Minimum Number of Edges of Polytopes with $2d + 2$ Vertices

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Abstract

We define two d-polytopes, both with $2d + 2$ vertices and $(d + 3)(d - 1)$ edges, which reduce to the cube and the 5-wedge in dimension three. We show that they are the only minimisers of the number of edges, amongst all d-polytopes with $2d+2$ vertices, when $d = 6$ or $d \geqslant 8$. We also characterise the minimising polytopes for $d = 4, 5$ or 7, where four sporadic examples arise.

Mathematics Subject Classifications: 52B05, 52B12

1 Background: excess, taxonomy and decomposability

This paper is concerned with graphs of polytopes with not too many vertices. Throughout, we will denote the number of vertices and edges of a polytope P by $v(P)$ and $e(P)$ respectively, or simply by v and e if P is clear from the context. The set of vertices is denoted as usual by $\mathrm{Vert}(P)$. Different letters will be used for the names of individual vertices. The dimension of the ambient space is denoted by d.

In 1967, Grünbaum [7, Sec. 10.2] made a conjecture concerning the minimum number of edges of d-polytopes with $v \leqslant 2d$ vertices, and confirmed it for $v \leqslant d + 4$. In [11], we

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confirmed it for $v \leqslant 2d$, and also characterised the minimising polytope, which is unique for each v (up to combinatorial equivalence). We also found the corresponding results for polytopes with $2d+1$ vertices. We extend this program here by calculating the minimum number of edges of polytopes with $2d+2$ vertices, also characterising the minimising polytopes. At the end, we make some remarks about the corresponding problem for polytopes with $2d+3$ vertices.

An important concept in resolving Grünbaum's conjecture was the *excess degree*. Recall that the *degree* of any vertex is the number of edges incident to it; this cannot be less than the dimension of the ambient polytope. We defined the *excess degree* of a vertex u as $\deg u - d$; thus a vertex is *simple* if its excess degree is zero. We then define the *excess* of a d-polytope P, denoted $\xi(P)$, as $\sum_{u \in \mathrm{Vert}(P)}(\deg u - d)$, i.e. the sum of the excess degrees of its vertices. Thus a polytope is *simple*, i.e. every vertex is simple, if $\xi(P) = 0$. A vertex is *non-simple* in a d-polytope P if its degree in P is at least $d+1$. A polytope with at least one non-simple vertex is called *non-simple*. It is easy to see that

$$\xi(P) = 2e(P) - dv(P). \tag{1}$$

A fundamental result about the excess degree is that it cannot take arbitrary values [10, Theorem 3.3].

Theorem 1. *Let P be a non-simple d-polytope. Then $\xi(P) \geqslant d-2$.*

Recall that the Minkowski sum of two polytopes Q, R is simply $Q + R = \{q + r : q \in Q, r \in R\}$. A *prism* based on a facet F is the Minkowski sum of F and a line segment, or any polytope combinatorially equivalent to it. The *simplicial d-prism* is any prism whose base is a $(d-1)$-simplex; we will often refer to these simply as prisms. Any d-dimensional simplicial prism has $2d$ vertices, d^2 edges, and $d+2$ facets. For $m, n > 0$, the polytope $\varDelta_{m,n}$ is defined as the Minkowski sum of an m-dimensional simplex and an n-dimensional simplex, lying in complementary subspaces. It is easy to see that it has dimension $m+n$, $(m+1)(n+1)$ vertices, $m+n+2$ facets, $\frac{1}{2}(m+n)(m+1)(n+1)$ edges, and is simple. For $n = 1$, $\varDelta_{d-1,1}$ is simply a prism. Being simple, all the polytopes just described have excess degree 0.

Remark 2 (Facets of $\varDelta_{m,n}$). The facets of the $(m+n)$-polytope $\varDelta_{m,n}$, $m+n+2$ in total, are:

- $m+1$ copies of $\varDelta_{m-1,n}$,
- $n+1$ copies of $\varDelta_{m,n-1}$.

In particular, $\varDelta_{m,n}$ contains no simplex facets at all if $m \geqslant 2$ or $n \geqslant 2$.

A *triplex* is defined as a multifold pyramid over a simplicial prism [11, p. 29]. More precisely a $(k, d-k)$-triplex, denoted $M(k, d-k)$ is a $(d-k)$-fold pyramid over the simplicial k-prism. Grünbaum [7, p 184] defined a quadratic polynomial by

$$\phi(v, d) = \binom{d+1}{2} + \binom{d}{2} - \binom{2d+1-v}{2} = \binom{v}{2} - 2\binom{v-d}{2},$$

and conjectured that its value is the minimum number of edges of d-polytopes with $v \leqslant 2d$ vertices.

Note also the equivalent expression for ϕ,

$$\phi(d+k, d) = \frac{1}{2}d(d+k) + \frac{1}{2}(k-1)(d-k).$$

The following result verifies Grünbaum's conjecture.

Theorem 3. *[11, Theorem 7] Let P be a d-polytope with $d+k$ vertices, where $1 \leqslant k \leqslant d$. Then P has at least $\phi(d+k, d) = \binom{d}{2} - \binom{k}{2} + kd$ edges, equivalently P has excess degree at least $(k-1)(d-k)$. Furthermore, equality is obtained only if P is a $(k, d-k)$-triplex, i.e a $(d-k)$-fold pyramid over the simplicial k-prism.*

A *missing edge* in a polytope is a pair of distinct vertices with no edge between them. Theorem 3 then says that a d-polytope with $v \leqslant 2d$ vertices has at most $2\binom{v-d}{2}$ missing edges, and that this maximum is attained only by the appropriate triplex.

For simplicial polytopes, the well known Lower Bound Theorem gives a stronger conclusion, without a restriction on the number of vertices.

Theorem 4. *[1] Let P be a d-polytope with $v = d+k$ vertices, and suppose that every facet of P is a simplex. Then P has at least $dv - \binom{d+1}{2}$ edges, equivalently P has excess degree at least $(k-1)d$.*

In describing a polytope, it is enough to know all the vertex-facet incidences; this determines the entire face lattice.

Let us recall the concept of truncation. If H is a hyperplane intersecting the interior of a polytope P and containing no vertex of P, denote by H^+ and H^- the two closed half-spaces bounded by H, and put $P' := H^+ \cap P$. In the case that there is a unique vertex of P lying in H^-, the polytope P' is said to have been obtained by *truncating* that vertex. Truncating a simple vertex of any polytope clearly yields a new polytope with $d-1$ more vertices than the original, but the same excess degree; thus the number of edges increases by $\binom{d}{2}$. In case there are exactly two vertices in H^-, they must be adjacent, and the polytope P' is said to have been obtained by *truncating* that edge. Truncating an edge joining two simple vertices will produce a polytope with $2d-4$ more vertices than before and the same excess degree.

