The root distributions of Ehrhart polynomials of free sums of reflexive polytopes

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Abstract

In this paper, we study the root distributions of Ehrhart polynomials of free sums of certain reflexive polytopes. We investigate cases where the roots of the Ehrhart polynomials of the free sums of $A_d^\vee$'s or A_d's lie on the canonical line $\mathrm{Re}(z) = -\frac{1}{2}$ on the complex plane $\mathbb{C}$, where A_d denotes the root polytope of type A of dimension d and $A_d^\vee$ denotes its polar dual. For example, it is proved that $A_m^\vee \oplus A_n^\vee$ with $\min\{m, n\} \leqslant 1$ or $m + n \leqslant 7$, $A_2^\vee \oplus (A_1^\vee)^{\oplus n}$ and $A_3^\vee \oplus (A_1^\vee)^{\oplus n}$ for any n satisfy this property. We also perform computational experiments for other types of free sums of $A_n^\vee$'s or A_n's.

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1 Introduction

A polytope $Q \subseteq \mathbb{R}^d$ is called *integral* if all the vertices are on $\mathbb{Z}^d$. For an integral polytope $Q \subset \mathbb{R}^d$ of dimension d and a positive integer k, $E_Q(k) = \#(kQ \cap \mathbb{Z}^d)$ is known to be a polynomial of degree d, where $kQ = \{kx : x \in Q\}$. This polynomial is called the *Ehrhart polynomial* of Q. Its generating function $\mathrm{Ehr}_Q(t)$, called the *Ehrhart series*, can be written as

$$\mathrm{Ehr}_Q(t) = \sum_{k=0}^{\infty} E_Q(k)t^k = \frac{\delta_0 + \delta_1 t + \cdots + \delta_d t^d}{(1-t)^{d+1}},$$

where the numerator is the *δ-polynomial* of Q, denoted by $\delta_Q(t)$, and the sequence of the coefficients $\delta(Q) = (\delta_0, \delta_1, \ldots, \delta_d)$ is the *δ-vector* of Q. (They are also known as h^*-polynomial and h^*-vector, respectively.) The δ-vector fully encodes the Ehrhart polynomial and $E_Q(k)$ can be recovered from $\delta(Q)$ as follows:

$$E_Q(k) = \sum_{j=0}^{d} \delta_j \binom{d+k-j}{d} =: f^{\mathrm{Ehr}}(\delta(Q)).$$

We refer the reader to [4] for the introduction to the Ehrhart polynomials and δ-polynomials of integral polytopes.

For a polytope $Q \subset \mathbb{R}^d$, the *polar dual* of Q is defined by

$$Q^\vee = \{x \in \mathbb{R}^d : \langle x, y \rangle \geqslant -1 \text{ for any } y \in Q\},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product of $\mathbb{R}^d$. Note that $(Q^\vee)^\vee = Q$ holds for polytopes containing the origin. An integral polytope containing the origin in its interior is *reflexive* if its polar dual is also an integral polytope ([3, 8]). Note that if Q is reflexive, then so is $Q^\vee$. It is known that an integral polytope is reflexive if and only if its δ-vector is palindromic, and correspondingly, the roots of the Ehrhart polynomial distribute symmetrically with respect to the line $\mathrm{Re}(z) = -\frac{1}{2}$ on the complex plane $\mathbb{C}$. (See, e.g., [9, Proposition 2.1].) Naturally, it is of interest when the roots of the Ehrhart polynomials all lie on the line $\mathrm{Re}(z) = -\frac{1}{2}$. Such reflexive polytopes are called "CL-polytopes" ([7]) and studied in several papers (e.g. [7, 9, 10, 11]). (In what follows, we call a reflexive polytope *CL* if it is a CL-polytope.)

