Extremal rays of the equivariant Littlewood-Richardson cone

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Abstract

We give an inductive procedure for finding the extremal rays of the equivariant Littlewood-Richardson cone, which is closely related to the solution space to S. Friedland's majorized Hermitian eigenvalue problem. In so doing, we solve the "rational version" of a problem posed by C. Robichaux, H. Yadav, and A. Yong. Our procedure is a natural extension of P. Belkale's algorithm for the classical Littlewood-Richardson cone. The main tools for accommodating the equivariant setting are certain foundational results of D. Anderson, E. Richmond, and A. Yong. We also study two families of special rays of the cone and make observations about the Hilbert basis of the associated lattice semigroup.

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1 Introduction

In this work we collide the two worlds of [3] and [1] in order to answer a question of C. Robichaux, H. Yadav, and A. Yong [14]. In [3], P. Belkale introduced an algorithm for finding the extremal rays of the Hermitian eigencone (also called the tensor cone or Littlewood-Richardson cone) – the pointed rational cone which among other things governs the nonvanishing of Littlewood-Richardson coefficients. In [1], D. Anderson, E. Richmond, and A. Yong proved that the *equivariant* Littlewood-Richardson nonvanishing problem is determined by a similar cone, of which the former is a facet, thereby proving the equivariant nonvanishing problem to be saturated. Here, we naturally adapt Belkale's algorithm to the equivariant setting, repeatedly making use of the core Proposition 2.1 from [1], thus finding *most* of the extremal rays of the equivariant Littlewood-Richardson cone. The missing rays are few in number and easily described.

To provide a little context, a famous problem from the 19$^{\text{th}}$ century is to determine the possible eigenvalues of three Hermitian matrices A, B, and C which satisfy $A + B = C$. Horn conjectured [9] that a certain recursive set of inequalities on the eigenvalues were necessary and sufficient for such matrices to exist. This turned out to be true, as established by Klyachko [11] and Totaro [16]. While Horn's list of constraints is overdetermined, the essential inequalities were found by Belkale [2] and proven to be minimal by Knutson, Tao, Woodward [13]. For a much more thorough treatment of this story, see [8].

There is a natural generalization to this problem. Recall that a Hermitian matrix A *majorizes* another, say B, if $A - B$ is positive semidefinite (written $A \geqslant B$). S. Friedland [6] studied the question: what are the possible eigenvalues of A, B, and C if $A + B \geqslant C$? Friedland proposed a set of inequalities, and W. Fulton [7] showed they correctly answered the problem but were redundant, providing a minimal set that is largely the same as the essential Horn inequalities from above!

In both the aforementioned problems, the sets of integer eigenvalue solutions have special significance. By the Saturation Theorem of [12], nonnegative integral solutions to the first problem parametrize the nonzero structure constants (Littlewood-Richardson coefficients) for multiplication in three equivalent settings: Schur polynomials, Grassmannian Schubert classes, and classes of irreducibles in the representation ring of GL_n. By [1, Theorem 1.3], the second problem is connected to *double* Schur polynomials and multiplication in the *equivariant* cohomology rings of Grassmannians.

The solutions to both problems are governed by rational, linear inequalities, so the solution sets, viewed inside the relevant real vector spaces, form convex rational polyhedral cones. These we denote by LR for the nonvanishing of Littlewood-Richardson problem and EqLR for the second (equivariant) problem. Such cones have finitely many *extremal rays* – faces of dimension 1 – and the associated semigroups of lattice points have a finite generating set, the *Hilbert basis*, which always includes the set of (primitive points on) extremal rays.

In [17, Problem A], A. Zelevinsky posed the question of finding the Hilbert basis for LR explicitly, and Robichaux, Yadav, and Yong [14, §6] asked the same question for EqLR. While both questions remain open, Belkale's algorithm [3] serves as a partial ("rational") solution to Zelevinsky's question in that it gives formulas for finding the extremal rays of LR. Our main result is an analogous solution to Robichaux, Yadav, and Yong's question, which we obtain by extending Belkale's algorithm to the equivariant setting.

After reviewing the various cones and their inequalities (§2), as well as recalling Belkale's algorithm in detail (§3), we show that for any Horn facet of the equivariant Littlewood-Richardson cone, our algorithm produces the extremal rays on that facet (Theorems 15, 18). Unlike in [3], there are some rays not on any Horn facet, and we explicitly enumerate these (Theorem 23). Finally, we show by example (§6) that there do in general ($r \geqslant 6$) exist Hilbert basis elements which do not lie on an extremal ray. This phenomenon has not been observed for the classical Littlewood-Richardson cones.

2 Notation and inequalities for the cones

Both of the eigenvalue problems above can be considered with a greater number of summands. Let $r \geqslant 1$ and fix some $s \geqslant 3$. Let us consider the space of possible eigenvalues of $r \times r$ matrices $A_1, \ldots, A_{s-1}$, and C such that $\sum A_j = C$ or $\sum A_j \geqslant C$. We will always list the eigenvalues of a matrix in decreasing order, counted with multiplicity, for example: $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_r) \in \mathbb{R}^r$. Define

$$\mathsf{C}_r^s = \left\{ (\lambda^1, \ldots, \lambda^{s-1}, \nu) \in (\mathbb{R}^r)^s \;\middle|\; \begin{array}{c} \exists\ r \times r \text{ Herm. matrices } A_j, C \text{ with e-vals. } \lambda^j, \nu \\ \text{s.t. } A_1 + \ldots + A_{s-1} = C \end{array} \right\}.$$

Similarly define

$$\mathsf{EqC}_r^s = \left\{ (\lambda^1, \ldots, \lambda^{s-1}, \nu) \in (\mathbb{R}^r)^s \;\middle|\; \begin{array}{c} \exists\ r \times r \text{ Herm. matrices } A_j, C \text{ with e-vals. } \lambda^j, \nu \\ \text{s.t. } A_1 + \ldots + A_{s-1} \geqslant C \end{array} \right\}.$$

2.1 Inequalities describing the cones

To describe the Horn inequalities, which cut out the above cones C_r^s, EqC_r^s inside $(\mathbb{R}^r)^s$, we first introduce the notation for Grassmannian Schubert calculus. Let n be a sufficiently large integer, to be specified as needed, and let $\mathrm{Gr}(r, \mathbb{C}^n)$ denote the Grassmannian of r-dimensional subspaces in $\mathbb{C}^n$. The cohomology ring $H^*(\mathrm{Gr}(r, \mathbb{C}^n))$ has a distinguished graded basis given by classes of Schubert varieties $[X_\lambda]$, one for each partition $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_r) \in \mathbb{Z}_{\geqslant 0}^r$ such that $\lambda_1 \leqslant n - r$ (i.e., the Young diagram for λ fits inside an $r \times (n-r)$ rectangle). Here

$$X_\lambda = \{V \in \mathrm{Gr}(r, \mathbb{C}^n) |\, \dim(V \cap F_{n-r+i-\lambda_i}) \geqslant i,\ \forall 1 \leqslant i \leqslant r\},$$

where $F_\bullet$ is the fixed flag on the standard basis of $\mathbb{C}^n$ defined by $F_i = \mathbb{C}e_n \oplus \ldots \oplus \mathbb{C}e_{n-i+1}$. The complex codimension of X_λ is $|\lambda| := \lambda_1 + \ldots + \lambda_r$.

We define the (multiple-factor) structure coefficients $c^\nu_{\lambda^1, \ldots, \lambda^{s-1}}$ by the rule

$$[X_{\lambda^1}] \cdots [X_{\lambda^{s-1}}] = \sum_\nu c^\nu_{\lambda^1, \ldots, \lambda^{s-1}} [X_\nu] \in H^*(\mathrm{Gr}(r, \mathbb{C}^n))$$

for n larger than $r + \sum_{j=1}^{s-1} \lambda_1^j$ (these classes and products are *stable*: it does not matter which n we take, as long as it is big enough.) When $s = 3$, these coefficients are the Littlewood-Richardson numbers.

Partitions λ whose Young diagrams fit in an $r \times (n-r)$ rectangle are in bijection with r-element subsets of $[n] := \{1, \ldots, n\}$. Such a subset we will typically write in increasing order as $I = \{i_1 < i_2 < \ldots < i_r\}$. The bijection is often denoted τ and goes as follows:

$$I = \{i_1 < i_2 < \ldots < i_r\} \overset{\tau}{\longmapsto} \tau(I) = (i_r - r \geqslant \ldots i_2 - 2 \geqslant i_1 - 1)$$

Note that the definition is independent of n. If no confusion is likely to arise, we'll just write $c^K_{I_1, \ldots, I_{s-1}}$ instead of $c^{\tau(K)}_{\tau(I_1), \ldots, \tau(I_{s-1})}$.

Now we can state the celebrated theorem resolving Horn's conjecture:

Theorem 1 ([11, 2, 13]). *Let* $\lambda^j = (\lambda^j_1, \ldots, \lambda^j_r)$, *for* $j \in [s-1]$, *and* $\nu = (\nu_1, \ldots, \nu_r)$ *lie in* $\mathbb{R}^r$. *Then* $(\lambda^1, \ldots, \lambda^{s-1}, \nu) \in \mathsf{C}^s_r$ *if and only if*

(i) $\lambda^j_1 \geqslant \ldots \geqslant \lambda^j_r$ *for each* $j \in [s-1]$, *and* $\nu_1 \geqslant \ldots \geqslant \nu_r$ *(since we write eigenvalues in nonincreasing order);*

(ii) $\sum_{j=1}^{s-1} |\lambda^j| = |\nu|$; *and*

(iii) for every $d \in [r-1]$ *and every collection* $I_1, \ldots, I_{s-1}, K$ *of* d*-element subsets of* $[r]$ *satisfying*

$$c^{\tau(K)}_{\tau(I_1),\ldots,\tau(I_{s-1})} = 1, \tag{1}$$

the inequality

$$\sum_{j=1}^{s-1} \sum_{a \in I_j} \lambda^j_a \geqslant \sum_{k \in K} \nu_k$$

holds.