We need to be familiar with some important examples. A *pentasm* (needed in case 3 of our main theorem) can be defined [10, p. 2015] as the result of truncating a simple vertex from the triplex $M(2, d-2)$. It has $2d+1$ vertices, d^2+d-1 edges, and hence excess degree $d-2$. Its facets are $d+3$ in number: $d-2$ pentasms of dimension $d-1$; two prisms; and three simplices. Another way to view the pentasm is as the convex hull of two disjoint faces: a pentagon and a $(d-2)$-dimensional prism.

Theorem 5. *[11, Thm. 13(iii)] Let P be a d-polytope with $2d+1$ vertices, where $d \geqslant 5$. Then P has at least d^2+d-1 edges, with equality only if P is a pentasm.*

Additional minimisers of the number of edges (of d-polytopes with $2d+1$ vertices) appear when $d = 3$ or 4; these will be discussed shortly.

Some further examples [11, §2.2] of polytopes with few vertices and edges, which occur repeatedly in our work, are as follows.

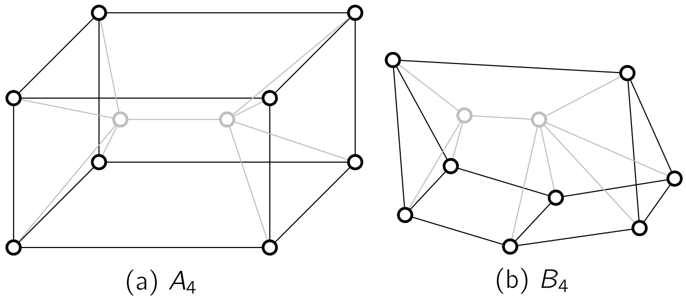

Figure 1: Polytopes A_4 and B_4.

Denote by A_d a polytope obtained by truncating a nonsimple vertex of the triplex $M(2, d-2)$. Since each nonsimple vertex of $M(2, d-2)$ is the apex of a pyramid, it makes no difference (up to combinatorial equivalence) which one we truncate. The polytope A_d can be also realised as a prism over a copy of $M(2, d-3)$. It has $2d+2$ vertices and excess degree $2d-6$ (Figure 1).

Remark 6 (Facets of A_d). The facets of the d-polytope A_d, $d+3$ in total, are as follows.

- $d-3$ copies of A_{d-1},
- 4 simplicial prisms, and
- 2 copies of $M(2, d-3)$.

Denote by B_d a polytope obtained by truncating a simple vertex of the triplex $M(3, d-3)$. The polytope B_3 is the well known *5-wedge*. The polytope B_d can also be visualised as the convex hull of B_3 and a simplicial $(d-3)$-prism K, with each vertex of one of the $(d-4)$-dimensional simplex faces in K being adjacent to each of the three vertices in a triangle of B_3, and each vertex of the other $(d-4)$–dimensional simplex face in K being adjacent to each of the remaining five vertices of B_3. It also has $2d+2$ vertices and excess degree $2d-6$ (Figure 1).

Remark 7 (Facets of B_d). The facets of the d-polytope B_d, $d+3$ in total, are as follows.

- $d-3$ copies of B_{d-1},
- 2 simplices,
- 1 simplicial prism,
- 1 copy of $M(2, d-3)$, and

- 2 pentasms.

Remark 8 (Similarity of A_d, B_d). There is a certain commonality in the structure of A_d and B_d. In both cases, the polytope can be described as the convex hull of three disjoint faces, namely two $(d-5)$-dimensional simplices S_1 and S_2 (whose convex hull constitutes a prism), and a simple 3-face (either a cube or a 5-wedge). The vertices of the 3-face can be partitioned into two subsets Q_1 and Q_2, in such a way that a vertex in S_i is adjacent to a vertex in Q_j if and only if $i = j$. In the case of the cube, Q_1 and Q_2 correspond to two opposite faces. In the case of the 5-wedge, Q_1 corresponds to a triangular face and Q_2 corresponds to the other triangular face, together with the quadrilateral with which it shares an edge.

We have presented the structure of A_d and B_d in some detail because, in most dimensions, these two examples are the minimisers of the number of edges, amongst all d-polytopes with $2d + 2$ vertices. As listed in Theorem 13 below, there are some exceptions in low dimensions, which we now describe.

Denote by C_d a polytope obtained by truncating such a *simple edge* in the triplex $M(2, d-2)$. It has $3d - 2$ vertices and excess degree $d - 2$. Obviously C_2 is just another quadrilateral.

Denote by Σ_d a certain polytope which is combinatorially equivalent to the convex hull of

$$\{0, e_1, e_1 + e_k, e_2, e_2 + e_k, e_1 + e_2, e_1 + e_2 + 2e_k : 3 \leqslant k \leqslant d\},$$

where $\{e_i\}$ is the standard basis of $\mathbb{R}^d$. It is easily shown to have $3d-2$ vertices; of these, one has excess degree $d-2$, and the rest are simple. It can be expressed as the Minkowski sum of two simplices. For consistency, we can also define Σ_2 as a quadrilateral. Note that $\Sigma_3 = C_3$, but the corresponding polytopes are distinct for $d \geqslant 4$.

Recall from [10, p. 2017] that J_d is the simple polytope obtained by truncating one vertex of a simplicial d-prism; it clearly has $3d-1$ vertices. Of course J_2 is just a pentagon and B_3 coincides with J_3. The facets of J_d are $d - 1$ copies of J_{d-1}, two prisms, and two simplices.

Now we present some technical results which will be necessary later.

It is well known that any simple d-polytope, other than a simplex or prism, has at least $3d - 3$ vertices. This follows easily from the g-theorem [18, §8.6], but elementary arguments are also available. More precisely, we have the following classification, which is a rewording of [15, Lemma 10(ii)].

Lemma 9. *Any simple d-polytope with strictly less than $3d-1$ vertices is either a simplex, a prism, $\Delta_{2,d-2}$ or $\Delta_{3,3}$. In particular, for every $d \neq 6$, the smallest vertex counts of simple d-polytopes are $d + 1, 2d, 3d - 3$ and $3d - 1$. In dimension 6 only, there is also a simple polytope with $3d - 2$ vertices. The only one of these which contains two disjoint simplex facets is the prism.*

The next result is surprisingly useful.

Lemma 10. *[10, Lemma 2.5] Let P be a polytope, F a facet of P and u a nonsimple vertex of P which is contained in F. If u is adjacent to a simple vertex x of P in $P \setminus F$, then u must be adjacent to a second vertex in $P \setminus F$, different from x.*

A very useful tool for us is the following concept: a polytope P is called *decomposable* if it can be expressed as the (Minkowski) sum of two polytopes which are not similar to it, i.e. not obtainable from P just by translation and scaling. Inevitably, all other polytopes are described as *indecomposable.* We refer to [14] and the references therein for a more detailed discussion of this topic. Kallay [8] showed that decomposability of a polytope can often be decided from properties of its graph. He introduced the concept of a *geometric graph*, as any graph G whose vertex set V is a subset of $\mathbb{R}^d$ and in which every edge is a line segment joining members of V; we find it convenient to add the restriction that no three vertices are collinear. Such a graph need not be the edge graph of any polytope. He then extended the notion of decomposability to geometric graphs in a consistent manner. We omit his definition; the important point [8, Theorem 1] is that a polytope is indecomposable as just defined if and only if its edge graph is indecomposable in his sense.

