Coloring Graph Classes with no Induced Fork via Perfect Divisibility

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Abstract

For a graph G, $\chi(G)$ will denote its chromatic number, and $\omega(G)$ its clique number. A graph G is said to be *perfectly divisible* if for all induced subgraphs H of G, $V(H)$ can be partitioned into two sets A, B such that $H[A]$ is perfect and $\omega(H[B]) < \omega(H)$. An integer-valued function f is called a χ*-binding function* for a hereditary class of graphs $\mathcal{C}$ if $\chi(G) \leqslant f(\omega(G))$ for every graph $G \in \mathcal{C}$. The *fork* is the graph obtained from the complete bipartite graph $K_{1,3}$ by subdividing an edge once. The problem of finding a quadratic χ-binding function for the class of fork-free graphs is open. In this paper, we study the structure of some classes of fork-free graphs; in particular, we study the class of (fork, F)-free graphs $\mathcal{G}$ in the context of perfect divisibility, where F is a graph on five vertices with a stable set of size three, and show that every $G \in \mathcal{G}$ satisfies $\chi(G) \leqslant \omega(G)^2$. We also note that the class $\mathcal{G}$ does not admit a linear χ-binding function.

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1 Introduction

For a positive integer n, K_n will denote the complete graph on n vertices, and P_n will denote the path on n vertices. For an integer $n > 2$, C_n will denote the cycle on n vertices. A *hole* in a graph is an induced cycle C_n with $n > 3$; an *antihole* is the complement of a hole. A hole or antihole is *odd* (*even*) if n is odd (even). The *union* of two vertex-disjoint graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The union of k copies of the same graph G will be denoted by kG. The complement of a graph G will be denoted by $\overline{G}$. A *stable set* (or an *independent set*) is a set of vertices that are pairwise nonadjacent.

A class of graphs $\mathcal{C}$ is *hereditary* if every induced subgraph of every graph in $\mathcal{C}$ is also in $\mathcal{C}$. An important and well studied type of hereditary class of graphs is the class of graphs which are defined by forbidden induced subgraphs. Given a graph H, we say that a graph G is *H-free* if G has no induced subgraph that is isomorphic to H. Given a class of graphs $\mathcal{H}$, we say that a graph G is *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$.

For a graph G, $\chi(G)$ will denote its chromatic number, and $\omega(G)$ its clique number. For every graph G, $\chi(G) \geqslant \omega(G)$. A graph G is called *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$.

A graph G is said to be *perfectly divisible* if for all induced subgraphs H of G, $V(H)$ can be partitioned into two sets A, B such that $H[A]$ is perfect and $\omega(H[B]) < \omega(H)$. Perfectly divisible graphs were introduced by Hoàng [12], and can be thought of as a generalization of perfect graphs in the sense that perfect graphs are perfectly divisible. However not all perfectly divisible graphs are perfect. For example, the vertex set of an odd hole can be partitioned into two sets such that the first set induces a perfect graph and the other is a stable set. So an odd hole is perfectly divisible but not perfect. Hoàng [13] observed that the class of $3K_1$-free graphs is perfectly divisible, and in [12] he showed that the class of (banner, odd hole)-free graphs is perfectly divisible. Chudnovsky and the third author showed that the class of (P_5, bull)-free graphs is perfectly divisible [6]. See Sections 2 and 4 for more on perfect divisibility.

A hereditary class of graphs $\mathcal{C}$ is *χ-bounded* if there is a function f (called a *χ-binding function*) such that $\chi(G) \leqslant f(\omega(G))$ for every graph $G \in \mathcal{C}$. In addition, if f is a polynomial function then the class $\mathcal{C}$ is *polynomially χ-bounded*. It has long been known that there are hereditary graph classes that are not χ-bounded (see [24] for examples) but it is not known whether there is a hereditary graph class that is χ-bounded but not polynomially χ-bounded. A recent survey of Scott and Seymour [24] gives a detailed overview of this area of research.

The *claw* is the complete bipartite graph $K_{1,3}$. The class of claw-free graphs is widely studied in a variety of contexts and has a vast literature; see [10] for a survey. A detailed and complete structural classification of claw-free graphs has been given by Chudnovsky and Seymour; see [9]. A result of Gyárfás [11] together with a result of Kim [16] show that the class of claw-free graphs is χ-bounded, and that every such graph G satisfies $\chi(G) \leqslant O(\omega(G)^2/\log \omega(G))$. It is also known that there is no linear χ-binding function even for a very special class of claw-free graphs; see [3]. Chudnovsky and Seymour [8] showed that every connected claw-free graph G with a stable set of size at least 3 satisfies $\chi(G) \leqslant 2\omega(G)$.

The *fork* is the graph obtained from $K_{1,3}$ by subdividing an edge once. The class of claw-free graphs is a subclass of the class of fork-free graphs. It is a natural line of research to see what properties of claw-free graphs are also enjoyed by fork-free graphs. A classic example is the polynomial-time solvability of the (weighted) stable set problem in the class of fork-free graphs [1, 18], generalizing the result for claw-free graphs [19, 21]. It has long been known that the class of fork-free graphs is χ-bounded [15], and it follows from a recent result of Scott, Seymour and Spirkl [25], the class of fork-free graphs is polynomially χ-bounded. However, it is not known whether the class of fork-free graphs admits a quadratic χ-binding function or not. Thus, we have the following:

Problem 1. Does there exist a quadratic χ-binding function for the class of fork-free graphs?

The third author (unpublished) has conjectured that the class of fork-free graphs is perfectly divisible which in turn will yield a quadratic χ-binding function. In this paper, we are interested in quadratic χ-binding functions for some classes of fork-free graphs, namely (fork, F)-free graphs $\mathcal{G}$, where F is any nontrivial graph on at most five vertices, and we give below some known results in this direction. The *paw* is the graph that consists of a K_3 with a pendant vertex attached to it. The *diamond* is the graph $K_4 - e$.

- If $F = K_3$, then it is observed in [23] that every $G \in \mathcal{G}$ satisfies $\chi(G) \leqslant 3$. Moreover if G is connected, then equality holds if and only if G is an odd hole.
- If $F \in \{P_3, \overline{P_3}\}$, then clearly every $G \in \mathcal{G}$ is perfect.
- If $F \in \{P_4, C_4, K_4, K_4 - e, K_3 \cup K_1, \text{paw}\}$, then $\mathcal{G}$ is linearly χ-bounded; see [5] and the reference therein.
- It follows from a result of Wagon [26] that, if $F = 2K_2$, then every $G \in \mathcal{G}$ satisfies $\chi(G) \leqslant \binom{\omega(G)+1}{2}$. Further, it is known that $\mathcal{G}$ does not admit a linear χ-binding function; see [3].
- If $F = K_{1,3}$, then every $G \in \mathcal{G}$ satisfies $\chi(G) \leqslant O(\omega(G)^2/\log \omega(G))$, and $\mathcal{G}$ does not admit a linear χ-binding function; see [11, 16].
- If $F \in \{P_3 \cup K_1, K_2 \cup 2K_1\}$, then it follows from Theorem 18 of [22] that $\mathcal{G}$ is quadratically χ-bounded.
- Randerath [20] showed that, if $F \in \{\overline{P_3 \cup 2K_1}, K_5 - e\}$, then every $G \in \mathcal{G}$ satisfies $\chi(G) \leqslant \omega(G) + 1$.
- Recently, Chudnovsky et al [4] proved a structure theorem for the class of (fork, antifork)-free graphs, and used it to prove that every (fork, antifork)-free graph G satisfies $\chi(G) \leqslant 2\omega(G)$. (Here, an *antifork* is the complement graph of a fork.)