Concerning Friedland's problem, we have the following result of W. Fulton:

Theorem 2 ([7]). *Let* $\lambda^j = (\lambda^j_1, \ldots, \lambda^j_r)$, *for* $j \in [s-1]$, *and* $\nu = (\nu_1, \ldots, \nu_r)$ *lie in* $\mathbb{R}^r$. *Then* $(\lambda^1, \ldots, \lambda^{s-1}, \nu) \in \mathsf{EqC}^s_r$ *if and only if (i) and (iii) above are satisfied, as well as*

(ii') $\sum_{j=1}^{s-1} |\lambda^j| \geqslant |\nu|$[1].

2.2 The Littlewood-Richardson cones

Amazingly, Horn's inequalities also provide the solutions to the following problems in Schubert calculus. One can ask for which tuples $(\lambda^1, \ldots, \lambda^{s-1}, \nu)$ the Littlewood-Richardson number $c^\nu_{\lambda^1,\ldots,\lambda^{s-1}}$ is nonzero, and the answer is in fact the same:

Theorem 3 ([11, 12]). *Let* $\lambda^j = (\lambda^j_1, \ldots, \lambda^j_r)$, *for* $j \in [s-1]$, *and* $\nu = (\nu_1, \ldots, \nu_r)$ *lie in* $\mathbb{Z}^r_{\geqslant 0}$. *Then* $c^\nu_{\lambda^1,\ldots,\lambda^{s-1}} \neq 0$ *if and only if (i), (ii), and (iii) hold above.*

This has the following equivariant analogue: $\mathrm{Gr}(r, \mathbb{C}^n)$ has an action of $T = (\mathbb{C}^*)^n$, and T fixes each Schubert variety. Moreover, the equivariant classes $[X_\lambda]_T$ once again form a basis for $H^*_T(\mathrm{Gr}(r, \mathbb{C}^n))$, as a module over $\mathbb{Z}[t_1, \ldots, t_n] = H^*_T(\{pt\})$. We therefore can define structure "coefficients" (polynomials in the variables t_i) $C^\nu_{\lambda^1,\ldots,\lambda^{s-1}}$ by the rule

$$[X_{\lambda^1}]_T \cdots [X_{\lambda^{s-1}}]_T = \sum_\nu C^\nu_{\lambda^1,\ldots,\lambda^{s-1}} [X_\nu]_T \in H^*_T(\mathrm{Gr}(r, \mathbb{C}^n)),$$

once again assuming n is large enough. The nonvanishing question for $C^\nu_{\lambda^1,\ldots,\lambda^{s-1}}$ was settled by D. Anderson, E. Richmond, and A. Yong[2]:

[1] in fact, this is the unique "Horn inequality" for $d = r$.

[2] technically, they cover the case $s = 3$. By induction, one obtains the statement for all $s > 3$.

Theorem 4 ([1]). *Let $\lambda^j = (\lambda^j_1, \ldots, \lambda^j_r)$, for $j \in [s-1]$, and $\nu = (\nu_1, \ldots, \nu_r)$ lie in $\mathbb{Z}^r_{\geqslant 0}$. Then $C^{\nu}_{\lambda^1,\ldots,\lambda^{s-1}} \neq 0$ if and only if (i), (ii'), and (iii) hold above, as well as*

(iv) $\nu_i \geqslant \lambda^j_i$ for every $i \in [r]$ and $j \in [s-1]$.

Criterion (iv) is also written $\lambda^j \subseteq \nu$ (for every $j \in [s-1]$); i.e., the Young diagram for λ^j fits inside the Young diagram for ν.

Let us therefore define two more cones, $\mathsf{LR}^s_r \subset \mathsf{C}^s_r$ and $\mathsf{EqLR}^s_r \subset \mathsf{EqC}^s_r$, as follows. Set

$$\mathsf{LR}^s_r = \{(\lambda^1, \ldots, \lambda^{s-1}, \nu) \in \mathsf{C}^s_r | \lambda^j_r \geqslant 0 \text{ for every } j \in [s-1]\}^3$$

and

$$\mathsf{EqLR}^s_r = \{(\lambda^1, \ldots, \lambda^{s-1}, \nu) \in \mathsf{EqC}^s_r | \lambda^j_r \geqslant 0 \text{ and } \lambda^j \subseteq \nu \text{ for every } j \in [s-1]\}.$$

Then Theorems 3 and 4 say that the partitions yielding nonvanishing (equivariant) structure coefficients are exactly the sets of lattice points $\mathsf{LR}^s_r \cap \mathbb{Z}^{rs}$ and $\mathsf{EqLR}^s_r \cap \mathbb{Z}^{rs}$, respectively. (On an important historical note, Klyachko showed that $(\lambda^1, \ldots, \lambda^{s-1}, \nu) \in \mathsf{LR}^s_r \cap \mathbb{Z}^{rs} \iff c^{N\nu}_{N\lambda^1,\ldots,N\lambda^{s-1}} \neq 0$ for some $N > 0$; the Saturation Theorem of Knutson and Tao resolved the conjecture that $c^{N\nu}_{N\lambda^1,\ldots,N\lambda^{s-1}} \neq 0 \iff c^{\nu}_{\lambda^1,\ldots,\lambda^{s-1}} \neq 0$, thus proving Theorem 3.)

3 Belkale's algorithm for the rays of LR^s_r

In [3], P. Belkale considers the following rational cone:

$$\Gamma_r(s) := \{(\lambda^1, \ldots, \lambda^{s-1}, \lambda^s) | (\lambda^1, \ldots, \lambda^{s-1}, -w_0\lambda^s) \in \mathsf{C}^s_r \text{ and each } |\lambda^j| = 0\},$$

which parametrizes the possible eigenvalues for *traceless* $r \times r$ Hermitian matrices $A_1, \ldots, A_s$ such that $A_1 + \ldots + A_s = 0$. Here w_0 is the involutive permutation

$$w_0(\lambda_1, \ldots, \lambda_r) = (\lambda_r, \ldots, \lambda_1).$$

Belkale describes the extremal rays that lie on an arbitrary *Horn facet*, i.e., the face of $\Gamma_r(s)$ where one of the Horn inequalities (iii) holds with equality. In order to recall that description in our present notation, we introduce the following cone:

$$\mathsf{C}^s_{SL_r} := \{(\lambda^1, \ldots, \lambda^{s-1}, \nu) \in \mathsf{C}^s_r | \lambda^j_r = 0 \text{ for every } j \in [s-1]\},$$

which is isomorphic to $\Gamma_r(s)$ via the Killing form isomorphism, after twisting the last entry by $-w_0$. For any $j \in [r]$, let ω_j denote the partition $(\underbrace{1, \ldots, 1}_{j}, \underbrace{0, \ldots, 0}_{r-j})$. Then the linear map

$$\begin{aligned} C^s_{SL_r} &\to \Gamma_r(s) \\ (\lambda^1, \ldots, \lambda^{s-1}, \nu) &\mapsto \left(\lambda^1 - \frac{|\lambda^1|}{r}\omega_r, \ldots, \lambda^{s-1} - \frac{|\lambda^{s-1}|}{r}\omega_r, -w_0\nu + \frac{|\nu|}{r}\omega_r\right) \end{aligned}$$

[3] the reader may notice we have omitted the requirement $\nu_r \geqslant 0$; this is because it follows from $\lambda^j_r \geqslant 0$, $\sum |\lambda^j| = |\nu|$, and one of the Horn inequalities.

is well-defined and an isomorphism of rational cones with inverse

$$(\lambda^1, \dots, \lambda^{s-1}, \lambda^s) \mapsto \left(\lambda^1 - \lambda^1_r \omega_r, \dots, \lambda^{s-1} - \lambda^{s-1}_r \omega_r, -w_0 \lambda^s - \sum_{j=1}^{s-1} \lambda^j_r \omega_r \right).$$

3.1 Relationships between the cones

So far we have accumulated several related cones, which fit in this diagram:

$$\begin{array}{ccccc} \mathsf{C}^s_{SL_r} & \subset & \mathsf{LR}^s_r & \subset & \mathsf{C}^s_r \\ & & \cap & & \cap \\ & & \mathsf{EqLR}^s_r & \subset & \mathsf{EqC}^s_r \end{array}$$

(Though one could define a sixth cone that "completes" the lower left corner of the diagram, that cone does not appear to be helpful for our purposes.)

The two vertical inclusions, as well as the left-most one, are inclusions of faces – that is, the smaller cone is defined inside the bigger one by taking one or more linear inequality valid on the bigger cone and forcing it/them to hold with equality. Moreover, we have the following structural results.

Proposition 5. *For each $j \in [s-1]$, let $x_j = (0, \dots, \underbrace{\omega_r}_{j^{th}\ position}, 0, \dots, 0, \omega_r) \in \mathbb{R}^{rs}$.*

We have the following internal decompositions, the first of them direct:

$$\mathsf{LR}^s_r = \mathsf{C}^s_{SL_r} \oplus \bigoplus_{j=1}^{s-1} \mathbb{R}_{\geqslant 0} x_j$$

$$\mathsf{C}^s_r = \mathsf{LR}^s_r + \bigoplus_{j=1}^{s-1} \mathbb{R} x_j$$

$$\mathsf{EqC}^s_r = \mathsf{EqLR}^s_r + \bigoplus_{j=1}^{s-1} \mathbb{R} x_j$$

We postpone the straightforward proof until Section 5.2. As a consequence, we get relationships among extremal rays: for example, the extremal rays of LR^s_r are the rays of $\mathsf{C}^s_{SL_r}$ together with $\{x_j | j \in [s-1]\}$. Since C^s_r and EqC^s_r are not pointed cones (they contain the linear subspaces $\mathbb{R}x_j$), their sets of extremal rays are not well-defined. Nonetheless, they are generated over $\mathbb{R}_{\geqslant 0}$ by the extremal rays of LR^s_r and EqLR^s_r (respectively) and $\{\pm x_j | j \in [s-1]\}$.

