Maximal Chains in Bond Lattices

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Abstract

Let G be a graph with vertex set $\{1, 2, \ldots, n\}$. Its bond lattice, $BL(G)$, is a sublattice of the set partition lattice. The elements of $BL(G)$ are the set partitions whose blocks induce connected subgraphs of G.

In this article, we consider graphs G whose bond lattice consists only of non-crossing partitions. We define a family of graphs, called *triangulation graphs*, with

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this property and show that any two produce isomorphic bond lattices. We then look at the enumeration of the maximal chains in the bond lattices of triangulation graphs. Stanley's map from maximal chains in the noncrossing partition lattice to parking functions was our motivation. We find the restriction of his map to the bond lattice of certain subgraphs of triangulation graphs. Finally, we show the number of maximal chains in the bond lattice of a triangulation graph is the number of ordered cycle decompositions.

Mathematics Subject Classifications: 05A10,05C99,06A07

1 Introduction

Rota investigated bond lattices of graphs in 1964 in connection with Möbius functions [6]. The bond lattice of a graph G is a sublattice of the lattice of set partitions – only set partitions where all blocks induce connected subgraphs of G are included in the bond lattice. It is in fact a geometric lattice and as such has a nice Möbius function. Furthermore, the characteristic polynomial of the bond lattice is a multiple of the chromatic polynomial of the graph G.

Many mathematicians have studied properties of bond lattices. Sachs gave necessary and sufficient conditions that a lattice be isomorphic to a bond lattice for some graph G, see [7]. Tsuchiya characterized graphs whose bond lattice is modular, supersolvable, or weakly modular [11]. More recently, Lazar and Wachs showed that the lattice of intersections of the homogenized Linial arrangement is isomorphic to the bond lattice of a certain bipartite graph [5]. They were then able to use its Möbius function to study the sequence of Genocchi numbers[1].

The Kreweras lattice, also known as the noncrossing partition lattice, is especially important in enumerative combinatorics. It is the sublattice of the set partition lattice consisting solely of noncrossing partitions. In work closely related to ours, Farmer, Hallam, and Smyth studied the intersection of a bond lattice with the Kreweras lattice [3]. They call this intersection a noncrossing bond poset and they determine conditions on the graph which ensure that the noncrossing bond poset is a lattice. Additionally, for several families of graphs, they give combinatorial descriptions of the Möbius function and characteristic polynomial of the noncrossing bond poset.

The maximal chains in the noncrossing partition lattice help explain the lattice's importance. Stanley defined a bijection between maximal chains in the noncrossing partition lattice $KL(n)$ and parking functions [9]. We recall that parking functions can be characterized as the collection of n-tuples with entries in $\{1, 2, \ldots, n\}$ whose nondecreasing rearrangements $(a_1, a_2, \ldots, a_n)$ satisfy the condition $a_i \leqslant i$ for all $1 \leqslant i \leqslant n$. Stanley's bijection motivates our research question: *What happens when we restrict the bijection to maximal chains in a bond lattice whose elements are noncrossing partitions?* In this paper, we consider this question and related ones for graphs whose bond lattice contains only noncrossing partitions; we do not investigate maximal chains in noncrossing bond posets in the sense of [3].

[1]Genocchi numbers are the sequence A036968 in the OEIS .

We organize the work as follows. After preliminaries in Section 2, we begin in Section 3 by characterizing graphs whose bond lattice is a sublattice of the Kreweras lattice. They are subgraphs of graphs which resemble triangulations of a polygon. We focus on a subfamily of these graphs, which we call triangulation graphs. In Section 4 we show that any two triangulation graphs produce isomorphic bond lattices. In Section 4, we consider some special cases. When the graph is a path P_n, we found that the image of the Stanley bijection are permutations of $\{1, \ldots, n-1\}$. We also look at cycles. In Section 5, we look at the bond lattices of maximal graphs which produce only noncrossing partitions. However, we note that characterizing the parking functions in the image remains an open problem. The final section ends with the proof that the number of maximal chains is the number of ordered cycle decompositions.

2 Preliminaries

2.1 Noncrossing partition lattice

A *(set) partition* of a set S is a collection $\pi = \{B_1, \ldots, B_k\}$ of disjoint, nonempty subsets of S whose union is S. The subsets making up the partition are called *blocks*. We may represent a set partition with an *arc diagram*, where we draw an arc between two elements if they are in the same block. A set partition of S is called *crossing* if there are elements $a < b < c < d$ of S such that a and c are in the same block and b and d are in the same block. The partition $\pi = 13 \mid 2 \mid 45$ is *noncrossing* and the partition $13 \mid 25 \mid 4$ is crossing. See Figure 1.

Figure 1: $13 \mid 24$ is crossing, $14 \mid 23$ is noncrossing

The usual partial order on set partitions is *refinement*, i.e., the partition σ is less than the partition τ if every block of σ is contained in a block of τ. For instance,

$$13 \mid 2 \mid 4 \mid 5 < 13 \mid 245 \text{ and } 13 \mid 2 \mid 4 \mid 5 \not< 12 \mid 345.$$

The partition σ is covered by the partition τ if we can obtain τ by combining exactly two blocks of σ. For example,

$$13 \mid 2 \mid 4 \mid 5 \lessdot 13 \mid 2 \mid 45.$$

It is traditional to refer to the set partitions of $\{1, 2, \ldots, n\}$, together with this partial order as Π_n. However, our interest is in the refinement order applied only to the noncrossing partitions of $\{1, 2, \ldots, n\}$; this is also a poset. In fact, the poset is a lattice called the noncrossing partition lattice or the Kreweras lattice and it is denoted $KL(n)$. See Figure 2 for examples.