Thus if $|V(F)| \leqslant 4$, then the class of (fork, F)-free graphs is known to be quadratically χ-bounded except when $F = 4K_1$, and not much is known when $|V(F)| = 5$. Here, we study the class of (fork, F)-free graphs, where F is a graph on five vertices with a stable set of size three. More precisely, we consider the class of (fork, F)-free graphs $\mathcal{F}$, where F is one of the following graphs: P_6, dart, co-dart, co-cricket, banner, and bull (see Figure 1), and show that the following hold:

(i) Every $G \in \mathcal{F}$ is perfectly divisible, when $F \in \{P_6, \text{co-dart}, \text{bull}\}$.

(ii) Every $G \in \mathcal{F}$ is either claw-free or perfectly divisible, when $F \in$ {dart, banner, co-cricket}.

(iii) Every $G \in \mathcal{F}$ satisfies $\chi(G) \leqslant \omega(G)^2$.

(iv) Since the class of $3K_1$-free graphs does not admit a linear χ-binding function [3], and since each graph F and the fork has a stable set of size 3, it follows that the class $\mathcal{F}$ does not admit a linear χ-binding function.

2 Preliminaries

We follow West [27] for standard notation and terminology which are not defined here. For a vertex v in a graph G, $N_G(v)$ is the set of vertices adjacent to v, $N_G[v]$ is the set $\{v\} \cup N_G(v)$, and $M_G(v)$ is the set $V(G) \setminus N_G[v]$. Given a subset $X \subseteq V(G)$, $N_G(X)$ is the set $\{u \in V(G) \setminus X : u$ is adjacent to a vertex of $X\}$, and $M_G(X)$ is the set $V(G) \setminus (X \cup N_G(X))$. We drop the subscript G in the above notations if there is no ambiguity. For a vertex set $X \subseteq V(G)$, $G[X]$ denotes the subgraph of G induced by X. We say that a graph G contains a graph H if H is an induced subgraph of G. Given disjoint vertex sets S, T, we say that S is *complete* to T if every vertex in S is adjacent to every vertex in T; we say S is *anticomplete* to T if every vertex in S is nonadjacent to every vertex in T; and we say S is *mixed* on T if S is not complete or anticomplete to T. When S has a single vertex, say v, we can instead say that v is complete to, anticomplete to, or mixed on T. A vertex v in G is *universal* if it is complete to $V(G) \setminus \{v\}$. A set $S \subseteq V(G)$ is a *homogeneous set* if $1 < |S| < |V(G)|$ and for every $v \in V(G) \setminus S$, v is either complete or anticomplete to S. We say that a graph G admits a homogeneous set decomposition if G has a homogeneous set. The *independence number* $\alpha(G)$ of a graph G is the size of a largest stable set in G. A *triad* in a graph G is a stable set of size 3.

For a vertex subset $S := \{v_1, v_2, \ldots, v_k\}$ of G, we write v_1-v_2-$\cdots$-v_k-v_1 to denote an induced cycle C_k in G with vertex set S and edge set $\{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_kv_1\}$, and we write v_1-v_2-$\cdots$-v_k to denote an induced path P_k in G with vertex set S and edge set $\{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\}$.

We say that a vertex v is a *center* of a claw in a graph G, if v has neighbors $a, b, c \in V(G)$ such that $\{a, b, c\}$ is a triad; and we call the vertices a, b, c the *leaves* of the claw.

We say that a vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ induces (see Figure 1):

- a *fork* if $\{v_2, v_3, v_4, v_5\}$ induces a claw with center v_3, and v_1 is a leaf adjacent to v_2.
- a *dart* if $\{v_1, v_2, v_3, v_4\}$ induces a claw with center v_1, and v_5 is adjacent to v_1, v_3, and v_4 but not to v_2.

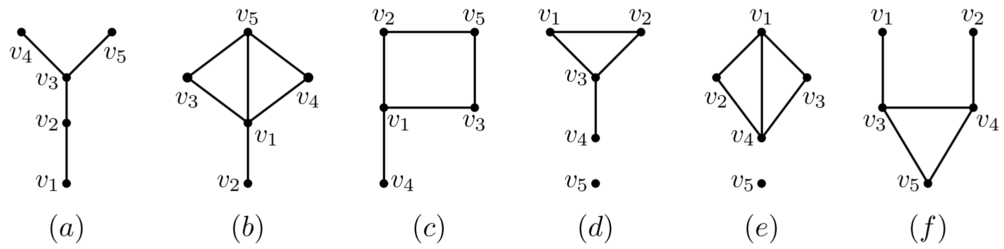

Figure 1: (a): Fork. (b): Dart. (c): Banner. (d): Co-dart. (e): Co-cricket. (f): Bull.

- a *banner* if $\{v_1, v_2, v_3, v_4\}$ induces a claw with center v_1, and v_5 is adjacent to v_2, v_3 but not to v_1, v_4.
- a *co-dart* if $\{v_1, v_2, v_3, v_4\}$ induces a paw, and v_5 is anticomplete to $\{v_1, v_2, v_3, v_4\}$.
- a *co-cricket* if $\{v_1, v_2, v_3, v_4\}$ induces a diamond, and v_5 is anticomplete to $\{v_1, v_2, v_3, v_4\}$.
- a *bull* if v_1-v_2-v_3-v_4 is a path, and v_5 is adjacent to v_2, v_3 but not to v_1, v_4.

We write $G \cong H$ to mean a graph G is isomorphic to a graph H. Note that a co-cricket $\cong$ diamond $\cup K_1$, and a co-dart $\cong$ paw $\cup K_1$ and is the complement graph of a dart.

We use the following known results. The class of perfect graphs admits a forbidden induced subgraph characterization, namely, *the strong perfect graph theorem*, given below.

Theorem 2 ([7]). *A graph is perfect if and only if it does not contain an odd hole or odd antihole as an induced subgraph.*

Chudnovsky and Seymour [8] give a simple proof of the following χ-bound for claw-free graphs in general.

Theorem 3 ([8]). *Every claw-free graph G satisfies $\chi(G) \leqslant \omega(G)^2$. Moreover, the bound is asymptotically tight.*

Although perfect divisibility is a structural property, it immediately implies a quadratic χ-binding function. Indeed, we have the following (see also [6]).

Lemma 4 ([12]). *Every perfectly divisible graph G satisfies $\chi(G) \leqslant \binom{\omega(G)+1}{2}$.*

A graph G is said to be *perfectly weight divisible* if for every nonnegative integer weight function w on $V(G)$, there is a partition of $V(G)$ into two sets S and T such that $G[S]$ is perfect and the maximum weight of a clique in $G[T]$ is smaller than the maximum weight of a clique in G. We will also use the following results.

