Maximizing 2-Independent Sets in 3-Uniform Hypergraphs

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Abstract

In this paper we solve three equivalent problems. The first is: what 3-uniform hypergraph on a ground set of size n, having at least e edges, has the most 2-independent sets? Here a 2-independent set is a subset of vertices containing fewer than 2 vertices from each edge. This is equivalent to the problem of determining the 3-uniform hypergraph for which the size of $\partial^+(\partial_2(\mathcal{H}))$ is minimized. Here $\partial_2(\cdot)$ is the down-shadow on level 2, and $\partial^+(\cdot)$ denotes the up-shadow on all levels. This in turn is equivalent to the problem of determining which graph on n vertices having at least e triangles has the most independent sets. The (hypergraph) answer is that, ignoring some transient and some persistent exceptions which we can classify completely, a $(2, 3, 1)$-lex style 3-graph is optimal.

We also discuss the general problem of maximizing the number of s-independent sets in r-uniform hypergraphs of fixed size and order, proving some simple results, and conjecture an asymptotically correct general solution to the problem.

Mathematics Subject Classifications: 05C35, 05C65, 05D05

1 Introduction

For many years, there has been interest in finding the maximum size of a variety of sub-structures (such as independent sets or matchings) in a graph satisfying certain conditions. In recent years, there has been increased interest in extremal questions about the *number* of these sub-structures. That is, rather than asking for the size of the largest independent

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set, one could ask which graph has the most independent sets, given some set of conditions. In fact, many extremal problems for the number of independent sets have been studied.

A simple example is the question of maximizing the number of independent sets in a graph on n vertices with e edges. A set is independent for a graph G precisely if it is not in the up-shadow of $E(G)$ on levels $2, 3, \ldots, n$. Thus the the Kruskal-Katona Theorem [6, 5] implies that the lex graph, $\mathcal{L}(n, e)$ is the optimal graph (see, e.g., [1] for details). The *lex graph* is the graph that has vertex set $[n]$ with edge set consisting of the first e sets in the lex (or dictionary) order[1] on $\binom{[n]}{2}$.

It is natural to try to extend these extremal results for the number of independent sets to hypergraphs. A *hypergraph* $\mathcal{H}$ is an ordered pair $(\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ where $\mathcal{V}(\mathcal{H})$ is a vertex set and $\mathcal{E}(\mathcal{H})$ is a set of edges where each edge is a subset of $\mathcal{V}(\mathcal{H})$. A hypergraph is r-*uniform* if all edges have size r. For convenience we'll often call an r-uniform hypergraph an *r-graph.*

In a graph, an independent set is a subset of vertices containing at most one vertex from each edge. In an r-graph for $r > 2$, it makes sense to consider allowing more than one vertex from the independent set to be in each edge.

Definition 1. For an r-graph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and an integer s with $1 \leqslant s \leqslant r$, a set $I \subset \mathcal{V}$ is *s-independent* if $|I \cap E| < s$ for all $E \in \mathcal{E}$. We let $\mathcal{I}_s(\mathcal{H})$ denote the set of s-independent sets of a hypergraph $\mathcal{H}$ and set $i_s(\mathcal{H}) = |\mathcal{I}_s(\mathcal{H})|$.

The r-independent set case, or the weak independent set case, and the 1-independent set case are much more straightforward than other cases. A 2-independent set is also known as a strong independent set. Most of the paper will focus on 2-independent sets in 3-graphs. Defining the *lex r-graph* $\mathcal{L}_r(n, e)$ to be the r-graph with vertex set $[n]$ and edge set the first e sets in the lex ordering on $\binom{[n]}{r}$, the Kruskal-Katona Theorem also implies the following:

Theorem 2. *Let $i_r(\mathcal{H})$ be the number of r-independent sets in $\mathcal{H}$. If $\mathcal{H}$ is an r-graph with n vertices and e edges then $i_r(\mathcal{H}) \leqslant i_r(\mathcal{L}_r(n, e))$.* □

For $s = 1$ the *colex r-graph* $\mathcal{C}_r(n, e)$ is optimal. $\mathcal{C}_r(n, e)$ is the r-graph with vertex set $[n]$ and edge set the first e sets in the colex order[2], on $\binom{[n]}{r}$.

Theorem 3. *If $\mathcal{H}$ is an r-graph on n vertices with e edges then $i_1(\mathcal{H}) \leqslant i_1(\mathcal{C}_r(n, e))$.* □

The proof is again Kruskal-Katona, only this time it's the down-shadow of our edges that we want to minimize. The 1-independent sets of $\mathcal{H}$ are precisely those not contained in any edge, i.e., they are the subsets of the set of isolated vertices of $\mathcal{H}$. This set of isolated vertices is exactly the complement of the down-shadow of $\mathcal{H}$ on level 1. To minimize the down-shadow the colex graph is optimal.

Remark 4. If e is not of the form $\binom{k}{r}$ for any k then there are many graphs having the same number of isolated vertices as the colex graph. In fact, if $\binom{k-1}{r} < e < \binom{k}{r}$ then any e-subset of $\binom{K}{r}$ for K a k-set has the maximum number of isolated vertices.

[1]The lex ordering, $<_L$, on $\binom{[n]}{r}$ is defined by $A <_L B$ if and only if $\min\{A \Delta B\} \in A$.

[2]The colex ordering, $<_L$, on $\binom{[n]}{r}$ is defined by $A <_L B$ if and only if $\max\{A \Delta B\} \in B$.

To talk about the general case, it will be helpful to introduce some notation for up- and down-shadows. If $\mathcal{H}$ is an r-graph on vertex set V and $s \leqslant r$, $t \geqslant r$, we define the *down-shadow on level* s to be

$$\partial_s(\mathcal{H}) = \{S \in \binom{V}{s} : \exists\, E \in \mathcal{H} \text{ s.t. } S \subseteq E\}$$

and the *up-shadow on level* t to be

$$\partial^t(\mathcal{H}) = \{T \in \binom{V}{t} : \exists\, E \in \mathcal{H} \text{ s.t. } E \subseteq T\}.$$

For an r-graph $\mathcal{H}$ on a ground set of size n we also write

$$\partial^+(\mathcal{H}) = \bigcup_{t=r}^{n} \partial^t(\mathcal{H})$$

for the entire up-shadow.

In general, a set is s-independent for $\mathcal{H}$ precisely if it is not in the up-shadow of R, where R is the down-shadow on level s of $\mathcal{H}$. That is $A \subseteq [n]$ is s-independent if and only if $A \notin \partial^+(\partial_s(\mathcal{H}))$. Maximizing the number of s-independent sets requires minimizing the size of the up-shadow of a down-shadow. This puts conflicting constraints on $\mathcal{H}$, and this makes the problem more complex. For $\mathcal{H}$ to have small lower shadow, it should look as much like a colex initial segment as possible. For R to have small upper shadow it should look as much like the lex hypergraph as possible. The case $s = 2$ is especially interesting, because in that case R is a graph.

1.1 Three equivalent problems

The problem we consider in this paper can be phrased in three ways. We determine the maximum number of 2-independent sets in a 3-graph on n vertices with e edges. Writing $\mathcal{H}_r(n, e)$ for the family of r-uniform hypergraphs with n vertices and e edges, we are determining

$$\max\{i_2(\mathcal{H}) : \mathcal{H} \in \mathcal{H}_3(n, e)\}$$

for all values of n and e. As above this is equivalent to finding

$$\min\{\partial^+(\partial_2(\mathcal{H})) : \mathcal{H} \in \mathcal{H}_3(n, e)\}.$$

The final perspective is a graph-theoretic one. If $\mathcal{H}$ is a 3-uniform hypergraph on vertex set V we can consider the graph $G = \partial_2(\mathcal{H})$. A set $I \subseteq V$ is 2-independent in $\mathcal{H}$ if and only if it does not overlap with any edge of $\mathcal{H}$ in at least 2 vertices. But this is precisely the same as requiring that I is an independent set of G. Each edge of $\mathcal{H}$ gives a triangle in G (though not necessarily *vice versa*). From this perspective we are determining

$$\max\{i(G) : n(G) = n,\ k_3(G) \geqslant e\},$$

where we write $k_3(G)$ for the number of triangles in G. For completeness we carefully prove the equivalence of the hypergraph problem and the graph problem.

Lemma 5. *For all $n, e \in \mathbb{N}$ we have*

$$\max\{i_2(\mathcal{H}) : \mathcal{H} \text{ is a 3-uniform hypergraph on vertex set } [n] \text{ with } e(\mathcal{H}) = e\}$$
$$= \max\{i(G) : G \text{ is a graph on vertex set } [n] \text{ with } k_3(G) \geqslant e\}.$$

Proof. To prove that the left hand side is at most the right we just take $\mathcal{H}$ to attain the maximum on the left and let $G = \partial_2(\mathcal{H})$. We have $k_3(G) \geqslant e(\mathcal{H}) = e$ and $i(G) = i_2(\mathcal{H})$. In the other direction, take a graph G maximizing the right hand side. Let $K_3(G)$ be the 3-uniform hypergraph on $[n]$ whose edges are the vertex sets of triangles in G. By hypothesis $e(K_3(G)) \geqslant e$, so we can take $\mathcal{H}$ to be an arbitrary sub-hypergraph of $K_3(G)$ (still on vertex set $[n]$) having exactly e edges. We get

$$i_2(\mathcal{H}) = i(\partial_2(\mathcal{H})) \geqslant i(G). \qquad \square$$

In [4], Huang, Linial, Naves, Peled, and Sudakov consider a similar problem. They give asymptotic results for the maximum number of independent sets of fixed size t in a graph on n vertices having at least e copies of K_r. Their results do not give asymptotic bounds for our problem, since they find that the optimal graph varies depending on the value of t.

We state here our main theorem using some terms that will be defined later. The edges in a $(2,3,1)$-lex hypergraph consist of an initial segments in $(2,3,1)$-lex order (described in Section 2), which combines features of both lex and colex order. See Section 4 for definitions of lexish and $(2,3,1)$-lexish. In contrast to the work of Huang *et. al.* our result describes the optimal hypergraphs/graphs precisely.

Main Theorem. *With a finite number of persistent exceptions (that appear for all values of n), and a finite number of transient exceptions (that only appear for $n \leqslant 32$) the maximum number of 2-independent sets in a 3-uniform hypergraph with e edges on n vertices is achieved either by the $(2,3,1)$-lex hypergraph or the $(2,3,1)$-lexish hypergraph having e edges.*

Equivalently, and with the same caveats, the maximum number of independent sets in a graph G, subject to having at least m triangles, is achieved either by a lex graph or a lexish graph (having at least m triangles).

We have chosen in this paper to present the hypergraph as our fundamental object for the purposes of proving the main theorem. Later we will meet the *downset* associated with a shifted hypergraph $\mathcal{H}$. This is (essentially) the edge set of $G = \partial_2(\mathcal{H})$.

In Section 2 we introduce π-lex hypergraphs (for any permutation π). In Section 3 we provide background on shifted hypergraphs and prove that an r-graph attaining the maximum number of s-independent sets can be found among the shifted hypergraphs. Section 4 states our main theorem more explicitly (Theorem 17), including specification of both the persistent and transient exceptions.

In Section 5 we introduce a way to draw a shifted 3-graph as a "nice" subset of a 3-dimensional cube and discuss a way to count the number of 2-independent sets lost when an edge is added to a shifted 3-graph. Using this we restate the problem yet again, in language useful for our proof. In Sections 6 and 7 we introduce a set of local moves that do not decrease the number of 2-independent sets. In Sections 8 and 9 we use these local moves to determine which cases are left to be proved by computation. Finally, we prove Theorem 17 in Section 10.

1.2 Conventions

We describe here some conventions that apply throughout our paper.

- Typically we abuse notation and refer to a hypergraph as its edge set, writing, for example, $E \in \mathcal{H}$ to mean $E \in \mathcal{E}(\mathcal{H})$ and $\mathcal{H} + E$ to mean $(\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}) \cup \{E\})$.
- It will be convenient for us to use a slightly non-standard ground set for our hypergraphs: we let $[n] = \{0, 1, \dots, n-1\}$, and we will consider all our hypergraphs to have vertex set $[n]$ for some n.
- We will often need to describe finite sets of integers by listing their elements. Whenever we do so we do so in increasing order. Thus when we write $A = \{a_1, a_2, \dots, a_k\}$ we will always assume that $a_1 < a_2 < \cdots < a_k$.
- We will use lg to mean $\log_2$ throughout.