While Belkale's algorithm was developed for the Horn facets of $\mathsf{C}^s_{SL_r}$, it applies equally well to the Horn facets of LR^s_r, as each x_j belongs to every Horn facet and is in the image of every induction map.

3.2 Belkale's algorithm

Definition 6. Suppose $I_1, \dots, I_{s-1}, K$ satisfy (1). The associated Horn facet $\mathcal{F}^K_{I_1,\dots,I_{s-1}}$ is

$$\mathcal{F}^K_{I_1,\dots,I_{s-1}} := \left\{ (\lambda^1, \dots, \lambda^{s-1}, \nu) | \sum_{j=1}^{s-1} \sum_{a \in I_j} \lambda^j_a = \sum_{k \in K} \nu_k \right\} \cap \mathsf{LR}^s_r.$$

We will often write $\mathcal{F}$ for short, if the d-element subsets of $[r]$ are understood.

Fix such a collection $I_1, \dots, I_{s-1}, K$ satisfying (1). The rays on $\mathcal{F}$ come in two types. The first are obtained by Algorithm 7 below and are called "type I" rays. The "type II" rays are by default the remaining extremal rays on $\mathcal{F}$. In fact, there is an internal direct sum decomposition $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ (see the isomorphism (2) below) such that the type I rays are the extremal rays of $\mathcal{F}_1$ and the type II rays are the rays of $\mathcal{F}_2$. The type II rays are obtained via a surjective linear map $\mathcal{C} \twoheadrightarrow \mathcal{F}_2$ from a smaller-dimensional cone $\mathcal{C}$; see Theorem 12 and the discussion following.

Each type I ray is the result of a "type I datum" as follows: Fix any $j \in [s-1]$ and an $a \in I_j$ such that $a+1 \notin I_j$, and $a < r$. Alternatively, let $j = s$ and find an $a \in K$ such that $a - 1 \notin K$, but $a > 1$. The pair (j, a) is a type I ray datum. Modify just I_j by swapping a for $a+1$ (or a for $a-1$ in case $j = s$) to produce a new collection $I'_1, \dots, I'_{s-1}, K'$. From the choice of j, a, we get a "type I" ray $r(j,a) = (\lambda^1, \dots, \lambda^{s-1}, \nu)$ by this procedural definition:

Algorithm 7.

1. First, set $\lambda^k_r = 0$ for every $k \in [s-1]$.
2. For every $k \in [s-1]$, if $b > 1$ satisfies $b \in I'_k$ and $b-1 \notin I'_k$, set $I''_k = I'_k \cup \{b-1\} \setminus \{b\}$ and $I''_\ell = I'_\ell$ for $\ell \neq k$, $K'' = K'$. Then set
$$\lambda^k_{b-1} - \lambda^k_b = c^{K''}_{I''_1,\dots,I''_{s-1}}.$$
If $b > 1$ satisfies $b \notin I'_k$ or $b - 1 \in I'_k$, then set $\lambda^k_{b-1} - \lambda^k_b = 0$. Thus we have determined $\lambda^1, \dots, \lambda^{s-1}$ completely.
3. If $c < r$ satisfies $c \in K'$ and $c+1 \notin K'$, set $I''_k = I'_k$ for all k and $K'' = K' \cup \{c+1\} \setminus \{c\}$. Then set
$$\nu_c - \nu_{c+1} = c^{K''}_{I''_1,\dots,I''_{s-1}}.$$
If $c < r$ satisfies $c \notin K'$ or $c+1 \in K'$, then set $\nu_c - \nu_{c+1} = 0$.
4. All that remains is to find ν_r. This can be done by using the requirement
$$|\lambda^1| + \dots + |\lambda^{s-1}| = |\nu|.$$
However, there is an alternative rule. If $r \notin K'$, then $\nu_r = 0$; otherwise, set $K'' = K' \cup \{r+1\} \setminus \{r\}$, and $I''_k = I'_k$ for all k. Then
$$\nu_r = c^{K''}_{I''_1,\dots,I''_{s-1}},$$
where technically this Littlewood-Richardson coefficient is calculated in the cohomology of a bigger Grassmannian ($\mathrm{Gr}(d, \mathbb{C}^{r+1})$ will suffice).

Remark 8. The author is indebted to G. Orelowitz for observing the alternative rule in step 4., which allows one to treat 4. as a new case of 3., defining "$\nu_{r+1} = 0$".

See Section 6.1.1 for examples of using Algorithm 7.

Theorem 9 (Belkale)**.** *Every $r(j,a)$ produced by Algorithm 7 generates an extremal ray of $\mathsf{C}^s_{SL_r} \subset \mathsf{LR}^s_r$ which lies on the face $\mathcal{F}$. They form a linearly independent set, enumerated $\{r_1, \ldots, r_q\}$, and they span a simplicial subcone $\mathcal{F}_1 = \prod_{i=1}^q \mathbb{R}_{\geqslant 0} r_i \subseteq \mathcal{F}$.*

In fact, linear independence follows from [3, Lemma 4.2]:

Lemma 10. *Say $r(j,a) = (\lambda^1, \ldots, \lambda^{s-1}, \nu)$ and $r(\widehat{j}, \hat{a}) = (\hat{\lambda}^1, \ldots, \hat{\lambda}^{s-1}, \hat{\nu})$ are two distinct type I rays on $\mathcal{F}$. (For simplicity assume $j, \widehat{j} < s$; analogous statements hold if $j = s$ or $\widehat{j} = s$ or both.) Then $\lambda^j_a - \lambda^j_{a+1} = 1$ and $\lambda^{\widehat{j}}_{\hat{a}} - \lambda^{\widehat{j}}_{\hat{a}+1} = 0$; likewise $\hat{\lambda}^{\widehat{j}}_{\hat{a}} - \hat{\lambda}^{\widehat{j}}_{\hat{a}+1} = 1$ and $\hat{\lambda}^j_a - \hat{\lambda}^j_{a+1} = 0$.*

This leads one to define the subcone

$$\mathcal{F}_2 := \left\{ (\lambda^1, \ldots, \lambda^{s-1}, \nu) \in \mathcal{F} \;\middle|\; \begin{array}{c} \lambda^j_a - \lambda^j_{a+1} = 0 \text{ for every type I datum } (j,a), j < s; \\ \nu_{a-1} - \nu_a = 0 \text{ for every type I datum } (s,a) \end{array} \right\}.$$

Clearly the addition map

$$\mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{F} \tag{2}$$

is an injection of rational cones, given the above lemma and the definition of $\mathcal{F}_2$. In [3, Proposition 4.3], we find that this map is also surjective. Therefore the remaining extremal rays of $\mathcal{F}$ are just the extremal rays of $\mathcal{F}_2$; these are called "type II". For this, we have a surjective linear map onto $\mathcal{F}_2$ from a cone whose rays we can determine inductively.

We'll define that map momentarily. First, for a d-element subset I of $[r]$ and $\lambda \in \mathbb{R}^r$, we define

$$\lambda_I = (\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_d}).$$

Note that if λ is a partition, so is λ_I. We also write $\bar{I}$ for the complement of I inside $[r]$. Next, define the invertible map

$$\begin{aligned} \pi : \mathbb{R}^{rs} &\to \mathbb{R}^{ds} \times \mathbb{R}^{(r-d)s} \\ (\lambda^1, \ldots, \lambda^{s-1}, \nu) &\mapsto (\lambda^1_{I_1}, \ldots, \lambda^{s-1}_{I_{s-1}}, \nu_K), (\lambda^1_{\bar{I}_1}, \ldots, \lambda^{s-1}_{\bar{I}_{s-1}}, \nu_{\bar{K}}). \end{aligned}$$

The proof of the following result can be found in [8, Proposition 8] and is also a consequence of the factorization rule of [10].

Proposition 11. *The above π restricts to a map*

$$\pi : \mathcal{F} \to \mathsf{LR}^s_d \times \mathsf{LR}^s_{r-d}.$$

Although this restriction of π is not necessarily surjective, we do clearly have

$$\pi^{-1}(\mathsf{LR}_d^s \times \mathsf{LR}_{r-d}^s) \subseteq \mathbb{R}\mathcal{F}.$$

Finally let

$$p_2 : \mathbb{R}\mathcal{F} \to \mathbb{R}\mathcal{F}_2$$

be the projection onto the second factor. Then define the induction map Ind to be $p_2 \circ \pi^{-1}$.

Theorem 12 (Belkale)**.** *The linear map*

$$\mathrm{Ind} = p_2 \circ \pi^{-1} : \mathsf{LR}_d^s \times \mathsf{LR}_{r-d}^s \to \mathcal{F}_2$$

is well-defined and surjective.

The surjectivity means that every extremal ray of $\mathcal{F}_2$ is the image of an extremal ray of $\mathsf{LR}_d^s \times \mathsf{LR}_{r-d}^s$. Moreover, the extremal rays of $\mathsf{LR}_d^s \times \mathsf{LR}_{r-d}^s$ are all of the form $a \times 0$ or $0 \times b$ where a (resp., b) is an extremal ray of LR_d^s (resp., LR_{r-d}^s). As both d and $r-d$ are strictly smaller than r, we get an inductive algorithm for finding the extremal rays of LR_r^s, starting with LR_1^s.