A strategy for proving indecomposability of a polytope is to prove that certain basic geometric graphs are indecomposable, and then to build up from them to deduce that the entire skeleton of our polytope is indecomposable. As in [15, p. 171], let us say that a geometric graph $G = (V, E)$ is a simple extension of a geometric graph $G_0 = (V_0, E_0)$ if G_0 is a subgraph of G, $V \setminus V_0$ contains just one vertex, and $E \setminus E_0$ comprises two (or more) edges containing that vertex. The following result summarizes everything we need in the sequel; we have not included stronger known statements about decomposability.

Theorem 11.

(i) *If G is a simple extension of G_0, and G_0 is an indecomposable geometric graph, then G is also indecomposable.*

(ii) *A single edge is indecomposable; any triangle is indecomposable.*

(iii) *A geometric graph isomorphic to the complete bipartite graph $K_{2,3}$ is decomposable if and only if it lies in a plane.*

(iv) *A polytope P is indecomposable, if (and only if) its graph contains an indecomposable subgraph G whose vertex set contains at least one vertex from every facet of P.*

(v) *If P is a pyramid, then it is indecomposable.*

(vi) *Any d-polytope with $2d$ or fewer vertices, other than the prism, is indecomposable.*

Proof. (indication)

(i) See [15, Prop. 1].

(ii) This is easy for an edge; the case of a triangle then follows from (i).

(iii) This is proved without statement in [14, Example 12].

(iv) This is [14, Theorem 8], which is an extension of [8, Theorem 1].

(v) Consider any edge containing the apex of the pyramid. This is an indecomposable subgraph which touches every facet, and the conclusion follows from (iv).

(vi) See [15, Theorem 9]. □

In the other direction, the following sufficient condition will be useful to us several times. It is essentially due to Shephard; for another proof, see [15, Prop. 5]. We say [11, p. 30] that a facet F of a polytope P has *Shephard's property* if for every vertex $u \in F$, there exists exactly one edge in P that is incident to u and does not lie in F. We also say that a polytope is a *Shephard polytope* if it has at least one facet with Shephard's property.

Theorem 12 ([16, Result (15)])**.** *If a polytope P has a facet F with Shephard's property, and there are at least two vertices outside F, then P is decomposable. In particular, any simple polytope other than a simplex is decomposable.*

2 Polytopes with $2d + 2$ vertices

As in [11], we define the set $E(v, d) = \{e$: there is a d-polytope with v vertices and e edges$\}$. Theorem 3 asserts, for each fixed $k \leqslant d$, that $\min E(d+k, d) = \frac{1}{2}d(d+k) + \frac{1}{2}(k-1)(d-k)$, and that the triplex $M(k, d-k)$ is the unique minimiser. So $\min E(v, d)$ is known, whenever $v \leqslant 2d$.

Theorem 5 asserts that $\min E(2d+1, d) = d^2 + d - 1$ for $d \geqslant 5$, and that the pentasm is the unique minimiser. For low dimensions, some sporadic examples occur. For $d = 3$, it is easy to check that there is a second minimiser, namely Σ_3. For $d = 4$, the pentasm is the only polytope with nine vertices and 19 edges, but $\Delta_{2,2}$ has nine vertices and 18 edges.

Similarly, we will show here that $\min E(2d+2, d) = d^2 + 2d - 3$ for all $d \geqslant 3$ except $d = 5$. (It is well known that $\min E(12, 5) = 30 < 32 = 5^2 + 2 \times 5 - 3$, and that $\Delta_{2,3}$ is the only 5-polytope with 12 vertices and 30 edges. It also follows from the Excess Theorem (Theorem 1) that no 5-polytope has 12 vertices and 31 edges.) Furthermore, we show that, for all $d \geqslant 3$ except $d = 4$ and $d = 7$, the only polytopes with $2d + 2$ vertices and $d^2 + 2d - 3$ edges are the polytopes A_d and B_d defined above.

Theorem 13. *For $d \geqslant 3$, the only d-polytopes with $2d+2$ vertices and precisely d^2+2d-3 edges, equivalently with excess degree $2d - 6$, are as follows.*

(i) *For $d = 3$, $d = 5$, $d = 6$ and all $d \geqslant 8$, only the two polytopes A_d and B_d.*

(ii) *For $d = 4$, the four polytopes A_4, B_4, C_4 and Σ_4.*

(iii) *For $d = 7$, the three polytopes A_7, B_7 and the pyramid over $\Delta_{2,4}$.*

Moreover, the 5-polytope $\Delta_{2,3}$ is the only polytope of any dimension with $2d + 2$ vertices and strictly fewer than $d^2 + 2d - 3$ edges.

The case $d = 3$ of Theorem 13 is easy to check using Steinitz' Theorem; one may also consult catalogues [2, 5]. The case $d = 4$ was established in [10, Theorem 6.1]. Some arguments in the sequel are simplified by considering only the case $d \geqslant 5$. We establish several special cases first in order to streamline the proof. Some of them are of independent interest.

Lemma 14. *Let P be a d-polytope with $2d + 2$ vertices and no more than $d^2 + 2d - 3$ edges. If P is a pyramid, then $d = 7$ and the base of P is $\Delta_{2,4}$.*

Proof. Our hypothesis amounts to saying that P has excess degree at most $2d - 6$. Let F denote the base, which has $2d + 1$ vertices. The apex of the pyramid has excess degree $d + 1$, and so F has excess degree at most $(2d - 6) - (d + 1)$. Since $d - 7 < d - 2$, the Excess Theorem informs us that F is simple and $d = 7$. By Lemma 9, the only simple 6-polytope with $15 = 3 \times 6 - 3$ vertices is $\Delta_{2,4}$. □

A fundamental property of polytopes is that for any facet F and any ridge R contained in F, there is a unique facet F' containing R and different from F. In this situation, we have $R = F \cap F'$, and we will F' the **other facet** for R.

Lemma 15. *[3, Theorem 15.5] Suppose P is a d-polytope and F is a proper face of P. Then the subgraph of the graph of P induced by the vertices outside F is connected.*

Lemma 16. *Suppose P is a d-polytope, F is a facet of P, and there are precisely three vertices, say u_1, u_2, u_3, outside F, all of them simple. Then, at least $d-4$ ridges contained in F have the property that their other facet contains all three of u_1, u_2, u_3.*

Proof. By Lemma 15, the subgraph containing u_1, u_2, u_3 is connected. There are two cases to consider. Either the three vertices are mutually adjacent, and each is adjacent to exactly $d - 2$ vertices in F. Or (after relabelling) u_1 is not adjacent to u_3, in which case u_1 and u_3 are both adjacent to u_2 and to $d - 1$ vertices in F, while u_2 is adjacent to $d - 2$ vertices in F.