2 Orderings on k-sets and π-lex Graphs

In order to state our results we need to describe a number of orderings on r-sets of integers and some associated r-graphs. These graphs are an extension of the idea of lex and colex graphs to r-graphs for $r > 2$. Recall that the *lex order*, $<_L$, on finite subsets of $\mathbb{N}$ is defined by $A <_L B$ if $\min(A \Delta B) \in A$. The *colex order*, $<_C$, is defined by $A <_C B$ if $\max(A \Delta B) \in B$. The *lex r-graph*, $\mathcal{L}_r(n, e)$, is the r-graph with vertex set $[n]$ and edge set the initial segment in the lex order on $\binom{[n]}{r}$ of length e. Similarly, the *colex r-graph*, $\mathcal{C}_r(n, e)$, is the r-graph with vertex set $[n]$ and edge set the initial segment in the colex order on $\binom{[n]}{r}$ of length e.

Example 6. The first few edges in the lex ordering on $\binom{[n]}{2}$ are

$$\{0,1\}, \{0,2\}, \dots, \{0, n-1\}, \{1,2\}, \{1,3\}, \dots, \{1, n-1\}, \{2,3\}, \dots$$

and the first few edges in the colex ordering are

$$\{0,1\}, \{0,2\}, \{1,2\}, \{0,3\}, \{1,3\}, \{2,3\}, \{0,4\}, \{1,4\}, \dots$$

Note that initial segments of colex do not depend on the size of the ground set, unlike those of the lex ordering. Sets that are early in the lex ordering have small least elements, and sets that are early in the colex ordering have small greatest elements. This idea will help in understanding π-lex graphs.

In r-graphs for $r > 2$ we can define other orders on $\binom{[n]}{r}$ leading to other r-graphs. In fact, we can define $r!$ orderings. While these seem natural we have not seen them introduced elsewhere.

Definition 7. Consider a permutation $\pi = (\pi_1, \dots, \pi_k)$ and let $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_k\}$ be sets in $\binom{[n]}{k}$. We define the *π-lex order* on $\binom{[n]}{k}$ by $A <_\pi B$ if for the least i for which $a_{\pi_i} \neq b_{\pi_i}$ we have $a_{\pi_i} < b_{\pi_i}$.

Given a permutation π, define the *π-lex r-graph with n vertices and e edges* to be the r-graph on vertex set $[n]$ with edge set forming an initial segment of the π-lex order on $\binom{[n]}{r}$ of length e.

Example 8. The lex ordering on $\binom{[n]}{3}$ is π-lex for $\pi = (1, 2, 3)$ and the colex ordering on $\binom{[n]}{3}$ is π-lex for $\pi = (3, 2, 1)$. The π-lex ordering that will be particularly important to us is the $(2, 3, 1)$-lex ordering. The first few sets in the $(2, 3, 1)$-lex ordering on $\binom{[n]}{3}$ are

$$\begin{aligned}\{0,1,2\}, \{0,1,3\}, \dots, \{0,1,n-1\}, \{0,2,3\}, \{1,2,3\}, \{0,2,4\}, \{1,2,4\}, \dots\\ \{0,2,n-1\}, \{1,2,n-1\}, \{0,3,4\}, \{1,3,4\}, \{2,3,4\}, \{0,3,5\},\\ \{1,3,5\}, \{2,3,5\}, \dots, \{0,3,n-1\}, \{1,3,n-1\}, \{2,3,n-1\},\\ \{0,4,5\}, \{1,4,5\}, \{2,4,5\}, \{3,4,5\}, \dots, \{0,4,n-1\},\\ \{1,4,n-1\}, \{2,4,n-1\}, \{3,4,n-1\}, \dots\end{aligned}$$

Notice that sets that are small in the $(2, 3, 1)$-lex ordering have their second greatest element being small.

There is a natural partial ordering on $\binom{[n]}{k}$ that will also be relevant, that we call the *compression* ordering. Given $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_k\}$ we let $A \preccurlyeq B$ if $a_i \leqslant b_i$ for all i. Equivalently, $A \preccurlyeq B$ if and only if for all $x \in \mathbb{R}$ we have $|A \cap (-\infty, x]| \geqslant |B \cap (-\infty, x]|$. The following simple lemma will be useful later.

Lemma 9. *If A_1, B_1 are s-sets with $A_1 \preccurlyeq B_1$, A_2, B_2 are $(r-s)$-sets with $A_2 \preccurlyeq B_2$ and $A_1 \cap A_2 = B_1 \cap B_2 = \varnothing$ then $A \preccurlyeq B$ where $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$.*

Proof. For all $x \in \mathbb{R}$ we have

$$\begin{aligned}|A \cap (-\infty, x]| &= |A_1 \cap (-\infty, x]| + |A_2 \cap (-\infty, x]|\\ &\geqslant |B_1 \cap (-\infty, x]| + |B_2 \cap (-\infty, x]| = |B \cap (-\infty, x]|.\end{aligned}$$ □

3 Shifted Hypergraphs

The notion of *shifted hypergraphs*, introduced in [3] will be crucial in our proof because s-independent sets in r-graphs are maximized by shifted hypergraphs. We will use this fact to restate the problem. First we carefully define what it means to be a shifted hypergraph.

Definition 10. Given a set $A \subset [n]$ and $i, j \in [n]$ such that $A \cap \{i, j\} = \{i\}$ define $A_{i \to j} = (A \setminus \{i\}) \cup \{j\}$. Consider a hypergraph $\mathcal{H}$ with vertex set $[n]$ and edge set $\mathcal{E}$. For $0 \leqslant j < i \leqslant n-1$ define $\mathcal{H}_{i \to j}$ to be the hypergraph with vertex set $[n]$ and edge set $\{S_{i\to j}(E) : E \in \mathcal{E}\} \cup \{E : E, S_{i \to j}(E) \in \mathcal{E}\}$ where for each $E \in \mathcal{E}$,

$$S_{i\to j}(E) = \begin{cases} E_{i \to j} & \text{if } E \cap \{i,j\} = \{i\} \\ E & \text{otherwise} \end{cases}.$$

A hypergraph $\mathcal{H} = ([n], \mathcal{E})$ is *shifted* if and only if $\mathcal{H}_{i\to j} = \mathcal{H}$ for all $0 \leqslant j < i \leqslant n-1$.

Thus, $\mathcal{H}_{i\to j}$ is a hypergraph with the same number of edges as $\mathcal{H}$ with the same sizes, but where we have replaced i with j whenever possible. We will extend the definition of $\mathcal{H}_{i\to j}$ slightly and set $\mathcal{H}_{i\to i} = \mathcal{H}$ for all $i \in [n]$. In the next definition we extend again to apply a number of shifts at once.

Definition 11. Given an r-graph $\mathcal{H}$ and k-sets $A \preccurlyeq B$ with $A = \{a_1, a_2, \ldots, a_k\}$, $B = \{b_1, b_2, \ldots, b_k\}$ we define

$$\mathcal{H}_{B\to A} = (\cdots((\mathcal{H}_{b_1\to a_1})_{b_2\to a_2})\cdots)_{b_k\to a_k}.$$

We will use this definition in Section 5. In particular, we will use the fact that if we apply a shift from all the vertices in one edge to another r-set of vertices, A, then A will be in the edge set of the shifted graph. We prove this in the next lemma.

Lemma 12. *If $\mathcal{H}$ is an r-graph on $[n]$ and $A \preccurlyeq B$ are r-sets with $B \in \mathcal{H}$ then $A \in \mathcal{H}_{B\to A}$.*

Proof. We'll prove it by (reverse) induction on the parameter

$$\ell = \max\{j \,:\, a_i = b_i \text{ for all } i \leqslant j\}.$$

If $\ell = r$ then $A = B$ and there is nothing to prove. If $\ell = r-1$ then $A = B_{b_r\to a_r}$ and $\mathcal{H}_{B\to A} = \mathcal{H}_{b_r\to a_r}$. It is clear from the definition of shifting that $A \in \mathcal{H}_{B\to A}$. Suppose then that $\ell < r-1$. Note that $a_{\ell+1} \neq b_{\ell+1}$. Consider $B' = B \,\triangle\{a_{\ell+1}, b_{\ell+1}\} = B_{b_{\ell+1}\to a_{\ell+1}}$. We have $A \preccurlyeq B' \preccurlyeq B$. Since all earlier compressions have no effect we have $\mathcal{H}_{B\to A} = (\mathcal{H}_{b_{\ell+1}\to a_{\ell+1}})_{B'\to A}$. By the definition of shifting we know that $B' \in \mathcal{H}_{b_{\ell+1}\to a_{\ell+1}}$ since $B \in \mathcal{H}$. This implies by induction that $A \in (\mathcal{H}_{b_{\ell+1}\to a_{\ell+1}})_{B'\to A} = \mathcal{H}_{B\to A}$, as required. $\square$

Proposition 13. *A hypergraph maximizing the number of s-independent sets among all hypergraphs with n vertices and e edges can be found among the shifted hypergraphs.*

Proof. Let $\mathcal{H}$ be a hypergraph with vertex set $[n]$ and let $0 \leqslant j < i < n$. First, one can define a function from $\mathcal{I}_s(\mathcal{H}) \setminus \mathcal{I}_s(\mathcal{H}_{i\to j}) \to \mathcal{I}_s(\mathcal{H}_{i\to j}) \setminus \mathcal{I}_s(\mathcal{H})$ by $I \mapsto I_{j\to i}$. By showing this map is an injection we conclude that for all s,

$$i_s(\mathcal{H}_{i\to j}) \geqslant i_s(\mathcal{H}).$$

Let $t(\mathcal{H}) = \sum_{E\in\mathcal{E}(\mathcal{H})} \sum_{i\in E} i$. Pick $\mathcal{H}$ with the maximal number of s-independent sets and $t(\mathcal{H})$ minimal. Let $0 \leqslant j < i \leqslant n$. Then $\mathcal{H}_{i\to j}$ has the same number of vertices and edges as $\mathcal{H}$ and $i_s(\mathcal{H}_{i\to j}) \geqslant i_s(\mathcal{H})$. Thus, we must have $\mathcal{H}_{i\to j} = \mathcal{H}$, else $t(\mathcal{H}_{i\to j}) < t(\mathcal{H})$ contradicting the definition of $\mathcal{H}$. So $\mathcal{H}$ is a shifted hypergraph maximizing the number of s-independent sets. $\square$

For the remainder of the paper we will focus on shifted hypergraphs.

4 Formal Statement of Main Result

Theorems 2 and 3 answer the question of which 3-graphs have the most 3-independent sets and 1-independent sets, respectively. Our main result answers the question of which 3-graphs have the most 2-independent sets. We need some preliminary definitions before we state the theorem.

As shown in Section 3, we need only consider shifted hypergraphs. It will turn out that the feature of a shifted 3-graph $\mathcal{H}$ that determines $i_2(\mathcal{H})$ is the collection of its edges that contain 0. We make the following definition so that we can state our main result, but we discuss the topic more extensively in Section 5.

Definition 14. Given a shifted 3-graph $\mathcal{H}$ the *downset* of $\mathcal{H}$ is the set

$$D(\mathcal{H}) = \{(i,j) : \{0,i,j\} \in \mathcal{H}\}.$$

This is indeed a downset in the poset

$$B_n = \{(i,j) : 1 \leqslant i < j \leqslant n-1\} \subseteq \{1,2,\ldots,n-1\}^2$$

with the product order. We call B_n the *base layer*.

Associating hypergraphs to downsets is a many to one relationship. A hypergraph $\mathcal{H}$ has exactly one downset, but given a downset D, there are often many (shifted) hypergraphs that have downset D. An example of how we visualize the downset is shown in Figure 1. A cell (i,j) is shaded provided that $\{0,i,j\} \in \mathcal{H}$. The downset of a hypergraph differs from the lower shadow $\partial_2(\mathcal{H})$ introduced in Section 1.1 in that the edges in $\partial_2(\mathcal{H})$ that contain 0 are not shown in the downset—they are implied.

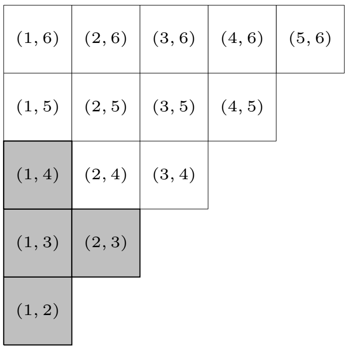

Figure 1: A visualization for a downset for a hypergraph with 7 vertices. The hypergraph could have edge set $\{\{0,1,2\},\{0,1,3\},\{0,1,4\},\{0,2,3\}\}$ or $\{\{0,1,2\},\{0,1,3\},\{0,1,4\},\{0,2,3\},\{1,2,3\}\}$.