Theorem 5 ([6]). *A minimal non-perfectly weight divisible graph does not admit a homogeneous set decomposition.*

The proof of the following theorem is similar to the proof of Theorem 3.7 in [6], and we give it here for completeness.

Theorem 6. *Let $\mathcal{C}$ be a hereditary class of graphs. Suppose that every graph $H \in \mathcal{C}$ has a vertex v such that $H[M_H(v)]$ is perfect. Then every $G \in \mathcal{C}$ is perfectly weight divisible, and hence perfectly divisible.*

Proof. Let $G \in \mathcal{C}$ be a minimal counterexample to the theorem. Then there is a nonnegative integer weight function w on $V(G)$ for which there is no partition of $V(G)$ as in the definition of perfectly weight divisibility. Let U be the set $\{v \in V(G) \mid w(v) > 0\}$, and let H be the graph induced on U. Since $\mathcal{C}$ is hereditary, $H \in \mathcal{C}$, and so by the hypothesis of the theorem, H has a vertex v such that $H[M_H(v)]$ is perfect. But now, since $w(v) > 0$, if we let $S := M_H(v) \cup \{v\}$ and $T := N_H(v) \cup (V(G) \setminus U)$, then we get a partition of $V(G)$ as in the definition of perfectly weight divisibility, a contradiction. This proves the theorem. □

3 Classes of fork-free graphs

3.1 The class of (fork, P_6)-free graphs

In this section we prove that (fork, P_6)-free graphs are perfectly divisible, and hence the class of (fork, P_6)-free graphs is quadratically χ-bounded. A vertex set X in a graph G is said to be *anticonnected* if the subgraph induced by X in $\overline{G}$ is connected. Also, a vertex v is an *anticenter* for a vertex set X if $N[v] \cap X = \varnothing$. An *antipath* in G is the complement of the path v_1-v_2-$\cdots$-v_k in G, for some k.

Theorem 7. *Let G be a (fork, P_6)-free graph which is not perfectly divisible. Then G has a homogeneous set.*

Proof. Since G is not perfectly divisible, given $v \in V(G)$, $G[M(v)]$ is not perfect, so it contains an odd hole C_n or an odd antihole $\overline{C_n}$, by Theorem 2. Note that $C_5 \cong \overline{C_5}$, and since G is P_6-free, G does not contain C_n for $n > 6$. So G contains an odd antihole induced by X_0 with anticenter v. We construct a sequence of vertex sets $X_0, X_1, \ldots, X_t, \ldots$ such that for each i, X_i is obtained from X_{i-1} by adding one vertex x_i that has a neighbor and a nonneighbor in X_{i-1}. Let X be the maximal vertex set obtained this way. By maximality, X is a homogeneous set. We show that $X \neq V(G)$; in particular, we show that X does not intersect the set A of anticenters for X_0.

Claim 8. *$N(A)$ is complete to X_0.*

Let $v_1, v_2, \ldots, v_n$ be the vertices of X_0, with edges v_iv_j whenever $|i-j| \neq 1$ (indices are modulo n). Suppose to the contrary that there exists a vertex $b \in N(A)$ that has a nonneighbor in X_0. By assumption, b has a neighbor in X_0, and a neighbor $a \in A$. Suppose that b has two consecutive neighbors in X_0. Then there is some i such that $v_i, v_{i+1} \in N(b)$ and $v_{i+2} \notin N(b)$. But then $\{v_{i+2}, v_i, b, v_{i+1}, a\}$ induces a fork. So we may assume that b does not have two consecutive neighbors in X_0. Then since n is odd, there must exist some i such that $v_i, v_{i+1}, v_{i+3} \notin N(b)$ and $v_{i+2} \in N(b)$. But now a-b-v_{i+2}-v_i-v_{i+3}-v_{i+1} is a P_6 which is a contradiction. This proves 8.

Claim 9. *For each $i \in \{0, 1, 2, \dots, \}$, X_i is complete to $N(A)$, and X_i does not intersect $A \cup N(A)$.*

We prove the assertion by induction on i. By 8, we may assume that $i \geqslant 1$. Suppose to the contrary that x_i has a nonneighbor $b \in N(A)$; then b is complete to X_{i-1}. Since x_i has a neighbor in X_{i-1} and $N(A) \cap X_{i-1} = \varnothing$, we have $x_i \notin A$. Since x_i has a nonneighbor in X_{i-1} and $N(A)$ is complete to X_{i-1}, we have $x_i \notin N(A)$. Since $x_i \notin A$, x_i has a neighbor in X_0, say v_1. Note that X_i is anticonnected for each i. Then if $\overline{P}$ is a shortest antipath from x_i to v_1 in $G[X_{i-1}]$, $\overline{P}$ contains at least three vertices; label its vertices $x_i = w_1$-w_2-$\cdots$-$w_t = v_1$ in order. Since $w_2, w_3 \in X_{i-1} \subseteq N(b)$, $\{x_i, w_3, b, w_2, a\}$ induce a fork which is a contradiction. So x_i has no nonneighbors in $N(A)$. This proves 9.

Now by 9, it follows that the vertex set X does not intersect $A \cup N(A)$. Since $A \neq \varnothing$, we have $X \neq V(G)$, as desired. □

Corollary 10. *Every (fork, P_6)-free graph is perfectly divisible.*

Proof. This follows from Theorems 5 and 7. □

We immediately have the following corollaries which generalize the result that the class of $3K_1$-free graphs is perfectly divisible [13].

Corollary 11. *Every $P_3 \cup K_1$-free graph is perfectly divisible.*

Corollary 12. *Every $P_2 \cup 2K_1$-free graph is perfectly divisible.*

Corollary 13. *Every (fork, P_6)-free graph G satisfies $\chi(G) \leqslant \binom{\omega(G)+1}{2}$.*

Proof. This follows from Lemma 4. □

The above corollary implies that every $P_2 \cup 2K_1$-free graph G satisfies $\chi(G) \leqslant \binom{\omega(G)+1}{2}$, and since the class of $3K_1$-free graphs does not admit a linear χ-binding function [3], the class of $P_2 \cup 2K_1$-free graphs too does not admit a linear χ-binding function. This partially answers a question of Gyárfás; see Problem 2.20 of [11].

3.2 The class of (fork, dart)-free graphs

In this section, we prove that the class of (fork, dart)-free graphs is quadratically χ-bounded.

Theorem 14. *Let G be a connected (fork, dart)-free graph. If G contains a claw with center v, then $G[M(v)]$ is perfect.*

Proof. Let G be a (fork, dart)-free graph containing a claw with center v. Let L be the set of leaves of claws in G with center v. Then L has a triad, and so $|L| \geqslant 3$. Let Y denote the set $M(v) \cap N(L)$, and X denote the set $M(v) \setminus N(L)$. Then $M(v)$ is the set $X \cup Y$.

Claim 15. *If $y \in Y$ has a neighbor in a triad T in L, then it has at least two neighbors in T. Moreover, every $y \in Y$ has a neighbor in a triad T, so has two nonadjacent neighbors in T.*

Let $\{a, b, c\}$ be a triad in L, and suppose that y is adjacent to a. Then since $\{y, a, v, b, c\}$ does not induce a fork, we see that y is adjacent to b or c. This prove the first assertion of 15. Note that by definition, any y in Y has a neighbor $a \in L$; then there exist $b, c \in L$ such that $\{a, b, c\}$ is a triad, and hence y is also adjacent to b or c. This proves 15.