For two integral polytopes $Q_1 \subset \mathbb{R}^{\dim Q_1}$ and $Q_2 \subset \mathbb{R}^{\dim Q_2}$, both containing the origins, the free sum $Q_1 \oplus Q_2$ is defined by

$$Q_1 \oplus Q_2 = \mathrm{conv}((Q_1 \times 0_{Q_2}) \cup (0_{Q_1} \times Q_2)) \subset \mathbb{R}^{\dim Q_1} \times \mathbb{R}^{\dim Q_2},$$

where 0_{Q_1} and 0_{Q_2} are the origins of $\mathbb{R}^{\dim Q_1}$ and $\mathbb{R}^{\dim Q_2}$, respectively. The operation of the free sum has importance since it is the polar dual of the Cartesian product in such a way that

$$(Q_1 \times Q_2)^\vee = Q_1^\vee \oplus Q_2^\vee.$$

Note that $Q_1 \oplus Q_2$ is reflexive if and only if both Q_1 and Q_2 are reflexive.

The δ-polynomial of the free sum has the following simple formula [5]:

$$\delta_{Q_1 \oplus Q_2}(t) = \delta_{Q_1}(t)\delta_{Q_2}(t). \tag{1}$$

On the other hand, the Ehrhart polynomial of $Q_1 \oplus Q_2$ can be given (see [5, 16]) but not so simple, and the root distribution of the Ehrhart polynomial of $Q_1 \oplus Q_2$ is not clear. Especially, as we will see later, $Q_1 \oplus Q_2$ is not always CL even if both Q_1 and Q_2 are CL. In this paper, we are interested in when $Q_1 \oplus Q_2$ becomes CL for CL polytopes Q_1 and Q_2.

A typical example of CL-polytopes is the following special case. For a reflexive polytope Q, when all the roots z of the δ-polynomial of Q satisfy $|z| = 1$, it follows from [14] that all the roots of the Ehrhart polynomial of Q are on the line $\mathrm{Re}(z) = -\frac{1}{2}$, i.e., Q is CL. For example, the following polytopes can be shown to be CL by this reasoning:

- A cross polytope $\mathrm{Cr}_d = \mathrm{conv}(\{e_1, \dots, e_d, -e_1, \dots, -e_d\})$, where $\delta_{\mathrm{Cr}_d}(t) = (1+t)^d$.
- A simplex $T_d = \mathrm{conv}(\{e_1, \dots, e_d, -(e_1 + \cdots + e_d)\})$, where $\delta_{T_d}(t) = 1 + t + \cdots + t^d$.

Here, e_i denotes the i-th unit vector of $\mathbb{R}^d$. If Q_i's are such polytopes, then we have $Q_1 \oplus \cdots \oplus Q_n$ is CL since the roots of the δ-polynomial of $Q_1 \oplus \cdots \oplus Q_n$ also satisfy $|z| = 1$ by (1). (Notice that Cr_d is unimodularly equivalent to $\underbrace{T_1 \oplus \cdots \oplus T_1}_{d}$.)

In this paper, we mainly discuss the case Q_i's are the dual of the classical root polytopes of type A. Here, the classical root polytope of type A is defined as

$$A_d = \mathrm{conv}(\{\pm(e_i + \cdots + e_j) : 1 \leqslant i \leqslant j \leqslant d\}),$$

and we consider its dual $A_d^\vee$. The Ehrhart polynomial of $A_d^\vee$ is known to be

$$E_{A_d^\vee}(k) = (k+1)^{d+1} - k^{d+1}$$

in [9, Lemma 5.3]. Reflexive polytopes A_d and $A_d^\vee$ are shown to be CL in [9], but we see the roots z of their δ-polynomials do not satisfy $|z| = 1$. The reason we consider this free sum of $A_d^\vee$'s is that it appears as the equatorial spheres of the complete graded posets. This will be discussed in Section 2. After that, we investigate the CL-ness of $A_{p_1}^\vee \oplus A_{p_2}^\vee \oplus \cdots \oplus A_{p_k}^\vee$ in the following sections.