In the first case, each of the three vertices will have degree $d - 1$ in any facet which contains it; such a facet must therefore contain one of the other two. So no ridge in F has the property that its other facet contains precisely one of u_1, u_2, u_3.

Again by the simplicity of u_1, there is only one facet F' of P which contains u_1 and u_2 but not u_3. Thus there is at most one ridge in F whose other facet is F'. Hence there are at most three ridges in F having the property that their other facet contains precisely two of u_1, u_2, u_3.

Since F contains at least d ridges of P, at least $d - 3$ of them must have the alleged property.

In the second case, the same reasoning shows that every facet containing u_2 also contains either u_1 or u_3; there is precisely one facet containing u_1 and u_2 but not u_3, and precisely one facet containing u_2 and u_3 but not u_1. There will also be precisely one facet containing u_1 but not u_2 or u_3, and precisely one facet containing u_3 but not u_1 or u_2. No facet can contain u_1 and u_3 but not u_2. Hence there are at most four ridges in F having the property that their other facet contains either one or two of u_1, u_2, u_3.

Since F contains at least d ridges of P, at least $d-4$ of them must have the alleged property. $\square$

A common situation for us will be the need to estimate the number of edges involving a particular set of vertices (often, but not always, the complement of a given facet).

Lemma 17. *[11, Lemma 4] Let S be a set of n vertices of a d-polytope P, with $n \leqslant d$. Then the total number of edges containing at least one vertex in S is at least $nd - \binom{n}{2}$. Moreover, this minimum is obtained precisely when every vertex in S is simple, and every two vertices in S are adjacent.*

We need some information about the structure of d-polytopes with $2d$ vertices and whose number of edges is close to minimal. It is known that such a polytope has d^2 edges only if it is a prism, and d^2+1 edges only if $d=3$ [15, Theorem 13].

Lemma 18. *Let P be a d-polytope with $2d$ vertices and d^2+2 edges. Then P is one of only seven examples, all with dimension at most five. More precisely*

(i) *If $d=5$, P is a pyramid over $\Delta_{2,2}$.*

(ii) *If $d=4$, P has at least two nonsimple vertices, and is a pyramid over either a pentasm or Σ_3, or one of the two polytopes detailed in the table below.*

(iii) *For $d=3$, P is the dual of either a pentasm or Σ_3.*

Proof. A special case of [11, Thm. 19] asserts that a d-polytope with $2d$ vertices which is not a prism must have at least d^2+d-3 edges, and this is $\geqslant d^2+3$ if $d \geqslant 6$. Thus $d<6$.

(i) If $d=5$, then P has 27 edges and excess degree $4=d-1$, so [10, Theorem 4.18] informs us that P is a Shephard polytope, in particular either decomposable or a pyramid. But P is not a prism, so must be indecomposable by Theorem 11(vi). Thus P is a pyramid, and its base must have nine vertices and 18 edges, making $\Delta_{2,2}$ the only option.

(ii) In the case $d=4$, P has eight vertices, 18 edges and excess degree four. Any vertex of P has degree at most seven and hence excess degree at most three. So there must be at least two nonsimple vertices. It is not hard to establish directly that there are only four examples, but we simply note that this can be verified from catalogues such as [6]. Let us now describe these four examples. Two obvious examples are the pyramid over a pentasm and the pyramid over Σ_3, which both have just two nonsimple vertices.

Another known example, which has three nonsimple vertices, is given in [12, Figure 1d]. Its facets are one prism, two tetragonal antiwedges, one pyramid and three simplices; the vertex-facet relations are detailed in the first column in Table 1. For a concrete representation, take the convex hull of $(\varepsilon,0,0,0)$, $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, $(1,0,0,1)$, $(0,1,0,1)$, and $(0,0,1,1)$, where $\varepsilon>0$ need not be too small.

The fourth example has a simple concrete representation, with vertices $(\pm1,\pm1,0,0)$, $(\pm1,0,1,0)$, and $(0,\pm1,0,1)$. It is not hard to verify that its facets are two prisms,

Facet	Polytope 1	Polytope 2
1:	{1 2 3 4 5 6}	{1 2 3 4 5 6}
2:	{1 2 3 4 7 8}	{1 2 3 4 7 8}
3:	{1 2 5 6 7 8}	{1 2 5 6 7}
4:	{3 4 5 6 7}	{1 3 5 7 8}
5:	{2 4 6 8}	{ 3 4 5 6 8}
6:	{1 3 5 7}	{2 4 6 7 8}
7:	{4 6 7 8}	{5 6 7 8}

Table 1: Vertex-facet incidences of nonpyramidal 4-polytopes with eight vertices and eighteen edges.

four quadrilateral pyramids and a simplex. There are four nonsimple vertices, and the vertex-facet relations are detailed in the second column in Table 1.

(iii) Suppose $d = 3$. Then P has six vertices and 11 edges, so Euler's formula ensures that its dual P^* must have seven vertices and 11 edges. Thus P^* is either a pentasm or Σ_3. $\square$

Lemma 19. *Let P be a d-polytope with $2d + 2$ vertices and no more than $d^2 + 2d - 3$ edges. Suppose that no facet of P has $2d$ vertices and no facet of P is a $(d-1)$-pentasm, but that some facet has $2d - 1$ vertices. Then either $d = 7$ and P is a pyramid over $\Delta_{2,4}$, or $d = 4$ and P is C_4 or Σ_4.*

Proof. In three dimensions, a facet with five vertices is a pentasm, so no polytope satisfies the hypotheses. If $d = 4$, we simply recall [10, Theorem 6.1], which asserts that the only 4-polytopes with 10 vertices and no more than 21 edges are A_4, B_4, C_4, and Σ_4. Note that both A_4 and B_4 contain facets with eight vertices, and do not satisfy the hypotheses.

So assume $d \geqslant 5$. We will show that P is a pyramid. If F is a facet having $2d - 1 = 2(d-1)+1$ vertices, Theorem 5 ensures that F has at least $d^2 - d$ edges, and consequently is not simple. Moreover, the three vertices u_1, u_2, u_3 outside F must be mutually adjacent and all simple, and so F actually has exactly $d^2 - d$ edges; otherwise by Lemma 17, P would have strictly more than $d^2 + 2d - 3$ edges.