Claim 16. *If $a \in L$, then $G[N(a) \cap Y]$ is P_3-free.*

If there is a P_3, say y_1-y_2-y_3, in $G[N(a) \cap Y]$, then $\{a, v, y_1, y_3, y_2\}$ induces a dart, a contradiction. This proves 16.

Claim 17. *$N(v) \setminus L$ is complete to L.*

Suppose to the contrary that $t \in N(v) \setminus L$ has a nonneighbor $a \in L$. Then since $|L| \geqslant 3$, there are vertices b and c in L such that $\{v, a, b, c\}$ is a claw. Now, if t is not adjacent to b, then t is a leaf of the claw induced by $\{v, a, b, t\}$, a contradiction to our assumption that $t \notin L$. So t is adjacent to b. Likewise, t is adjacent to c. But then $\{v, a, b, c, t\}$ induces a dart which is a contradiction. This proves 17.

Claim 18. *$N[v]$ is anticomplete to X.*

Suppose to the contrary that there exists a vertex $t \in N(v)$ such that t has a neighbor, say $x \in X$. Since L is anticomplete to X (by the definition), we may assume that $t \in N(v) \setminus L$. By 17, $\{t\}$ is complete to L. Then since $|L| \geqslant 3$, there exist $a, b \in L$ such that $\{t, x, a, b, v\}$ induces a dart which is a contradiction. This proves 18.

Claim 19. *Let $y \in Y$ and let X' be a component of X. Then y is not mixed on $V(X')$.*

Suppose not. We may assume that y is mixed on an edge xx' in X'. Then since $y \in Y$, by 15, there exist nonadjacent vertices $a, b \in L$ such that $a, b \in N(y)$. But then $\{x', x, y, a, b\}$ induces a fork, a contradiction. This proves 19.

Claim 20. *X is complete to Y.*

By 18, $N(X) \subseteq Y$. Since G is connected, it follows from 19 that every vertex in X has a neighbor in Y. Suppose to the contrary that $x \in X$ is mixed on Y. Let $y \in N(x) \cap Y$ and let $y' \notin N(x) \cap Y$. By 15, let a and b be the neighbors of y in L. Recall that, by 18, x is anticomplete to L. Suppose that y' is adjacent to y. Then since $\{v, a, y, y', x\}$ does not induce a fork, y' is adjacent to a. Likewise, y' is adjacent to b. But then $\{y, x, a, b, y'\}$ induces a dart, a contradiction. So we may assume that y' is not adjacent to y. Then since $\{x, y, a, y', v\}$ does not induce a fork, y' is not adjacent to a. Likewise, y' is not adjacent to b. Then by 15, y' has nonadjacent neighbors $a', b' \in L \setminus \{a, b\}$. Then since $\{y', a', v, a, b\}$ does not induce a fork or a dart, we may assume that a' is adjacent to a, but not to b. Then since $\{a', v, y, y', a\}$ does not induce a dart, y is not adjacent to a'. Now, $\{a', a, y, x, b\}$ induces a fork which is a contradiction. This proves 20.

Claim 21. *X is a clique.*

Suppose not. Let x and x' be two nonadjacent vertices in X. By 18, $N(X) \subseteq Y$. Now choose any $y \in Y$ and any $a \in L$ adjacent to y. Then, by 18 and 20, $\{v, a, y, x, x'\}$ induces a fork which is a contradiction. This proves 21.

Claim 22. *If C is an odd hole or an odd antihole in $G[M(v)]$, then $V(C) \subseteq Y$.*

By 20 and 21, every vertex in X is universal in $G[X \cup Y]$. Since odd holes and odd antiholes have no universal vertices, we see that $V(C) \cap X = \varnothing$. So $V(C) \subseteq Y$. This proves 22.

Claim 23. *Let $C := y_1\text{-}y_2\text{-}\cdots\text{-}y_n\text{-}y_1$ be an odd hole in $G[Y]$. Then every vertex in L which has a neighbor in C is adjacent to exactly two consecutive vertices of C.*

Let $a \in L$. We may assume that y_1 is a neighbor of a in C. Then since $\{v, a, y_1, y_2, y_n\}$ does not induce a fork or a dart, we may assume that a is adjacent to y_2, and is nonadjacent to y_n. Then since $\{a, v, y_1, y_3, y_2\}$ does not induce a dart, a is not adjacent to y_3. If a has a neighbor in $C \setminus \{y_1, y_2\}$, say y_i with the largest index i, then $3 < i < n$. By the choice of i, we have $y_{i+1} \notin N(a)$, and then $\{y_{i+1}, y_i, a, y_2, v\}$ induces a fork. Thus a is anticomplete to $C \setminus \{y_1, y_2\}$. Hence every vertex in L which has a neighbor in C is adjacent to exactly two consecutive vertices of C. This proves 23.

Claim 24. *$G[M(v)]$ is C_{2k+1}-free, where $k \geqslant 2$.*

Suppose to the contrary that $G[M(v)]$ contains an odd hole, say $C := y_1\text{-}y_2\text{-}\cdots\text{-}y_{2k+1}\text{-}y_1$. By 22, $V(C) \subseteq Y$. By 15, let $\{a, b, c\}$ be a triad in L, and let a and b be the neighbors of y_1 in L. Then by 23, $\{a, b\}$ is anticomplete to y_3, and we may assume that $N(a) \cap V(C) = \{y_1, y_2\}$. If b is adjacent to y_2, then $\{y_2, y_3, a, b, y_1\}$ induces a dart. So we may assume that b is not adjacent to y_2. Then by 15, c is a neighbor of y_2. As earlier, we see that c is not adjacent to y_1, and hence by 23, c is adjacent to y_3. But now $\{y_3, c, v, a, b\}$ induces a fork which is a contradiction. This proves 24.

Claim 25. *$G[M(v)]$ is $\overline{C_{2k+1}}$-free, where $k \geqslant 3$.*

Suppose to the contrary that $G[M(v)]$ contains a $\overline{C_{2k+1}}$, say C with vertices $y_1, y_2, \ldots, y_{2k+1}$ and edges y_iy_j whenever $|i-j| \neq 1$ (indices are modulo $2k+1$). By 22, $V(C) \subseteq Y$. Let $a \in L$ be a neighbor of y_2. Consider any consecutive pair of vertices $y_i, y_{i+1} \in C \setminus \{y_1, y_2, y_3\}$. Then since $\{v, a, y_2, y_i, y_{i+1}\}$ does not induce a fork or a dart, a is adjacent to exactly one of y_i, y_{i+1}. Therefore, if a is adjacent to y_4, then y is adjacent to precisely the vertices with even index $i > 3$ in C, and if a is not adjacent to y_4, then y is adjacent to precisely the vertices with odd index $i > 3$ in C. We may assume that a is adjacent to y_4. Then a is not adjacent to y_{2k+1}. Then since $y_2\text{-}y_4\text{-}y_1$ is a P_3, by 16, a is not adjacent to y_1. But then $\{v, a, y_4, y_1, y_{2k+1}\}$ induces a fork which is a contradiction. This proves 25.