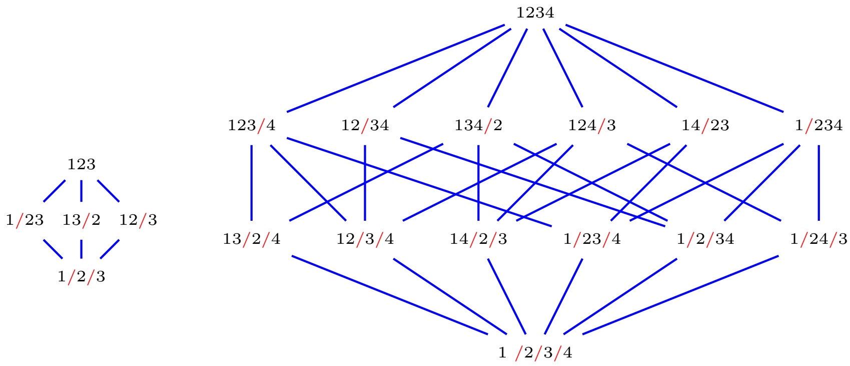

Figure 2: The Kreweras lattices $KL(3)$ and $KL(4)$.

2.2 Bond lattices

2.2.1 Graph terminology

We review a few graph theoretic definitions here; for more on this topic we refer the interested reader to [2]. Graphs are denoted by their vertices and edges: $G = (V, E)$. The vertices V will be $[n] := \{1, \ldots, n\}$ unless otherwise indicated and the edges E are subsets $e = \{x, y\}$ from the vertices. The *endpoints* of the edge $\{x, y\}$ are x and y. Suppose W is a subset of V. The *induced subgraph* $G\mid_W$ is the graph whose vertices are W and all edges in E which connect two vertices of W. A (possibly non-induced) *subgraph* $H = (W, F)$ of G is a graph where $W \subseteq V$ and $F \subseteq E$. A *path* in a graph is a subgraph given by a sequence of vertices $x_0, x_1, \ldots, x_k$ such that $\{x_i, x_{i+1}\}$ is an edge for i from 0 to $k-1$. If $\{x_0, x_k\}$ is also an edge in E, then the path $x_0, x_1, \ldots, x_k$ together with the edge $\{x_0, x_k\}$ is a *cycle* in the graph. A cycle with k distinct vertices is called a k-cycle. We will also refer to paths and cycles by their edges. Two vertices are *connected* in a graph if we can find a path from one to the other and a graph is connected if every pair of its vertices is connected. A maximal connected subgraph of G is called a *component* of G.

There are two equivalent definitions of the bond lattice of a graph $G = (V, E)$ and we will use both. First, we define the lattice in terms of edges. Let $S \subseteq E$ and $H = (V, S)$. Define $\bar{S}$ to be the set of all edges in $e \in E$ such that both endpoints of e belong to the same component of H. A set $S \subset E$ is *closed* if $S = \bar{S}$. The lattice $BL(G)$ of closed subsets of E ordered by inclusion is called the *bond lattice* of G [6]. Since each subset of edges S determines components in $H = (V, S)$, it also determines a partition of the vertex set. Given a partition σ of the vertices, the corresponding edge set is denoted $E(\sigma)$. Likewise, the vertex partition corresponding to the edge set S is $\pi(S)$. Alternatively, we may define the lattice in terms of vertices. Suppose $\sigma = B_1 \mid B_2 \mid \cdots \mid B_k$ is a partition of V. If $G\mid_{B_i}$ is a connected graph for $1 \leqslant i \leqslant k$, then σ is an element of $BL(G)$. The order on the lattice is refinement.

Example 1. Let G be the graph with vertices $V = \{1, 2, 3, 4\}$ and edges $E = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$. Since $G\mid_{\{1,3\}}$ is not connected, $13 \mid 2 \mid 4$ is not an element of $BL(G)$. Since both $G\mid_{\{1,2\}}$ and $G\mid_{\{3,4\}}$ are connected, the partition $12 \mid 34$ is an element. See Figure 3 for an illustration of the bond lattice of this graph.

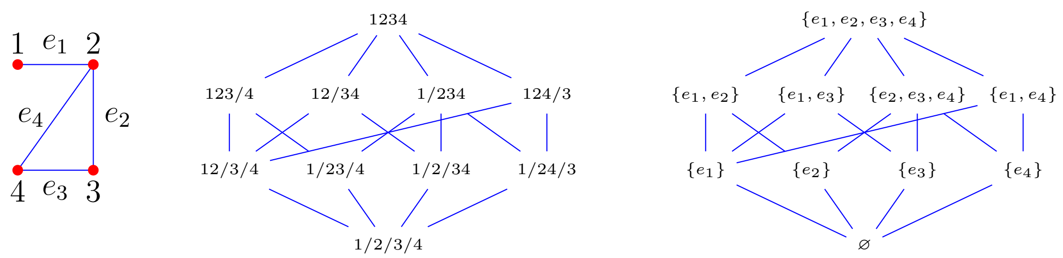

Figure 3: Figure for Example 1. On the left is the graph G, in the middle $BL(G)$ in terms of vertices, and on the right $BL(G)$ in terms of edges.

In what follows we restrict ourselves to graphs G where all elements of $BL(G)$ are noncrossing.