Lemma 16 ensures that some ridge R contained in F is such that its other facet F' contains all three of u_1, u_2, u_3. Each u_i is simple in F', and so has $d-3$ edges running into R, two edges running into the other u_j and exactly one edge running into $P \setminus F'$. But each vertex in $F \setminus R$ is adjacent to at least one u_i; this implies that $P \setminus F'$ contains at most three vertices. If P is a pyramid over F', Lemma 14 completes the proof. Otherwise, F' has $2d-1$ vertices and $P \setminus F'$ contains exactly three vertices. Then there are exactly three vertices in $F \setminus R$, say w_1, w_2, w_3, each of them simple, and three edges joining them to u_1, u_2, u_3. Then R has $2d-4$ vertices, and there are $3(d-3)$ edges between u_1, u_2, u_3 and R, likewise $3d-9$ edges between w_1, w_2, w_3 and R, and nine edges between $u_1, u_2, u_3, w_1, w_2, w_3$. Thus the number of edges in R is exactly $(d-2)^2 + 2 = \phi(2d-4, d-2) + 2$. According to Lemma 18, this implies that $d - 2 < 6$.

Since every vertex in $F \setminus R$ is simple, Lemma 10 ensures that every vertex in R which is not simple in F has at least two neighbours in $F \setminus R$. This implies that the number of nonsimple vertices in R is at most $(3d-9)-(2d-4)=d-5$. Since F is not simple, R must contain a nonsimple vertex, whence $d \geqslant 6$. But if $d=6$, then R is 4-dimensional with eight vertices, 18 edges, and a unique nonsimple vertex, which is impossible by Lemma 18(ii).

The only remaining possibility is that $d=7$. By Lemma 18(i), the only 5-polytope with 10 vertices and 27 edges is the pyramid over $\Delta_{2,2}$; this must be R. The apex of R will then be the only nonsimple vertex in P. With excess degree eight, it must be adjacent to every other vertex in P. A special case of [10, Corollary 2.2] asserts that a polytope with a unique nonsimple vertex, which is adjacent to every other vertex, must be a pyramid. Again, the base can be only $\Delta_{2,4}$. □

Lemma 20. *Let P be a d-polytope, with two disjoint faces F_1 and F_2 whose union contains* Vert(P)*. Suppose that F_1 is a facet, and that $F_2=[w_0,w_1]$ is an edge. Then P is decomposable if, and only if, F_1 has Shephard's property in P. In this case, denoting $V_i=\{u \in \mathrm{Vert}(F_1) : u \text{ is adjacent to } w_i\}$, every vertex in V_0 is adjacent to at most one vertex in V_1 and vice versa. Moreover, F_1 is also decomposable.*

Proof. Any facet disjoint from F_2 must be contained in, and hence equal to, F_1.

If F_1 fails Shephard's property, there will be a vertex $u \in F_1$ adjacent to both vertices in F_2. The resulting triangle will be an indecomposable graph touching every facet, which implies indecomposability of P by Theorem 11(iv).

If F_1 has Shephard's property, then P is decomposable, according to Theorem thm:shp. Now suppose there is a vertex $a \in V_0$ which is adjacent to two distinct vertices $b,c \in V_1$. Then the five vertices a,b,c,w_0,w_1 are not contained in any plane. The graph G comprising these six edges is isomorphic to the complete bipartite graph $K_{3,2}$. According to Theorem 11(iii), G is indecomposable. By the remark at the beginning of this proof, G touches every facet, contradicting the decomposability of P. Decomposability of F_1 now follows from [10, Lemma 5.5], but we repeat the short argument: the graph of F_1 also touches every facet, so if F_1 were indecomposable, Theorem 11(iv) would imply indecomposability of P. □

We remark that in Lemma 20, the edge F_2 will actually be a summand of a polytope combinatorially equivalent to F_1. This can be deduced from the proof of [15, Proposition 5]. Moreover, a generalisation of this result remains valid when F_2 is merely assumed to be an indecomposable face. We don't need these stronger versions, so we omit the details.

The next result improves [10, Lemma 5.6(ii) and Remark 5.7].

Corollary 21. *Let F be a facet of a d-polytope P, with only two vertices u_0,u_1 of P being outside F. Suppose F is either C_4, Σ_4, a pyramid over $\Delta_{2,4}$, or $\Delta_{n,m}$ with $m \geqslant n \geqslant 2$. Then F fails Shephard's property in P, and P is indecomposable. In case F is $\Delta_{n,m}$, there are at least $2(m+1)n$ edges running out of F.*

Proof. The two vertices outside F must be adjacent, and so constitute an edge. By Lemma 20, failure of Shephard's property for F is equivalent to indecomposability of P.

For the case of a pyramid over $\Delta_{2,4}$, F is indecomposable by Theorem 11(v), and then P is indecomposable by Theorem 11(iv).

Suppose next that F is C_4 or Σ_4. Inspection of the graphs (Figure 2) reveals that in both cases there are four triangles T_1, T_2, T_3, T_4, with $T_i \cap T_{i+1}$ nonempty for $i = 1, 2, 3$, whose union contains at least seven vertices. Each of the three (or fewer) remaining vertices must then be adjacent to at least two vertices in this collection of triangles. If F had Shephard's property, then Lemma 20 would allow us to colour the vertices of F with two colours in such a way that every vertex is adjacent to at most one vertex of the other color. In particular, any three mutually adjacent vertices would have the same colour. In our situation, all vertices of F would have the same colour, i.e. no such 2-colouring is possible.

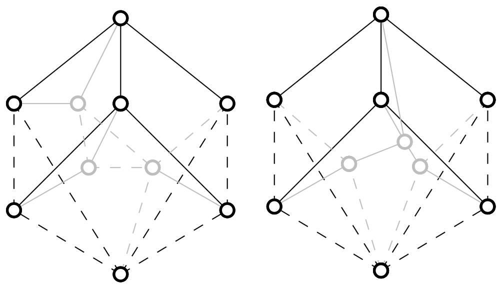

Figure 2: Polytopes C_4 and Σ_4. In each polytope, four triangles T_1, T_2, T_3, T_4 with $T_i \cap T_{i+1}$ nonempty for $i = 1, 2, 3$ are highlighted in dashed lines.

Finally, suppose $F = \Delta_{m,n}$; then $m+n = d-1$. We claim that there are least $2(m+1)n$ edges between F and u_0, u_1. Fix a facet F' of P containing u_0 but not u_1. Let R be an arbitrary ridge contained in F' but not containing u_0. Clearly $R \subset F$ and $R \subset F'$, which forces $R = F \cap F'$. Thus R is the unique ridge of F' not containing u_0. This implies F' is a pyramid over R and u_0 is adjacent to every vertex in R. Being contained in F, R must be either of the form $\Delta_{m-1,n}$, with $mn + m$ vertices, or of the form $\Delta_{m,n-1}$, with $mn + n$ vertices (Remark 2). So u_0 has at least $(m + 1)n$ edges running into F. Likewise for u_1. Since $n > 1$, we have $2mn + 2n > (m + 1)(n + 1) = v(F)$, meaning that F fails Shephard's property. □

We now have enough machinery to prove Theorem 13. We reformulate it slightly to streamline the proof.