Now by claims 24 and 25, and by Theorem 2, we conclude that $G[M(v)]$ is perfect. This completes the proof. □

Theorem 26. *Let G be a connected (fork, dart)-free graph. Then G is either claw-free or for any claw in G with center, say v, $G[M(v)]$ is perfect.*

Proof. This follows from Theorem 14. □

The following corollary generalizes the result known for the class of claw-free graphs (Theorem 3).

Corollary 27. *Every (fork, dart)-free graph G satisfies $\chi(G) \leqslant \omega(G)^2$.*

Proof. Let G be a (fork, dart)-free graph. We may assume that G is connected. If G is claw-free, then the desired result follows from Theorem 3. So let us assume that G contains a claw with center, say v. Then by Theorem 26, $G[M(v)]$ is perfect. Now since $\omega(G[N(v)]) \leqslant \omega(G) - 1 < \omega(G)$ and since $G[\{v\} \cup M(v)]$ is perfect, we see that G is perfectly divisible, and hence the result follows from Lemma 4. □

3.3 The class of (fork, co-dart)-free graphs

In this section, we prove that (fork, co-dart)-free graphs are perfectly divisible, and hence the class of (fork, co-dart)-free graphs is quadratically χ-bounded.

Theorem 28. *Let G be a connected (fork, co-dart)-free graph. Then either G admits a homogeneous set decomposition or for each vertex v in G, $G[M(v)]$ is perfect.*

Proof. Suppose to the contrary that G does not admit a homogeneous set decomposition and that there is a vertex v in G such that $G[M(v)]$ is not perfect. So by Theorem 2, $G[M(v)]$ contains an odd hole or an odd antihole. Since G has no co-dart, $G[M(v)]$ is paw-free, and so $G[M(v)]$ has no odd antiholes except $\overline{C_5}$. So suppose that $G[M(v)]$ contains an odd hole. Let $C := v_1\text{-}v_2\text{-}\cdots\text{-}v_\ell\text{-}v_1$ be a shortest odd hole in $G[M(v)]$ for some $\ell \geqslant 5$ with vertex set $S := \{v_1, v_2, \ldots, v_\ell\}$.

Claim 29. *If $x \in N(S)$, then $x \in N(v)$.*

Suppose to the contrary that x is nonadjacent to v. First suppose that x has two adjacent neighbours in C. We may assume that $v_1, v_2 \in N(x)$. Then since $\{v_1, v_2, v_3, x, v\}$ does not induce a co-dart, we see that x is adjacent to v_3. Then by similar arguments, we conclude that $N(x) \cap S = S$. But now $\{v_1, v_2, v_{\ell-1}, x, v\}$ induces a co-dart, a contradiction. So suppose that x is nonadjacent to any two consecutive vertices in C. Since x has a neighbor in C, we may assume that x is adjacent to v_1. Then x is nonadjacent to both v_2 and v_ℓ. Then since $\{v_\ell, v_1, v_2, v_3, x\}$ and $\{v_2, v_1, v_\ell, v_{\ell-1}, x\}$ do not induce forks, x is adjacent to both v_3 and $v_{\ell-1}$. Since x is nonadjacent to two consecutive vertices in C, this implies that $\ell \geqslant 7$, and x is nonadjacent to v_4. But now $\{v_{\ell-1}, x, v_3, v_4, v_2\}$ induces a fork, a contradiction. This proves 29.

Claim 30. *Any vertex in $N(S)$ is complete to S.*

Let x be a vertex in $N(S)$. Then by 29, x is adjacent to v. We may assume that x is adjacent to v_1. Then since $\{v, x, v_1, v_2, v_\ell\}$ does not induce a fork, x is adjacent to either v_2 or v_ℓ. We may assume that x is adjacent to v_2. Then for $j \in \{4, 5, \ldots, \ell - 1\}$, since $\{v_1, x, v, v_2, v_j\}$ does not induce a co-dart, x is complete to $\{v_4, v_5, \ldots, v_{\ell-1}\}$. Now suppose to the contrary that x is nonadjacent to one of v_3 or v_ℓ, say v_ℓ. Then since $\{x, v, v_1, v_4, v_3\}$

does not induce a fork, x is adjacent to v_3. But then $\{v, x, v_2, v_3, v_\ell\}$ induces a co-dart, a contradiction. So x is adjacent to both v_3 and v_ℓ, and hence x is complete to S. This proves 30.

By 30, we see that S is a homogenous set, a contradiction. This proves Theorem 28. $\square$

Corollary 31. *Every (fork, co-dart)-free graph is perfectly weight divisible, and hence perfectly divisible.*

Proof. Let G be a minimal counterexample to the theorem. Then, by Theorem 5, G does not admit a homogeneous set decomposition. So, by Theorem 28, there is a vertex v in G such that $G[M(v)]$ is perfect. Then, by Theorem 6, it follows that G is perfectly weight divisible, and hence perfectly divisible, a contradiction. This proves Corollary 31. $\square$

Corollary 32. *Every (fork, co-dart)-free graph G satisfies $\chi(G) \leqslant \binom{\omega(G)+1}{2}$.*

Proof. This follows from Corollary 31, and from Lemma 4. $\square$

3.4 The class of (fork, banner)-free graphs

In this section, we prove that (fork, banner)-free graphs are either claw-free or perfectly divisible, and hence the class of (fork, banner)-free graphs is quadratically χ-bounded. We use the following lemma.

Lemma 33 ([2])**.** *If G is a banner-free graph that does not admit a homogeneous set decomposition, then G is $K_{2,3}$-free.*

Theorem 34. *Let G be a (fork, banner)-free graph that contains a claw. Then either G admits a homogeneous set decomposition or there is a vertex v in G such that $G[M(v)]$ is perfect.*

Proof. Let G be a (fork, banner)-free graph that contains a claw. Suppose that G does not admit a homogeneous set decomposition. Then G is connected, and, by Lemma 33, we may assume that G is $K_{2,3}$-free. Let v be a vertex in G such that $\alpha(G[N(v)])$ is maximized. Let L be a maximum stable set in $N(v)$, and let Q denote the set $N(v) \setminus L$. Since G contains a claw, we see that $|L| \geqslant 3$ and so L has a triad.

Claim 35. *$M(v)$ is anticomplete to L.*

Suppose $x \in M(v)$ has a neighbor a in a triad $\{a, b, c\} \subseteq L$. Then since $\{v, a, b, c, x\}$ does not induce an $K_{2,3}$ or a banner, x is not adjacent to b and c. But then $\{v, a, b, c, x\}$ induces a fork, a contradiction. This proves 35.

Claim 36. *$G[M(v)]$ is a stable set.*

Suppose to the contrary that $G[M(v)]$ has a component, say C with more than one vertex. Then, by 35, $V(C)$ is anticomplete to L. Let $x, y \in V(C)$ be neighbors, and suppose $t \in Q$ is adjacent to x. Then t is adjacent to at least two vertices in any given triad $\{a, b, c\} \subseteq L$ (otherwise, $G[\{x, t, v, a, b, c\}]$ contains a fork, a contradiction). We may assume $a, b \in N(t)$. Then since $\{y, x, t, a, b\}$ does not induce a fork, t is adjacent to y. Thus we conclude that every vertex in $N[v]$ is either complete or anticomplete to $V(C)$, and so $V(C)$ is a homogeneous set, a contradiction to our assumption. This proves 36.

