young diagram for λ with a single number from the set $[m]$ such that numbers are weakly increasing across rows (reading from left to right) and strictly increasing down columns (reading from top to bottom). Denote by SSYT_λ the set of all SSYT of shape λ . The *Schur polynomial* for λ in the variables $(\mathbf{x}) = x_1, \dots, x_m$ is then defined to be

$$s_\lambda(\mathbf{x}) = \sum_{T_\lambda \in \text{SSYT}_\lambda} \prod_{\beta \in T_\lambda} x_{T_\lambda(\beta)}, \quad (12)$$

where $T_\lambda(\beta)$ denotes the number filling box β in tableaux T_λ . The elementary symmetric polynomial $e_i(\mathbf{x})$ is the Schur polynomial $s_{(1)^i}(\mathbf{x})$, and the homogeneous symmetric polynomial $h_i(\mathbf{x})$ equals $s_{(i)}(\mathbf{x})$. The classical cohomology $H^*(\text{Gr}(m, n))$ has a well-known presentation as a polynomial ring generated by the elementary symmetric polynomials $e_1(\mathbf{x}), \dots, e_m(\mathbf{x})$, in which the Schubert class σ_λ is identified with the Schur polynomial $s_\lambda(\mathbf{x})$.

There is a corresponding presentation for $H_T^*(\text{Gr}(m, n))$ in terms of the equivariant analogs of the elementary and homogeneous symmetric polynomials; namely the *factorial elementary symmetric polynomials* $e_i(\mathbf{x}|\mathbf{t})$ and the *factorial homogeneous symmetric polynomials* $h_i(\mathbf{x}|\mathbf{t})$ in the variables (\mathbf{x}) and $(\mathbf{t}) = t_1, \dots, t_n$. If we denote by $\Lambda := \mathbb{Z}[t_1, \dots, t_n]$, then the T -equivariant cohomology ring $H_T^*(\text{Gr}(m, n))$ has the following presentation

$$H_T^*(\text{Gr}(m, n)) \cong \frac{\Lambda[e_1(\mathbf{x}|\mathbf{t}), \dots, e_m(\mathbf{x}|\mathbf{t})]}{\langle h_{n-m+1}(\mathbf{x}|\mathbf{t}), \dots, h_n(\mathbf{x}|\mathbf{t}) \rangle};$$

see [Mih2, Corollary 5.1]. Moreover, under this isomorphism, the equivariant Schubert class σ_λ is represented by the equivariant generalization of the Schur polynomial $s_\lambda(\mathbf{x})$, given by the associated factorial Schur polynomial; see [Mih2, Proposition 5.2]. For our purposes, we define the *factorial Schur polynomial* $s_\lambda(\mathbf{x}|\mathbf{t})$ analogously to (12) by

$$s_\lambda(\mathbf{x}|\mathbf{t}) = \sum_{T_\lambda \in \text{SSYT}_\lambda} \prod_{\beta \in T_\lambda} x_{T_\lambda(\beta)} - t_{T_\lambda(\beta)+j-i}, \quad (13)$$

where the box $\beta \in T_\lambda$ is located in row i and column j of the tableaux, indexing rows from top to bottom and columns from left to right. As in the classical case, we have $e_i(\mathbf{x}|\mathbf{t}) = s_{(1)^i}(\mathbf{x}|\mathbf{t})$ and $h_i(\mathbf{x}|\mathbf{t}) = s_{(i)}(\mathbf{x}|\mathbf{t})$.

4.2 Specializations of factorial Schur polynomials

The localization $\xi^\gamma(\eta)$ of the equivariant Schubert class σ_γ at the T -fixed point η coincides with a certain specialization of the factorial Schur polynomial $s_\gamma(\mathbf{x}|\mathbf{t})$. We follow the treatment in [IN], though they credit Theorem 31 below to earlier work of Knutson and Tao [KT] and Lakshmibai, Raghavan, and Sankaran [LRS].

Any partition $\eta \in P_{mn}$ can be associated to a permutation in S_n , as we now review. The *window* notation for a permutation $\pi \in S_n$ records the action of π on the set $[n]$ as $\pi = [\pi_1 \cdots \pi_n]$, where $\pi_i := \pi(i)$. A permutation $\pi \in S_n$ has a *descent in position* k if $\pi(k) > \pi(k+1)$, and a permutation with at most one descent is called a *Grassmann permutation*. The set of all Grassmann permutations with a descent in position m , together with the identity permutation, is denoted by S_n^m .