Theorem 22. *For $d \geqslant 3$, the only d-polytopes with $2d + 2$ vertices and $d^2 + 2d - 3$ or fewer edges are A_d, B_d, C_4, Σ_4, $\Delta_{2,3}$, and the pyramid over $\Delta_{2,4}$. Each of these examples, except $\Delta_{2,3}$, has precisely $d^2 + 2d - 3$ edges.*

Proof. The case $d = 3$ is well known and easy to prove: A_3 is the cube and B_3 is the 5-wedge. See also [2] or [5]. The case $d = 4$ was established in [10, §6]. We proceed by induction on d, henceforth assuming $d \geqslant 5$.

Let P be a d-polytope with $2d+2$ vertices, by hypothesis with excess degree at most $2d-6$. We distinguish a number of cases based on the maximum number of vertices of the facets in P. We will show in each case that P either has one of the required forms or has strictly more than d^2+2d-3 edges.

Case 1: Some facet has $\mathbf{2d+1}$ vertices. In this case, P is a pyramid and Lemma 14 informs us that the base is $\varDelta_{2,4}$.

Case 2: $\boldsymbol{P}$ is not a pyramid, but some facet $\boldsymbol{F}$ has $\mathbf{2d = 2(d-1)+2}$ vertices. There are two adjacent vertices outside F, which we will call u_1 and u_2, and at least $2d$ edges running out of F. Thus F has at most $d^2+2d-3-(2d+1)=(d-1)^2+2(d-1)-3$ edges. By induction, F must be A_{d-1}, B_{d-1}, $\varDelta_{2,3}$, C_4, $\varSigma_4$ or a pyramid over $\varDelta_{2,4}$. If F were $\varDelta_{2,3}$, Corollary 21 would ensure at least 16 edges running out of F, giving P at least $47 > 6^2+2\times 6-3$ edges. In all the other cases, F has exactly $(d-1)^2+2(d-1)-3$ edges, and so there are exactly $2d$ edges running out of F. Thus F has Shephard's property, and Corollary 21 rules out C_4, $\varSigma_4$ and the pyramid over $\varDelta_{2,4}$ as options.

So F is of the form A_{d-1} or B_{d-1}: Recall their structure from Remarks 6 and 7. In both cases, there are $d+2$ ridges (of P) contained in F and $d-4$ of them have same form, i.e. either A_{d-2} or B_{d-2} respectively. If the other facet F' for one of these $d-4$ ridges were a pyramid, its apex, say u_1, would be adjacent to all $2d-2$ vertices in the ridge, giving $d^2-4+2d-2$ edges in the union of the two facets. But u_2 has degree at least d, which would give P at least $d^2+3d-6 > d^2+2d-3$ edges. So each such "other facet" must contain both u_1 and u_2, i.e. F' has $2(d-1)+2$ vertices. It follows that F' also has at most $(d^2+2d-3)-(2d+1)$ edges. The induction hypothesis tells us that F' is also of the form A_{d-1} or B_{d-1}, respectively.

Let S_i and Q_i be as in Remark 8. The $d-4$ ridges referred to above each omit one of the $d-4$ edges linking S_1 and S_2. Given that their other facets have the same form as F (A_{d-1} or B_{d-1} respectively), we can suppose that u_1 is adjacent to every vertex in S_1 and Q_1, and that u_2 is adjacent to every vertex in S_2 and Q_2. Thus P has the same graph as A_d or B_d. We claim this ensures that P is A_d or B_d respectively.

If F is A_{d-1}, the six ridges of P contained in F which are not of the form A_{d-2} correspond to the six faces of the cube, i.e. each is the convex hull of one face of the cube and the $(d-3)$-prism. They are two copies of $M(2,d-4)$ (the convex hull of $Q_i\cup S_i$ for $i=1,2$); and four $(d-2)$-prisms (the convex hull of $S_1\cup S_2\cup E$, for each of the four edges E linking Q_1 and Q_2). The other facets for these ridges are now easy to see. The other facet corresponding to each copy of $M(2,d-4)$ is a copy of $M(2,d-3)$, namely the convex hull Q_i, S_i and u_i. For each prism facet, the other facet is a $(d-2)$-prism, containing both u_1 and u_2. This completely describes the facet-vertex incidences of P; they are the same as A_d.

Likewise, if F is B_{d-1}, the other six ridges of P contained in F correspond to the six faces of the 5-wedge. They are one copy of $M(2,d-4)$, one $(d-2)$-prism, two simplices and two pentasms. A similar argument investigating their other facets determines the facet-vertex incidences. (Alternatively, since P has only $d-3$ nonsimple vertices, we could apply [4, Theorem 3.1], which asserts that the face lattice in this case is determined by the 2-skeleton.)

Case 3A: No facet has $2d$ or more vertices, and some facet F is a $(d-1)$-pentasm.

We show that this case cannot arise. Recall that a $(d-1)$-pentasm has $2d-1 = 2(d-1)+1$ vertices and $d^2-d-1 = (d-1)^2+(d-1)-1$ edges.

One of the $(d-2)$-faces of F is a $(d-2)$-pentasm R, which has $(d-2)^2+(d-2)-1$ edges; there will then be $2d-2$ edges incident with vertices in $F \setminus R$, and one of the vertices in $F \setminus R$ will be nonsimple. We consider the other facet F' of P containing R. As a pentasm, R has excess degree $d-4$. If F' were a pyramid, its apex would have degree $2d-3$, and hence excess $d-2$, in F'. This would give F' excess degree $2d-6$. Now the excess degree of any facet of a nonsimple polytope is strictly less than the excess degree of the whole polytope [10, Lemma 3.2]; thus P would have excess degree $> 2d-6$, contrary to hypothesis. So we can assume that F' is not a pyramid.

The hypotheses of this case ensure that the facet F' contains only two of the three vertices t, u, w outside F, say t, u. Furthermore, the vertices u, t must be adjacent. Then by Theorem 5, $F \cup F'$ contains at least $(d^2-d-1)+(2d-3)+1 = d^2+d-3$ edges; note that this total does not include any edges between $F \setminus R$ and $F' \setminus R$. If w, the unique vertex outside $F \cup F'$, is not simple, then P will have at least $(d^2+d-3)+(d+1) = d^2+2d-2$ edges, contrary to hypothesis. So we assume that w is simple.

Clearly the nonsimple vertex in $F \setminus R$ is adjacent to some vertex outside F; if it is not adjacent to w, then it must be adjacent to u or t. But if it is adjacent to w, it must, by Lemma 10, be adjacent to another vertex outside F, which can only be one of t, u. In either case, this gives us an edge between $F \setminus R$ and $F' \setminus R$. Of course there are d edges containing w, so again P has at least $(d^2+d-3)+d+1 = d^2+2d-2$ edges.

Case 3B: No facet has $2d$ or more vertices, no facet is a pentasm, but some facet F has $2d-1 = 2(d-1)+1$ vertices.

This case is settled by Lemma 19.

Case 4A. No facet has $2d-1$ or more vertices, some facet F has $2d-2 = 2(d-1)$ vertices, but is not a prism.