In Section 2 we introduced $(2,3,1)$-lex 3-graphs. The maximizers of 2-independent sets in $\mathcal{H}_3(n,e)$ are generally $(2,3,1)$-lex graphs. We describe the 3-graphs that are maximizers by their downsets in the following definition. In the introduction we referred to these graphs as $(2,3,1)$-lexish.

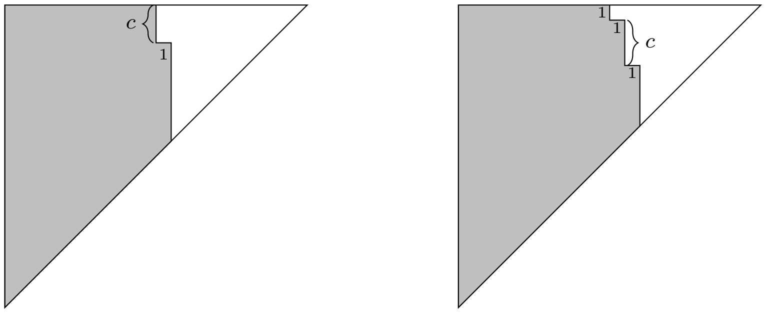

Figure 2: The (zoomed out) downset of a $(2,3,1)$-lex 3-graph at left and a $(2,3,1)$-lex style 3-graph that is not $(2,3,1)$-lex at right. The vertical drops are length c for some $c \geqslant 0$.

Definition 15. We say that a shifted 3-graph $\mathcal{H}$ is $(2,3,1)$-*lex style* if its downset $D = D(\mathcal{H})$ satisfies

- D is an initial segment in lex order, or
- D is a downset in B_n that is an initial segment in lex order missing one edge.

The possible downsets of $(2,3,1)$-lex style 3-graphs are shown in Figure 2.

Remark 16. A $(2,3,1)$-lex graph will have a downset that is an initial segment in lex order. Thus all $(2,3,1)$-lex graphs are $(2,3,1)$-lex style. Also, if D is a downset in B_n that is an initial segment in lex order missing one edge then that edge must correspond to the top cell in the second to last column. This is shown in the right downset in Figure 2.

Theorem 17 says, roughly, that hypergraphs that have downsets that are $(2,3,1)$-lex style maximize 2-independent sets. Additionally, we describe the exceptions by their lower shadow graph.

Theorem 17. *Let $\mathcal{H}$ be a 3-graph on n vertices with e edges where $n \geqslant 33$. Then there exists a shifted 3-graph $\mathcal{G}$ with n vertices and e edges such that*

$$i_2(\mathcal{H}) \leqslant i_2(\mathcal{G}),$$

where $\mathcal{G}$ is either $(2,3,1)$-lex style or $\mathcal{G}$ has $\partial_2(\mathcal{G})$ coming from one of the following set of 5 persistent exceptions:

$$\mathcal{P}_n = \{K_4 \cup E_{n-4}, (K_3 \vee E_{n-6}) \cup E_3, (K_3 \vee E_{n-6}) \cup E_2, (K_4 \vee E_{n-5}) \cup E_1, (K_5 \vee E_{n-6}) \cup E_1\}.$$

where E_j is an independent set of size j (here we write $G \vee H$ for the join of two graphs G and H). When $n < 33$ there are 16 possible downsets of hypergraphs that maximize 2-independent sets that are not $(2,3,1)$-lex style or in $\mathcal{P}_n$. These downsets are shown in Table 1.

n	$\partial_2(\mathcal{H})$
7	K_5
8	K_5, K_6, K_7
9	K_5, K_6, K_7, K_8, $K_9 - K_{1,6}$
10	K_9
11	K_{10}, $K_{11} - K_{1,9}$
12	K_{11}
14	K_{13}, $K_{13} - e$
16	K_{15}

Table 1: All exceptions to the maximizer being $(2, 3, 1)$-lex style or in $\mathcal{P}_n$ when $n < 33$.

To complete the picture, we state the equivalent theorem for the graph problem. We need a definition first.

Definition 18. A graph with n vertices and e edges is *lexish* if it is either the lex graph $\mathcal{L}(n, e)$ or else $\mathcal{L}(n, e) - f$ where f is the edge $(i-1)n$ where i is such that $\{1, 2, \dots, i+1\}$ is the unique largest clique in $\mathcal{L}(n, e)$.

Theorem 19. *Let H be a graph on n vertices with t triangles where $n \geqslant 33$. Then there exists a graph G on n vertices such that $k_3(G) \geqslant t$ and $i(G) \geqslant i(H)$ and moreover G is either a lex graph, a lexish graph, or $G \in \mathcal{P}_n$.*

5 Counting 2-independent Sets in Shifted 3-graphs

In this section we will develop a way to count 2-independent sets in shifted 3-graphs. This will result in a translation of the problem to an optimization problem that is easier to visualize.

Definition 20. Given $r \geqslant s \geqslant 2$, suppose $I \subseteq [n]$ is a set of size at least s. Let I_s be the s-set consisting of the s smallest elements of I, and let J be the $r - s$ smallest elements of $[n] \setminus I_s$. Define the *minimal edge of* I to be $E_0(I) = I_s \cup J$. Note that $E_0(I)$ is the unique $\preccurlyeq$-minimal set in $\binom{[n]}{r}$ that has $|E \cap I| \geqslant s$.

Remark 21. For $r = 3, s = 2$ and $I \subset [n]$ of size at least 2, the minimal edge of I is $E_0(I) = \{a_1, a_2, b\}$ where a_1 and a_2 are the two smallest elements of I and $b = \min\{i \in [n] : i \neq a_1, a_2\}$.

The purpose of defining the minimal edge of a set I is that I is s-independent in a shifted r-graph $\mathcal{H}$ exactly when $E_0(I)$ is not in $\mathcal{H}$.

Lemma 22. *Let $\mathcal{H}$ be a shifted r-graph and consider a set $I \subseteq [n]$ with $|I| \geqslant s$. The set I is s-independent in $\mathcal{H}$ if and only if $E_0(I) \notin \mathcal{E}(\mathcal{H})$.*

Proof. Suppose that I is an s-independent set. Then $E_0(I) \notin \mathcal{E}(\mathcal{H})$ since $|I \cap E_0(I)| \geqslant s$. Suppose now that I is not an s-independent set. There exists an edge $E \in \mathcal{H}$ such that $|E \cap I| \geqslant s$. Let E_s be the set of the s smallest elements of $E \cap I$, and F be $E \setminus E_s$. Note that, with the notation of the previous definition, $I_s \preccurlyeq E_s$, since I_s is the unique $\preccurlyeq$-minimal s set in I. It is also true that $J \preccurlyeq F$. To see this note first that $F \subseteq [n] \setminus I_s$; any $x \in F \cap I_s$ would have to be one of the s smallest elements of $E \cap I$, hence in E_s, a contradiction. Now $J \preccurlyeq F$ since J is the unique $\preccurlyeq$-minimal $(r-s)$-set in $[n] \setminus I_s$. By Lemma 9 we have $E_0(I) = I_s \cup J \preccurlyeq E_s \cup F$. Now by Lemma 12, since $E \in \mathcal{H}$, we have

$$E_0(I) \in \mathcal{H}_{E \to E_0(I)} = \mathcal{H},$$

the last equality holding since $\mathcal{H}$ is shifted. $\square$

Corollary 23. *Let $I \subset [n]$ with $|I| \geqslant s$. Suppose $\mathcal{H}' = \mathcal{H} + E$ and that $\mathcal{H}'$ and $\mathcal{H}$ are shifted r-graphs. Then $I \in \mathcal{I}_s(\mathcal{H}) \setminus \mathcal{I}_s(\mathcal{H}')$ if and only if $E_0(I) = E$.*

Proof. By Lemma 22, $I \in \mathcal{I}_s(\mathcal{H})$ if and only if $E_0(I) \notin \mathcal{H}$ and $I \notin \mathcal{I}_s(\mathcal{H}')$ if and only if $E_0(I) \in \mathcal{H}'$. Thus, $I \in \mathcal{I}_s(\mathcal{H}) \setminus \mathcal{I}_s(\mathcal{H}')$ if and only if $E_0(I) = E = \mathcal{H}' \setminus \mathcal{H}$. $\square$

Now we are able to calculate the number of sets that are lost when an edge is added to a shifted hypergraph.

Lemma 24. *Let $\mathcal{H}$ be a shifted 3-graph on vertex set $[n]$, let $E = \{i, j, k\}$ and suppose that $\mathcal{H}' = \mathcal{H} + E$ is also shifted. Then*

$$i_2(\mathcal{H}') = i_2(\mathcal{H}) - c_{ijk}$$

where

$$c_{ijk} = \begin{cases} 2^{n-1} & \text{if } \{i,j,k\} = \{0,1,2\}, \\ 2^{n-k} & \text{if } i = 0, j = 1 \text{ and } k \neq 2, \\ 2^{n-k-1} & \text{if } i = 0 \text{ and } j > 1, \\ 0 & \text{if } i \neq 0. \end{cases}.$$

Remark 25. We will refer to c_{ijk} as the *cost* of the edge $\{i, j, k\}$.

Proof. By Corollary 23, $I \in i_2(\mathcal{H}) \setminus i_2(\mathcal{H}')$ if and only if $E_0(I) = E$. Thus, to determine the cost of an edge E we must count the number of sets I such that $E_0(I) = E$.

If $E = \{0, 1, 2\}$ we are counting sets such that $E_0(I) = \{0, 1, 2\}$. These are exactly those sets having two smallest elements 0 and 1, 0 and 2, or 1 and 2. The number of sets with this property is $2^{n-2} + 2^{n-3} + 2^{n-3} = 2^{n-1}$. Thus, $c_{012} = 2^{n-1}$.

Suppose that $\{0, 1, k\}$ is added to a hypergraph where $k \neq 2$. Here we count sets I such that $E_0(I) = \{0, 1, k\}$. These are the sets with smallest elements 0 and k or 1 and k. The number of sets with this property is $2^{n-k-1} + 2^{n-k-1} = 2^{n-k}$. Thus $c_{01k} = 2^{n-k}$ for $k \neq 2$.

Suppose now $E = \{0, j, k\}$ with $j > 1$. Here, $E_0(I) = E$ if and only if the two smallest elements of I are j and k. There are 2^{n-k-1} of these meaning $c_{0jk} = 2^{n-k-1}$ when $j > 1$.

Finally, if $0 \notin E$ then it is not one of the edges of the form $E = \{a_1, a_2, b\}$ where $b = \min\{i \in [n] : i \neq a_1, a_2\}$. Thus, the cost of $\{i, j, k\}$ where $i \neq 0$ is 0. $\square$

Note that $\sum_{i<j<k} c_{ijk} = 2^n - (n+1)$ meaning that $i_2(\mathcal{K}_n^3) = n+1$ where $\mathcal{K}_n^3$ is the complete 3-graph on n vertices. The 2-independent sets in $\mathcal{K}_n^3$ are the empty set and all the singletons.

Let $\mathcal{H}$ be a 3-graph with vertex set $[n]$. We will visualize $\mathcal{H}$ by letting its edges be $1 \times 1 \times 1$ cubes labeled by the vertices in the edge in increasing order. Then we can think of these $1 \times 1 \times 1$ cubes inside an $(n-2) \times (n-2) \times (n-2)$ cube labeled as in Figure 3. Figure 4 shows the edges of the complete hypergraph on 7 vertices inside a $5 \times 5 \times 5$ cube with the visible cubes labeled.

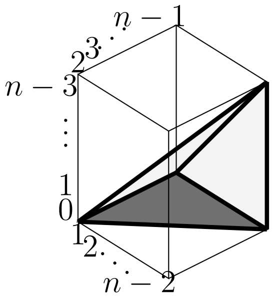

Figure 3: The labeling of the cube. The shaded tetrahedron represents the collection of $1 \times 1 \times 1$ cubes that have labels in increasing order.