Given $\eta \in P_{mn}$, record the up-steps and side-steps of the partition from the bottom of the rectangle as $\mathbf{U}(\eta) = \{u_1, \dots, u_m\}$ and $\mathbf{S}(\eta) = \{s_1, \dots, s_{n-m}\}$. Define the associated permutation $\pi_\eta \in S_n^m$ by the following window

$$\pi_\eta := [u_1 \cdots u_m | s_1 \cdots s_{n-m}]. \quad (14)$$

Note that if η is nonzero, then π_η has exactly one descent in position m , as suggested by the vertical line in the window above. We define a collection (x_{π_η}) of m variables associated to the permutation π_η by

$$(x_{\pi_\eta}) := (t_{\pi_\eta(i)}) \quad (15)$$

for $i \in [m]$. In particular, note that $(x_{\pi_\eta})_i = t_{u_i}$ by definition.

Theorem 31 (Theorem 5.4 [IN]). *For any partitions $\gamma \subseteq \eta \in P_{mn}$, we have*

$$\xi^\gamma(\eta) = s_\gamma(x_{\pi_\eta}|\mathbf{t})$$

We remark that the above statement differs from [IN, Theorem 5.4] by the sign $(-1)^{|\gamma|}$ because our convention for torus weights follows [BMT], in order to apply the equivariant rim hook rule in the proof of Theorem 1 later in Section 4.3. Our next and final lemma provides the critical link required to prove Proposition 30, and thus the main theorem.

Lemma 32. *Fix any integers $1 \leq p \leq m$ and $0 \leq r \leq p$. For any partition $\eta \in P_{m',n+1}$, we have*

$$\Psi(\eta, p') = e_{p'}(x_{\pi_\eta}|\mathbf{t}),$$

where $e_{p'}(\mathbf{x}|\mathbf{t})$ is the factorial elementary symmetric polynomial in $x_1, \dots, x_{m'}$ and t_1, \dots, t_{n+1} .

Note that Example 33 below illustrates both the statement of Lemma 32, as well as the weight-preserving bijection which is the key ingredient in the proof.

Proof. For notational convenience throughout this proof alone, we shall write $k := p'$. We thus aim to show that $\Psi(\eta, k) = e_k(x_{\eta\pi}|t)$ for any $\eta \in P_{m', n+1}$.

Recalling that $e_k(x|t)$ is the factorial Schur polynomial $s_{(1)^k}(x|t)$, formula (13) gives us

$$e_k(x|t) = \sum_{\vec{\ell} \in \text{SSYT}_{(1)^k}} \prod_{j=1}^k x_{\ell_j} - t_{\ell_j - j + 1}. \quad (16)$$

Here $\text{SSYT}_{(1)^k}$ denotes the set of all strictly increasing fillings of the column shape containing k boxes using the alphabet $[m']$. By recording only the fillings, we may thus identify $\text{SSYT}_{(1)^k}$ with the set $\mathcal{A}_k^\delta = \{\vec{\ell} \in [m']^k \mid \ell_1 < \dots < \ell_k\}$ from Remark 25.

For any $\vec{\ell} \in \text{SSYT}_{(1)^k}$, denote the corresponding summand by $e_k^{\vec{\ell}}(x|t) = \prod_{j=1}^k x_{\ell_j} - t_{\ell_j - j + 1}$. Enumerate the up-steps of η as $\mathbf{U}(\eta) = \{\mathbf{u}_1, \dots, \mathbf{u}_{m'}\}$, recorded from the bottom of the rectangle. Specializing the variables $x_i \mapsto t_{\eta\pi(i)} = t_{\mathbf{u}_i}$ as prescribed by $x_{\eta\pi}$ via (15), we obtain

$$e_k^{\vec{\ell}}(x_{\eta\pi}|t) = \prod_{j=1}^k t_{\mathbf{u}_{\ell_j}} - t_{\ell_j - j + 1}. \quad (17)$$