Now it follows from 36 that $G[M(v)]$ is perfect. This completes the proof. □

Corollary 37. *Let G be a (fork, banner)-free graph. Then either G is claw-free or G admits a homogeneous set decomposition or there is a vertex v in G such that $G[M(v)]$ is perfect.*

Proof. This follows from Theorem 34. □

Corollary 38. *Let G be a (fork, banner)-free graph. Then either G is claw-free or G is perfectly weight divisible, and hence perfectly divisible.*

Proof. This follows from Theorems 5 and 6, and from Corollary 37. □

Corollary 39. *Every (fork, banner)-free graph G satisfies $\chi(G) \leqslant \omega(G)^2$.*

Proof. This follows from Corollary 38, Theorem 3, and from Lemma 4. □

3.5 The class of (fork, co-cricket)-free graphs

In this section, we prove that (fork, co-cricket)-free graphs are either claw-free or perfectly divisible, and hence the class of (fork, co-cricket)-free graphs is quadratically χ-bounded.

Theorem 40. *Let G be a (fork, co-cricket)-free graph. Then either G is claw-free or G admits a homogeneous set decomposition or for each vertex u in G, $G[M(u)]$ is perfect.*

Proof. Let G be a (fork, co-cricket)-free graph. Suppose to the contrary that none of the assertions hold. We may assume that G is connected. Let x be a vertex in G such that $G[M(x)]$ is not perfect. Since G is co-cricket-free, we see that $G[M(x)]$ does not contain a diamond, and hence does not contain an odd antihole except $\overline{C_5}$. So by Theorem 2, $G[M(x)]$ contains an odd hole. Let $C := v_1\text{-}v_2\text{-}\cdots\text{-}v_\ell\text{-}v_1$ be a shortest odd hole in $G[M(x)]$ for some $\ell \geqslant 5$, and let S denote the vertex set of C.

Claim 41. *If v is a vertex in G which has three consecutive neighbors in C, then v is complete to $S \cup M(S)$.*

We may assume that v is adjacent to the vertices v_1, v_2 and v_3. Now if there is a vertex in $M(S)$ that is nonadjacent to v, say a, then $\{v_1, v_2, v_3, v, a\}$ induces a co-cricket, a contradiction. So v is complete to $M(S)$. In particular, v is adjacent to x. Next suppose to the contrary that v is not complete to C. Let $k \in \{4, 5, \ldots, \ell\}$ be the least positive integer such that v is adjacent to v_{k-1}, and v is nonadjacent to v_k. Now if $k \neq \ell$, then $\{x, v, v_{k-1}, v_k, v_1\}$ induces a fork, and if $k = \ell$, then $\{v_\ell, v_{\ell-1}, v, x, v_2\}$ induces a fork, a contradiction. So v is complete to S. This proves 41.

Claim 42. *Let v be a vertex in G which has a neighbor in C. Then the following hold:*

(a) *If* $\ell = 5$*, then* $N(v) \cap S$ *is either* $\{v_j, v_{j+1}\}$ *or* $\{v_j, v_{j+1}, v_{j+3}\}$ *or* S*, for some* $j \in \{1, 2, \ldots, 5\}, j$ *mod* 5.

(b) *If* $\ell \geqslant 7$*, then* $N(v) \cap S$ *is either* $\{v_j, v_{j+1}\}$ *or* S*, for some* $j \in \{1, 2, \ldots, \ell\}$*,* j *mod* ℓ.

If v has three consecutive vertices of C as neighbors, then by 41, $N(v) \cap S = S$, and we conclude the proof. So we may assume that no three consecutive vertices of C are neighbors of v.

Now suppose that 42:(a) does not hold. So by our assumption, there is an index $j \in \{1, 2, \ldots, 5\}$, j mod 5 such that v is adjacent to v_j, and v is anticomplete to $\{v_{j+1}, v_{j-1}, v_{j-2}\}$. Then $\{v_{j-2}, v_{j-1}, v_j, v_{j+1}, v\}$ induces a fork, a contradiction. So 42:(a) holds.

Next suppose that 42:(b) does not hold. First let us assume that no two consecutive vertices of C are neighbors of v. Since v has a neighbor in C, we may assume that v is adjacent to v_1. By assumption, v is not adjacent to both v_2 and v_ℓ. Then since $\{v_3, v_2, v_1, v_\ell, v\}$ does not induce a fork, v is adjacent to v_3. Likewise, v is adjacent to $v_{\ell-1}$. Also by our assumption, v is not adjacent to v_4. Now $\{v_4, v_3, v, v_1, v_{\ell-1}\}$ induces a fork, a contradiction. So we may assume that there is an index $j \in \{1, 2, \ldots, \ell\}$, j mod ℓ such that v is adjacent to both v_j and v_{j+1}, say $j = \ell$. Moreover, by our contrary assumption, v has a neighbor in $\{v_3, v_4, \ldots, v_{\ell-2}\}$. Also by our earlier arguments, v is nonadjacent to both v_2 and $v_{\ell-1}$. Suppose that v has a neighbor in $\{v_3, v_4, \ldots, v_{\lceil \frac{\ell}{2} \rceil - 1}\}$; let t be the least possible integer in $\{3, 4, \ldots, \lceil \frac{\ell}{2} \rceil - 1\}$ such that v is adjacent to v_t. Now since $\{v_{\ell-1}, v_\ell, v, x, v_t\}$ does not induce a fork, $v \in M(x)$. Then since v-v_1-v_2-$\cdots$-v_t-v is not an odd hole in $G[M(x)]$ which is shorter than C, we see that t is odd. Also since $\{v_\ell, v, v_t, v_{t-1}, v_{t+1}\}$ does not induce a fork, v is adjacent to v_{t+1}. So by our assumption, v is nonadjacent to v_{t+2}. Moreover, since v-v_{t+1}-v_{t+2}-$\cdots$-$v_{\ell-1}$-v_ℓ-v is not an odd hole in $G[M(x)]$ which is shorter than C, v has a neighbor in $\{v_{t+3}, v_{t+4}, \ldots, v_{\ell-2}\}$, say v_k. But now $\{v_2, v_1, v, v_{t+1}, v_k\}$ induces a fork, a contradiction. Thus, by using symmetry, we may assume that v has no neighbor in $\{v_2, v_3, \ldots, v_{\lceil \frac{\ell}{2} \rceil - 1}, v_{\lceil \frac{\ell}{2} \rceil + 1}, \ldots, v_{\ell-1}\}$. So by our contrary assumption, v is adjacent to $v_{\lceil \frac{\ell}{2} \rceil}$. But then $\{v_1, v, v_{\lceil \frac{\ell}{2} \rceil}, v_{\lceil \frac{\ell}{2} \rceil - 1}, v_{\lceil \frac{\ell}{2} \rceil + 1}\}$ induces a fork, a contradiction. So 42:(b) holds. This proves 42.