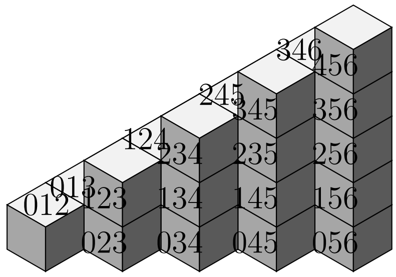

Figure 4: Edges of the complete hypergraph on 7 vertices.

Lemma 24 says that, assuming the hypergraph is shifted, any edge that does not contain 0 is "free", i.e., adding such an edge does not cost us any independent sets. More rigorously, if $E = \{i, j, k\}$ with $i \neq 0$ we have $i_2(\mathcal{H}) = i_2(\mathcal{H} + E)$. In the cube picture this means that any edge that is not in the bottom layer is free. For this reason, we focus on the downset of $\mathcal{H}$. The downset of $\mathcal{H}$ corresponds to edges in the base layer. Figure 5 shows the cube where we have suppressed the first dimension and show only the edges with non-zero costs.

016	026	036	046	056
015	025	035	045	
014	024	034		
013	023			
012				

Figure 5: Edges in base layer, B_7.

We will call each of the squares in B_n a *cell* and label it (a, b) if the edge associated to that square is $\{0, a, b\}$.

Recall that we are restricting ourselves to shifted hypergraphs as we can find a maximizer among the shifted hypergraphs. By definition a shifted hypergraph $\mathcal{H}$ on vertex set $[n]$ satisfies the following condition: if $\{a, b, c\} \in \mathcal{E}(\mathcal{H})$ then $\{i, j, k\} \in \mathcal{E}(\mathcal{H})$ whenever $i \leqslant a$, $j \leqslant b$, and $k \leqslant c$. In B_n this says that if $\{0, b, c\} \in \mathcal{E}(\mathcal{H})$ then $\{0, j, k\} \in \mathcal{E}(\mathcal{H})$ for all $j \leqslant b$ and $k \leqslant c$. That is, if we include a cell (b, c) in our hypergraph, we must also include all cells that are to the left or below.

Each cell has an associated cost as given in Lemma 24 and an associated amount of *space*: the number of edges we could get for that cost, given that taking those edges results in a shifted hypergraph. The cost and space for cells in B_7 are given in Figure 6.

2	1	1	1	1
4	2	2	2	
8	4	4		
16	8			
64				

1	2	3	4	5
1	2	3	4	
1	2	3		
1	2			
1				

Figure 6: At left the cost of each cell in B_7, at right the space in each cell.

For D, a collection of cells, let $C(D)$ be the cost of those cells and $S(D)$ be the amount of space in those cells.

Remark 26. The space of a cell (i, j) is i. We chose $[n] = \{0, 1, \ldots, n-1\}$ for this reason.

Our goal, finding a 3-graph on n vertices having e edges with the maximum number of 2-independent sets, can be rephrased as follows: find a downset D in B_n such that $C(D)$ is minimized subject to the condition that $S(D) \geqslant e$.

For the rest of the paper we will only be concerned with the shape of the downset in the base layer. Given a downset in B_n that has enough space to accommodate the number of edges we need we can arrange the edges in higher layers to get a shifted 3-graph (often in several ways). When we discuss the number of 2-independent sets in $D \subseteq B_n$ we mean the number of 2-independent sets in any $\mathcal{H}$ that has downset D.

Finally we introduce an order on downsets in B_n. For downsets D and D' we say that D is lex-less than D', or $D <_L D'$, if

$$\min_{\text{Lex}} D \Delta D' \in D.$$

Here $\min_{\text{Lex}} D\Delta D'$ means the minimum cell in $D\Delta D'$ under the lex ordering on cells in B_n.

Definition 27. A downset D in B_n is an *optimal downset* if, for some e, D minimizes $C(D)$ among all downsets with space at least e and it is the earliest downset in lex order to do so.

6 Local Moves

In this section we show certain downsets in B_n do not have as many 2-independent sets as the downset associated to a $(2,3,1)$-lex style 3-graph. Our strategy is to show that, given a downset D that is not $(2,3,1)$-lex style, there exists a downset D' such that $S(D') \geqslant S(D)$, $C(D') \leqslant C(D)$, and $D' <_L D$. That is, we will show that some downsets that are not $(2,3,1)$-lex style are not optimal downsets. We'll call the switch from D to D' a *local move*. To talk about the local moves we first need the definition of corner.

Definition 28. For a downset D the cell (a,b) is a *corner* of D if it is a maximal element of D with respect to the compression ordering.

The rest of this section is organized into three subsections, one for each of the three types of local moves we will perform. In Section 6.1 we will perform "one cell moves", that is, local moves in which we remove only one cell from D. In Section 6.2 we will perform "column moves" which are local moves in which we remove a column-like subset of the downset D. Finally in Section 6.3 we consider a local move that moves a large subset of cells.

6.1 One Cell Moves

First we will consider some local moves where we exchange one cell of a downset D for two cells in $B_n \setminus D$. To do this, we first define the horizontal distance vector of a downset.

Definition 29. For a downset D, let $o_1, o_2, \ldots, o_k$ be the horizontal positions of the corners written in increasing order. We define the *horizontal distance vector* be $(o_2 - o_1, o_3 - o_2, \ldots, o_k - o_{k-1})$.

Lemma 30. *Let D be a downset with corners at positions $o_1, o_2, \ldots, o_k$ and with horizontal distance vector $(d_1, d_2, \ldots, d_{k-1})$. If $3 \leqslant d_i \leqslant \frac{o_{i+1}+3}{2}$ for at least one $1 \leqslant i \leqslant k-1$, then D is not optimal.*

Proof. Let (a,b) and (c,d) be consecutive corners such that $3 \leqslant c - a \leqslant \frac{c+3}{2}$. Since the previous corner is (a,b) we can remove cell (c,d) and replace it with cells $(a+1, d+1)$ and $(a+2, d+1)$ and still have a downset. Let $D' = D - (c,d) + (a+1,d+1) + (a+2,d+1)$. The move from D to D' is illustrated in Figure 7. The dark gray cell is removed and replaced with the two cells with checkmarks.

Note the space of cell (c,d) is c and the space in the replacement cells is collectively $2a+3$. Since $c - a \leqslant \frac{c+3}{2}$ we have $c \leqslant 2a+3$ and so there is at least as much space in D'. Moreover, the cost of each of the replacement cells is half the cost of (c,d) and so $C(D) = C(D')$. Finally $D' <_L D$. Therefore such a D is not optimal. $\square$

Lemma 30 says that in an optimal downset the horizontal distance between two corners is either small (less than 3) or is large (about half the larger amount of space). Let's consider first when the horizontal distance between corners is small. When the horizontal distance between two corners is 1 we will say there is a *short stair* and when the horizontal distance between two consecutive corners is 2 we will say there is a *long stair*.

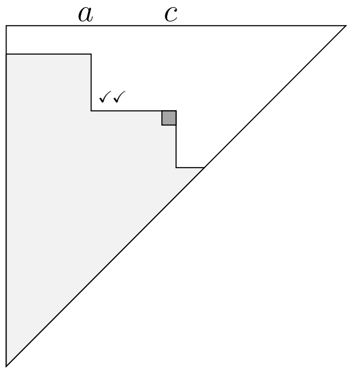

Figure 7: Move occurring in the proof of Lemma 30 for consecutive corners.

Lemma 31. *If D is a downset whose horizontal distance vector has three consecutive 1's, two consecutive 2's, or an adjacent 1 and 2 then D is not optimal.*

Proof. In Figure 8 we show the downsets resulting from the horizontal distance vectors having three consecutive 1's, two consecutive 2's, a 2 followed by a 1, and a 1 followed by a 2. In each case we can show that there is a downset with at least as much space and less cost that is earlier in lex order.

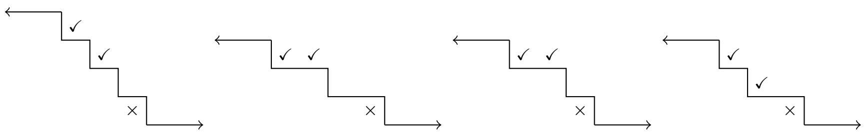

Figure 8: From left to right, 3 short stairs, 2 long stairs, 1 long stair followed by a short stair, and 1 short stair followed by a long stair. The vertical drops may be of any height at least 1. We create downsets that are earlier in lex order by removing cells marked $\times$ and replacing them with cells marked $\checkmark$.

Suppose that the horizontal distance vector has three consecutive 1's. Name the corresponding corners (i, a), $(i+1, b)$, $(i+2, c)$, and $(i+3, d)$ and note $a > b > c > d$. Consider the downset $D' = D - (i+3, d) + (i+1, b+1) + (i+2, c+1)$. Since $(i+1)+(i+2) = 2i+3 > i+3$ we have $S(D') > S(D)$. Moreover, since $a > b > c$, the cost of $(i+2, c+1)$ is at most a fourth the cost of the cell $(i+3, d)$ and the cost of the cell $(i+1, b+1)$ is at most an eighth of the cost of the cell $(i+3, d)$. Therefore $C(D') < C(D)$. Finally, $D' <_L D$ and so D is not optimal. The proof for each of the other cases is similar. $\square$

From Lemma 31 we know that in an optimal downset the only possible "staircases" are 1 long stair, 1 short stair, or 2 short stairs. Note that these are exactly the types of staircases that appear at the end of a downset of a $(2, 3, 1)$-lex style hypergraph. Lemma 32 describes the types of vertical drops that can appear in these transitions.

Lemma 32. *Suppose D is a downset with corners (a, b), $(a+1, c)$ and $(a+2, d)$. If $b - c > 1$ then D is not an optimal downset. Similarly, if D is a downset with corners (a, b) and $(a+2, c)$ where $b - c > 1$ then D is not an optimal downset.*

Proof. First consider a downset D with corners (a, b), $(a+1, c)$, and $(a+2, d)$. If $b-c > 1$ then $D' = D - (a+2, d) + (a+1, c+1) + (a+1, c+2)$ is a downset with $C(D') < C(D)$, $S(D') > S(D)$, and $D' <_L D$. For a downset D with corners (a, b) and $(a+2, c)$ if $b-c > 1$ then $D' = D - (a+2, c) + (a+1, c+1) + (a+1, c+2)$ is a downset with $C(D') < C(D)$, $S(D') > S(D)$, and $D' <_L D$. □

Lemmas 30, 31, and 32 allow us to say that optimal downsets have small groups of corners that are "far" apart. The small groups (or "transitions") look like those in Figure 9 where the uncolored drops have arbitrary length. We refer to the leftmost in this group of corners as the *top stair*.

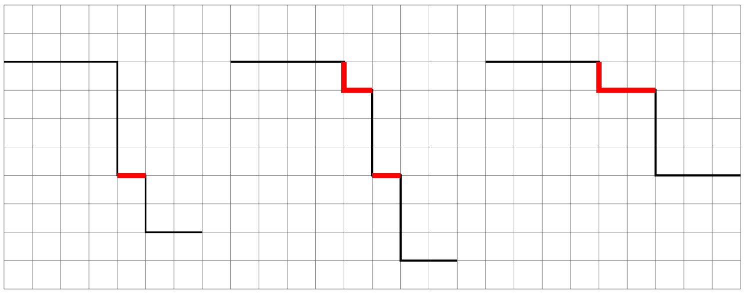

Figure 9: From left to right: one short stair, two short stairs, and one long stair. The bolded red segments are of fixed length, all other segments are arbitrarily long. These are the possible transitions in an optimal downset.

We will say that a downset *ends with stairs* if the last entry of the horizontal distance vector is a 1 or a 2. Lemmas 30 and 31 say that if a downset ends with stairs, then it ends with 2 short stairs, 1 short stair, or 1 long stair. In the next lemma we address downsets that end with 2 short stairs or 1 long stair and are not $(2, 3, 1)$-lex style.

Lemma 33. *Suppose that D is not $(2, 3, 1)$-lex style. If D ends with 2 short stairs or 1 long stair then D is not an optimal downset.*

Proof. In each case we replace the dark gray cell with the check-marked cells shown in Figure 10.