Remark 13. Actually, Belkale's Ind has a proper subspace of $\mathsf{LR}_d^s \times \mathsf{LR}_{r-d}^s$ as its domain, but it nonetheless surjects onto $\mathcal{F}_2 \cap \mathsf{C}_{SL_r}^s$. With the larger domain comes a larger kernel (see Corollary 20) but a more transparent generalization to EqLR_r^s.

4 Adaptation of the algorithm for EqLR_r^s

Theorems 9 and 12 can be straightforwardly adapted to find the extremal rays of the Horn facets of EqLR_r^s. Once again assume that $I_1, \ldots, I_{s-1}, K$ satisfy (1). Define

$$\hat{\mathcal{F}} = \left\{ (\lambda^1, \ldots, \lambda^{s-1}, \nu) \;\middle|\; \sum_{j=1}^{s-1} \sum_{a \in I_j} \lambda_a^j = \sum_{k \in K} \nu_k \right\} \cap \mathsf{EqLR}_r^s.$$

Note that $\mathcal{F} = \hat{\mathcal{F}} \cap \mathsf{LR}_r^s$, and since $\mathcal{F}$ is a facet of $\hat{\mathcal{F}}$, every extremal ray of $\mathcal{F}$ is a ray of $\hat{\mathcal{F}}$ as well.

Therefore type I rays on $\mathcal{F}$ are naturally considered type I rays on $\hat{\mathcal{F}}$; i.e., Theorem 9 holds verbatim with $\hat{\mathcal{F}}$ instead of $\mathcal{F}$. It is the set of type II rays which will increase from $\mathcal{F}$ to $\hat{\mathcal{F}}$. Define

$$\hat{\mathcal{F}}_2 := \left\{ (\lambda^1, \ldots, \lambda^{s-1}, \nu) \in \hat{\mathcal{F}} \;\middle|\; \begin{array}{c} \lambda_a^j - \lambda_{a+1}^j = 0 \text{ for every type I datum } (j,a), j < s; \\ \nu_{a-1} - \nu_a = 0 \text{ for every type I datum } (s,a) \end{array} \right\}.$$

Extremal rays of $\hat{\mathcal{F}}_2$ will continue to be called type II rays for $\hat{\mathcal{F}}$.

Before we come to the proof of the decomposition $\hat{\mathcal{F}} = \mathcal{F}_1 \times \hat{\mathcal{F}}_2$, let us recall an important consequence of [1, Proposition 2.1(B)] (they state it for $s = 3$ and for integer vectors, but the statement below follows easily). For a pair of vectors λ, μ we use $\lambda \subseteq \mu$ to mean $\lambda_i \leqslant \mu_i$ for every $i \in [r]$.

Proposition 14 (Anderson-Richmond-Yong)**.** *Suppose* $(\lambda^1, \dots, \lambda^{s-1}, \nu) \in \mathsf{EqLR}_r^s$*. Then for any* $j \in [s-1]$*, one can find a* $\lambda^{j,\downarrow}$ *such that*

1. $\lambda^{j,\downarrow} \subseteq \lambda^j$,
2. $(\lambda^1, \dots, \lambda^{j,\downarrow}, \dots, \lambda^{s-1}, \nu) \in \mathsf{EqLR}_r^s$*, and*
3. $\sum_{k \neq j} |\lambda^k| + |\lambda^{j,\downarrow}| = |\nu|$.

In other words, every element of EqLR_r^s possesses several "shadows" in LR_r^s obtainable by shrinking any one of the first $s-1$ partitions.

Theorem 15. *The addition map*

$$\mathcal{F}_1 \times \hat{\mathcal{F}}_2 \to \hat{\mathcal{F}}$$

is an isomorphism of cones.

Proof. Once again, this map is clearly injective given Lemma 10 and the definition of $\hat{\mathcal{F}}_2$. Now let us show it is surjective.

Let $x = (\lambda^1, \dots, \lambda^{s-1}, \nu) \in \hat{\mathcal{F}}$ be arbitrary. If x already belongs to $\hat{\mathcal{F}}_2$, then we are done. Otherwise, there exists some type I datum (j, a) such that $\lambda_a^j - \lambda_{a+1}^j > 0$ (in case $j < s$) or $\nu_{a-1} - \nu_a > 0$ (in case $j = s$). Set $\beta = \lambda_a^j - \lambda_{a+1}^j$ or $\beta = \nu_{a-1} - \nu_a$, depending on the case. We will show that

$$x - \beta r(j, a) \in \hat{\mathcal{F}}; \tag{3}$$

then by induction on the number of such type I data, we can assume $x - \beta r(j,a) \in \mathcal{F}_1 \times \hat{\mathcal{F}}_2$, from which the result follows. In fact, to show (3) it suffices to show $x - \beta r(j, a) \in \mathsf{EqLR}_r^s$, since their difference clearly lies in the hyperplane spanned by $\hat{\mathcal{F}}$.

Toward that end, we begin with a simple observation. Write $r(j, a) = (\bar{\lambda}^1, \dots, \bar{\lambda}^{s-1}, \bar{\nu})$. Fix any $\ell \in [s-1]$ and let $\tilde{x}_\ell = (\lambda^1, \dots, \lambda^{\ell,\downarrow}, \dots, \lambda^{s-1}, \nu)$ be an element of LR_r^s as guaranteed by Proposition 14. Set $\mu^k = \lambda^k$ for $k \neq \ell$ and $\mu^\ell = \lambda^{\ell,\downarrow}$, so in this notation

$$\tilde{x}_\ell = (\mu^1, \dots, \mu^{s-1}, \nu).$$

Observe that $\tilde{x}_\ell$ is automatically on $\hat{\mathcal{F}}$ since

$$\sum_{k=1}^{s-1} \sum_{b \in I_k} \lambda_b^k \geqslant \sum_{k=1}^{s-1} \sum_{b \in I_k} \mu_b^k \geqslant \sum_{k \in K} \nu_k$$

(the second inequality holds due to $\tilde{x}_\ell \in \mathsf{LR}_r^s$) and

$$\sum_{k=1}^{s-1} \sum_{b \in I_k} \lambda_b^k = \sum_{k \in K} \nu_k$$

due to $x \in \hat{\mathcal{F}}$; hence

$$\sum_{k=1}^{s-1} \sum_{b \in I_k} \mu_b^k = \sum_{k \in K} \nu_k.$$

Moreover, this shows us that $\lambda_a^\ell = \mu_a^\ell$ whenever $a \in I_\ell$. This means that $\mu_a^j - \mu_{a+1}^j \geqslant \beta$ (equality holds if $\ell \neq j$, but if $\ell = j$ then μ_{a+1}^ℓ could be smaller while μ_a^ℓ is unchanged.)

So $\tilde{x}_\ell$ still has $\mu_a^j - \mu_{a+1}^j$ (or $\nu_{a-1} - \nu_a$) at least equal to β. By [3, Proposition 4.3], we know that

$$\tilde{x}_\ell - \beta r(j,a) \in \mathcal{F} \subset \hat{\mathcal{F}},$$

and this holds for every $\ell \in [s-1]$.

Finally, we show that $x - \beta r(j,a) \in \mathsf{EqLR}_r^s$ by verifying each of inequalities (i) (along with nonnegativity), (ii'), (iii), and (iv).

(i) Let $k \in [s-1]$. Then $\lambda^k = \mu^k$ from $\tilde{x}_\ell$ as long as $\ell \neq k$, so choose some such ℓ. Because $\tilde{x}_\ell - \beta r(j,a)$ belongs to LR_r^s, $\mu^k - \beta\bar{\lambda}^k$ is a partition; therefore $\lambda^k - \beta\bar{\lambda}^k$ is a partition. For any ℓ, $\nu - \beta\bar{\nu}$ is a partition for the same reason.

(ii') This inequality follows by subtracting $\beta \sum_{k=1}^{s-1} |\bar{\lambda}^k| = \beta|\bar{\nu}|$ from $\sum_{k=1}^{s-1} |\lambda^k| \geqslant |\nu|$.

(iii) Let $(J_1, \ldots, J_{s-1}, L)$ parametrize a Horn inequality (i.e., $c^L_{J_1,\ldots,J_{s-1}} = 1$). Fix an arbitrary $\ell \in [s-1]$. Then since $\tilde{x}_\ell - \beta r(j,a)$ satisfies this Horn inequality, we have

$$\sum_{k=1}^{s-1} \sum_{b \in J_k} \lambda_b^k - \beta\bar{\lambda}_b^k \geqslant \sum_{k=1}^{s-1} \sum_{b \in J_k} \mu_b^k - \beta\bar{\lambda}_b^k \geqslant \sum_{k \in L} \nu_k.$$

(iv) Let $k \in [s-1]$ and choose $\ell \neq k$. Since $\tilde{x}_\ell - \beta r(j,a) \in \mathsf{LR}_r^s$, we know that

$$\mu^k - \beta\bar{\lambda}^k \subseteq \nu - \beta\bar{\nu}.$$

Since $\lambda^k = \mu^k$, the needed inequalities follow.

□

Remark 16. The proof shows that the addition map is surjective even on lattice points.

So just like in the classical setting, to find the remaining extremal rays of $\hat{\mathcal{F}}$ it now suffices to find the extremal rays of $\hat{\mathcal{F}}_2$ (the type II rays). Recall the definition of π from earlier:

$$\begin{aligned} \pi : \mathbb{R}^{rs} &\to \mathbb{R}^{ds} \times \mathbb{R}^{(r-d)s} \\ (\lambda^1, \ldots, \lambda^{s-1}, \nu) &\mapsto (\lambda^1_{I_1}, \ldots, \lambda^{s-1}_{I_{s-1}}, \nu_K), (\lambda^1_{\bar{I}_1}, \ldots, \lambda^{s-1}_{\bar{I}_{s-1}}, \nu_{\bar{K}}). \end{aligned}$$

Akin to Proposition 11, we have the following generalization in the equivariant setting.