We collect the results which show the CL-ness for the free sums of $A_d^\vee$'s or A_d's in what follows:

- $A_m^\vee \oplus A_n^\vee$ with $\min\{m, n\} \leqslant 1$ or $m + n \leqslant 7$ (Theorem 5);
- $A_m^\vee \oplus (A_1^\vee)^{\oplus n}$ for any $n \geqslant 1$ with $m = 1, 2, 3$ (Proposition 6, Theorems 8 and 9);
- $A_1 \oplus A_n$ for any $n \geqslant 1$ (Theorem 12);
- $A_m \oplus A_1^{\oplus n}$ for any $n \geqslant 1$ with $m = 1, 2, 3$ (Proposition 13 and Theorem 14).

We also perform other types of free sums of $A_n^\vee$'s or A_n's and describe the computational results.

2 Ehrhart polynomials of equatorial spheres of graded posets

Let $(P, \preceq)$ be a finite partially ordered set, or a poset, with $|P| = d$. The *order polytope* O_P of P is given by

$$O_P = \{x \in [0,1]^d : x_a \leqslant x_b \ \text{ for } \ b \prec a \ (a, b \in P)\ \},$$

where the coordinates of $\mathbb{R}^d$ are indexed by the elements of P. This is an integral polytope whose vertices correspond to the order ideals of P ([15]).

As another polytope arising from posets closely related to the order polytope, the *chain polytope* of P is defined by

$$C_P = \{x \in \mathbb{R}^d : \ x_a \geqslant 0 \ (a \in P), \ \ x_{a_1} + \cdots + x_{a_k} \leqslant 1 \ \text{ for } \ a_1 \prec \cdots \prec a_k \ (a_i \in P)\ \}.$$

This is an integral polytope whose vertices correspond to the antichains of P, and it is shown in [15] that the Ehrhart polynomials of O_P and C_P coincide: $E_{O_P}(k) = E_{C_P}(k)$, so we also have $\delta_{O_P}(t) = \delta_{C_P}(t)$.

For the poset P on $[n] = \{1, 2, \ldots, n\}$, the *$P$-Eulerian polynomial* is

$$W(P) = \sum_{\pi \in \mathcal{L}(P)} x^{\mathrm{des}(\pi)+1},$$

where $\mathcal{L}(P)$ is the set of all linear extensions of P and $\mathrm{des}(\pi)$ is the size of the descent set of w with respect to P. That is, $\mathcal{L}(P)$ is the set of permutations $w = (w_1, w_2, \ldots, w_n)$ of $[n]$ such that $w_i \prec w_j$ implies $i < j$, and $\mathrm{des}(w) = \#\{i \in [n-1] : w_i > w_{i+1}\}$. This polynomial is equal to the δ-polynomial of O_P, i.e., $W(P) = \delta_{O_P}(t)$.

When the poset P is graded of rank r, the result of [13] shows that the δ-vector can be written as

$$\delta(O_P) = h(\Delta_{\mathrm{eq}}(P) * \sigma^r),$$

where σ^r is the r-dimensional standard simplex and $\Delta_{\mathrm{eq}}(P)$ is the equatorial sphere of P, which will be explained below. Here, the operator $*$ is the simplicial join of simplicial complexes and $h(\Delta_{\mathrm{eq}}(P)*\sigma^r)$ represents the h-vector of the simplicial complex $\Delta_{\mathrm{eq}}(P)*\sigma^r$.

For a poset P, a *P-partition* is a function $f : P \to \mathbb{R}$ such that $f(a) \geqslant 0$ for all $a \in P$ and $f(a) \geqslant f(b)$ for all $a \prec b$. When P is a graded poset of rank r, let $P^{(i)}$ denote the set of the elements of P of rank i. We say that a P-partition is *equatorial* if $\min_{a \in P} f(a) = 0$ and for every $2 \leqslant j \leqslant r$ there exists $a_{j-1} \prec a_j$ with $a_{j-1} \in P^{(j-1)}$, $a_j \in P^{(j)}$ and $f(a_{j-1}) = f(a_j)$. An order ideal I of P is *equatorial* if its characteristic vector χ_I is equatorial. A chain of order ideals $I_1 \subset I_2 \subset \cdots \subset I_t$ is *equatorial* if $\chi_{I_i} + \cdots + \chi_{I_t}$ is equatorial. The *equatorial complex* $\Delta_{\mathrm{eq}}(P)$ of P is the simplicial complex whose vertex set is the equatorial ideals of P and faces are equatorial chains of order ideals of P. The result of [13] shows that $\Delta_{\mathrm{eq}}(P)$ is a (polytopal) simplicial sphere and it is called the *equatorial sphere* of P. Since the h-vector of a simplicial sphere is palindromic by the Dehn-Sommerville equations, this implies that the δ-vector of O_P for a graded poset P is palindromic followed by r 0's as follows:

$$\delta(O_P) = (h_0, h_1, \ldots, h_1, h_0, \underbrace{0, 0, \ldots, 0}_{r}).$$

The palindromic part $(h_0, h_1, \ldots, h_1, h_0) = h(\Delta_{\text{eq}})$ of $\delta(O_P)$ corresponds to the equatorial sphere $\Delta_{\text{eq}}(P)$, so it will make sense to consider the corresponding polynomial as follows.

$$E_P^{\text{eq}}(k) = f^{\text{Ehr}}(h(\Delta_{\text{eq}})) = f^{\text{Ehr}}((h_0, h_1, \ldots, h_1, h_0)).$$

We call this $E_P^{\text{eq}}(k)$ the *equatorial Ehrhart polynomial* of the graded poset P. In [13], the equatorial sphere is constructed as a quotient polytope from the order polytope, that is, as a quotient polytope $O_P^{\text{eq}} = O_P/V^{\text{rc}}$, where V^{rc} is the rank-constant subspace, the subspace consisting of partition functions that are rank-constant (i.e., $f(x) = f(y)$ whenever x and y are of the same rank in P). The polynomial $E_P^{\text{eq}}(k)$ corresponds to the Ehrhart polynomial of this polytope.

Since $h(\Delta_{\text{eq}})$ is palindromic, the roots of $E_P^{\text{eq}}(k)$ distribute symmetrically with respect to the line $\text{Re}(z) = -\frac{1}{2}$. It is of our interest for which graded poset P all the roots of $E_P^{\text{eq}}(k)$ lie on the line $\text{Re}(z) = -\frac{1}{2}$. We call such $E_P^{\text{eq}}(k)$ to be CL analogously to the CL-polytopes among reflexive polytopes.

A *complete graded poset* $P_{n_1,n_2,\ldots,n_r}$ stands for a graded poset of rank r such that the set $P^{(i)}$ of the elements of rank i consists of n_i elements for every i and $a_i \prec a_j$ holds for every $a_i \in P^{(i)}$ and $a_j \in P^{(j)}$ with $i < j$. For complete graded posets, we can easily calculate the δ-polynomials as follows. Since the antichains of $P_{n_1,n_2,\ldots,n_r}$ are subsets $X \subset P^{(i)}$ for some i, we have

$$C_{P_{n_1,n_2,\ldots,n_r}} = [0,1]^{n_1} \oplus [0,1]^{n_2} \oplus \cdots \oplus [0,1]^{n_r}.$$

The δ-polynomial of $[0,1]^n$ is given by the Eulerian polynomial $S_n(t) = \sum_{j=0}^{n-1} \left\langle {n \atop j} \right\rangle t^j$, where $\left\langle {n \atop j} \right\rangle$ is the Eulerian number, and hence we have

$$\delta_{O_{P_{n_1,n_2,\ldots,n_r}}}(t) = \delta_{C_{P_{n_1,n_2,\ldots,n_r}}}(t) = \prod_{i=1}^{r} S_{n_i}(t).$$