On the other hand, since $\text{SSYT}_{(1)^k}$ and \mathcal{A}_k^δ are in bijection, the set $\text{SSYT}_{(1)^k}$ is also in bijection with $\mathcal{A}_k = \{\vec{\iota} \in [m - p + 1]^k \mid \iota_1 \leq \dots \leq \iota_k\}$ by Remark 25. If we fix any $\vec{\iota} = (\iota_1, \dots, \iota_k) \in \mathcal{A}_k$, then the corresponding summand in expression (5) for $\Psi(\eta, k)$ may be recorded via (4) as

$$\Psi^{\vec{\iota}}(\eta, k) = \prod_{j=1}^k t_{\mathbf{u}_{\iota_j^\delta}} - t_{\iota_j},$$

where we recall from Remark 25 that $\iota_j^\delta = \iota_j + j - 1$, and more generally that $\vec{\iota}^\delta = (\iota_1^\delta, \dots, \iota_k^\delta) = \vec{\iota} + (0, 1, \dots, k - 1)$. In particular, the map $\vec{\iota} \mapsto \vec{\iota}^\delta$ gives the bijection from \mathcal{A}_k to $\text{SSYT}_{(1)^k}$ noted above.

Now given any $\vec{\iota} \in \mathcal{A}_k$ and $j \in [k]$, consider the corresponding multiplicand $t_{\mathbf{u}_{\iota_j^\delta}} - t_{\iota_j}$ in the product $\Psi^{\vec{\iota}}(\eta, k)$. The j^{th} multiplicand in the associated product $e_k^{\vec{\ell}^\delta}(x_{\eta\pi}|t)$ is then $t_{\mathbf{u}_{\iota_j^\delta}} - t_{\iota_j^\delta - j + 1} = t_{\mathbf{u}_{\iota_j^\delta}} - t_{\iota_j}$, since $\iota_j^\delta = \iota_j + j - 1$. Therefore, $\Psi^{\vec{\iota}}(\eta, k) = e_k^{\vec{\ell}^\delta}(x_{\eta\pi}|t)$ for any $\vec{\iota} \in \mathcal{A}_k$. Summing over all \mathcal{A}_k and $\text{SSYT}_{(1)^k}$, respectively, we then have $\Psi(\eta, k) = e_k(x_{\eta\pi}|t)$, which proves the lemma. \square

We now revisit our running example from Section 3, to illustrate both the statement of Lemma 32 and the method of proof.

Example 33. Fix $p = 5$ and $r = 3$, in which case $p' = 5 - 3 = 2$. Consider $\eta = (8, 8, 3, 0) \in P_{4, 15}$, and recall from Example 19 that $\eta = \lambda_\mu$ for partitions $\mu \subseteq \lambda \in P_{7, 15}$ so that here $m' = 7 - 3 = 4$. Recall by (11) in Example 29 that

$$\Psi(\eta, 2) = (t_5 - t_2)(t_{11} - t_2) + (t_5 - t_2)(t_{12} - t_3) + (t_{11} - t_3)(t_{12} - t_3). \quad (18)$$

On the other hand, we now compute $e_2(x_{\pi_\eta}|t)$ directly from the definition. The set $\text{SSYT}_{(1)^2}$ consists of the 6 strictly increasing fillings of the column shape $(1)^2$ using the alphabet $[m'] = [4]$, which we depict as

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \end{array} .$$

If we encode each of these SSYT as a vector $\vec{\ell} = (\ell_1, \ell_2)$ where $\ell_1 < \ell_2$ denotes the filling, then by (16) we have

$$e_2(x|t) = \sum_{\vec{\ell} \in \text{SSYT}_{(1)^2}} \prod_{j=1}^2 x_{\ell_j} - t_{\ell_j - j + 1}. \quad (19)$$

Recall that for $\eta = (8, 8, 3, 0) \in P_{4,15}$, we have $U(\eta) = \{1, 5, 11, 12\} = \{u_1, u_2, u_3, u_4\}$ so that

$$\pi_\eta = [1, 5, 11, 12 \mid 2, 3, 4, 6, 7, 8, 9, 10, 13, 14, 15]$$

by formula (14). Specializing the variables $x_i \mapsto t_{\pi_\eta(i)}$ in this example thus sends $x_1 \mapsto t_1$. Therefore, the 3 left-hand SSYT, all of which satisfy $\ell_1 = 1$, correspond to a product having $(t_1 - t_1)$ as a factor; that is, these 3 left-hand SSYT contribute zero to the sum in expression (19) for $e_2(x_{\pi_\eta}|t)$. Recall from Example 29 that the polynomial $\Psi(\eta, 2)$ also contains 3 zero terms corresponding to the vectors $(1, 1), (1, 2), (1, 3) \in \mathcal{A}_2$. Moreover, for these 3 vectors $\vec{\tau} \in \mathcal{A}_2$, the associated vector $\vec{t}_\delta \in \mathcal{A}_2^\delta$ gives the fillings of the 3 left-hand SSYT.