2.3 Parking functions

An n-tuple of positive integers $(a_1, a_2 \ldots, a_n) \in [n]^n$ is called a *parking function* if its nondecreasing rearrangement $a'_1 \leqslant a'_2 \leqslant \cdots \leqslant a'_n$ satisfies $a'_i \leqslant i$. Parking functions have been studied in the context of hashing functions and it is known that there are $(n+1)^{n-1}$ parking functions [4]. Moreover many generalizations of these combinatorial objects exist and their study has received ample attention in the literature. For a general reference we point the interested reader to [12].

In 1997, Stanley [9] found a bijection between maximal chains in the Kreweras lattice and parking functions. We describe it here. The idea is to label each edge in the lattice and then collect those labels as you trace the chain from bottom to top. The sequence of labels forms a parking function. Suppose $\sigma \lessdot \tau$ in $KL(n)$. Then two blocks of σ, say B_i and B_j are merged to form a single block of τ. Assume the minimum element of B_i is less than the minimum element of B_j. Let λ be the biggest element of B_i that is less than the minimum element of B_j. This number λ is the label of the edge from σ to τ in $KL(n)$. Here is an example from $KL(4)$. The maximal chain

$$1 \mid 2 \mid 3 \mid 4 \quad \lessdot \quad 12 \mid 3 \mid 4 \quad \lessdot \quad 124 \mid 3 \quad \lessdot \quad 1234$$

is labeled with the parking function $(1, 2, 2)$.

3 Triangulation graphs and their bond lattices

We are interested in the graphs whose bond lattice only contains noncrossing partitions. We say a graph on vertices $1, \ldots, n$ is *properly displayed* if its vertices are arranged on a circle in increasing, clockwise order, from 1 to n, and all edges are drawn inside the circle.

In [3], this is called the *graphical representation* of G. We assume all graphs are properly displayed.

Two edges *cross* if the lines representing them cross when the graph is properly displayed. The graph G is therefore *noncrossing* if it does not have vertices $u < v < w < x$ such that both $\{u, w\}$ and $\{v, x\}$ are edges. This is similar to outerplanar graphs [2], but here the order of the vertices is important. A properly displayed graph has an *inner triangle* if it has three pairwise adjacent vertices $1 \leqslant a < b < c \leqslant n$ such that $a + 1 < b$, $b + 1 < c$, and if $c = n$, then $1 < a$. A noncrossing graph with the maximum number of edges and with no inner triangles is called a *triangulation graph*. See Figure 4 for an example. In a triangulation graph, $\{i, i + 1\}, 1 \leqslant i \leqslant n - a$, and $\{1, n\}$ are edges. This follows from the definition; in particular, from the requirements that the vertices of G are arranged in clockwise order and that G is maximal. Triangulations of a polygon with no internal triangles may be viewed as triangulation graphs–the diagonals and boundary edges of the polygon correspond to edges of the graph. In a triangulation graph, we call the n edges $\{i, i+1\}$, for $1 \leqslant i \leqslant n$, and $\{1, n\}$ *external* and the $n-3$ other edges *internal*. We refer to the 3-cycles as *triangles*.

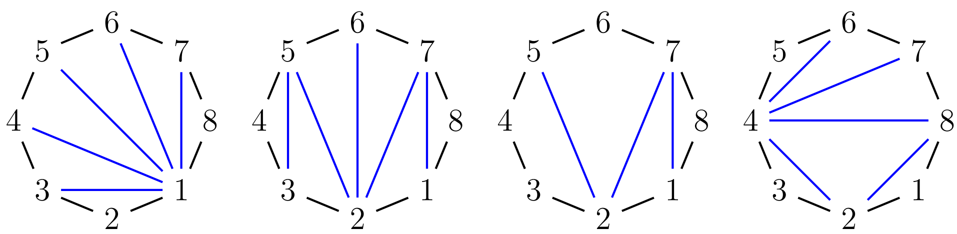

Figure 4: The first and second graphs from the left are triangulation graphs. The third graph is not, because it is not maximal. The graph on the right has an inner triangle, so it is also not a triangulation graph.

Claim 2. *The vertices of the bond lattice $BL(G)$ consists solely of noncrossing partitions if and only if G is a subgraph of a noncrossing graph.*

Proof. A graph is a subgraph of a noncrossing graph if and only if it does not have a pair of edges of the form $\{a, c\}, \{b, d\}$, where $a < b < c < d$. If $BL(G)$ consists solely of noncrossing partitions, then such a pair cannot exist and G is a subgraph of a noncrossing graph. Conversely, suppose there exists $a < b < c < d$ such that a and c belong to a block B and b and d belong to block $B' \neq B$ of partition π. Then there exists a path from a to c with all vertices in B and a path from b to d with all vertices in B'. When the graph G is properly displayed, these paths must cross. They cannot meet at a vertex, since B and B' are disjoint. Therefore, there are edges that cross. $\square$

3.1 Isomorphism

In this section, we show that any two triangulation graphs on $[n]$ produce isomorphic bond lattices. We begin presenting a specific labelling of the edges and 3-cycles of triangulation graphs.

Definition 3 (Triangulation graph labelling)**.** Let G be a triangulation graph on n vertices. It has $n-2$ *triangles*; that is, $n-2$ 3-cycles. A straightforward counting argument shows that there are two triangles with two external edges and one internal edge and the other $n-4$ triangles have one external and two internal edges. Set T_1 to one of the triangles with two external edges in G, T_2 to the triangle which shares T_1's single internal edge, T_3 to the triangle which shares T_2's other internal edge, and so on. Since G has no internal triangles, this labelling is well-defined. The edges of T_1 are labelled e_1, e_2, and f_1, where e_1 and e_2 are the external edges, in clockwise order, and f_1 is T_1's internal edge. The edges of T_{n-3} are labelled e_{n-1}, e_n, and f_{n-3}. Finally, label T_i's edges with e_{i+1}, f_{i-1}, and f_i, for $2 \leqslant i < n-3$. For $1 \leqslant i \leqslant j \leqslant n-2$, set $T_{[i,j]}$ to be the induced subgraph of G whose vertex set is the union of the vertex sets of $T_i, T_{i+1}, \dots, T_j$.