Note in the bottom two cases, where there is no earlier corner, we can guarantee that the top cell in the first column is available, else the hypergraph would already be $(2, 3, 1)$-lex style. □

6.2 Column Moves

In this section we apply moves in which a subset of the cells in the last column of the downset are traded for a row. These moves will be used on downsets that have that their

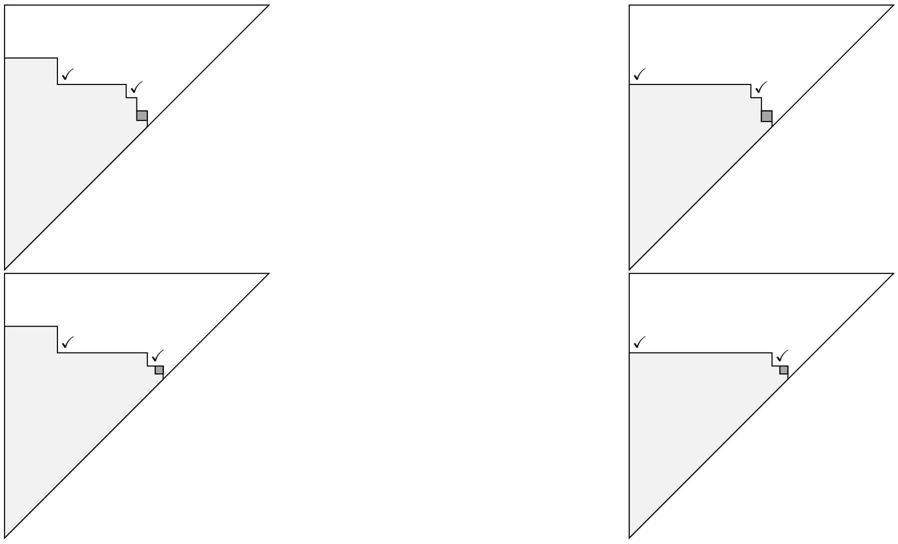

Figure 10: Downsets that are not $(2,3,1)$-lex style that end in 2 short stairs or 1 long stair, with or without previous corners, are not optimal.

last corner (i,j) satisfies $j-i \geqslant \lfloor \lg(i) \rfloor$. Since having a corner (i,j) means the number of cells in column j is $j-i$ this is ensuring that the last column of the downset has at least $\lfloor \lg(i) \rfloor$ cells.

Lemma 34. *If D is not $(2,3,1)$-lex style, the only corner is (i,j) where $j-i \geqslant \lfloor \lg(i) \rfloor$, and $i \geqslant 5$, then D is not an optimal downset.*

Proof. Since D is not $(2,3,1)$-lex style, $j < n-1$. Let $t = \lfloor \lg(i) \rfloor$ and define $L = \{(i,h) : j-t+1 \leqslant h \leqslant j\}$ and $R = \{(h, j+1) : 1 \leqslant h \leqslant i-2\}$. Consider $D' = D - L + R$. Note that we add all possible cells in the row except for one (see Figure 11).

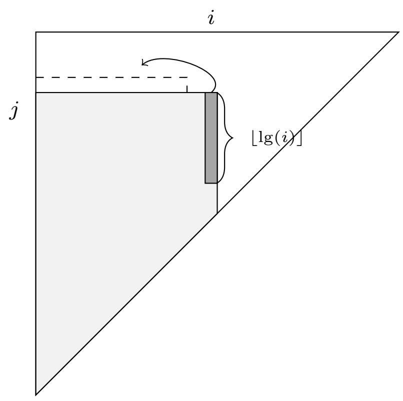

Figure 11: Column move in the proof of Lemma 34.

Computing the cost of L and R we have

$$\begin{aligned} C(L) &= 2^{n-j-1} + 2^{n-j} + \cdots + 2^{n-j-1+t-1} \\ &= 2^{n-j-1}(2^t - 1) \\ &= 2^{n-j-2}(2^{\lfloor \lg(i) \rfloor + 1} - 2) \end{aligned}$$

and

$$C(R) = 2^{n-j-2}(i-3) + 2^{n-j-1} = 2^{n-j-2}(i-1).$$

Since $2^{\lfloor \lg(i) \rfloor + 1} - 2 \geqslant i - 1$, we have $C(D) \geqslant C(D')$. Moreover,

$$S(L) = i \cdot \lfloor \lg(i) \rfloor$$

and

$$S(R) = \frac{(i-2)(i-1)}{2}.$$

So $S(D') \geqslant S(D)$ when $i \geqslant 9$ or $i = 7$. Since $D' <_L D$ we are done if $i \geqslant 9$ or $i = 7$.

In the cases where $i = 5, 6$ or 8 we add all possible cells in the row. That is, we let $R = \{(h, j+1) : 1 \leqslant h \leqslant i-1\}$ and leave L the same. The downsets $D' = D - L + R$ each have at most the cost of D, at least the space of D, and $D' <_L D$. □

Lemma 35. *Suppose a downset D does not end in stairs and has last corner (i, j) where $j - i \geqslant \lfloor \lg(i) \rfloor$. If there is an earlier corner then D is not an optimal downset.*

Proof. Since D does not end in stairs, all previous corners (k, m) have $1 \leqslant k < \frac{i-3}{2}$. This implies that $i \geqslant 6$. Choose (k, m) to be the second to last corner. Let $t = \lfloor \lg(i) \rfloor$ and consider

$$D' = D - \{(i, h) : j - t + 1 \leqslant h \leqslant j\} + \{(h, j+1) : k+1 \leqslant h \leqslant i-1\}.$$

That is, we consider the downset D' in which we remove t cells from the last column and replace them with the available cells at height $j + 1$. This move is shown in Figure 12.

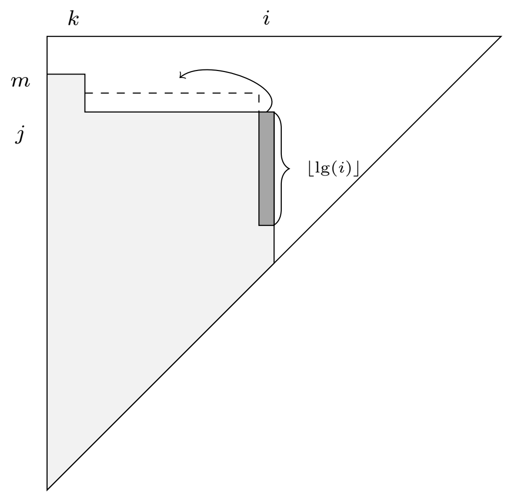

Figure 12: Column move in the proof of Lemma 35.

The cost of the column is

$$2^{n-j-1}+\cdots+2^{n-j-1+t-1}=2^{n-j-1}(2^t-1)=2^{n-j-2}(2^{\lfloor \lg(i)\rfloor+1}-2)>2^{n-j-2}(i-2).$$

Note there are at most $i-2$ cells in the row (since there is a previous corner) and the cost of each cell is 2^{n-j-2}. Thus, the cost of the row is strictly less than the cost of the column.

The space in the column is exactly $i\lfloor \lg(i)\rfloor$ and the space in the row is

$$\begin{aligned}S(\{(h,j+1):k+1\leqslant h\leqslant i-1\})&=(k+1)+(k+2)+\cdots+(i-1)\\&=\frac{(i-1)i}{2}-\frac{k(k+1)}{2}\\&\geqslant\frac{(i-1)i}{2}-\frac{\frac{i-4}{2}\cdot\frac{i-2}{2}}{2}\\&=\frac{3}{8}i^2+\frac{i}{4}-1.\end{aligned}$$

So $S(D')\geqslant S(D)$. Therefore D is not an optimal downset. □

By Lemmas 34 and 35 we conclude the following.

Corollary 36. *Suppose that a downset D does not end in stairs, is not $(2,3,1)$-lex style, and has last corner (i,j) where $j-i\geqslant\lfloor\lg(i)\rfloor$ and $i\geqslant 5$. Then D is not an optimal downset.*

Corollary 36 deals with downsets that do not end in stairs and have that the last column is tall. In the next lemmas, we will deal with downsets that end with stairs and the column of the top stair is tall. By Lemmas 31 and 33 we only need to consider downsets that end in one short stair.

Lemma 37. *Suppose that D is not $(2,3,1)$-lex style, the last corner of a downset D is (i',j'), and the first corner is (i,j) with $i=i'-1$. If $j-i\geqslant\lfloor\lg(i)\rfloor+1$ and $i\geqslant 6$ then D is not an optimal downset.*

Proof. Since D is not $(2,3,1)$-lex style, $j<n-1$. Let $t=\lfloor\lg(i)\rfloor$. For $h\in\{j-t+1,\dots,j\}$ let $\ell(h)$ be the greatest integer such that $(\ell(h),h)\in D$. Note $\ell(h)\in\{i,i'\}$. Let $L=\{(\ell(h),h):j-t+1\leqslant h\leqslant j\}$. These are cells in B_n since $j-i\geqslant\lfloor\lg(i)\rfloor+1$ and $i'=i+1$. Let $R=\{(m,j+1):1\leqslant m\leqslant i-2\}$. Consider $D'=D-L+R$ as shown in Figure 13.

Since the cost of a cell (with the exception of those in the first column) only depends on the height of the cell, the cost argument is exactly the same as that of Lemma 34. Moreover,

$$S(L)\leqslant i+(i+1)(\lfloor\lg(i)\rfloor-1)=(i+1)\lfloor\lg(i)\rfloor-1$$

and

$$S(R)=\frac{(i-2)(i-1)}{2}.$$

Thus, $S(D')\geqslant S(D)$ when $i\geqslant 10$ or $i=7$.

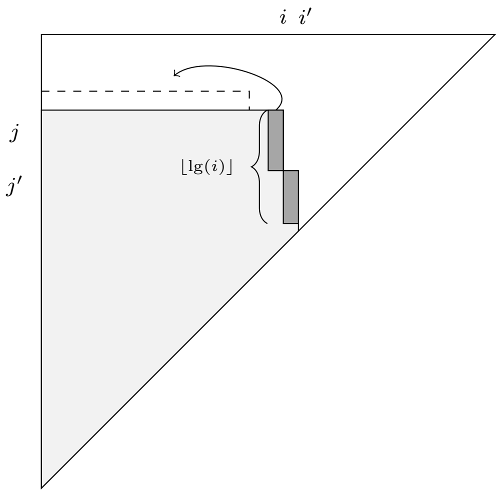

Figure 13: Column move in the proof of Lemma 37

For $i = 6, 8$ or 9 we add all possible cells in the row. That is, we let $R = \{(h, j+1) : 1 \leqslant h \leqslant i-1\}$ and leave L the same. The downsets $D' = D - L + R$ each have at most the cost and at least the space of D and are earlier in lex order. Therefore, for $i \geqslant 6$, such a D is not optimal. □

The next lemma considers the case where a downset ends with one short stair, there is an earlier corner, and the column of the top stair is tall. The proof combines the ideas of those in Lemmas 37 and 35 and has similar counting arguments.

Lemma 38. *Suppose that the last corner of a downset is (i', j'), the second to last corner is (i, j) where $i = i' - 1$ and there is an earlier corner. If $j - i \geqslant \lfloor \lg(i) \rfloor + 1$ then D is not an optimal downset.*

Proof. As in the proof of Lemma 35, an earlier corner has space less than $\frac{i-3}{2}$ which means $i \geqslant 6$. As in the proof of Lemma 37, let $t = \lfloor \lg(i) \rfloor$, for each $h \in \{j-t+1, \ldots, j\}$ let $\ell(h)$ be the greatest integer such that $(\ell(h), h) \in D$, let $L = \{(\ell(h), h) : j-t+1 \leqslant h \leqslant j\}$, and let $R = \{(h, j+1) : 1 \leqslant h \leqslant i-1\} \cap (B_n \setminus D)$. Consider $D' = D - L + R$. The cost argument is exactly the same as that of Lemma 35. By similar arguments to that of Lemma 37 and 35 we find

$$S(L) \leqslant (i+1)(\lfloor \lg(i) \rfloor - 1) + i = (i+1)\lfloor \lg(i) \rfloor - 1$$

and

$$S(R) \geqslant \frac{3i^2}{8} + \frac{i}{4} - 1.$$

So $S(R) > S(L)$ for $i \geqslant 6$ and $i \neq 8$. If $i = 8$ one can take $\lfloor \lg(i) \rfloor - 1$ cells for L to show D is not optimal. □

By Lemmas 37 and 38 we conclude the following.

Corollary 39. *Suppose that a downset D is not $(2,3,1)$-lex style, ends in one short stair, and the top stair (i,j) has $j-i \geqslant \lfloor \lg(i) \rfloor + 1$ with $i \geqslant 6$. Then D is not an optimal downset.*

6.3 Larger Moves

In this section we will consider moves that are very similar to those in the previous section. We will trade a number of cells from the right side of a downset for the cells in the next row up. The difference is that we allow the removed cells to come from multiple columns. The removed cells will be those that are largest in the lex order on cells. For two cells (i,j) and (m,k) we say $(i,j) \leqslant (m,k)$ in lex order if and only if $i < m$ or $i = m$ and $j \leqslant k$. First we include a lemma about the average cost of the removed cells.