For the 3 right-hand SSYT, note more generally that $x_i \mapsto t_{u_i}$ when we specialize $e_2(x_{\pi_\eta}|t)$. Indexing the summands of $e_2(x_{\pi_\eta}|t)$ as in (17) then gives us

$$\begin{aligned} e_2^{(2,3)}(x_{\pi_\eta}|t) &= (t_{u_2} - t_{2-1+1})(t_{u_3} - t_{3-2+1}) = (t_5 - t_2)(t_{11} - t_2), \\ e_2^{(2,4)}(x_{\pi_\eta}|t) &= (t_{u_2} - t_{2-1+1})(t_{u_4} - t_{4-2+1}) = (t_5 - t_2)(t_{12} - t_3), \\ e_2^{(3,4)}(x_{\pi_\eta}|t) &= (t_{u_3} - t_{3-1+1})(t_{u_4} - t_{4-2+1}) = (t_{11} - t_3)(t_{12} - t_3). \end{aligned}$$

Taking the sum of the three above expressions and comparing with (18), we see that indeed $e_2(x_{\pi_\eta}|t) = \Psi(\eta, 2)$ term by term, confirming the statement of Lemma 32 in this example. Moreover, the 3 nonzero terms in $\Psi(\eta, 2)$ correspond to the 3 vectors $(2, 2), (2, 3), (3, 3) \in \mathcal{A}_2$ from Example 29, whose respective associates in \mathcal{A}_2^δ via Remark 25 give the fillings $(2, 3), (2, 4), (3, 4)$ of the 3 right-hand SSYT above, completing our illustration of the weight-preserving bijection in the proof of Lemma 32.

Amusingly, there is a dual version of Lemma 32, obtained by noting that \mathcal{A}_k is also in natural bijection with the set $\text{SSYT}_{(k)}$ of row-shaped semi-standard Young tableaux. This observation gives us a similar statement for the complete homogeneous factorial Schur functions, although the localization is dual in a sense, and the geometric interpretation is less clear.

Remark 34. Fix any integers $1 \leq p \leq m$ and $0 \leq r \leq p$. For any partition $\eta \in P_{m', n+1}$, we have

$$\Psi(\eta, p') = (-1)^{p'} h_{p'}(x_{\text{id}}|t_{\pi_\eta}),$$

where $h_{p'}(x|t)$ is the factorial homogeneous symmetric polynomial in $x_1, \dots, x_{m'}$ and t_1, \dots, t_{n+1} , and we instead specialize the torus weights via $t_i \mapsto t_{\pi_\eta(i)}$, whereas $x_i \mapsto t_i$.

4.3 Proof of the equivariant quantum Pieri rule on cylindric shapes

We are now able to prove the key proposition upon which the proof of Theorem 1 relies.

Proof of Proposition 30. Fix an integer $1 \leq p \leq m$ and a partition $\mu \in P_{mn}$. Consider any partition $\lambda \in P_{m,n+1}$ such that the skew shape $\lambda/\mu = \mathbf{v}^r$ for some integer $0 \leq r \leq p$. Apply Lemma 32 to $\lambda_\mu \in P_{m',n+1}$ to obtain

$$\Psi(\lambda_\mu, \mathbf{p}') = \mathbf{e}_{\mathbf{p}'}(\mathbf{x}_{\pi(\lambda_\mu)}|\mathbf{t}). \quad (20)$$

In the case $(1)^{p'} \subseteq \lambda_\mu$, applying Theorem 31 to (20), we thus have

$$\Psi(\lambda_\mu, \mathbf{p}') = \mathbf{e}_{\mathbf{p}'}(\mathbf{x}_{\pi(\lambda_\mu)}|\mathbf{t}) = \mathbf{s}_{(1)^{p'}}(\mathbf{x}_{\pi(\lambda_\mu)}|\mathbf{t}) = \xi_{m'}^{(1)^{p'}}(\lambda_\mu),$$

confirming Proposition 30 in this case.