Proposition 17. *The map π restricts to*

$$\pi : \hat{\mathcal{F}} \to \mathsf{LR}_d^s \times \mathsf{EqLR}_{r-d}^s.$$

The proof of this fact can be found in [7, Claim, pg. 30]. We also provide a short argument based on Proposition 11 and [1, Proposition 2.1].

Proof. Suppose $(\lambda^1, \ldots, \lambda^{s-1}, \nu) \in \mathsf{EqLR}_r^s$ satisfies the Horn inequality given by $(I_1, \ldots, I_{s-1}, K)$ with equality. By Proposition 14 we know that we can find an $\tilde{x} = (\lambda^1, \ldots, \lambda^{j,\downarrow}, \ldots, \lambda^{s-1}, \nu) \in \mathsf{LR}_r^s$. Moreover, we showed in the previous proof that $\tilde{x} \in \mathcal{F}$, and that $\lambda^{j,\downarrow}_{I_j} = \lambda^j_{I_j}$. Applying Proposition 11, we have

$$\pi(\tilde{x}) = (\lambda^1_{I_1}, \ldots, \lambda^{s-1}_{I_{s-1}}, \nu_K) \times (\lambda^1_{\bar{I}_1}, \ldots, \lambda^{j,\downarrow}_{\bar{I}_j}, \ldots, \lambda^{s-1}_{\bar{I}_{s-1}}, \nu_{\bar{K}}) \in \mathsf{LR}_d^s \times \mathsf{LR}_{r-d}^s.$$

Since $(\lambda^1_{\bar{I}_1}, \ldots, \lambda^{j,\downarrow}_{\bar{I}_j}, \ldots, \lambda^{s-1}_{\bar{I}_{s-1}}, \nu_{\bar{K}}) \in \mathsf{LR}_{r-d}^s$, and since $\lambda^j_{\bar{I}_j} \subseteq \nu_{\bar{K}}$ (by [1, Lemma 2.7]), we can apply [1, Proposition 2.1(A)] to conclude that $(\lambda^1_{\bar{I}_1}, \ldots, \lambda^{s-1}_{\bar{I}_{s-1}}, \nu_{\bar{K}}) \in \mathsf{EqLR}_{r-d}^s$. ☐

Once again, even though π is not surjective, it is true that $\pi^{-1}(\mathsf{LR}_d^s \times \mathsf{EqLR}_{r-d}^s) \subseteq \mathbb{R}\hat{\mathcal{F}}$. Let

$$\hat{p}_2 : \mathbb{R}\hat{\mathcal{F}} \to \mathbb{R}\hat{\mathcal{F}}_2$$

be the second projection, and define $\widehat{\mathrm{Ind}} = \hat{p}_2 \circ \pi^{-1}$. Note that, if ι denotes the inclusion map $\mathcal{F}_2 \subset \hat{\mathcal{F}}_2$, we have $\widehat{\mathrm{Ind}}\big|_{\mathsf{LR}_d^s \times \mathsf{LR}_{r-d}^s} = \iota \circ \mathrm{Ind}$.

Theorem 18. *The linear map*

$$\widehat{\mathrm{Ind}} = \hat{p}_2 \circ \pi^{-1} : \mathsf{LR}_d^s \times \mathsf{EqLR}_{r-d}^s \to \hat{\mathcal{F}}_2$$

is well-defined and surjective.

In order to save some on notation in what follows, we will use $v^{(k)}$ to denote the k^{th} entry (itself a vector in $\mathbb{R}^r$) of v; i.e., $v = (v^{(1)}, v^{(2)}, \ldots, v^{(s)}) \in \mathbb{R}^{rs}$.

Proof. Linearity is obvious and surjectivity follows since Proposition 17 implies $\widehat{\mathrm{Ind}}$ has a section given by π.

It remains to show that the image of $\widehat{\mathrm{Ind}}$ is contained in $\hat{\mathcal{F}}_2$. Let $x = (\mu^1, \ldots, \mu^{s-1}, \kappa) \in \mathsf{LR}_d^s$ and $y = (\alpha^1, \ldots, \alpha^{s-1}, \gamma) \in \mathsf{EqLR}_{r-d}^s$. Write $\pi^{-1}(x, y)$ as $(\lambda^1, \ldots, \lambda^{s-1}, \nu)$, but remember that $\lambda^1, \ldots, \lambda^{s-1}, \nu$ may not be partitions.

For each $\ell \in [s-1]$, find a $y_\ell = (\beta^1, \ldots, \beta^{s-1}, \gamma)$ with $\beta^\ell = \alpha^{\ell,\downarrow}$, $\beta^k = \alpha^k$ for $k \neq \ell$ as in the conclusion of Proposition 14. Let $\mathfrak{J}$ be the set of all the type I pairs (j, a).

Write

$$\pi^{-1}(x, y) = \sum_{\mathfrak{J}} c_{j,a} r(j, a) + z$$

where $c_{j,a} \in \mathbb{R}$ and $z = \widehat{\mathrm{Ind}}(x, y) \in \mathbb{R}\hat{\mathcal{F}}_2$. Likewise, express

$$\pi^{-1}(x, y_\ell) = \sum_{\mathfrak{J}} d^\ell_{j,a} r(j, a) + z_\ell.$$

We claim that for every $(j,a) \in \mathfrak{J}$, $d^\ell_{j,a} \geqslant c_{j,a}$.

First examine the case $j < s$. Then $a \in I_j$ and $a+1 \in \bar{I}_j$. Therefore $\lambda^j_a = \mu_p$ for some p and $\lambda^j_{a+1} = \alpha_q$ for some q, and (by Lemma 10 and the definition of $\hat{\mathcal{F}}_2$) $c_{j,a} = \lambda^j_a - \lambda^j_{a+1} = \mu_p - \alpha_q$. Likewise, $d^\ell_{j,a} = \mu_p - \beta_q$. Since $\beta_q \leqslant \alpha_q$, we get $d^\ell_{j,a} \geqslant c_{j,a}$.

Second, for $j = s$, we have $a \in K$ and $a-1 \in \bar{K}$. So $\nu_{a-1} = \gamma_q$ for some q and $\nu_a = \kappa_p$ for some p, and $c_{j,a} = \gamma_q - \kappa_p$. But since the last coordinates of y and y_ℓ agree, we also have $d^\ell_{j,a} = \gamma_q - \kappa_p$, hence $c_{j,a} = d^\ell_{j,a}$.

Now we verify that $z \in \mathsf{EqLR}^s_r$ by verifying the inequalities (i) (plus nonnegativity), (ii'), (iii), and (iv).

(i) If $k \neq \ell$, then $\pi^{-1}(x, y_\ell)^{(k)} = \pi^{-1}(x,y)^{(k)}$. Therefore

$$z^{(k)} = z^{(k)}_\ell + \sum_{\mathfrak{J}} (d^\ell_{j,a} - c_{j,a}) r(j,a)^{(k)}. \tag{4}$$

It follows at once that each $z^{(k)}$ is a partition (recall that the above claim holds for arbitrary ℓ).

(ii') We have $\sum_{j=1}^{s-1} |z^{(j)}| - |z^{(s)}| = \sum_{j=1}^{s-1} |\mu^j| - |\kappa| + \sum_{j=1}^{s-1} |\alpha^j| - |\gamma| = \sum_{j=1}^{s-1} |\alpha^j| - |\gamma| \geqslant 0$.

(iii) Fix any $\ell \in [s-1]$. Even though $\pi^{-1}(x,y)^{(\ell)}$ and $\pi^{-1}(x,y_\ell)^{(\ell)}$ are not likely to be partitions, they do still satisfy the entrywise bound

$$\pi^{-1}(x, y_\ell)^{(\ell)} \subseteq \pi^{-1}(x,y)^{(\ell)}.$$

Therefore $z^{(\ell)} \supseteq z^{(\ell)}_\ell + \sum_{\mathfrak{J}} (d^\ell_{j,a} - c_{j,a}) r(j,a)^{(\ell)}$. Since $z_\ell = \mathrm{Ind}(x, y_\ell)$ satisfies all the Horn inequalities, as of course do the $r(j,a)$, and since every $d^\ell_{j,a} - c_{j,a} \geqslant 0$, z must satisfy the Horn inequalities as well.

(iv) If $k_0 \neq \ell$, then since $z^{(k_0)}_\ell \subseteq z^{(s)}_\ell$ and each $r(j,a)^{(k_0)} \subseteq r(j,a)^{(s)}$, we get from (4) applied to both $k = k_0$ and $k = s$ that $z^{(k_0)} \subseteq z^{(s)}$.

We have just shown that $z = \widehat{\mathrm{Ind}}(x,y)$ belongs to EqLR^s_r. Since $z \in \mathbb{R}\hat{\mathcal{F}}_2$, we get that $z \in \hat{\mathcal{F}}_2$ as desired. ☐

So each type II ray on $\widehat{\mathcal{F}}_2$ is the image of a ray from $\mathsf{LR}^s_d \times \mathsf{EqLR}^s_{r-d}$ under the map $\widehat{\mathrm{Ind}}$. In Section 6.1.2, we give an example of finding the type II rays on a face $\widehat{\mathcal{F}}$. For this, it is helpful to have a formula for $\widehat{\mathrm{Ind}}$, or really for $\widehat{p}_2$, so we record that here.