Lemma 40. *Let D be a downset. Suppose (i,j) is the last corner if D is does not end in stairs, or the top stair if D ends in stairs, and T consists of the t lex-last cells in D and doesn't contain a previous corner of D. In general the average cost of the cells is at least $2C$ where C is the cost of the cell at (i,j). The exceptions when D does not end in stairs are $t \leqslant 2$, or $j-i=1$ and $t \leqslant 9$, or $j-i=2$ and $t=1,2,3,4,6,7$. The exceptions when D does end in stairs are when the last column is exactly one cell at height $j-1$ and $t=2$ or $t=3$.*

Proof. In outline, we use the formula for the sum of arithmetico-geometric progression to compute the total cost of the complete columns in T. If there are m complete columns and the first has height h then the total cost of the complete columns is $C(2^h(2^m-1)-m)$. If there is a partial column on the right the average cost is always at least C. If there is a partial column on the left its average cost is at least C unless it only has 1 or 2 cells. The result follows from case analysis. ☐

Lemma 41. *Suppose that D is a downset that does not end in stairs. If the last corner of D is (i,j) such that $j-i < \lfloor \lg(i) \rfloor$, $i \geqslant 16$ and $j \leqslant n-3$ then D is not an optimal downset.*

Proof. We will prove that there exists a downset D' such that $C(D') \leqslant C(D)$, $S(D') \geqslant S(D)$, and that $D' <_L D$. First we consider the case where $2 \leqslant j-i$. Let T be the $t = \left\lfloor \frac{i}{2} \right\rfloor$ lex-last cells of D. Let ℓ be such that T occupies ℓ columns. Note that $i-\ell+1$ is the minimum space in any cell of T.

Let R be the cells in $B_n \setminus D$ at height $j+1$ and $j+2$ and in columns labeled $1,\ldots,c=i-\ell$. Since $j \leqslant n-3$, there are available cells at both height $j+1$ and $j+2$. So

$$R = \{(h,k) : 1 \leqslant h \leqslant c, j+1 \leqslant k \leqslant j+2\} \cap (B_n \setminus D).$$

The regions R and T are shown in Figure 14. Let $D' = D - T + R$.

First we will compare $C(T)$ and $C(R)$. When $i \geqslant 16$ we have $t \geqslant 8$. By Lemma 40, the average cost in T is at least 2^{n-j} and

$$C(T) \geqslant 2^{n-j} \cdot \left\lfloor \frac{i}{2} \right\rfloor \geqslant 2^{n-j}\left(\frac{i}{2} - \frac{1}{2}\right) = (i-1) \cdot 2^{n-j-1}.$$

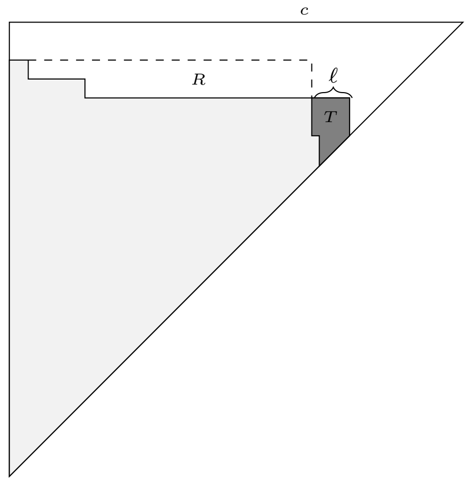

Figure 14: A downset D with R and T as described in the proof of Lemma 41.

Since $(\lfloor \lg i \rfloor - 1) + \lfloor \lg i \rfloor < \left\lfloor \frac{i}{2} \right\rfloor$ for $i \geqslant 16$, T must occupy at least 3 columns. So the cost of R is greatest if there are no previous corners and $c = i - 3$. This gives

$$C(R) \leqslant 3[(i-4)2^{n-j-3} + 2^{n-j-2}] = 3(i-2)2^{n-j-3}.$$

Since $C(T) \geqslant 4(i-1)2^{n-j-3} \geqslant 3(i-2)2^{n-j-3} \geqslant C(R)$ we have $C(D') \leqslant C(D)$ when $i \geqslant 16$.

Now we compare $S(T)$ and $S(R)$. Since T must occupy at least 3 columns, T has exactly $j - i$ cells in the rightmost column, and $j - i + 1$ cells in the second rightmost column, so

$$\begin{aligned} S(T) &\leqslant it - (t - (j-i)) - (t - (j-i) - (j-i+1)) \\ &\leqslant it - (t - \lfloor \lg i \rfloor + 1) - (t - 2\lfloor \lg i \rfloor + 1) \\ &= (i-2)\left\lfloor \frac{i}{2} \right\rfloor + 3\lfloor \lg i \rfloor - 2. \end{aligned} \tag{†}$$

To compute $S(R)$ we first find an upper bound on ℓ. Suppose, for the sake of contradiction, $\ell > \sqrt{i}$. The set of cells T consists of a $(\ell - 1) \times (j - i - 1)$ rectangle, a triangle, and at least one cell in the leftmost column. So we would have, since $j - i \geqslant 2$,

$$|T| \geqslant (j-i-1)(\ell-1) + \frac{\ell(\ell-1)}{2} + 1 > (\sqrt{i} - 1) + \frac{\sqrt{i}(\sqrt{i}-1)}{2} + 1 = \frac{\sqrt{i}}{2} + \frac{i}{2}. \tag{‡}$$

So, $\ell \leqslant \lfloor \sqrt{i} \rfloor$ and $c = i - \ell \geqslant i - \lfloor \sqrt{i} \rfloor$.

Let a and b be the horizontal coordinates of the previous corners at height $j + 1$ and $j + 2$, respectively. Let $a = 0$ or $b = 0$ if there is no previous corner at the appropriate height. By Lemma 30, $a, b \leqslant \lfloor \frac{i-4}{2} \rfloor$ and thus

$$\begin{aligned} S(R) &\geqslant [(a+1)+(a+2)+\cdots+c]+[(b+1)+(b+2)+\cdots+c] \\ &\geqslant \left(i-\lfloor\sqrt{i}\rfloor\right)\left(i-\lfloor\sqrt{i}\rfloor+1\right)-\left\lfloor\frac{i-4}{2}\right\rfloor\left\lfloor\frac{i-2}{2}\right\rfloor. \end{aligned}$$

So $S(R) \geqslant S(T)$ when $i > 16$. When $i = 16$, T uses at most 3 columns and so our lower bound for $S(R)$ can be improved and still $S(R) \geqslant S(T)$. Moreover, $D' <_L D$. Therefore, D is not an optimal downset in the case where $j - i \geqslant 2$.

When $j - i = 1$ we let T be the $t = \left\lfloor \frac{i}{2} \right\rfloor - 1$ greatest cells in lex ordering and keep R the same. Since $j - i = 1$ and $i \geqslant 16$ we get $\ell \geqslant 4$. Similar to the calculation in (†),

$$S(T) \leqslant i+2(i-1)+3(i-2)+(t-6)(i-3)=(i-3)t+10 \leqslant (i-3)\left\lfloor\frac{i}{2}\right\rfloor-i+13.$$

Since $j - i = 1$ it is sometimes the case that T occupies more columns than our previous bound. Similarly to the calculation in ‡ we get $\ell \leqslant \lceil\sqrt{i}\rceil$ and thus,

$$S(R) \geqslant (i-\lceil\sqrt{i}\rceil)(i-\lceil\sqrt{i}\rceil+1)-\left\lfloor\frac{i-4}{2}\right\rfloor\left\lfloor\frac{i-2}{2}\right\rfloor.$$

So we find $S(R) \geqslant S(T)$ for $i \geqslant 16$, except when $i = 18$ which one can check by hand. Finally, we check cost. The upper bound on the cost of R remains the same. By Lemma 40 the average cost of T when $i \geqslant 22$ is 2^{n-j} and we find,

$$C(T) \geqslant 2^{n-j}\left(\left\lfloor\frac{i}{2}\right\rfloor-1\right) \geqslant 3(i-2)2^{n-j-3} \geqslant C(R).$$

When $16 \leqslant i \leqslant 21$ we compare the actual cost of T to the bound on the cost of R and get $C(T) \geqslant C(R)$. Since $D' <_L D$, we get that D is not optimal when $j - i = 1$. □

Lemma 41 dealt with downsets that did not end in stairs, but the last column was short. We now do a similar move when there is a short stair and the top stair's column is short.

Lemma 42. *Suppose that a downset D ends in one short stair and is not $(2,3,1)$-lex style. If the last two corners (i', j') and (i, j) with $i = i' - 1$ satisfy $j - i \leqslant \lfloor \lg(i) \rfloor$, $j \leqslant n - 3$, and $i \geqslant 16$ then D is not an optimal downset.*

Proof. Again we will prove that there exists a downset D' such that $C(D') \leqslant C(D)$, $S(D') \geqslant S(D)$, and that $D' <_L D$. Let T be the $t = \left\lfloor \frac{i}{2} \right\rfloor$ greatest cells of D under the lex order. Let ℓ be such that T occupies $\ell + 1$ columns. So $i - \ell + 1$ is the least amount of space in any cell of T. So Let $R = \{(h, k) : 1 \leqslant h \leqslant i - \ell, j + 1 \leqslant k \leqslant j + 2\} \cap B_n \setminus D$. The regions R and T are shown in Figure 15. Let $D' = D - T + R$.

By Lemma 40 the average cost of a cell in T is at least 2^{n-j}. As in the proof of Lemma 41, T occupies at least 3 columns and $C(D') \leqslant C(D)$.

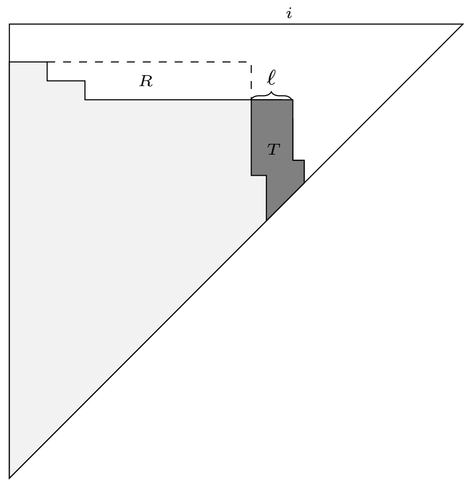

Figure 15: An example of a downset D with R and T as described in the proof of Lemma 42.

We use a cruder argument than in Lemma 41 to compare $S(T)$ and $S(R)$. This time we can simply use that the maximum space in any cell of T is $i+1$ so $S(T) \leqslant (i+1)\lfloor \frac{i}{2} \rfloor$ and $S(R) \geqslant c(c+1) - \lfloor \frac{i-4}{2} \rfloor \lfloor \frac{i-2}{2} \rfloor$ where $c = i - \ell \geqslant i - (\lfloor \sqrt{i} \rfloor - 1)$. (Compare to (‡).) We conclude that $S(R) \geqslant S(T)$. Since we also have $D' <_L D$, D is not an optimal downset. □

In the previous lemmas, we moved $\lfloor \frac{i}{2} \rfloor$ cells to two rows in the case that two rows were available. In the next lemmas, we address if there is only one available row by moving $\lfloor \frac{i}{4} \rfloor$ cells to 1 row.

Lemma 43. *Suppose that D is a downset that does not end in stairs with last corner (i, j) such that $j - i < \lfloor \lg(i) \rfloor$, $i \geqslant 24$ and $j = n - 2$. Then D is not an optimal downset.*

Proof. We will prove that there exists a downset D' such that $C(D') \leqslant C(D)$, $S(D') \geqslant S(D)$, and $D' <_L D$. Let T be the $t = \lfloor \frac{i}{4} \rfloor$ greatest cells of D under the lex ordering. Let ℓ be the number of columns occupied by T. So $i - \ell + 1$ is the least amount of space in any cell of T.

Let R be the cells in $B_n \setminus D$ at height $n-1$ and with space at most $c = \min\{i-\ell, i-4\}$. Also, let a be the space in the previous corner (and $a = 0$ if there is no previous corner) and $R = \{(h, n-1) : a+1 \leqslant h \leqslant c\}$. The regions R and T are shown in Figure 16. Let $D' = D - T + R$.