Let X be the set $\{v \in V(G) \setminus S \mid N(v) \cap S = S\}$, and let Y be the set $\{v \in V(G) \setminus S \mid N(v) \cap S = \{v_j, v_{j+1}\}$, for some $j \in \{1, 2, \ldots, \ell\}, j \mod \ell\}$. Moreover, if $\ell = 5$, then let Z be the set $\{v \in V(G) \setminus S \mid N(v) \cap S = \{v_j, v_{j+1}, v_{j+3}\}$, for some $j \in \{1, 2, \ldots, \ell\}, j$ mod $\ell\}$, otherwise let $Z = \varnothing$. Then by 42, we immediately have the following assertion.

Claim 43. $N(S) = X \cup Y \cup Z$*, and so* $V(G) = S \cup X \cup Y \cup Z \cup M(S)$.

Claim 44. $X = \varnothing$.

Suppose to the contrary that X is nonempty. We claim that X is complete to $(V(G) \setminus X) = S \cup Y \cup Z \cup M(S)$. Suppose there are nonadjacent vertices, say $p \in X$ and $q \in V(G) \setminus X$. Recall that, by 41, X is complete to $S \cup M(S)$; in particular, x is complete to X. So $q \in Y \cup Z$. Then by our definitions of Y and Z, there is an index

$j \in \{1,2,\ldots,\ell\}, j \mod \ell$ such that q is complete to $\{v_j, v_{j+1}\}$, and anticomplete to $\{v_{j-1}, v_{j+2}\}$, say $j = 1$. Now if q is adjacent to x, then $\{q, x, p, v_3, v_\ell\}$ induces a fork, and if q is nonadjacent to x, then $\{q, v_1, p, x, v_3\}$ induces a fork. These contradictions show that X is complete to $Y \cup Z$. Thus X is complete to $V(G) \setminus X$. But then since $S \subseteq V(G) \setminus X$, we see that $V(G) \setminus X$ is a homogeneous set in G, which is a contradiction. This proves 44.

Claim 45. *Z is anticomplete to $M(S)$.*

Suppose to the contrary that there are adjacent vertices, say $p \in Z$ and $q \in M(S)$. Since $p \in Z$, by the definition of Z, we may assume that $\ell = 5$, and there is an index $j \in \{1,2,\ldots,5\}, j \mod 5$ such that $N(p) \cap S = \{v_j, v_{j+1}, v_{j+3}\}$, say $j = 1$. But now $\{q, p, v_4, v_5, v_3\}$ induces a fork, a contradiction. This proves 45.

Claim 46. *If $Z \neq \varnothing$, then $N(S) = Z$.*

Let $p \in Z$. Then by our definition of Z, we may assume that $\ell = 5$, and so there is an index $j \in \{1,2,\ldots,5\}, j \mod 5$ such that $N(p) \cap S = \{v_j, v_{j+1}, v_{j+3}\}$, say $j = 1$. Moreover, by 45, p is nonadjacent to x. Recall that $N(S) = X \cup Y \cup Z$, and by 44, we know that $X = \varnothing$. So we show that $Y = \varnothing$. Suppose to the contrary that Y is nonempty, and let $q \in Y$. Then by the definition of Y, there is an index $k \in \{1,2,\ldots,5\}, k \mod 5$ such that $N(q) \cap S = \{v_k, v_{k+1}\}$. Then, up to symmetry, we have three cases:

- $k = 1$: If p is adjacent to q, then $\{q, p, v_4, v_5, v_3\}$ induces a fork, a contradiction, and so p is nonadjacent to q. Then since $\{p, v_1, q, v_2, x\}$ does not induce a co-cricket, q is adjacent to x. But then $\{x, q, v_1, v_5, p\}$ induces a fork, a contradiction.
- $k = 2$: If p is nonadjacent to q, then $\{q, v_3, v_4, v_5, p\}$ induces a fork, a contradiction, and so p is adjacent to q. Then since $\{p, q, v_1, v_2, x\}$ does not induce a co-cricket, q is adjacent to x. But then $\{x, q, p, v_1, v_4\}$ induces a fork, a contradiction.
- $k = 3$: If p is nonadjacent to q, then $\{v_5, v_4, p, v_2, q\}$ induces a fork, a contradiction, and so p is adjacent to q. Then since $\{p, q, v_3, v_4, x\}$ does not induce a co-cricket, q is adjacent to x. But then $\{v_1, p, q, v_3, x\}$ induces a fork, a contradiction.

The above contradictions show that Y is empty. This proves 46.

Claim 47. *$Z = \varnothing$.*

Suppose to the contrary that Z is nonempty. So $\ell = 5$. Moreover, by 46, we have $N(S) = Z$, and by 45, Z is anticomplete to $M(S)$. But then, since $x \in M(S)$, we conclude that the graph is not connected, a contradiction. This proves 47.

Now by 43, 44 and 47, we conclude that $N(S) = Y$, and hence we have the following.

Claim 48. *If v is a vertex in G which has a neighbor in C, then there is an index $j \in \{1,2,\ldots,\ell\}$, $j \mod \ell$ such that $N(v) \cap S = \{v_j, v_{j+1}\}$.*

Now let K be an induced claw with vertex set $\{a, b, c, d\}$ and edge set $\{ab, ac, ad\}$. By 48, K cannot have more than two vertices on C. Also, at most one vertex in $\{b, c, d\}$ belongs to C. Then, up to symmetry, we have the following cases.

• $V(K) \cap S = \{a, d\}$: Let $a = v_1$ and $d = v_\ell$. Then by 48, $bv_2, cv_2 \in E$. But then again by 48, $\{v_{\ell-1}, d, a, b, c\}$ induces a fork, a contradiction.

• $V(K) \cap S = \{a\}$: Let $a = v_1$. Then by 48, at least two vertices in $\{b, c, d\}$ are adjacent to either v_2 or v_ℓ, say b and c are adjacent to v_2. Then again by 48, $\{v_4, v_3, v_2, b, c\}$ induces a fork, a contradiction.

• $V(K) \cap S = \{d\}$: Let $d = v_1$. Then by 48, we may assume that $av_2 \in E$. Then by 48, since $\{v_\ell, d, a, b, c\}$ does not induce a fork, v_ℓ has a neighbor in $\{b, c\}$. Also, to avoid an induced claw with center in C, v_ℓ has a nonneighbor in $\{b, c\}$. So we may assume that v_ℓ is adjacent to b, and nonadjacent to c. Thus by 48, $N(b) \cap S = \{v_{\ell-1}, v_\ell\}$. Then since $\{v_3, v_2, a, b, c\}$ does not induce a fork, by 48, c is adjacent to v_3. But then, by 48, $\{v_3, c, a, b, d\}$ induces a fork which is a contradiction.

• $V(K) \cap S = \varnothing$ *and* a *has a neighbor on* C: By 48, we may assume that $N(a) \cap S = \{v_1, v_2\}$. To avoid an induced claw intersecting C, both v_1 and v_2 have exactly two neighbors among b, c, d, and thus we may assume that v_1 is adjacent to b and c, and not adjacent to d. Again to avoid an induced claw intersecting C, exactly one of b, c is adjacent to v_ℓ, say b is adjacent to v_ℓ, and so by 48, $N(b) \cap S = \{v_1, v_\ell\}$. Moreover, by 48, $N(c) \cap S = \{v_1, v_2\}$. Then since $\{v_3, v_2, a, b, d\}$ does not induce a fork, by 48, d is adjacent to v_3. But then $\{v_3, d, a, b, c\}$ induces a fork, a contradiction.