Lemma 19. *Once again let $\mathfrak{J} = \{(j,a)\}$ be the collection of type I data on the facet $\hat{\mathcal{F}}$, with associated rays $r(j,a)$. The map $\widehat{p}_2 : \mathbb{R}\hat{\mathcal{F}} \to \mathbb{R}\hat{\mathcal{F}}_2$ sends $(\lambda^1, \ldots, \lambda^{s-1}, \nu)$ to*

$$(\lambda^1, \ldots, \lambda^{s-1}, \nu) - \sum_{\substack{(j,a) \in \mathfrak{J} \\ j < s}} (\lambda^j_a - \lambda^j_{a+1}) r(j,a) - \sum_{\substack{(j,a) \in \mathfrak{J} \\ j = s}} (\nu_{a-1} - \nu_a) r(j,a). \tag{5}$$

It is possible, in general, for $\widehat{\mathrm{Ind}}$ to take extremal rays to non-extremal rays, or even to 0. In fact, by [3, Proposition 9.3], we have a good understanding of the kernel of $\widehat{\mathrm{Ind}}$.

Corollary 20. *The kernel of* $\widehat{\mathrm{Ind}}$ *is spanned by the elements* $\pi(r(j,a))$ *as* (j,a) *ranges over* $\mathfrak{J}$. *Moreover, the number of extremal rays of* $\mathsf{LR}_d^s \times \mathsf{EqLR}_{r-d}^s$ *which map to* 0 *under* $\widehat{\mathrm{Ind}}$ *is equal to* $|\mathfrak{J}|$, *and these rays therefore also form a basis of* $\ker \widehat{\mathrm{Ind}}$.

Proof. Clearly each $\pi(r(j,a))$ is in the kernel. Since π is an invertible map, the collection $\{\pi(r(j,a)) | (j,a) \in \mathfrak{J}\}$ is a linearly independent set. If it has the cardinality of $\dim \ker \widehat{\mathrm{Ind}}$ we will have shown that they are a basis. For this we observe that

$$\begin{aligned}\dim \ker \widehat{\mathrm{Ind}} &= \dim(\mathbb{R}\mathsf{LR}_r^s \times \mathbb{R}\mathsf{EqLR}_r^s) - \dim \mathbb{R}\hat{\mathcal{F}}_2 \\ &= \dim \mathbb{R}\hat{\mathcal{F}} - \dim \mathbb{R}\hat{\mathcal{F}}_2 \\ &= \dim \mathbb{R}\mathcal{F}_1 = |\mathfrak{J}|.\end{aligned}$$

Now, the only rays of $\mathsf{LR}_d^s \times \mathsf{EqLR}_{r-d}^s$ which map to 0 are in fact rays of $\mathsf{LR}_d^s \times \mathsf{LR}_{r-d}^s$, since $\widehat{\mathrm{Ind}}$ preserves the extent to which inequality (ii') is strict.

From [3, Proposition 9.3(3) and Corollary 9.4], there are exactly $|\mathfrak{J}| - (s-1)$ rays of $\mathsf{C}_{SL_d}^s \times \mathsf{C}_{SL_{r-d}}^s$ which map to 0. Moreover $x_j \times 0$ maps to 0 for every $j \in [s-1]$, as follows from [4, Corollary 60]. For dimension reasons, the elements $0 \times x_j$ cannot map to 0; otherwise $\ker \widehat{\mathrm{Ind}}$ would have dimension greater than $|\mathfrak{J}|$. ☐

For an illustration of Corollary 20, see Section 6.1.2.

5 Special rays

5.1 Rays of EqLR_r^s not on any Horn facet

In this section, we find the exceptional rays that are not on any Horn facet. We begin with a couple of lemmas on certain rays.

Lemma 21. *Suppose* $x = (\omega_{k_1}, \omega_{k_2}, \ldots, \omega_{k_{s-1}}, \omega_\ell)$ *belongs to* EqLR_r^s. *Then* $\mathbb{R}_{\geqslant 0}x$ *is an extremal ray.*

Proof. By hypothesis, the inequalities (iv) imply that each $k_i \leqslant \ell$. The only way to write x as a sum of partitions is

$$(\omega_{k_1}, \ldots, \omega_{k_{s-1}}, \omega_\ell) = (q_1\omega_{k_1}, \ldots, q_{s-1}\omega_{k_{s-1}}, q_s\omega_\ell) + (r_1\omega_{k_1}, \ldots, r_{s-1}\omega_{k_{s-1}}, r_s\omega_\ell),$$

where $q_i + r_i = 1$ for each i. If both elements on the RHS belong to EqLR_r^s, then we have both $q_s \geqslant q_i$ and $r_s \geqslant r_i$ for each i by (iv), which forces equalities to hold since the two sides each add to 1. So the summands are parallel. ☐

Lemma 22. *The tuple* $(\omega_{k_1}, \ldots, \omega_{k_{s-1}}, \omega_\ell) \in \mathsf{EqLR}_r^s$ *if and only if the inequalities*

$$\forall i \in [s-1], \quad k_i \leqslant \ell \text{ and}$$
$$\sum_{i=1}^{s-1} k_i \geqslant \ell$$

are satisfied.

Proof. (⇒) These are just inequalities (iv) and (ii').

(⇐) Observe that $(\omega_\ell, 0, \ldots, 0, \omega_\ell) \in \mathsf{EqLR}_r^s$ since the corresponding LR coefficient is 1. By [1, Proposition 2.1(A)], we also have $(\omega_\ell, \omega_{k_2}, \omega_{k_3}, \ldots, \omega_{k_{s-1}}, \omega_\ell) \in \mathsf{EqLR}_r^s$.

By [1, Proposition 2.1(B)], for any value t between $\ell - \sum_{i\geqslant 2} k_i = |\omega_\ell| - \sum_{i\geqslant 2} |\omega_{k_i}|$ and $\ell = |\omega_\ell|$ we may find λ such that $(\lambda, \omega_{k_2}, \ldots, \omega_\ell) \in \mathsf{EqLR}_r^s$, $\lambda \subset \omega_\ell$, and $|\lambda| = t$. By assumption, $\ell - \sum_{i\geqslant 2} k_i \leqslant k_1 \leqslant \ell$, so we may find such a λ with $|\lambda| = k_1$. Of course, the only partition of k_1 fitting inside ω_ℓ is ω_{k_1}, so $(\omega_{k_1}, \ldots, \omega_{k_{s-1}}, \omega_\ell) \in \mathsf{EqLR}_r^s$ as promised. □

Theorem 23.

(I) Suppose $(\lambda^1, \ldots, \lambda^{s-1}, \nu)$ gives an extremal ray of EqLR_r^s satisfying

(A) $\sum_{j=1}^{s-1} |\lambda^j| > |\nu|$;

(B) each inequality of (iii) holds strictly.

Then $\lambda^1 = \ldots = \lambda^{s-1} = \nu$ and this common partition is ω_ℓ for some ℓ.

(II) Furthermore, every element of the form $(\omega_\ell, \omega_\ell, \ldots, \omega_\ell)$ is an extremal ray of EqLR_r^s, and such a ray lies on no Horn facet if and only if $\ell \geqslant r/(s-1)$.

Proof of (I). Assume for the sake of contradiction that $\nu_j > \lambda_j^1$ for some j, and assume j is as small as possible. Then since $\lambda_{j-1}^1 = \nu_{j-1} \geqslant \nu_j > \lambda_j^1$, we know that $\lambda^1 \pm \epsilon\omega_{j-1}$ is still a partition for $\epsilon > 0$ a small enough real number.

Let k be the greatest index satisfying $k \geqslant j$ and $\nu_k = \nu_j$, and for each $p = 2, \ldots, s-1$, let i_p be the smallest index satisfying $i_p \leqslant k$ and $\lambda_{i_p+1}^p = \lambda_{k+1}^p$ (where $\lambda_{r+1}^p = 0$ by definition). Thus in particular $\lambda_{i_p}^p > \lambda_{i_p+1}^p$, so both $\nu \pm \epsilon\omega_k$ and $\lambda^p \pm \epsilon\omega_{i_p}$ are also partitions for small ϵ.

Set
$$z_{\pm\epsilon} := (\lambda^1, \ldots, \lambda^{s-1}, \nu) \pm \epsilon \left(\omega_{j-1}, \omega_{i_2}, \ldots, \omega_{i_{s-1}}, \omega_k\right).$$

If we show that for small enough ϵ both $z_{+\epsilon}$ and $z_{-\epsilon}$ belong to EqLR_r^s, we will have a contradiction regarding the extremality of $(\lambda^1, \ldots, \lambda^{s-1}, \nu)$. Since $(\lambda^1, \ldots, \lambda^{s-1}, \nu)$ satisfies (ii') and (iii) strictly, we are guaranteed that $z_{\pm\epsilon}$ satisfy (ii') and (iii) for small enough ϵ. The preceding paragraph showed that $z_{\pm\epsilon}$ satisfy (i) (and nonnegativity) for small ϵ as well. So it suffices to show they satisfy (iv) for small enough ϵ:

- We first show $z_{\pm\epsilon}^{(1)} \subseteq z_{\pm\epsilon}^{(s)}$. If $i \leqslant j$, the inequality $\lambda_i \pm \epsilon \leqslant \nu_i \pm \epsilon$ is satisfied. For $k \geqslant i > j$, we have $\lambda_i < \nu_j = \nu_i$, so $\lambda_i < \nu_i \pm \epsilon$ for small enough ϵ. For $i > k$, the inequality $\lambda_i \leqslant \nu_i$ is unchanged.
- Now let $p \in \{2, \ldots, s-1\}$; we'll verify that $z_{\pm\epsilon}^{(p)} \subseteq z_{\pm\epsilon}^{(s)}$. If $i \leqslant i_2$, the inequality $\lambda_i^p \pm \epsilon \leqslant \nu_i \pm \epsilon$ is satisfied. For $i_2 < i \leqslant k$, we have $\lambda_i^p = \lambda_{k+1}^p \leqslant \nu_{k+1} < \nu_k \leqslant \nu_i$, so $\lambda_i^p < \nu_i \pm \epsilon$ for small enough ϵ. For $i > k$, the inequality $\mu_i \leqslant \nu_i$ is unchanged.