There is another explanation for this. For the complete graded poset $P_{n_1,n_2,\ldots,n_r}$, an equatorial ideal is a proper subset of $P^{(i)}$ for some $0 \leqslant i \leqslant r$ together with all $P^{(j)}$'s with $j < i$, hence we observe that $\Delta_{\text{eq}}(P_{n_1,n_2,\ldots,n_r})$ is isomorphic to the order complex of $\check{B}_{n_1} \uplus \cdots \uplus \check{B}_{n_r}$, where $\check{B}_n$ is the poset removing the top element from the boolean lattice of order n (= the ordered set consisting of all the strict subsets of $\{1, \ldots, n\}$ ordered by inclusion), and $\uplus$ is the operator of the ordinal sum of the posets (i.e., $P \uplus P'$ is the poset over $P \cup P'$ with an order relation $\preceq_{P \uplus P'}$ such that $u \preceq_{P \uplus P'} v$ if $u, v \in P$ and $u \preceq_P v$, $u, v \in P'$ and $u \preceq_{P'} v$, or $u \in P$ and $v \in P'$). This shows that the equatorial sphere of $P_{n_1,n_2,\ldots,n_r}$ is isomorphic to $\text{sd}(\Delta_{n_1}) * \cdots * \text{sd}(\Delta_{n_r})$, where $\text{sd}(\Delta)$ is the barycentric subdivision of Δ. Since the h-polynomial of $\text{sd}(\Delta_n)$ is given by the Eulerian polynomial (see, e.g., [12, Sec. 9.2]), we have the same conclusion.

The equatorial Ehrhart polynomial for P_n, which is just an antichain with n elements, can be calculated as follows. Since we have $\delta_i = \left\langle {n \atop i} \right\rangle$ for $0 \leqslant i \leqslant n-1$, where $\delta(O_{P_n}) =$

$(\delta_0, \delta_1, \ldots, \delta_{n-1}, 0)$, we obtain that

$$\begin{aligned}
E^{\mathrm{eq}}_{P_n}(k) &= \sum_{j=0}^{n-1} \left\langle {n \atop j} \right\rangle \binom{n-1+k-j}{n-1} = \sum_{j=0}^{n-1} \left\langle {n \atop n-1-j} \right\rangle \binom{k+(n-1-j)}{n-1} \\
&= \sum_{j'=0}^{n-1} \left\langle {n \atop j'} \right\rangle \binom{k+j'}{n-1} \quad (j' = n-1-j) \\
&= \sum_{j'=0}^{n-1} \left\langle {n \atop j'} \right\rangle \left(\binom{k+j'+1}{n} - \binom{k+j'}{n} \right) = (k+1)^n - k^n.
\end{aligned}$$

Here, the last equality is derived from Worpitzky's identity (e.g. [6, Sec. 6.2]): $x^n = \sum_{j=0}^{n-1} \left\langle {n \atop j} \right\rangle \binom{x+j}{n}$. This polynomial $(k+1)^n - k^n$ equals the Ehrhart polynomial of $A^\vee_{n-1}$ as shown in [9]. That is, we have $E^{\mathrm{eq}}_{P_n}(k) = E_{A^\vee_{n-1}}(k)$. In fact, more strongly, we observe that the equatorial polytope $O^{\mathrm{eq}}_{P_n}$ is unimodularly equivalent to $A^\vee_{n-1}$ as follows.

Proposition 1. *$O^{\mathrm{eq}}_{P_n} = O_{P_n}/V^{rc}$ is unimodularly equivalent to $A^\vee_{n-1}$.*

Proof. The subspace V^{rc} is the space of rank-constant partitions, and in this case, it is a one-dimensional space $V^{\mathrm{rc}} = \operatorname{span}\{\sum_{i\in[n]} e_i\}$. Let π be the projection map from O_{P_n} to $O^{\mathrm{eq}}_{P_n}$. By letting $f = \sum_{i\in[n]} e_i$, for any $v \in \mathbb{R}^n$, we can uniquely write $v = \sum_{i=1}^{n-1} r_i e_i + sf \in V$ $(r_i, s \in \mathbb{R})$, then we have $\pi(v) = \sum_{i=1}^{n-1} r_i e_i$. The vertex set of O_{P_n} is $\{\sum_{i\in S} e_i : S \subseteq [n]\}$, and they are mapped to the following:

$$\pi\left(\sum_{i\in S} e_i\right) = \begin{cases} \sum_{i\in S} e_i & \text{if } n \notin S, \\ -\sum_{i\notin S} e_i & \text{if } n \in S. \end{cases}$$