Please see Figure 5 for an example of the labelling defined in Definition 3.

There are a few properties worth noting.

- $T_{[i,j]}$ is a triangulation graph in its own right, with $j-i+3$ vertices, $j-i+1$ 3-cycles, and $j-i$ internal edges.
- The 3-cycles of T_i and T_{i+1} share the edge f_i.
- The vertices of $T_{[i,j]}$ form a $(j-i+3)$-cycle.

The following lemma is a property of the labelling.

Lemma 4. *Let $C = \{w_0, \dots, w_{k-1}, w_k = w_0\}$ be a k-cycle in a triangulation graph G and let $H = G\mid_C$ be the induced subgraph of G on C. If the graph G is labelled as in Definition 3, then there exists i, j, $1 \leqslant i \leqslant j \leqslant n-2$ such that the vertices of $H = T_{[i,j]}$.*

Proof. We use induction on k. For $k = 3$, the result holds for $i = j$, the index of the 3-cycle with vertices w_0, w_1, w_2 in G. Suppose $k > 3$ and consider the pair w_0 and w_2. If $\{w_0, w_2\}$ is an edge of G, then let C' be the $(k-1)$-cycle on $\{w_0, w_2, \dots, w_{k-1}\}$ and $H' = G\mid_{C'}$. There are $i \leqslant j$ such that $H' = T_{[i,j]}$. The 3-cycle $T = \{w_0, w_1, w_2\}$ shares an edge with either T_i or T_j, since any triangle T_h, $1 < h < n-2$ shares an edge with T_{h-1} and an edge with T_{h+1}. We have that either $H = T_{[i-1,j]}$ or $H = T_{[i,j+1]}$. If $\{w_0, w_2\}$ is not an edge of G, then by maximality of G, there must be an edge from w_1 to say w_h, where $2 < h < k$. In this case, we make two smaller cycles $C' = \{w_1, w_2, \dots, w_h\}$ and $C'' = \{w_0, w_1, w_h, w_{h+1}, \dots, w_{k-1}\}$. There exists i_1, j_1, i_2, j_2 such that $G\mid_{C'} = T_{[i_1,j_1]}$ and $G\mid_{C''} = T_{[i_2,j_2]}$. Since $T_{[i_1,j_1]}$ and $T_{[i_2,j_2]}$ share the edge $\{w_1, w_h\}$, either $j_1 = i_2 - 1$ or $j_2 = i_1 - 1$; in either case, we see that there exists $i < j$ such that $G\mid_C = T_{[i,j]}$, as desired. □

With the labelling and its properties in hand, we can go on to define the isomorphism between the bond lattices of two triangulation graphs.

Let $G_1 := G_1(V_1, E_1)$ and $G_2 := G_2(V_2, E_2)$ be two triangulation graphs on $n \geqslant 3$ vertices. The map ϕ from the edges of G_1 to the edges of G_2 simply matches up edges with the same label. We then use ϕ to define a poset isomorphism, $\psi : BL(G_1) \to BL(G_2)$,

Cover σ	$E(\sigma)$	$\psi(\sigma)$
12678 | 34 | 5	$\{e_1, e_2, e_3, e_6, e_7, f_1, f_2\}$	123458 | 6 | 7
134678 | 2 | 5	$\{e_1, e_2, e_3, e_6, f_1, f_2, f_4, f_5\}$	123468 | 5 | 7
15678 | 2 | 34	$\{e_1, e_2, e_3, e_4, e_6, f_1, f_2, f_3\}$	123478 | 5 | 6
1678 | 234 | 5	$\{e_1, e_2, e_3, e_6, e_8, f_1, f_2\}$	12348 | 56 | 7
1678 | 2 | 345	$\{e_1, e_2, e_3, e_5, e_6, f_1, f_2\}$	12348 | 5 | 67

Table 1: Table of data for Example 5.

from the bond lattice of G_1 to the bond lattice of G_2. Note that the map ψ is defined on the elements of the poset $BL(G_1)$ considered as set partitions, not as closed sets of edges.

To make this precise, we label G_1 as in Definition 3 and label G_2 similarly with $T'_1, \ldots, T'_{n-2}$, $e'_1, \ldots, e'_n$, $f'_1, \ldots, f'_{n-3}$. We set $\phi(e_i) = e'_i$, $\phi(T_i) = T'_i$, and $\phi(f_i) = f'_i$. Every set partition π in $BL(G_1)$ corresponds to a subset $E(\pi)$ of E_1, where $x \in E_1$ is an element of $E(\pi)$ if and only if both endpoints of x are in the same block of π. We define the map $\psi : BL(G_1) \to BL(G_2)$ by $\psi(\pi) = \pi'$ if and only if $\phi(E(\pi)) = E(\pi') \subseteq E_2$.