So we have a contradiction and it must instead be true that $\lambda^1 = \nu$. The same argument applies to any λ^j.

If our common partition $\lambda^1 = \ldots = \lambda^{s-1} = \nu$ is expressed $\sum c_i \omega_i$ in the $\{\omega_i\}$ basis, note that for any $c_i \neq 0$, $(\lambda^1, \ldots, \lambda^{s-1}, \nu) \pm \epsilon(\omega_i, \ldots \omega_i, \omega_i)$ satisfies (ii') and (iii) for small enough ϵ and maintains (iv) with equalities, so there must be only one such nonzero coefficient. □

Proof of (II). Each $(\omega_\ell, \ldots, \omega_\ell, \omega_\ell)$ is an extremal ray by Lemma 22.

Finally, we show that $\ell < r/(s-1) \iff (\omega_\ell, \ldots, \omega_\ell, \omega_\ell)$ lies on a Horn facet.

First, suppose $\ell < r/(s-1)$. Find integers $a_1, \ldots, a_{s-1}$ such that[4]

$$r - 1 = a_1 + \ldots + a_{s-1},$$
$$\text{each } a_j \geqslant \left\lfloor \frac{r-1}{s-1} \right\rfloor.$$

Set $I_p = \{a_p + 1\}$ and $K = \{r\}$. Then $c^{\tau(K)}_{\tau(I_1),\ldots,\tau(I_{s-1})} = 1$. Since

$$\ell < r/(s-1) \leqslant \left\lfloor \frac{r-1}{s-1} \right\rfloor + 1 \leqslant a_p + 1$$

for each p, the associated Horn inequality, applied to $(\omega_\ell, \ldots, \omega_\ell, \omega_\ell)$, is $0 + \ldots + 0 \geqslant 0$, and thus is satisfied with equality.

Second, suppose $\ell \geqslant r/(s-1)$. Assume that $(\omega_\ell, \ldots, \omega_\ell, \omega_\ell)$ lies on a Horn facet associated to d-element subsets $I_1, \ldots, I_{s-1}, K$. So

$$\sum_{a \in I_1, a \leqslant \ell} 1 + \ldots + \sum_{a \in I_{s-1}, a \leqslant \ell} 1 = \sum_{k \in K, k \leqslant \ell} 1.$$

Since $c^{\omega_\ell}_{\omega_\ell, 0, \ldots, 0} = 1$, it must also be true that

$$\sum_{a \in I_1, a \leqslant \ell} 1 + 0 \geqslant \sum_{k \in K, k \leqslant \ell} 1,$$

in which case the above must hold with equality and the sets $\{a \in I_p | a \leqslant \ell\}$ must be empty for $p \geqslant 2$. By symmetry, the set $\{a \in I_1 : a \leqslant \ell\}$ is also empty, so every I_p consists only of elements $> \ell$.

Now, the stipulation $|\tau(I_1)| + \ldots + |\tau(I_{s-1})| = |\tau(K)|$ forces

$$\sum_{p=1}^{s-1} \sum_{a \in I_p} a = \sum_{k \in K} k + (s-2)d(d+1)/2.$$

However, a lower bound for the LHS (i.e., each I_p is as small as possible at $\{\ell+1, \ldots, \ell+d\}$) is $(s-1)d\ell + (s-1)d(d+1)/2$, while an upper bound for the RHS (i.e., where $K =$

[4]unless $r-1$ is divisible by $s-1$, choices abound here. Say $r - 1 \equiv b \bmod s-1$, where $0 \leqslant b < s-1$. Then one can take $a_i = (r-1-b)/(s-1)$ for $i \geqslant 2$ and $a_1 = (r-1+b(s-2))/(s-1)$.

$\{r-d+1,\ldots,r-d+d\}$) is $d(r-d)+(s-1)d(d+1)/2$. Therefore we get $(s-1)d\ell \leqslant dr-d^2$. Assuming $r \leqslant (s-1)\ell$, this forces

$$(s-1)d\ell + d^2 \leqslant rd \leqslant (s-1)d\ell,$$

an obvious contradiction to $d > 0$. □

5.2 Rays of EqLR^s_r on every Horn facet

In contrast, there are some extremal rays of EqLR^s_r that lie on every Horn facet. If $r = 1$, then there are no inequalities (iii), so to make the current discussion more uniform, we will treat (ii') as a Horn inequality, at least if $r = 1$.

Proposition 24. *Suppose that $(\lambda^1,\ldots,\lambda^{s-1},\nu)$ belongs to EqLR^s_r and lies on every Horn facet (so really belongs to LR^s_r). Then each λ^p is a scalar multiple of ω_r, as is ν.*

Proof. Actually, we will need surprisingly few of the Horn inequalities. We begin with $\sum_{p=1}^{s-1} \lambda^p_1 = \nu_1$. Now for any $1 < k \leqslant r$, there is also the equality

$$\lambda^1_k + \sum_{p=2}^{s-1} \lambda^p_1 = \nu_k.$$

Combining these two equations, we get $\lambda^1_1 - \lambda^1_k = \nu_1 - \nu_k$. The choice of λ^1 was arbitrary, so we get

$$\lambda^1_1 - \lambda^1_k = \ldots = \lambda^{s-1}_1 - \lambda^{s-1}_k = \nu_1 - \nu_k.$$

Letting a represent that common difference, we wish to show $a = 0$. Consider the quantities

$$0 = \sum_{p=1}^{s-1} \lambda^p_1 - \nu_1,$$

$$b = \sum_{p=1}^{s-1}\sum_{i=1}^{k-1} \lambda^p_i - \sum_{i=1}^{k-1} \nu_i,$$

$$c = \sum_{p=1}^{s-1}\sum_{i=1}^{k} \lambda^p_i - \sum_{i=1}^{k} \nu_i.$$

Of course b and c are also 0 by assumption, but even if we had not assumed that these Horn inequalities held with equality, we could use the following argument to show $a = c = 0$, given that (inducting on k) $b = 0$, because

$$0 + b - c = \underbrace{a + \ldots + a}_{s-1} - a$$

and at the very least $c \geqslant 0, a \geqslant 0$.

Therefore all partitions λ^p and ν are multiples of ω_r. Furthermore, even if we had not assumed $\sum |\lambda^p| = |\nu|$, we would have proved it along the way, except in the single case $r = 1$. □

Corollary 25. *The extremal rays of* EqLR_r^s *lying on every Horn facet are spanned by the collection*

$$x_j = (0, \ldots, 0, \underbrace{\omega_r}_{j}, 0, \ldots, \omega_r),\ \ j \in [s-1].$$

Proposition 26. *Consider the addition map* $f : \bigoplus \mathbb{R}_{\geqslant 0} x_j \oplus \mathsf{C}_{SL_r}^s \to \mathsf{LR}_r^s$. *We claim* f *is an additive,* $\mathbb{R}_{\geqslant 0}$*-linear bijection.*

Proof. Clearly there are no dependencies among the direct summands on the left. Let $(\lambda^1, \ldots, \lambda^{s-1}, \nu) \in \mathsf{LR}_r^s$ be arbitrary. Since $\lambda_r^1 \leqslant \nu_r$, $(\lambda^1 - \lambda_r^1\omega_r, \lambda^2, \ldots, \lambda^{s-1}, \nu - \lambda_r^1\omega_r)$ still satisfies (i), nonnegativity, (ii), and (iii).

But therefore $\lambda_r^2 \leqslant \nu_r - \lambda_r^1$, from which we deduce that

$$(\lambda^1 - \lambda_r^1\omega_r, \lambda^2 - \lambda_r^2\omega_r, \ldots, \lambda^{s-1}, \nu - \lambda_r^1\omega_r - \lambda_r^2\omega_r)$$

once again satisfies (i), nonnegativity, (ii), and (iii). Continuing in this manner we arrive at an element of $\mathsf{C}_{SL_r}^s$, each time subtracting $\lambda_r^j x_j$. ☐

Proposition 27. *Any element of* C_r^s *can be written as a sum* $z + \sum a_j x_j$*, where* $z \in \mathsf{LR}_r^s$ *and* $a_j \in \mathbb{R}$*. Likewise, any element of* EqC_r^s *can be written as a sum* $\hat{z} + \sum \hat{a}_j x_j$*, where* $\hat{z} \in \mathsf{EqLR}_r^s$ *and* $\hat{a}_j \in \mathbb{R}$*.*

Proof. Let $x \in \mathsf{C}_r^s$ be arbitrary. If all entries of x are nonnegative, then $x \in \mathsf{LR}_r^s$. Otherwise, $x + B(x_1 + x_2 + \ldots + x_{s-1})$ will have nonnegative entries for $B \gg 0$, and will still belong to C_r^s, so

$$x = (x + B(x_1 + x_2 + \ldots + x_{s-1})) - B(x_1 + x_2 + \ldots + x_{s-1}).$$

A similar argument works for EqC_r^s, recognizing that the containment inequalities (iv) will also be satisfied for $B \gg 0$. ☐

6 Examples and Counterexamples

6.1 Using the algorithm

To illustrate the rays algorithm, let us take a small example where $s = 3$, $r = 3$. Consider the Horn facet $\hat{\mathcal{F}}$ given by $I_1 = I_2 = \{2\}, K = \{3\}$, with associated equality

$$\lambda_2^1 + \lambda_2^2 = \nu_3.$$

6.1.1 Type I rays

One choice of type I datum is $j = 1, a = 2$. Using these, we get $I_1' = \{3\}, I_2' = \{2\}, K' = \{3\}$. Now we follow Algorithm 7 to determine $r(1, 2) = (\lambda^1, \lambda^2, \nu)$. The ways to decrement either of the first two sets or increment K' are as follows:

- $I_1'' = \{2\}, I_2'' = \{2\}, K'' = \{3\} \rightsquigarrow c_{\{2\},\{2\}}^{\{3\}} = 1 \implies \lambda_2^1 - \lambda_3^1 = 1.$

- $I_1'' = \{3\}, I_2'' = \{1\}, K'' = \{3\} \rightsquigarrow c^{\{3\}}_{\{3\},\{1\}} = 1 \implies \lambda^2_1 - \lambda^2_2 = 1.$
- $I_1'' = \{3\}, I_2'' = \{2\}, K'' = \{4\} \rightsquigarrow c^{\{4\}}_{\{3\},\{2\}} = 1 \implies \nu_3 = 1.$

All other consecutive differences in $\lambda^1, \lambda^2, \nu$ are 0 and we get the ray

$$((1,1,0),(1,0,0),(1,1,1)).$$

A second type I datum is $j = 2, a = 2$. By symmetry, we know from our first calculation that the ray is $((1,0,0),(1,1,0),(1,1,1))$.