Note $C(R) \leqslant i - 3$ since $c \leqslant i - 4$. Since $i \geqslant 24$, $t \geqslant 6$. By Lemma 40 the average cost of a cell in T is at least 4 except when $j - i = 1$ and $t \in \{6, 7, 8, 9\}$ or $j - i = 2$ and $t \in \{6, 7\}$. In the unexceptional cases, $C(T) \geqslant 4\lfloor \frac{i}{4} \rfloor \geqslant C(R)$. We check the exceptional cases computationally.[3]

[3]The code is available at http://gvsu.edu/s/1S0.

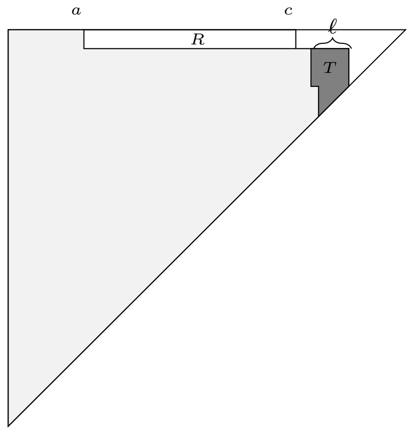

Figure 16: A downset D with R and T as described in the proof of Lemma 43.

Since the space in any cell of T is at most i and $t = \lfloor \frac{i}{4} \rfloor$, $S(T) \leqslant \frac{i^2}{4}$. Also $S(R) \geqslant \frac{c(c+1)}{2} - \frac{a(a+1)}{2}$. By Lemma 30 $a \leqslant \lfloor \frac{i-4}{2} \rfloor$ and using an argument similar to (‡) we prove $\ell \leqslant \left\lfloor \sqrt{\frac{i}{2}} \right\rfloor + 1$. We find $S(R) \geqslant S(T)$ when $i \geqslant 24$. Since $D' <_L D$, such a D is not an optimal downset. □

In the final lemma for this section we consider downsets that are similar to those in Lemma 43, but that end in one short stair.

Lemma 44. *Suppose that a downset D ends in one short stair. If the last two corners (i', j') and (i, j) with $i = i' - 1$ satisfy $j - i \leqslant \lfloor \lg(i) \rfloor$, $i \geqslant 20$, and $j = n - 2$ then D is not an optimal downset.*

Proof. Using the same setup as in the proof of Lemma 43, let T be the $t = \lfloor \frac{i}{4} \rfloor$ greatest cells of D under the lex ordering, and ℓ such that T occupies $\ell + 1$ columns. Let R be the cells in $B_n \setminus D$ at height $n-1$ and with space at most $c = \min\{i - \ell, i - 4\}$. Allowing a previous corner to have space a (and setting $a = 0$ if there is no previous corner), $R = \{(h, n-1) : a + 1 \leqslant h \leqslant c\}$. The regions R and T are shown in Figure 17. Let $D' = D - T + R$. By Lemma 40, the average cost of a cell in T is at least 4 and so $C(D') \leqslant C(D)$.

By an argument similar to (‡) we have $\ell \leqslant \left\lfloor \sqrt{\frac{i}{2}} \right\rfloor$, noting that in this case we also have $j - i \geqslant 2$. We get $S(D') \geqslant S(D)$ and $D' <_L D$, and thus such a D is not an optimal downset. □

7 Narrow Downsets and Persistent Exceptions

Many of our lemmas thus far required that the downset was "wide", that is the last corner (i, j) had $i \geqslant c$ for some c. In this section we will deal with the "narrow" cases.

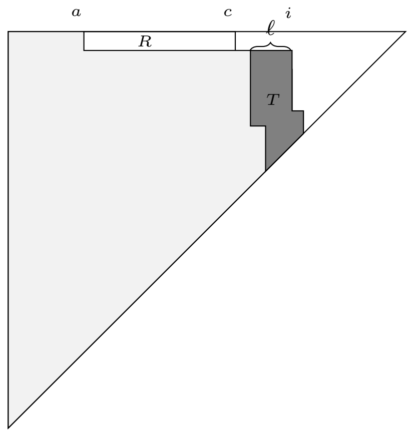

Figure 17: An example of a downset D with R and T as described in the proof of Lemma 44.

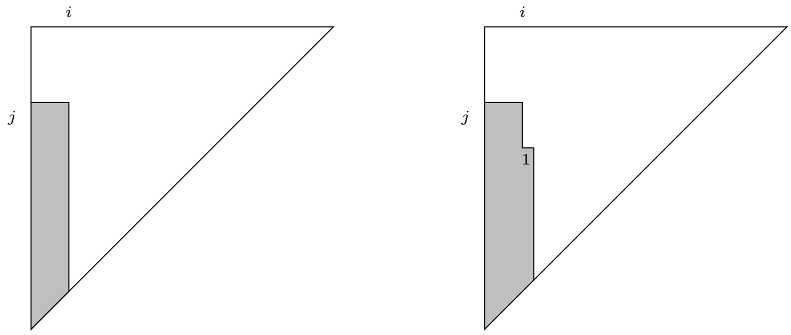

Figure 18: Representative downsets for Lemmas 45 and Lemma 46.

Recall from Theorem 17, there are 5 persisent exceptions to the maximizing hypergraph being $(2, 3, 1)$-lex style. Their shadows are given in $\mathcal{P}_n$ in Theorem 17. The corresponding downsets are

$$\mathcal{D}_n = \left\{\{(2, 3)\}^*, \{(2, n-3)\}^*, \{(2, n-4)\}^*, \{(3, n-2)\}^*, \{(4, n-2)\}^*\right\}$$

where we write S^* to be the smallest downset containing S. The following two lemmas discuss the cases where a narrow downset does not end in stairs and does end in stairs, respectively. See Figure 18 for representative pictures.

Lemma 45. *Suppose that D is a downset in B_n for $n \geqslant 10$. Suppose D does not end in stairs, D is not $(2, 3, 1)$-lex style, $D \notin \mathcal{D}_n$, and the last corner of D is (i, j). If $i \leqslant 4$ then D is not optimal.*

Proof. By Lemma 30 there can be no previous corners. If $i = 1$ then D is $(2, 3, 1)$-lex style. If $i = 2$ and $j = 3$ or $j \geqslant n-4$ then D is $(2, 3, 1)$-lex style or $D \in \mathcal{D}_n$. If $i = 2$ and

$4 \leqslant j < n-4$ then

$$D' = D - \{(2,j),(2,j-1)\} + \{(1,j+1),(1,j+2),(1,j+3),(1,j+4)\}$$

shows D is not optimal.

Suppose $i = 3$. If $j \geqslant n-2$ then D is $(2,3,1)$-lex style or $D \in \mathcal{D}_n$. If $5 \leqslant j \leqslant n-3$ then

$$D' = D - \{(3,j),(3,j-1)\} + \{(1,j+1),(1,j+2),(2,j+1),(2,j+2)\}$$

shows that D is not optimal. If $j = 4$ then, recalling that $n \geqslant 10$, we see that

$$D' = D - \{(2,4),(3,4)\} + \{(1,k) : 5 \leqslant k \leqslant 9\}$$

shows D is not optimal.

Finally, suppose $i = 4$. If $j \geqslant n-2$ then D is $(2,3,1)$-style or $D \in \mathcal{D}_n$. If $6 \leqslant j \leqslant n-3$ then

$$D' = D - \{(4,j),(4,j-1)\} + \{(1,j+1),(1,j+2),(2,j+1),(2,j+2),(3,j+1)\}$$

shows D is not optimal. If $j = 5$ then

$$D' = D - \{(4,5),(3,5),(3,4)\} + \{(m,n) : 1 \leqslant m \leqslant 2 \text{ and } 6 \leqslant n \leqslant 9\}$$

shows D is not optimal. □

Lemma 46. *Suppose that D is a downset that is not $(2,3,1)$-lex style, that D ends in one short stair, and the second to last corner is (i,j). If $i \leqslant 5$ then D is not optimal.*

Proof. Let (i,j) be the second to last corner and (i',j') be the last corner. By Lemma 30 there are no previous corners. Since D is not $(2,3,1)$-lex style we know $j < n-1$. If $i = 1$ and $j = n-2$ then D is $(2,3,1)$-lex style. If $i = 1$ and $j < n-2$ then there are at least two empty rows and

$$D' = D - (2,j') + \{(1,j+1),(1,j+2)\}$$

shows D is not optimal. If $i = 2$ then

$$D' = D - (3,j') + \{(1,j+1),(2,j+1)\}$$

shows D is not optimal. If $3 \leqslant i \leqslant 5$ then

$$D' = D - (i',j') + \{(1,j+1),(2,j+1),(3,j+1)\}$$

shows D is not optimal. □

8 Downset Extensions

In this section we will consider a downset D in B_n inside B_ℓ for $\ell > n$. We will show that if D is not optimal in B_n then D is not optimal in B_ℓ either, and a similar lemma for when D is optimal.

Remark 47. Given a downset D in B_n, we let the *extension* of D, denoted $\overline{D}$, be the downset in B_{n+1} where $(i,j) \in \overline{D}$ if and only if $(i,j) \in D$. For an extension $\overline{D}$ we have $S(\overline{D}) = S(\overline{D})$ and $C(\overline{D}) = 2C(D)$ (since the cost of a cell depends exponentially on n, see Lemma 24).

Lemma 48. *Suppose D is not an optimal downset in B_n. Then the extension of D is not optimal in B_{n+1}.*

Proof. Suppose that D is not an optimal downset in B_n and let $\overline{D}$ be the extension of D. Then there exists a downset D' in B_n such that $S(D') \geqslant S(D)$, $C(D') \leqslant C(D)$, and D' is earlier in lex order. We claim that $\overline{D}$ is not an optimal downset in B_{n+1}. Consider $\overline{D'}$. Then

$$S(\overline{D'}) = S(D') \geqslant S(D) = S(\overline{D}) \qquad \text{and} \qquad C(\overline{D'}) = 2C(D') \geqslant 2C(D) = C(\overline{D}).$$

Moreover, $\overline{D'} <_L \overline{D}$ since $D' <_L D$ and our definition for the lex ordering on downsets is independent of n. Therefore, if D is not an optimal downset in B_n then its extension is not an optimal downset in B_{n+1}. □

So if D is not optimal in B_n then it is not optimal in B_ℓ for $\ell > n$. The next lemma shows that when we have at least 4 empty rows in D then D is not optimal, which allows us to conclude that an optimal downset in B_n will not be optimal in B_{n+4}.

Lemma 49. *Let D be a downset in B_n with first corner (a,b) where $n-1-b \geqslant 4$ and last corner (i,j) where $i \geqslant 5$. Then D is not optimal.*

Proof. Let D be such a downset. We will construct a downset D' that has at least as much space and costs at most as much. Let T be the $t = \lceil \frac{i}{2} \rceil$ greatest cells of D under lex ordering. We will build our set R of replacement cells by taking 4 cells in each column for columns 1 through t. To be precise, let (c,d) be the top cell in column c with $c \leqslant t$, let $R(c) = \{(c,d+1),(c,d+2),(c,d+3),(c,d+4)\}$ and set

$$R = \bigcup_{c=1}^{t} R(c).$$

Let $D' = D - T + R$. This is a downset: T occupies less than t columns since some column of T has at least 2 cells as $t \geqslant 3$ when $i \geqslant 5$. Now we claim that $C(D') \leqslant C(D)$. We can pair each column of R with a distinct cell of T. If (c,d) is the top cell in column c and (e,f) is the corresponding cell in T we have $f \leqslant d$ and so for $c \neq 1$, we have

$$C(R(c)) = 2^{n-d-2} + 2^{n-d-3} + 2^{n-d-4} + 2^{n-d-5} = 2^{n-d-5}(2^4 - 1) < 2^{n-d-1} \leqslant 2^{n-f-1}.$$

and so the cost of $R(c)$ is less than that of (e, f) when $c \neq 1$. The cost of the first column is double, so we must pair it with a cell at height at most $b-1$ in T. As above, this exists since $t \geqslant 3$.