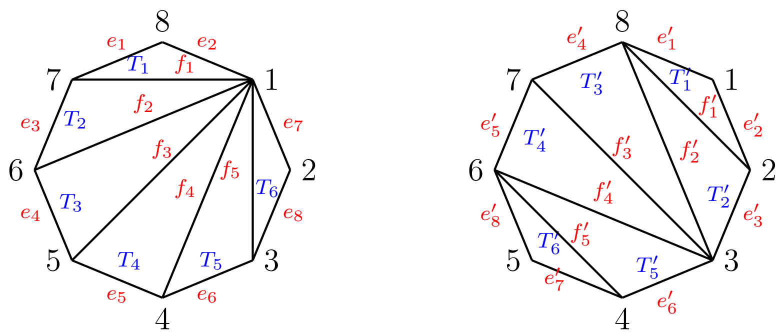

Figure 5: The labelling used to define ϕ. Let G_1 be the triangulation graph on the left, and G_2 on the right.

Example 5. Let G_1 and G_2 be as in Figure 5 and let $\pi \in BL(G_1)$ be 1678 | 2 | 34 | 5. The edges corresponding to π are $E(\pi) = \{e_1, e_2, e_3, e_6, f_1, f_2\}$ and $\psi(\pi)$ is 12348 | 5 | 6 | 7. Table 1 lists the covers of π in $BL(G_1)$, together with their edges set and image under ψ.

Based on Lemma 4 and the fact that ψ maps the 3-cycle T_i to the 3-cycle T'_i, we have the following corollary.

Corollary 6. *Under ψ, the image of a k-cycle is a k-cycle.*

Note that Corollary 6 implies that for $S \subseteq E$, S is closed if and only if $\phi(S)$ is closed.

Theorem 7. *Let G and G' be two triangulation graphs with vertex set $[n]$. Suppose the set partitions π and σ are elements of $BL(G)$ and $\pi' = \psi(\pi)$ and $\sigma' = \psi(\sigma)$ of $BL(G')$. Then $\pi \lessdot_{BL(G)} \sigma$ if and only if $\pi' \lessdot_{BL(G')} \sigma'$.*

Proof. Since ψ is a bijection, we need only prove that $\pi \lessdot_{BL(G)} \sigma$ implies $\pi' \lessdot_{BL(G')} \sigma'$. Denote the two blocks which merge in π to form σ by B_1 and B_2, the edges in $E(\sigma) \setminus E(\pi)$ by $x_1, \ldots, x_k$, and the images under ϕ of $x_1, \ldots, x_k$ by $x'_1, \ldots, x'_k$. Each edge in $E(\sigma) \setminus E(\pi)$ has one endpoint in B_1 and one in B_2. We must show there exists blocks B'_1 and B'_2 such that x'_i has one endpoint in B'_1 and the other in B'_2 for $1 \leqslant i \leqslant k$.

Consider x'_1 and x'_2. The edges x_1 and x_2 both have an endpoint in B_1, so there are edges $y_1, \ldots, y_{h_1}$ in $E(\pi)$ such that

- both endpoints of y_i are in B_1, for $1 \leqslant i \leqslant h_1$,
- y_1 shares an endpoint with x_1, and
- y_{h_1} shares an endpoint with x_2.

Similarly, there are edges $z_1, \ldots, z_{h_2}$ in $E(B_2)$ from the endpoint of x_2 in B_2 to the endpoint of x_1 in B_2. The edges $x_1, y_1, \ldots, y_{h_1}, x_2, z_1, \ldots, z_{h_2}$ form a cycle in G. See Figure 6 and Example 8 for an illustration. There may be other cycles containing x_1 and x_2, but that does not affect this argument. Denote the cycle's vertices by C. By Lemma 4, there exists $i < j$ such that $G \mid_C = T_{[i,j]}$. The image of $T_{[i,j]}$ is $T'_{[i,j]}$ and the image of C, denoted C', is the perimeter of $T'_{[i,j]}$. The edges x'_1 and x'_2 separate C' into at most two pieces. One of those pieces connects endpoints of x'_1 and x'_2 within one block of π' we'll call B'_1 and the other piece connects endpoints within what we will call B'_2. Since we may replace x'_2 by x'_i and repeat for x'_1 and x'_i, $2 < i \leqslant k$, the proof is complete. ☐

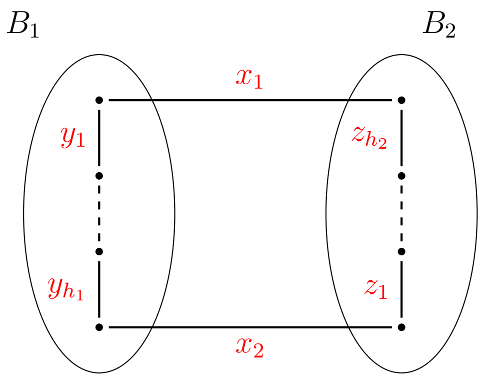

Figure 6: Cycle based on two edges between blocks.

Example 8. Let G be G_1 and G' be G_2 from Example 5 and as depicted in Figure 5. Let $\pi = 168 \mid 2 \mid 3 \mid 4 \mid 5 \mid 7$ and $\sigma = 1678 \mid 2 \mid 3 \mid 4 \mid 5$. We have $E(\pi) = \{e_2, f_2\}$ and $x_1, x_2, x_3 = e_1, e_3, f_1$. The cycle C defined in the proof of Theorem 7 is e_1, e_3, e_2, f_2 and the cycle C' is e'_1, e'_2, e'_3, f'_2.

4 Subgraphs

Certain subgraphs of triangulation graphs produce particularly nice bond lattices. In this section, we collect some of these results.

Proposition 9. *The elements of the bond lattice for the properly displayed path P_n are of the form:*

$$1\cdots a_1|a_1+1\cdots a_2|a_2+1\cdots a_3|a_3+1\cdots a_k|a_k+1\cdots n$$

where $a_1 < a_2 < a_3 < \cdots < a_k$ and a_i is the largest element of the block B_i.