• $V(K) \cap S = \varnothing$ *and* b *has a neighbor on* C: By 48, we may assume that $N(b) \cap S = \{v_1, v_2\}$, and we may assume that a has no neighbors on C. Then since $\{v_1, b, a, c, d\}$ does not induce a fork, we may assume that c is adjacent to v_1. Then to avoid an induced claw intersecting C, c is adjacent to v_ℓ, and so by 48, $N(c) \cap S = \{v_1, v_\ell\}$. Then since $\{v_2, b, a, c, d\}$ does not induce a fork, d is adjacent to v_2. So by 48, d is not adjacent to v_ℓ. But then $\{v_\ell, c, a, b, d\}$ induces a fork, a contradiction.

• $V(K) \cap S = \varnothing$ *and no vertex of* K *has a neighbor on* C: Since G is connected, there exists a $j \in \{1, 2, \dots, \ell\}$ and a shortest path p_1-p_2-$\cdot$-p_t-a, say P, such that $t \geqslant 2$, $v_j = p_1$, and p_2 has a neighbor on C. By the choice of P, no vertex of this path has a neighbor on C except p_2. We may assume that $j = 1$. Then by 48, we may further assume that $N(p_2) \cap S = \{v_1, v_2\}$. First suppose that $p_t = b$; so $t \geqslant 3$. Since $\{p_{t-1}, b, a, d, c\}$ does not induce a fork, we may assume that $cp_{t-1} \in E(G)$. Then by the choice of P, $cp_{t-2} \notin E(G)$, and then $\{v_\ell, v_1(= p_1), p_2, \dots, p_t(= b), c\}$ induces a graph containing a fork, a contradiction. So we may assume that $p_t \notin \{b, c, d\}$. Now if p_t has two or more neighbors in $\{b, c, d\}$, then $\{a, b, c, d, p_t, v_\ell\}$ induces a graph containing a co-cricket, a contradiction, and if p_t has exactly one neighbor in $\{b, c, d\}$, say b or if p_t has no neighbor in $\{b, c, d\}$, then $\{a, c, d, p_t, p_{t-1}\}$ induces a fork, a contradiction.

This completes the proof of the theorem. □

Corollary 49. *Let G be a (fork, co-cricket)-free graph. Then either G is claw-free or G is perfectly weight divisible, and hence perfectly divisible.*

Proof. This follows from Theorems 5 and 6, and from Theorem 40. □

Corollary 50. *Every (fork, co-cricket)-free graph G satisfies $\chi(G) \leqslant \omega(G)^2$.*

Proof. This follows from Corollary 49, Theorem 3, and Lemma 4. □

3.6 The class of (fork, bull)-free graphs

In this section, we observe that (fork, bull)-free graphs are perfectly divisible, and hence the class of (fork, bull)-free graphs is quadratically χ-bounded. We use the following theorem.

Theorem 51 ([14]). *If G is a (fork, bull)-free graph that does not admit a homogeneous set decomposition, then for every vertex v in G, $G[M(v)]$ is odd hole-free and $\overline{P_5}$-free, and hence perfect.*

Corollary 52. *Let G be a (fork, bull)-free graph. Then G is perfectly weight divisible, and hence perfectly divisible.*

Proof. This follows from Theorems 5 and 6, and from Theorem 51. □

Corollary 53. *Let G be a (fork, bull)-free graph. Then $\chi(G) \leqslant \binom{\omega(G)+1}{2}$.*

Proof. This follows from Corollary 52 and Lemma 4. □

4 Concluding remarks and open problems

We have studied the structure of (fork, F)-free graphs in the context of perfect divisibility, where F is some graph on five vertices with a stable set of size 3, and obtained quadratic χ-binding functions for such classes of graphs. Recall that if $|V(F)| \leqslant 4$, then the class of (fork, F)-free graphs is known to be quadratically χ-bounded except when $F = 4K_1$.

Problem 54. What is the smallest χ-binding function for the class of (fork, $4K_1$)-free graphs?

Further, it will be interesting to study χ-binding functions for the class of (fork, F)-free graphs, where F is a graph on five vertices with stable sets of size at most 2, in particular, for the class of (fork, C_5)-free graphs and for the class of (fork, $\overline{P_5}$)-free graphs.

The notion of perfect divisibility played a key role in proving quadratic χ-binding functions for some classes of fork-free graphs. In this paper, we showed that class of (fork, F)-free graphs is perfectly divisible, where $F \in \{P_6, \text{co-dart}, \text{bull}\}$, and the class of (fork, H)-free graphs is either claw-free or perfectly divisible, when $F \in$ {dart, banner, co-cricket}. Indeed, the third author conjectured the following.

Conjecture 55. Every fork-free graph is perfectly divisible.

The above conjecture is not even known to be true for a very special subclass of fork-free graphs, namely claw-free graphs. It is conceivable that the proof of perfect divisibility for claw-free graphs could be based on the detailed structure theorem for such graphs due to Chudnovsky and Seymour [9]. However, the proof of perfect divisibility for a subclass of claw-free graphs, namely, the class of line graphs seems to be easy as given below. The line graph of a graph G, denoted by $L(G)$, is the simple graph whose vertex-set is $E(G)$, with $e, f \in V(L(G))$ adjacent in $L(G)$ if they have a common endpoint in G. It is well-known that, from a theorem of König, line graphs of bipartite graphs are perfect; see [27].

Proposition 56. *Every line graph is perfectly divisible.*

Proof. Let G be a connected graph. If G is a tree, then $L(G)$ is perfect, and hence perfectly divisible. Hence we may assume that G is not a tree. Let T be a spanning tree of G. Consider the partition of $E(G) := E(T) \cup (E(G) \setminus E(T))$. In $L(G)$, $E(T)$ induces a perfect graph since it is the line graph of a tree. Also, since the maximum degree of $G-T$ is strictly less than that of G, $\omega(L(G)-E(T)) \leqslant \omega(L(G))$. Hence $L(G)$ is perfectly divisible. Since perfect divisibility is preserved under disjoint union, we are done. $\square$

It is well known that the complement of a perfect graph is perfect [17]. How about perfectly divisible graphs? The following proposition shows that it fails badly.

Proposition 57. *The class of graphs whose complements are perfectly divisible is not χ-bounded.*

Proof. We know that the class of $3K_1$-free graphs is perfectly divisible [13]. Hence the class of triangle-free graphs is contained in the class of graphs whose complements are perfectly divisible. It is well known that the class of triangle-free graphs have unbounded chromatic number, establishing the result. $\square$

But what about graphs G such that both G and its complement $\overline{G}$ are perfectly divisible?

Problem 58. What is the smallest χ-binding function for the class of graphs $\mathcal{G}$ such that for each $G \in \mathcal{G}$, both G and $\overline{G}$ are perfectly divisible?

Not much is known about the class of perfectly divisible graphs in general. Perhaps determining graphs that are not in the class and minimal with respect to that property will shed light on this.

Problem 59. Determine the set of forbidden induced subgraphs for the class of perfectly divisible graphs.

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