Next we claim $S(D') \geqslant S(D)$. Computing the space of R we have

$$S(R) = 4\frac{t(t+1)}{2} = 2 \cdot \lceil i/2 \rceil (\lceil i/2 \rceil + 1)$$

and $S(T) \leqslant i \left\lceil \frac{i}{2} \right\rceil$ so $S(D') \geqslant S(D)$. Finally, $D' <_L D$ so D is not optimal when $i \geqslant 5$. □

Corollary 50. *For $n \geqslant 10$, if D is optimal in B_n and $D \neq \{(2,3)\}^*$ then D is not optimal in B_{n+4}.*

Proof. Suppose D is optimal in B_n, the first corner of D is (a, b), and the last corner of D is (i, j). Note that $(n+4) - 1 - b \geqslant 4$ and so if $i \geqslant 5$ then D is not optimal in B_{n+4} by Lemma 49. Suppose now that $i \leqslant 4$. Then D is not $(2, 3, 1)$-lex style in B_{n+4} and $D \notin \mathcal{D}_{n+4}$ since then D would not fit inside B_n. Thus, by Lemma 45 and Lemma 46, D is not optimal in B_{n+4}. □

9 Upper Bounds on n

Lemma 51. *Suppose that D does not end in stairs and has last corner (i, j) with $j - i < \lfloor \lg i \rfloor$, $j = n - 2$. Then D is not optimal for any $n \geqslant 32$.*

Proof. By Lemma 43 we know such a D is not optimal when $i \geqslant 24$. Suppose $i \leqslant 23$. Since $j - i < \lfloor \lg i \rfloor$ then $j - i \leqslant 3$ and $j \leqslant 26$. Since $j = n - 2$ we know $n \leqslant 28$. If D is not optimal in B_{28} then D is not optimal in B_n for any $n \geqslant 29$ by Lemma 48. If D is optimal in B_{28} then D is not optimal in B_n for $n \geqslant 32$ by Corollary 50. □

Lemma 52. *Suppose that D does not end in stairs and has last corner (i, j) with $j \leqslant n-3$ and $j - i < \lfloor \lg i \rfloor$. Then $D = \{(2,3)\}^*$ or D is not optimal for any $n \geqslant 29$.*

Proof. By Lemma 41 we know such a D is not optimal for $i \geqslant 16$. Suppose $i \leqslant 15$ and so $j - i \leqslant 2$ and $j \leqslant 17$. If there is no previous corner then D fits inside B_{18} so isn't optimal for $n \geqslant 22$. If there is a previous corner, let (i', j') be the first corner. Let $T = \{(i, j)\}$ and let R consist of the leftmost cell not in D for each row at heights between $j+1$ and j'. Set $D' = D - T + R$. We save on cost. If D is optimal, we must have $S(R) < i$. Thus, since each cell in R has space at least 2, $2(j' - j) < i$ and so $j' < j + i/2 \leqslant 17 + (15/2) = 24.5$. So D fits inside B_{25} and thus is not optimal for $n \geqslant 29$ by one of Lemma 48 and Corollary 50. □

Lemma 53. *Suppose D is a downset that ends with one short stair with the top stair at (i, j). Additionally assume $j - i < \lfloor \lg i \rfloor + 1$ and $j = n - 2$. If $n \geqslant 29$ then D is not optimal.*

Proof. By Lemma 44 we know such a D is not optimal if $i \geqslant 20$. Suppose $i \leqslant 19$. Since $j - i < \lfloor \lg i \rfloor + 1$, $j - i \leqslant 4$. So $j \leqslant 23$ and $n \leqslant 25$. So D is not optimal in B_n for $n \geqslant 29$ by one of Lemma 48 and Corollary 50. □

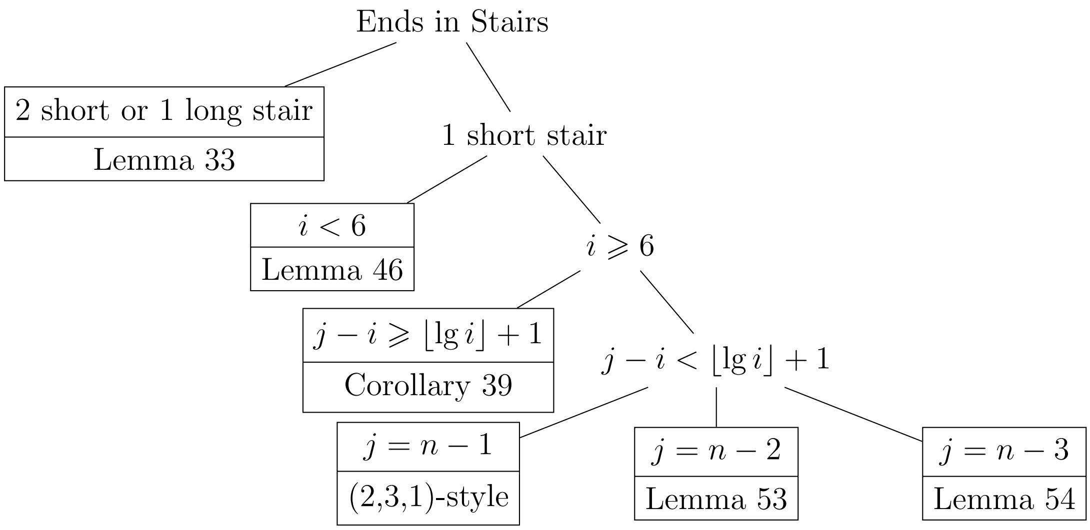

Figure 19: The cases in the proof of Proposition 55. Here (i, j) is the top stair.

Lemma 54. *Suppose D is a downset that ends with one short stair with the top stair being (i, j). Additionally, assume $j - i < \lfloor \lg i \rfloor + 1$ and $j \leqslant n - 3$. If $n \geqslant 30$ then D is not optimal.*

Proof. By Lemma 42 we know such a downset is not optimal if $i \geqslant 16$. Suppose $i \leqslant 15$. Since $j - i < \lfloor \lg i \rfloor + 1$ we know $j \leqslant 18$. If there is no previous corner then D fits inside B_{19} and so isn't optimal for $n \geqslant 22$. Suppose there is a previous corner. As in Lemma 52, we can replace (i, j) with one cell at each height above j where D already has a cell and still save on cost. So again $2(j' - j) < i$ and $j' < j + i/2 \leqslant 18 + 15/2 = 25.5$. and D fits inside B_{26}. By Lemma 48 and Corollary 50 we know D is not optimal if $n \geqslant 30$. □

10 Proof of Theorem

Proposition 55. *Suppose $\mathcal{H} \in \mathcal{H}(n, e)$ is not $(2, 3, 1)$-lex style. Let $D = D(\mathcal{H})$ and suppose D ends in stairs. If $\mathcal{H}$ is optimal then $n < 30$.*

Proof. The cases for this proof are outlined in Figure 19. By Lemma 31, D ends in one short stair, two short stairs, or one long stair. By Lemma 33, no optimal D ends in two short stairs or one long stair.

Suppose that D ends in one short stair and let (i, j) be the second to last corner. If $i < 6$ then D is not optimal by Lemma 46. Suppose that $i \geqslant 6$. If $j - i \geqslant \lfloor \lg i \rfloor + 1$ then D is not optimal by Corollary 39.

So, suppose $j - i < \lfloor \lg(i) \rfloor + 1$. If $j = n - 1$ then D is $(2, 3, 1)$-lex style. If $j = n - 2$ then $n < 29$ by Lemma 53. Finally, if $j \leqslant n - 3$ then $n < 30$ by Lemma 54. Therefore, if such an $\mathcal{H}$ is optimal then $n < 30$. □

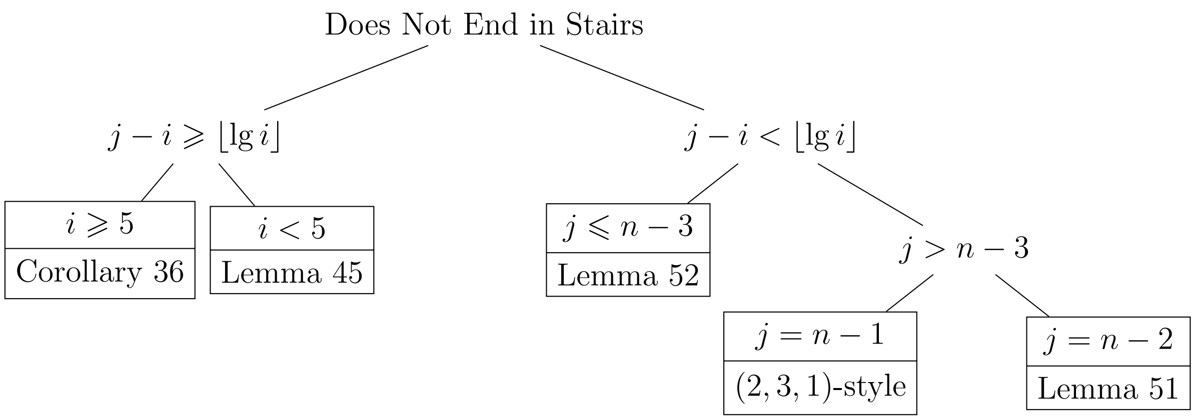

Figure 20: The cases in the proof of Proposition 56. Here (i, j) is the last corner.

Proposition 56. *Suppose $\mathcal{H} \in \mathcal{H}(n, e)$ is not $(2, 3, 1)$-lex style. Let $D = D(\mathcal{H})$ and suppose D does not end in stairs. If $\mathcal{H}$ is optimal then $D \in \mathcal{D}_n$ or $n < 32$.*

Proof. The cases for this proof are outlined in Figure 20. Let (i, j) be the last corner of D. Suppose that $j - i \geqslant \lfloor \lg i \rfloor$. When $i \geqslant 5$, Corollary 36 tells us that D is not optimal. If $i < 5$ then, by Lemma 45, $D \in \mathcal{D}_n$, D is not optimal, or $n < 10$.

Now suppose that $j - i < \lfloor \lg i \rfloor$. If $j = n - 1$ then D must be $(2, 3, 1)$-lex style. If $j = n - 2$ then by Lemma 51, $n < 32$. Finally, if $j \leqslant n - 3$ then $n < 29$ or $D \in \mathcal{D}_n$ by Lemma 52. □

Proof of Theorem 17. When $n \geqslant 33$ we find that the only shifted optimal 3-graphs are $(2, 3, 1)$-lex style or have downsets in $\mathcal{D}_n$ (see Section 7) by Propositions 55 and 56. These correspond to the shadows given in $\mathcal{P}_n$ (see Theorem 17). When $n < 33$ we find all hypergraphs that maximize 2-independent sets using a computer search[4] which leads us to $(2, 3, 1)$-lex style graphs or those with shadow graphs shown in Table 1, all of which have $n \leqslant 16$. □

11 Conclusion

We have found the maximum number of s-independent sets in n vertex 3-uniform hypergraphs with e edges for all possible n, e and s. While the answer is straightforward for $s = 1$ and $s = 3$, the answer for $s = 2$ requires a generalization of lex and colex graphs to π-lex graphs. Sadly the result is not as straightforward as saying that the optimal hypergraphs are $(2, 3, 1)$-lex initial segments. Even the generalization to $(2, 3, 1)$-lex style doesn't cover all the cases. There are both transient and persistent exceptions that are not $(2, 3, 1)$-lex style.

It still seems to us possible that asymptotically we can give a good characterization of the r-graph on n vertices having e edges having the fewest s-independent sets. The

[4]The code is available at http://gvsu.edu/s/1RZ.

following conjecture is a strengthened version of the main theorem (Theorem 5) of [2], where the factor of $1 + o(1)$ is in the exponent.

Conjecture 57. Fix $1 \leqslant s \leqslant r$ and $\eta > 0$. Let $\mathcal{H}$ be a hypergraph on n vertices with e edges (where $\eta\binom{n}{r} < e < (1-\eta)\binom{n}{r}$) having the maximum number of s-independent sets. Let $\mathcal{R}(e)$ be the initial segment of $\binom{[n]}{r}$ in the $(r-s+1, r-s+2, \ldots, r, 1, 2, \ldots, s)$-lex order. Then

$$i_s(\mathcal{H}) \leqslant (1 + o(1))i_s(\mathcal{R}(e)).$$

The case $r = 3, s = 2$ is a consequence of our main theorem. For all r, the cases $s = r$ and $s = 1$ are known. The case $s = r$ is a special case of Theorem 2. The case $s = 1$ is true because the argument just before Theorem 3 applies equally well to the initial segments in $(r, 1, 2, \ldots, r-1)$-lex.

As noted in the introduction, asymptotic results for the maximum number of 2-independent sets of size t in an r-graph with e edges are known [4]. Determining the exact answer seems difficult.

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