Proof. Suppose we have a set partition in the bond lattice of a path P_n. We may assume the blocks are ordered by their minimal elements. It is enough to show that elements within the same block of a set partition in the bond lattice of the path P_n must be consecutive. We prove this by contradiction. Suppose that a_i and a_j are two elements in the same block B, where $a_i < a_j \leqslant n$ and a_i, a_j are not consecutive. Since a_i and a_j are in the same block of a set partition in the bond lattice of P_n, we know that the induced subgraph containing a_i and a_j is connected. In order for it to be a connected induced subgraph, there must exist a walk from a_i to a_j using only vertices in the induced subgraph $P_n|_B$. Since the only walk in P_n from a_i to a_j is $a_i, a_{i+1}, a_{i+2}, \ldots, a_{j-1}, a_j$, these vertices must all be in B, contradicting the existence of a gap. □

Although the Boolean lattice is self-dual, we distinguish between it and its dual because the order on $BL(P_n)$ corresponds to reverse inclusion.

Theorem 10. *There exists a poset isomorphism between the bond lattice $BL(P_n)$ and the dual of the Boolean lattice $\mathcal{B}^*_{n-1}$.*

Proof. We construct the bijection

$$\vartheta : BL(P_n) \to \mathcal{B}^*_{n-1}.$$

Let $1\cdots a_1|a_1+1\cdots a_2|a_2+1\cdots a_3|\cdots a_k|a_k+1\cdots n$ be a set partition in the bond lattice of P_n, where the blocks are ordered by their minimal elements and elements within the same block are consecutive, resulting from Proposition 9. Given the partition above, let the subset $\{a_1, a_2, a_3, \ldots, a_k\}$, where $k \leqslant (n-1)$, correspond to it. That is, we define the function as:

$$\vartheta(1\ \cdots a_1|a_1+1\ \cdots a_2|\cdots a_k|a_k+1\ \cdots n)\ = \{a_1, a_2, a_3, \ldots, a_k\}$$

We show that ϑ is injective by contradiction. Suppose there are two elements of $BL(P_n)$, α and β, such that $\alpha \neq \beta$, but $\vartheta(\alpha) = \vartheta(\beta)$. Let

$$\alpha = 1\cdots a_1|a_1+1\cdots a_2|\cdots a_k|a_k+1\cdots n$$

and

$$\beta = 1\ \cdots b_1|b_1+1\ \cdots b_2|\cdots b_m|b_m+1\ \cdots n.$$

Then, $\vartheta(\alpha) = \{a_1, a_2, \ldots, a_k\}$, and $\vartheta(\beta) = \{b_1, b_2, \ldots, b_m\}$. Since the set partitions of a bond lattice are distinct, $\{a_1, a_2, \ldots, a_k\} \neq \{b_1, b_2, \ldots, b_m\}$. Therefore, $\vartheta(\alpha) \neq \vartheta(\beta)$, which contradicts the initial assumption. Hence, this shows that ϑ is injective.

Since the elements in the blocks of elements of $BL(P_n)$ are consecutive, we only need to record the breaks in the blocks to determine the set partition. Therefore, if $\vartheta(\alpha) = \vartheta(\beta)$, then $\alpha = \beta$ and ϑ is surjective.

The set partition α' covers α if two blocks of α are joined to form a block of α'. If we join the blocks $\{a_{i-1}+1, \ldots, a_i\}$ and $\{a_i + 1, \ldots, a_{i+1}\}$ of α, then $\vartheta(\alpha) \setminus \{a_i\} = \vartheta(\alpha')$. We have $\alpha \lessdot \alpha'$ if and only if $\vartheta(\alpha) \gtrdot \vartheta(\alpha')$. □

Proposition 11. *The image of the maximal chains of $BL(P_n)$ under Stanley's bijection in Section 2.3 is the set of permutations of $\{1, \ldots, n-1\}$.*

Proof. Since $BL(P_n)$ is isomorphic to the dual of the Boolean algebra on $n-1$ elements, we know that it has $(n-1)!$ maximal chains (see, for example Chapter 5 of [8]). When the blocks $\{a_{i-1}+1, \ldots, a_i\}$ and $\{a_i+1, \ldots, a_{i+1}\}$ are joined, the bijection records a_i. Each element of $\{1, \ldots, n-1\}$ will appear exactly once, for each chain. □

Lemma 12. *Let C_n be the properly displayed cycle on n vertices. The bond lattice $BL(C_n)$ has $\frac{n!}{2}$ maximal chains.*

Proof. For this proof, we consider the elements of $BL(C_n)$ as closed subsets of the edges of C_n. Write $E = \{e_1, \ldots, e_n\}$ for the set of edges. All subsets of E of size k are closed, for $k = 1, \ldots, n-2$ and $k = n$. If $S = E - \{e_i, e_j\}$, then $\overline{S \cup \{e_i\}} = \overline{S \cup \{e_j\}} = E$. This pairs maximal chains of the Boolean algebra of E which differ only in the order of last two subsets. There are $n!$ maximal chains of subsets of E, but as a result of this pairing, we have $\frac{n!}{2}$ maximal chains of closed subsets. □

5 Maximal chains in the bond lattice of a triangulation graph

In this section, we prove the following main result, Theorem 13. Throughout this section, we let $g(n)$ be the number of maximal chains in the bond lattice of a triangulation graph. Recall that $g(n)$ is well-defined by Theorem 7. Moreover, we let $c(n,k)$ be the number of permutations of n which have k cycles; that is, $c(n,k)$ is the signless Stirling number of the first kind. Following [10, Lemma 1.3.6] we have

$$c(n,k) = \begin{cases} (n-1)c(n-1,k) + c(n-1,k-1) & n,k \geqslant 1 \\ 1 & n = k = 0 \\ 0 & n < k \text{ or } k = 0. \end{cases}$$

With this definition at hand we are now ready to state and prove our main result of this section.