The final type I datum is $j = 3, a = 3$, resulting in $I_1' = I_2' = K' = \{2\}$. The possible ways to decrement/increment are:

- $I_1'' = \{1\}, I_2'' = \{2\}, K'' = \{2\} \rightsquigarrow c^{\{2\}}_{\{1\},\{2\}} = 1 \implies \lambda^1_1 - \lambda^1_2 = 1.$
- $I_1'' = \{2\}, I_2'' = \{1\}, K'' = \{2\} \rightsquigarrow c^{\{2\}}_{\{2\},\{1\}} = 1 \implies \lambda^2_1 - \lambda^2_2 = 1.$
- $I_1'' = \{2\}, I_2'' = \{2\}, K'' = \{3\} \rightsquigarrow c^{\{3\}}_{\{2\},\{2\}} = 1 \implies \nu_2 - \nu_3 = 1.$

So the ray produced is

$$((1,0,0),(1,0,0),(1,1,0)).$$

6.1.2 Type II rays

To find the type II rays on $\hat{\mathcal{F}}_2$, we need to know the rays of $\mathsf{LR}^3_1 \times \mathsf{EqLR}^3_2$, which are

$$((1),(0),(1)) \times ((0,0),(0,0),(0,0))$$
$$((0),(1),(1)) \times ((0,0),(0,0),(0,0))$$

together with the 10 rays of the form $((0),(0),(0)) \times z$ where z belongs to the $r = 2$ table in Section 6.2. For example, let's apply $\widehat{\mathrm{Ind}}$ to $((0),(0),(0)) \times ((1,0),(1,1),(1,1))$. First π^{-1} takes it to $((1,0,0),(1,0,1),(1,1,0))$. Then using formula (5), $\hat{p}_2$ sends this to

$$((1,0,0),(1,0,1),(1,1,0)) + ((1,0,0),(1,1,0),(1,1,1)) - ((1,0,0),(1,0,0),(1,1,0))$$
$$= ((1,0,0),(1,1,1),(1,1,1)),$$

which is one of the rays in Table 2.

For another example, let's apply the induction map to $((1),(0),(1)) \times ((0,0),(0,0),(0,0))$. Applying π^{-1} we get $((0,1,0),(0,0,0),(0,0,1))$. Then $\hat{p}_2$ takes this to

$$((0,1,0),(0,0,0),(0,0,1)) - ((1,1,0),(1,0,0),(1,1,1)) + ((1,0,0),(1,0,0),(1,1,0)) = 0.$$

This we expect as noted in the proof of Corollary 20.

$r=1$
((1),(0),(1))
((0),(1),(1))
((1),(1),(1))

$r=2$
((0,0),(1,0),(1,0))
((0,0),(1,1),(1,1))
((1,0),(0,0),(1,0))
((1,1),(0,0),(1,1))
((1,0),(1,0),(1,1))
((1,0),(1,0),(1,0))
((1,0),(1,1),(1,1))
((1,1),(1,0),(1,1))
((1,1),(1,1),(1,1))
((1,1),(1,1),(2,1))

Table 1: Extremal rays of EqLR_1^3 and EqLR_2^3

The total outputs produced include the following rays:

$$((0,0,0),(1,0,0),(1,0,0)) \qquad ((0,0,0),(1,1,1),(1,1,1))$$
$$((1,0,0),(0,0,0),(1,0,0)) \qquad ((1,1,1),(0,0,0),(1,1,1))$$
$$((1,0,0),(1,0,0),(1,0,0)) \qquad ((1,0,0),(1,1,1),(1,1,1))$$
$$((1,1,1),(1,0,0),(1,1,1))$$

Additionally, 0 is output three times in accordance with Corollary 20 (in our example $|\mathfrak{J}| = 3$), and two elements are produced which are not on an extremal ray:

$$((0),(0),(0)) \times ((1,1),(1,1),(1,1)) \xmapsto{\widehat{\mathrm{Ind}}} ((2,1,1),(2,1,1),(2,2,2))$$
$$((0),(0),(0)) \times ((1,1),(1,1),(2,1)) \xmapsto{\widehat{\mathrm{Ind}}} ((2,1,1),(2,1,1),(3,2,2))$$

6.2 Data for small r

To give a sense of the extremal rays of EqLR_r^3, we have recorded in Tables 1 and 2 the complete list of rays for $r \leqslant 3$. The "strictly equivariant" rays (those violating (ii)) are below the dashed line; those which lie on LR_r^3 are above.

In Table 3, we have recorded the number of extremal rays of the cones LR_r^3 and EqLR_r^3 for the first few values of r. Calculations were done using `Sage` [15] and `Normaliz` [5].

6.3 Extra Hilbert basis elements

The semigroup of lattice points $\mathsf{EqLR}_r^s \cap \mathbb{Z}^{rs}$ has a finite list of indecomposable elements – those which are not the sum of two nonzero elements – called the *Hilbert basis*. Equivalently, the Hilbert basis is the (unique) minimal generating set of $\mathsf{EqLR}_r^s \cap \mathbb{Z}^{rs}$ over $\mathbb{Z}_{\geqslant 0}$.

$r = 3$	
((0,0,0),(1,0,0),(1,0,0))	((1,0,0),(1,0,0),(1,1,0))
((0,0,0),(1,1,0),(1,1,0))	((1,0,0),(1,1,0),(1,1,1))
((0,0,0),(1,1,1),(1,1,1))	((1,1,0),(0,0,0),(1,1,0))
((1,0,0),(0,0,0),(1,0,0))	((1,1,0),(1,0,0),(1,1,1))
((1,1,0),(1,1,0),(2,1,1))	((1,1,1),(0,0,0),(1,1,1))
((1,0,0),(1,0,0),(1,0,0))	((1,1,0),(1,1,0),(2,1,0))
((1,0,0),(1,1,0),(1,1,0))	((1,1,0),(1,1,1),(1,1,1))
((1,0,0),(1,1,1),(1,1,1))	((1,1,0),(1,1,1),(2,1,1))
((1,1,0),(1,0,0),(1,1,0))	((1,1,1),(1,0,0),(1,1,1))
((1,1,0),(1,1,0),(1,1,0))	((1,1,1),(1,1,0),(1,1,1))
((1,1,0),(1,1,0),(1,1,1))	((1,1,1),(1,1,0),(2,1,1))
((1,1,1),(1,1,1),(1,1,1))	((1,1,1),(1,1,1),(2,1,1))
((1,1,1),(1,1,1),(2,2,1))	((1,1,1),(2,1,1),(2,2,1))
((2,1,1),(1,1,1),(2,2,1))	

Table 2: Extremal rays of EqLR_3^3

r	# rays of LR_r^3	# rays of EqLR_r^3
1	2	3
2	5	10
3	10	27
4	20	72
5	44	195
6	114	532
7	362	1469

Table 3: Extremal rays of LR_r^3 and EqLR_r^3

r	# rays of EqLR_r^3	# H.b. elts. of $\mathsf{EqLR}_r^3 \cap \mathbb{Z}^{3r}$
1	3	3
2	10	10
3	27	27
4	72	72
5	195	195
6	532	**535**
7	1469	**1500**

Table 4: Extremal rays and Hilbert basis elements of EqLR_r^3

Now every extremal ray affords us with exactly one Hilbert basis element, namely the first lattice point along that ray. This element is indecomposable since any summands must be parallel (by extremality) and therefore one of them equal to 0 (by being the first lattice point).

For general pointed rational cones, the Hilbert basis can be much larger than the set of extremal rays. We observe that for EqLR_r^3, this does not happen until $r = 6$; see Table 4. Since the natural inclusions $\mathsf{EqLR}_r^s \subset \mathsf{EqLR}_{r+1}^s$ given by appending 0's preserve the properties of extremal ray and Hilbert basis element, we conclude that for $r \geqslant 6$ the Hilbert basis of $\mathsf{EqLR}_r^3 \cap \mathbb{Z}^{3r}$ is greater in size than the set of extremal rays. To produce Table 4, calculations were once again done using `Sage` and `Normaliz`. Here are the three "extra" Hilbert basis elements at $r = 6$:

$$((2,1,1,1,1,1),(2,2,2,1,1,1),(3,3,2,2,2,1))$$
$$((2,2,1,1,1,1),(2,2,1,1,1,1),(3,2,2,2,2,1))$$
$$((2,2,2,1,1,1),(2,1,1,1,1,1),(3,3,2,2,2,1))$$

The phenomenon of extra Hilbert basis elements has neither been observed nor ruled out for the cones LR_r^3, having checked the cases $r \leqslant 9$ by computer.

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