If $(1)^{p'} \not\subseteq \lambda_\mu$, then by Lemma 17, we know that $\xi_{m'}^{(1)^{p'}}(\lambda_\mu) = 0$. The condition $(1)^{p'} \not\subseteq \lambda_\mu$ implies that the number of nonzero parts of $\lambda_\mu \in P_{m',n+1}$ is strictly less than p' . In particular, for all $\vec{\tau} \in \mathcal{A}_{p'}$, we have $(\lambda_\mu)_{\vec{\tau}} = 0$. By (9) and (10) in the proof of Proposition 28, we know that $\Psi(\lambda_\mu, \mathbf{p}') = 0$ in this case. Therefore, if $(1)^{p'} \not\subseteq \lambda_\mu$, we have

$$0 = \Psi(\lambda_\mu, \mathbf{p}') = \xi_{m'}^{(1)^{p'}}(\lambda_\mu),$$

as required to complete the proof of Proposition 30. \square

Equipped with Proposition 30, we are now prepared to prove the main theorem of this paper.

Proof of Theorem 1. We first prove the rule for multiplying by a Schubert class indexed by a column-shaped partition. Fix an integer $1 \leq p \leq m$ and a partition $\mu \in P_{mn}$. Consider any partition $\lambda \in P_{m,n+1}$ such that the skew shape $\lambda/\mu = \mathbf{v}^r$ for some integer $0 \leq r \leq p$. Applying Proposition 28, Proposition 30, and Theorem 20 in turn, we obtain

$$\sum_{\mathbf{v}^r \rightarrow \mathbf{v}^p} \prod_{\alpha \in \mathbf{v}^p \setminus \mathbf{v}^r} \text{wt}_\mu^\mathbf{v}(\alpha) = \Psi(\lambda_\mu, \mathbf{p}') = \xi_{m'}^{(1)^{p'}}(\lambda_\mu) = \mathbf{c}_{(1)^p, \mu}^\lambda,$$

where $\mathbf{c}_{(1)^p, \mu}^\lambda$ is an equivariant Littlewood-Richardson coefficient in $H_T^*(\text{Gr}(m, n+1))$. Further, note that in the Pieri case, $\sigma_{(1)^p} \circ \sigma_\mu \in H_T^*(\text{Gr}(m, n+1))$ is the same as the product $\sigma_{(1)^p} \circ \sigma_\mu \in H_T^*(\text{Gr}(m, 2n-1))$ required to apply the equivariant rim hook rule from [BMT], reviewed here as Theorem 5.

The argument now naturally divides into two cases: $\lambda \in P_{mn}$ and $\lambda \notin P_{mn}$. If $\lambda \in P_{mn}$, then by the equivariant rim hook rule, $\mathbf{c}_{(1)^p, \mu}^\lambda = \mathbf{c}_{(1)^p, \mu}^{\lambda, 0}$ for $\sigma_{(1)^p} \star \sigma_\mu \in \text{QH}_T^*(\text{Gr}(m, n))$ as well, since rim hook reduction acts as the identity both on σ_λ and the torus weights $\mathbf{t}_1, \dots, \mathbf{t}_n$ in this case. The formula for the classical terms in Theorem 1 thus follows.

Now suppose that $\lambda \notin P_{mn}$. Since the parts of $\lambda = (\lambda_1, \dots, \lambda_m)$ are weakly decreasing and $\lambda \in P_{m,n+1} \setminus P_{mn}$, we must have $\lambda_1 = n - m + 1$. Recall by Theorem 5 that if the n -core \mathbf{v} of λ is not in P_{mn} , then the equivariant rim hook rule sends $\sigma_\lambda \mapsto 0$, and this term does not contribute to the product $\sigma_{(1)^p} \star \sigma_\mu \in \text{QH}_T^*(\text{Gr}(m, n))$. We thus reduce to the case where $\mathbf{v} \in P_{mn}$. The n -core $\mathbf{v} \in P_{mn}$ in this case if and only if the partition

$\lambda \in \mathcal{P}_{m,n+1}$ contains a removable n -rim hook, in which case we must have $\lambda_m \geq 1$ as well. In the Pieri case, we can remove exactly one n -rim hook of height m , so that in the statement of Theorem 5, we have $\varepsilon_1 = m$. The equivariant rim hook rule thus sends $\sigma_\lambda \mapsto (-1)^{\varepsilon_1 - m} \mathbf{q} \sigma_\nu = \mathbf{q} \sigma_\nu$. Further, recall from our discussion in Section 2.2 that since λ and ν are related by removing a single n -rim hook with these coordinates, then the closed loops $\lambda[0] = \nu[1]$ on the cylinder $\mathcal{C}_{m,n+1}$.