From this, we observe that the vertex set of $O^{\mathrm{eq}}_{P_n}$ is $\{\pm \sum_{i\in S} e_i : S \subseteq [n-1]\}$. Hence

$$(O^{\mathrm{eq}}_{P_n})^\vee = \left\{ x \in \mathbb{R}^{n-1} : \left\langle \pm \sum_{i\in S} e_i, x \right\rangle \leqslant 1,\ S \subseteq [n-1] \right\}.$$

On the one hand, it is easy to see that A_{n-1} is unimodularly equivalent to

$$\operatorname{conv}(\{\pm e_i : 1 \leqslant i \leqslant n-1\} \cup \{e_i - e_j : 1 \leqslant i \neq j \leqslant n-1\}).$$

Since we have

$$\left\langle \sum_{i\in S} e_i, \pm e_j \right\rangle = \begin{cases} \pm 1 & \text{if } j \in S, \\ 0 & \text{if } j \notin S, \end{cases} \quad \text{and} \quad \left\langle \sum_{i\in S} e_i, e_j - e_k \right\rangle = \begin{cases} 1 & \text{if } j \in S, k \notin S, \\ -1 & \text{if } j \notin S, k \in S, \\ 0 & \text{if } j,k \in S \text{ or } j,k \notin S, \end{cases}$$

we see that $A_{n-1} \subseteq (O^{\mathrm{eq}}_{P_n})^\vee$. On the other hand, let $w = (w_1, \ldots, w_{n-1}) \in \mathbb{Z}^{n-1}$ satisfying that $\langle w, v \rangle \leqslant 1$ for any $v \in A_{n-1}$. If there is i with $|w_i| \geqslant 2$, then $|\langle w, e_i \rangle| \geqslant 2$, a contradiction. Thus, $w \in \{0, \pm 1\}^{n-1}$. Moreover, if there are i and i' with $w_i = 1$ and $w_{i'} = -1$, then $\langle w, e_i - e_{i'} \rangle = 2$, a contradiction. Hence, $w \in \{0,1\}^{n-1}$ or $w \in \{0,-1\}^{n-1}$. This means that w is always of the form $w = \pm \sum_{i \in S} e_i$. This implies that $(O^{\mathrm{eq}}_{P_n})^\vee \subset A_{n-1}$, as required. □

Corollary 2. *We have*

$$E^{\mathrm{eq}}_{P_{n_1,n_2,\dots,n_r}}(k) = E_{A^\vee_{n_1-1} \oplus A^\vee_{n_2-1} \oplus \cdots \oplus A^\vee_{n_r-1}}(k).$$

Proof. Since we have $\delta_{O^{\mathrm{eq}}_{P_n}}(t) = \delta_{A^\vee_{n-1}}(t)$ from Proposition 1,

$$\delta_{O^{\mathrm{eq}}_{P_{n_1,n_2,\dots,n_r}}}(t) = \delta_{O_{P_{n_1,n_2,\dots,n_r}}}(t) = \prod_{i=1}^{r} S_{n_i}(t) = \prod_{i=1}^{r} \delta_{O^{\mathrm{eq}}_{P_{n_i}}}(t) = \prod_{i=1}^{r} \delta_{A^\vee_{n_i-1}}(t) = \delta_{A^\vee_{n_1-1} \oplus \cdots \oplus A^\vee_{n_r-1}}(t).$$

The statement follows from $\dim O^{\mathrm{eq}}_{P_{n_1,n_2,\dots,n_r}} = \dim A^\vee_{n_1-1} \oplus A^\vee_{n_2-1} \oplus \cdots \oplus A^\vee_{n_r-1}$. □

By this, the CL-ness of $E^{\mathrm{eq}}_{P_{n_1,n_2,\dots,n_r}}(k)$ is equivalent to the CL-ness of $A^\vee_{n_1-1} \oplus A^\vee_{n_2-1} \oplus \cdots \oplus A^\vee_{n_r-1}$.