Theorem 13. *The number of maximal chains in the bond lattice of a triangulation graph is related to the Stirling numbers as follows:*

$$g(n) = \sum_{k=1}^{n-1} c(n-1,k)k!. \tag{1}$$

The sum (1) of signless Stirling numbers appears in [1] in connection with the enumeration of certain trees.

The proof of Theorem 13 makes up the rest of Section 5. Section 5.1 is devoted to a recursion for $g(n)$, the number of maximal chains in a bond lattice for a triangular graph with n vertices. In Section 5.2 we show that $h(n) = \sum_{k=0}^{n-1} c(n-1,k)k!$ satisfies the same recursion. The proof of Theorem 13 then follows by checking initial values.

5.1 Recursion for the number of maximal chains

We find a recursion for $g(n)$ in this subsection.

Lemma 14. *Let G be a triangulation graph with n vertices. The number $g(n)$ of maximal length chains in $BL(G)$ satisfies*

$$g(n) = (n-2)! \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} \frac{g(i)g(n-j-i+1)}{(i-1)!(n-j-i)!}. \tag{2}$$

For the proof, we will need a special family of triangulation graphs, which we call spider graphs.

Definition 15. For $n \geqslant 1$, we define the spider triangulation graph S_n on vertex set $[n]$ to have all internal edges of the form $\{i,n\}$ for $1 < i < n-1$ and no others.

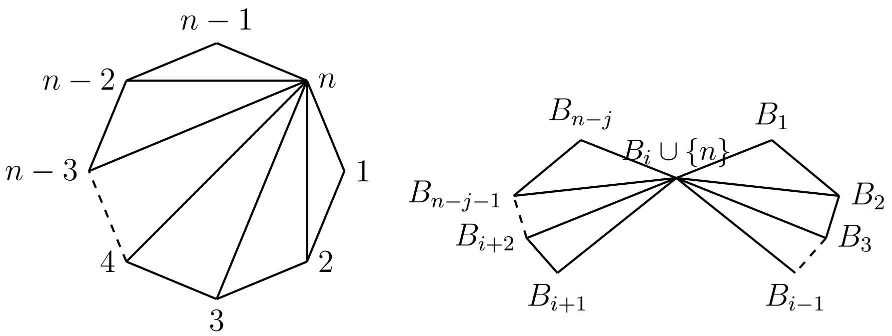

Figure 7: The graph on the left is the spider S_n. The graph on the right is a "double spider."

This graph S_n is particularly nice: if the element n is a singleton block in $\pi \in BL(S_n)$, then π is of the form $1 \cdots a_1 \mid a_1+1 \cdots a_2 \mid \cdots \mid a_{k-2}+1 \cdots a_{k-1} \mid n$.

Proof. We may assume our graph G is S_n by Theorem 7. Let c be an arbitrary chain

$$\pi_0 = 1 \mid 2 \mid \cdots \mid n-1 \mid n \lessdot \pi_1 \lessdot \pi_2 \lessdot \cdots \lessdot \pi_{n-1} = 123\cdots n. \tag{3}$$

First, we define $J(c)$ to be j if n is a singleton in π_{j-1} and not a singleton in π_j.

We want to count the number of chains c such that $J(c) = j$, for fixed j, $1 \leqslant j \leqslant n-1$. We count in stages:

Stage 1 Count the chains $\pi_0 \lessdot \pi_1 \lessdot \cdots \lessdot \pi_{j-1}$, where $\pi_{j-1} = B_1 \mid B_2 \mid \cdots \mid B_{n-j} \mid n$,

Stage 2 sum over i, where B_i is the block of π_{j-1} for n to join, and

Stage 3 count ways to build the rest of the chain, where each set partition coarsens $B_1 \mid B_2 \mid \cdots \mid B_i \cup \{n\} \mid \cdots \mid B_{n-j}$.

There are $\frac{(n-2)!}{(n-(j+1))!}$ chains of the type described in **Stage 1**: there are $n-2$ possible ways to merge two blocks of $\pi_0 = 1 \mid 2 \mid \cdots \mid n-1 \mid n$ which keep n a singleton, then $n-3$ ways to merge two blocks of the resulting π_1, etc.

There are $n-j$ blocks other than $\{n\}$ in π_{j-1}. We may pick B_i for $1 \leqslant i \leqslant n-j$, so we'll sum over i for **Stage 2**. The partition π_j has the form $B_1 \mid B_{i-1} \mid B_i \cup \{n\} \mid B_{i+1} \mid B_{n-j}$, where each B_k consists of consecutive elements.