Since $\lambda \in \mathcal{P}_{m,n+1}$, for any addable box α in an extension $\nu^r \rightarrow \nu^p$, the row number satisfies $r(\alpha) \leq m < n$, and the number of boxes below α also satisfies $b(\alpha) < m < n$. In addition, since $\mu \in \mathcal{P}_{mn}$, while $\lambda_1 = n - m + 1$, we know that the skew shape λ/μ contains a box in the top row of the rectangle. In particular, any addable box $\alpha \in \nu^p \setminus \nu^r$ sits in or below row $m - 1$, indexing rows from the bottom of the rectangle. Therefore, all up-step indices satisfy $u(\alpha) \leq n - 1 < n$. Since the equivariant rim hook rule sends $t_i \mapsto t_{i(\bmod n)}$ and each of these three statistics is strictly less than n for any addable box $\alpha \in \nu^p \setminus \nu^r$, each torus weight $t_{u(\alpha)} - t_{r(\alpha) - b(\alpha)}$ occurring in $c_{(1)^p, \mu}^\lambda$ is mapped to itself by the equivariant rim hook rule. Therefore, in case $\lambda \notin \mathcal{P}_{mn}$, we have

$$c_{(1)^p, \mu}^\lambda = \sum_{\nu^r \rightarrow \nu^p} \prod_{\alpha \in \nu^p \setminus \nu^r} \text{wt}_\mu^\nu(\alpha) = c_{(1)^p, \mu}^{\nu, 1}$$

where the skew shape $\nu^r = \lambda/\mu = \nu[1]/\mu[0] = \nu/1/\mu$. The formula for the quantum terms in the product $\sigma_{(1)^p} \star \sigma_\mu \in \text{QH}_T^*(\text{Gr}(m, n))$ in Theorem 1 thus follows by re-indexing.

We now deduce the rule for multiplying by a Schubert class indexed by a row-shaped partition from the column-shape rule via level-rank duality. Fix an integer $1 \leq k \leq n - m$ and a partition $\mu \in \mathcal{P}_{mn}$. Consider any partition $\lambda \in \mathcal{P}_{mn}$ such that the skew shape $\lambda/d/\mu = h^r$ for some integer $0 \leq r \leq k$ and $d \in \{0, 1\}$. Let $\alpha \in h^k \setminus h^r$ be an addable box in an extension $h^r \rightarrow h^k$.

Since $\lambda/d/\mu$ is a horizontal r -strip, then the transposed skew shape $\lambda'/d'/\mu'$ is a vertical r -strip ν^r with a corresponding extension $\nu^r \rightarrow \nu^k$. Moreover, the addable box $\alpha \in h^k \setminus h^r$ corresponds to a unique addable box $\beta \in \nu^k \setminus \nu^r$ in the image of the transpose map. Since α and β are related by the transpose, by Definition 10 we automatically have $r(\beta) = c(\alpha)$ and $b(\beta) = \text{rt}(\alpha)$, and therefore $r(\beta) - b(\beta) = c(\alpha) - \text{rt}(\alpha)$. In addition, the up-step $u(\beta)$ corresponds precisely to the side-step $s(\alpha)$ under the transpose map, both of which are enumerated from the bottom of the rectangle, and thus $u(\beta) + s(\alpha) = n + 1$.

Under the involution on torus weights from level-rank duality as in Theorem 6 we now have

$$-w_0 : \begin{cases} t_{u(\beta)} \mapsto -t_{s(\alpha)} \\ -t_{r(\beta) - b(\beta)} \mapsto t_{n+1 - (c(\alpha) - \text{rt}(\alpha))}. \end{cases}$$

Comparing with Definition 11 gives us $-w_0 : \text{wt}_{\mu'}^\nu(\beta) \mapsto \text{wt}_\mu^h(\alpha)$. Having already proved above that

$$c_{(1)^k, \mu'}^{\lambda', d} = \sum_{\nu^r \rightarrow \nu^k} \prod_{\beta \in \nu^k \setminus \nu^r} \text{wt}_{\mu'}^\nu(\beta)$$

in $\text{QH}_T^*(\text{Gr}(n - m, n))$, applying the level-rank duality from Theorem 6 to both sides of this expression yields

$$c_{(k), \mu}^{\lambda, d} = \sum_{h^r \rightarrow h^k} \prod_{\alpha \in h^k \setminus h^r} \text{wt}_\mu^h(\alpha),$$

as required to complete the proof of the formula for $\sigma_{(k)} \star \sigma_{\mu} \in \mathrm{QH}_{\mp}^*(\mathrm{Gr}(m, n))$ in Theorem 1. \square

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