We show that this case cannot arise; our argument depends on the dimension. First note that F has at least $(d-1)^2+(d-1)-3$ edges [11, Theorem 19], and by Lemma 17 the four vertices outside F belong to at least $4d-6$ edges. Thus P has at least d^2+3d-9 edges, and for $d > 6$, this exceeds d^2+2d-3.

If $d = 6$, and F has 28 edges or more, then P has at least 46 edges by Lemma 17. Otherwise, F has $27 = 5^2+2$ edges and must, by Lemma 18(i), be a pyramid over $\Delta_{2,2}$. The other facet containing the ridge $\Delta_{2,2}$ has only 10 vertices, so must also be a pyramid. Then these two facets have 36 edges between them, and the three remaining vertices must belong to at least 15 edges. This gives P at least 51 edges.

Finally if $d = 5$, then F will have eight vertices and hence more than sixteen edges. It cannot have seventeen edges, thanks to [7, Thm. 10.4.2]; see also [12, remarks after Prop. 2.7]. So F has at least eighteen edges, and excess degree at least four. The excess degree of P is strictly greater than that of F [10, Lemma 3.2], and even, hence at least $6 > 2d-6$, contrary to hypothesis.

Case 4B. No facet has $2d-1$ or more vertices, some facet F has $2d-2 =$

$2(d-1)$ vertices, and every such facet is a prism.

We show that this case does not arise either. Suppose F is a simplicial $(d-1)$-prism. We can label its vertices as $\{u_1, \cdots, u_{d-1}, w_1, \cdots, w_{d-1}\}$, so that u_i is adjacent to u_j and w_i is adjacent to w_j for all i, j, but u_i is adjacent to w_j if and only if $i = j$. The graph and face lattice are clear from this. Then the ridge R contained in F with vertices $\{u_1, \cdots, u_{d-2}, w_1, \cdots, w_{d-2}\}$ is a simplicial $(d-2)$-prism. Consider the other facet F' containing R. If F' were a pyramid, then $F \cup F'$ would contain $(d-1)^2 + 2(d-2) = d^2 - 3$ edges, and by Lemma 17 the remaining three vertices would belong to at least $3d-3$ edges. But then P would have at least $d^2 + 3d - 6 > d^2 + 2d - 3$ edges. Then F' must be another prism. We can label the two vertices of $F' \setminus R$ as u_0 and w_0, with the adjacency relationships clear from the notation. Now the graph of $F \cup F'$ has $d^2 - 2$ edges, and by Lemma 17 again, there are at least $2d - 1$ edges involving the other two vertices, say a, b. Since $d^2 - 2 + 2d - 1 = d^2 + 2d - 3$, there must be precisely $2d - 1$ edges involving a and b, and no edge between u_0 and u_{d-1}, w_{d-1}, nor between w_0 and u_{d-1}, w_{d-1}. In particular, $u_0, w_0, u_{d-1}, w_{d-1}$ each have degree $d-1$ in $F \cup F'$, so each of them must be connected to either a or b. On the other hand, there are at most $2d-1$ vertices in $F \cup F'$ adjacent to either a or b. Without loss of generality, we suppose that u_{d-3} is adjacent neither to a nor to b.

Now consider the ridge R' in F' with vertices $\{u_0, \cdots, u_{d-3}, w_0, \cdots, w_{d-3}\}$. The other facet F'' containing R' must be a prism. It cannot contain u_{d-2} or w_{d-2} because they belong to F'. Nor can it contain u_{d-1} or w_{d-1}, because they are not adjacent to u_0 and w_0. And it cannot contain a or b, because they are not adjacent to u_{d-3}. This is absurd.

Case 5. Some facet F has between $d+3 = d-1+4$ and $2d-3 = d-1+d-2$ vertices, but no facet has $2d-2$ or more vertices.

Denote by n the number of vertices outside F; then $5 \leqslant n \leqslant d-1$. Then the facet F has $(d-1) + (d-n+3)$ vertices and $d-n+3 < d-1$, so [10, Theorem 8(iii)] ensures that F has excess at least $(d-n+2)(n-3-1)$. By Theorem 1, the total number of edges in F is at least $\frac{1}{2}((d-1)(2d+2-n) + (d-n+2)(n-4))$, with equality only if F is a triplex. Lemma 17 informs us that the total number of edges outside F is at least $\frac{1}{2}(2nd - n^2 + n)$. Thus the number of edges in P is at least $d^2 + (n-2)d - (n^2 - 4n + 5)$, which is $> d^2 + 2d - 3$ provided

$$d > n + \frac{2}{n-4}.$$

If $n \geqslant 7$, then $n + \frac{2}{n-4} < n + 1 \leqslant d$, so we are fine. If $n = 6$, we need to consider the case $d = 7$ separately. If $n = 5$, we need to consider the cases $d = 6, 7$ separately.

Note that $d - n + 3 = 4$ or 5 in all three cases. If F is not a triplex, then [11, Theorems 19 and 20] guarantee F has at least $\phi(2d+2-n, d-1) + 2$ edges. So the total number of edges in P is at least $d^2 + (n-2)d - (n^2 - 4n) - 3 > d^2 + 2d - 3$.

If F is a triplex, it is a pyramid over some ridge R which is also a triplex. No facet has more vertices than F, so the other facet F' must also be a triplex. Now R has $(d-2) + (d-n+3)$ vertices, and hence $\phi(2d-n+1, d-2) = d^2 - 4d - \frac{1}{2}(n^2 - 9n + 12)$ edges. The two apices of F, F' belong to $2(2d-n+3)$ edges in $F \cup F'$. The $n-1$ vertices outside $F \cup F'$ must belong to at least $(n-1)d - \binom{n-1}{2}$ edges. Adding these up gives a total of at least $d^2 + (n-1)d - (n^2 - 4n + 1)$ edges, and this exceeds $d^2 + 2d - 3$ in each case of interest, namely $(n, d) = (5, 6)$, $(5, 7)$ and $(6, 7)$.

Case 6. Some facet $\boldsymbol{F}$ has $\boldsymbol{d+2 = d-1+3}$ vertices, and no facet has more vertices.

First suppose F is a pyramid. Then the base is a ridge R with exactly $d+1 = d-2+3$ vertices, and so has at most six missing edges. The other facet F' containing R must also be a pyramid over R, and so the union of the two facets $F \cup F'$ has at most seven missing edges. The union has at least $\binom{d+3}{2} - 7 = \frac{1}{2}(d^2 + 5d) - 4$ edges. The $d-1$ vertices outside $F \cup F'$ belong to at least $(d-1)d - \binom{d-1}{2} = \frac{1}{2}(d^2 + d) - 1$ edges. Thus P has at least $d^2 + 3d - 5 > d^2 + 2d - 3$ edges.

Next suppose that F is simplicial. The Lower Bound Theorem ensures that F has at least $\binom{d+2}{2} - 3 = \frac{1}{2}(d^2 + 3d) - 2$ edges. The d vertices outside F belong to at least $d^2 - \binom{d}{2} = \frac{1}{2}(d^2 + d)$ edges. Thus P has at least $d^2 + 2d - 2$ edges.