Let $\overline{G}$ be the spider graph which has i vertices labelled by $B_1, \ldots, B_{i-1}, B_i \cup \{n\}$ and such that all internal edges have $B_i \cup \{n\}$ as an endpoint. Let $\widetilde{G}$ be the spider graph which has $(n-i-j+1)$ vertices labelled by $B_i \cup \{n\}, B_{i+1}, \ldots, B_{n-j}$ and such that all internal edges have $B_i \cup \{n\}$ as an endpoint. Finally, let $\overline{\widetilde{G}}$ be the union of the $\overline{G}$ and $\widetilde{G}$. See Figure 7 on the right. A set partition π is above π_j in $BL(G)$ if and only if π can be seen as an element of $BL(\overline{\widetilde{G}})$, so we may count maximal chains in $BL(\overline{\widetilde{G}})$ in **Stage 3**. A maximal chain in $BL(\overline{\widetilde{G}})$ may be decomposed into two maximal chains, one from from $BL(\overline{G})$ and one from $BL(\widetilde{G})$. At each cover in the $BL(\overline{\widetilde{G}})$-chain either two blocks from an element of $BL(\overline{G})$ are merged or two blocks from an element of $\widetilde{G}$ are merged. Therefore, in **Stage 3** for a fixed i and j, there are

$$g(i)g(n-i-j+1)\binom{n-j-1}{i-1}$$

chains starting at π_j. We finish the proof by combining counts from **Stages 1** and **3** and summing over i and j. $\square$

5.2 Ordered cycle decompositions

In this section, we find a recursion similar to Lemma 14 for the number of ordered cycle decompositions. Furthermore, this recursion enables us to prove Theorem 13.

Lemma 16. *Let* $h(n) = \sum_{k=0}^{n-1} k!c(n-1,k)$. *We have*

$$h(n) = (n-2)! \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} \frac{h(i)h(n-j-i+1)}{(i-1)!(n-j-i)!}. \tag{4}$$

Proof. The proof is through exponential generating functions. We begin with the identity found in Chapter 4 of [8]:

$$k!\sum_{n\geqslant 0} c(n,k)\frac{x^n}{n!} = \left(\ln\frac{1}{1-x}\right)^k. \tag{5}$$

Now define $H(x)$ as $\sum_{k\geqslant 0} k!\sum_{n\geqslant 0} c(n,k)\frac{x^n}{n!}$. By switching the order of summation, using the fact that $c(n,k) = 0$ if $n < k$, and the definition of $h(n)$, we have

$$H(x) = \sum_{n\geqslant 0} h(n+1)\frac{x^n}{n!}. \tag{6}$$

Additionally, by (5), we have

$$H(x) = \sum_{k\geqslant 0}\left(\ln\frac{1}{1-x}\right)^k. \tag{7}$$

Our strategy is to use (7) to show

$$\frac{H^2(x)}{1-x} = H'(x), \tag{8}$$

then apply (8) to (6) to obtain (4).

Consider the left side of (8), using (7) for $H(x)$:

$$\begin{aligned}
\frac{H^2(x)}{1-x} &= \frac{1}{1-x}\left(\sum_{k\geqslant 0}\left(\ln\frac{1}{1-x}\right)^k\right)^2 \\
&= \frac{1}{1-x}\left(\sum_{n\geqslant 0}\sum_{k=0}^{n}\left(\ln\frac{1}{1-x}\right)^{k+n-k}\right) \\
&= \frac{1}{1-x}\left(\sum_{n\geqslant 0}(n+1)\left(\ln\frac{1}{1-x}\right)^n\right).
\end{aligned}$$

When we consider the right hand side of (8), again using (7) for $H(x)$:

$$\begin{aligned}
H'(x) = \frac{d}{dx}H(x) &= \sum_{k\geqslant 0}\frac{d}{dx}\left(\ln\frac{1}{1-x}\right)^k \\
&= \sum_{k\geqslant 1} k\left(\ln\frac{1}{1-x}\right)^{k-1}\cdot\frac{1}{1-x} \\
&= \frac{1}{1-x}\sum_{k\geqslant 0}(k+1)\left(\ln\frac{1}{1-x}\right)^k,
\end{aligned}$$

we obtain the same result and (8) holds.

Now use (6) in (8):

$$\begin{aligned}
\frac{H^2(x)}{1-x} &= \frac{1}{1-x}\left(\sum_{n\geqslant 0} h(n+1)\frac{x^n}{n!}\right)^2 \\
&= \frac{1}{1-x}\left(\sum_{n\geqslant 0}\sum_{i=0}^{n} \frac{h(i+1)h(n-i+1)}{i!(n-i)!}x^n\right) \\
&= \left(\sum_{j\geqslant 0} x^j\right)\cdot\left(\sum_{n\geqslant 0}\sum_{i=0}^{n} \frac{h(i+1)h(n-i+1)}{i!(n-i)!}x^n\right) \\
&= \sum_{m\geqslant 0}\sum_{j=0}^{m}\sum_{i=0}^{m-j} \frac{h(i+1)h(m-j-i+1)}{i!(m-j-i)!}x^m \\
&= \sum_{m\geqslant 0} m!\sum_{j=0}^{m}\sum_{i=0}^{m-j} \frac{h(i+1)h(m-j-i+1)}{i!(m-j-i)!}\cdot\frac{x^m}{m!}.
\end{aligned}$$

Since $H'(x) = \sum_{m\geqslant 1} h(m+1)\frac{x^{m-1}}{(m-1)!} = \sum_{m\geqslant 0} h(m+2)\frac{x^m}{m1}$, by comparing coefficients we have

$$h(m+2) = m!\sum_{j=0}^{m}\sum_{i=0}^{m-j} \frac{h(i+1)h(m-j-i+1)}{i!(m-j-i)!}.$$

By substituting in $n = m+2$ and replacing $i+1$ by i and $j+1$ by j, we obtain (4). $\square$

The proof of Theorem 13 is now routine. The initial values of $h(n)$ and $g(n)$ are the same: for $n \geqslant 3$ we have $3, 14, 88, 694, \ldots$. Since both obey the same recursion (2) and (4), the theorem is proved.

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