If F is neither simplicial nor a pyramid, then some ridge R in F has exactly d vertices. Let F' be the other facet containing R. We need to distinguish two cases, depending whether F' has $d+1$ or $d+2$ vertices. In either case, F has at least $\phi(d-1+3, d-1) = \frac{1}{2}(d^2 + 3d) - 5$ edges.

If F' has $d+1$ vertices, it is a pyramid over R, and its apex belongs to d edges in F'. Again by Lemma 17, the $d-1$ vertices outside $F \cup F'$ belong to at least $d(d-1) - \binom{d-1}{2} = \frac{1}{2}(d^2 + d) - 1$ edges. Thus P has at least $d^2 + 3d - 6 > d^2 + 2d - 3$ edges.

If F' has $d+2$ vertices, there are two vertices in $F' \setminus F$, which must belong to at least $2d - 3$ edges in F'. Moreover, the $d-2$ vertices outside $F \cup F'$ belong to at least $d(d-2) - \binom{d-2}{2} = \frac{1}{2}(d^2 + d) - 3$ edges. Thus P has at least $d^2 + 4d - 11 > d^2 + 2d - 3$ edges.

Case 7. Some facet $\boldsymbol{F}$ has $\boldsymbol{d+1 = (d-1)+2}$ vertices, and no facet has more vertices.

First consider the case when F is simplicial; then it has at most one missing edge. Let R be any ridge in F, and let G be the other facet facet containing R. Of course R is a simplex, while F' may have either d or $d+1$ vertices.

If F' has just d vertices, then it is also a pyramid over R, whose apex is adjacent to every vertex in R but possibly not adjacent to the two vertices in $F \setminus R$. With at most three missing edges, $F \cup F'$ has at least $\binom{d+2}{2} - 3$ edges and Lemma 17 ensures that the d vertices outside $F \cup F'$ belong to at least $d^2 - \binom{d}{2}$ edges. This gives a total of at least $d^2 + 2d - 2$ edges in P.

If on the other hand F' has $d+1$ vertices, then the two vertices in $F' \setminus R$ belong to at least $2d - 3$ edges in F', giving $F \cup F'$ at least $\frac{1}{2}(d^2 + 5d) - 4$ edges. Again, the $d-1$ vertices outside $F \cup F'$ contribute at least $\frac{1}{2}(d^2 + d) - 1$ edges. Thus P has at least $d^2 + 3d - 5 > d^2 + 2d - 3$ edges.

Now suppose that F is not simplicial; then it is $M(2, d-3)$. In particular, it contains a ridge R with d vertices, and so must be a pyramid over R. Then F' must have $d+1$ vertices and also be a pyramid. Now R has at least $\phi(d, d-2) = \frac{1}{2}(d^2 - d) - 2$ edges, the two apices belong to $2d$ edges in $F \cup F'$, and the d vertices outside $F \cup F'$ belong to at least $\frac{1}{2}(d^2 + d)$ edges. This gives a total of at least $d^2 + 2d - 2$ edges.

Case 8. Every facet F has just d vertices.

Then P is simplicial and the conclusion follows from the Lower Bound Theorem (Theorem 4): P has at least $\binom{d}{1}(2d + 3) - \binom{d+1}{2} = \frac{3}{2}d^2 + \frac{5}{2}d > d^2 + 2d - 3$ edges. $\square$

3 Further research

One obvious extension of this research is to consider the same problem for polytopes with $2d + 3$ or more vertices. The preceding techniques are more difficult to apply in this case. The problem of minimising the number of edges, over a family of all d-polytopes which all have the same number of vertices, is the same as minimising the excess degree over the same family. Accordingly, we find it convenient to consider this problem in terms of the excess degree.

Truncating a simple vertex of $M(4, d - 4)$ yields a new polytope with $2d + 3$ vertices, and excess degree $3d - 12$. It seems plausible that this will be the unique minimiser of the number of edges for polytopes with $2d + 3$ vertices, at least in sufficiently high dimensions. This is not true for $d = 8$, where this truncated polytope has excess 12, but a pyramid over $\varDelta_{2,5}$ has $3d - 5$ vertices and excess 10.

Likewise for $4 \leqslant k \leqslant d - 4$, truncating a simple vertex of $M(k + 1, d - k - 1)$ yields a new polytope with $2d + k$ vertices, and excess degree $k(d - k - 1)$. These are candidates for minimal excess amongst polytopes with v vertices, for $2d + 4 \leqslant v \leqslant 3d - 6$.

But for $v = 3d - 5$, the corresponding truncated polytope has excess $4d - 20$, which is not minimal, because a pyramid over $\varDelta_{2,d-3}$ has $3d - 5$ vertices and excess $2d - 6$. Recall also that there are simple polytopes in all dimensions with $3d - 3$ vertices, and $3d - 1$ vertices, while C_d and $\varSigma_d$ have $3d - 2$ vertices and excess $d - 2$.

Another natural extension is to consider the minimum number of k-dimensional faces, where $1 \leqslant k < d$, for all d-polytopes with a fixed number v of vertices. This problem was also considered by Grünbaum [7], for d-polytopes with $v \leqslant 2d$ vertices. He formulated a conjecture about the minimum values, and verified it in the case $v \leqslant d + 4$. Further special cases were solved later:

- $k = d - 1$ by McMullen in [7],
- $k = 1$ and $k \geqslant 0.62d$ in [11],
- all k by Xue in [17].

Moving beyond $2d$ vertices, it is noteworthy that McMullen's result about facets is actually valid for all $v \leqslant 2d + \frac{1}{4}d^2$. For d-polytopes with $2d + 1$ vertices, we proved in [13] that the pentasm is the unique minimiser of the number of k-faces, for $1 \leqslant k \leqslant d - 2$, when d is prime, but not when d is composite.

Returning to d-polytopes with $2d + 2$ vertices, it is not hard to show that both A_d and B_d have exactly

$$\binom{d+1}{k+1} + 2\binom{d}{k+1} - \binom{d-2}{k+1}$$

k-dimensional faces, for $1 < k < d$. Our guess is that when $d+1$ is prime, these two polytopes are the only minimisers of the number of k-faces, for all such k. But when $d+1$ is composite, there are d-polytopes with $2d+2$ vertices and fewer k-faces than A_d and B_d, for most values of k. More precisely, let p be the smallest prime factor of $d+1$, and set $q = (d+1)/p$, $t = d-p-q$. Then a t-fold pyramid over $\mathit{\Delta}(p,q)$ has exactly

$$\binom{d+2}{k+2} - \binom{pq-p}{k+2} - \binom{pq-q}{k+2} + \binom{pq-p-q}{k+2}$$

k-dimensional faces, for $1 < k < d$. Numerical evidence suggests that this is less than the previous value, at least for $k \geqslant 0.4d$, but we have only been able to prove this for $k = d-1$ and $d-2$.

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