Remark 3. The discussion of this section gives that the δ-polynomial of $A^\vee_d$ equals to

$$\delta_{A^\vee_d}(t) = \sum_{j=0}^{d} \left\langle \begin{matrix} d+1 \\ j \end{matrix} \right\rangle t^j .$$

3 CL-ness of $A^\vee_m \oplus A^\vee_n$

For the case of the free sum $A^\vee_1 \oplus A^\vee_n$, we have the following.

Proposition 4. *We have*

$$E_{A^\vee_1 \oplus A^\vee_n}(k) = (k+1)^n + k^n,$$

and $A^\vee_1 \oplus A^\vee_n$ is a CL-polytope.

Proof. Since $\delta_{A^\vee_1}(t) = 1 + t$ and $\delta_{A^\vee_n} = \sum_{i=1}^{n} \left\langle \begin{matrix} n+1 \\ i \end{matrix} \right\rangle t^i$, we have

$$\delta_i(A^\vee_1 \oplus A^\vee_n) = \left\langle \begin{matrix} n+1 \\ i \end{matrix} \right\rangle + \left\langle \begin{matrix} n+1 \\ i-1 \end{matrix} \right\rangle \quad (0 \leqslant i \leqslant n+1)$$

using the convention that $\left\langle \begin{matrix} n \\ i \end{matrix} \right\rangle = 0$ when $i < 0$ or $i \geqslant n$. Thus,

$$E_{A^\vee_1 \oplus A^\vee_n}(k) = \sum_{j=0}^{n+1} \left(\left\langle \begin{matrix} n+1 \\ j \end{matrix} \right\rangle + \left\langle \begin{matrix} n+1 \\ j-1 \end{matrix} \right\rangle \right) \binom{n+1+k-j}{n+1}$$

$$
\begin{aligned}
&= \sum_{j=0}^{n} \left\langle {n+1 \atop j} \right\rangle \binom{n+1+k-j}{n+1} + \sum_{j=1}^{n+1} \left\langle {n+1 \atop j-1} \right\rangle \binom{n+1+k-j}{n+1} \\
&= \sum_{j=0}^{n} \left\langle {n+1 \atop n-j} \right\rangle \binom{1+k+(n-j)}{n+1} + \sum_{j=1}^{n+1} \left\langle {n+1 \atop n-j+1} \right\rangle \binom{k+(n-j+1)}{n+1} \\
&= \sum_{j'=0}^{n} \left\langle {n+1 \atop j'} \right\rangle \binom{1+k+j'}{n+1} + \sum_{j''=0}^{n} \left\langle {n+1 \atop j''} \right\rangle \binom{k+j''}{n+1} \qquad (j' = n-j, \ j'' = n-j+1) \\
&= (k+1)^{n+1} + k^{n+1}.
\end{aligned}
$$

Here, the last equality is derived by Worpitzky's identity.

This polynomial $(k+1)^{n+1} + k^{n+1}$ equals the Ehrhart polynomial of the polar dual $C_{n+1}^{\vee}$ of the classical root polytope of type C and it is shown to be CL in [10]. □

This theorem shows $A_1^{\vee} \oplus A_n^{\vee} = (A_1 \times A_n)^{\vee}$ and $C_{n+1}^{\vee}$ have the same Ehrhart polynomial, though $A_1 \times A_n$ and C_{n+1} are not unimodularly equivalent since C_{n+1} does not have the structure of the product of two polytopes.

The CL-ness of $A_m^{\vee} \oplus A_n^{\vee}$ with small m and n are calculated by computer using Pari/GP. See appendix for the detail. The results are summarized as shown in Table 1. From the table, we have the following theorem.

Table 1: CL-ness of $A_m^{\vee} \oplus A_n^{\vee}$ with $m, n \leqslant 20$

$m \setminus n$	0	1	2	3	4	5	6	$7 \sim 20$	$\geqslant 21$
0	CL	CL	CL	CL	CL	CL	CL	CL	CL
1	CL	CL	CL	CL	CL	CL	CL	CL	CL
2	CL	CL	CL	CL	CL	CL	not CL	not CL	
3	CL	CL	CL	CL	CL	not CL	not CL	not CL	
4	CL	CL	CL	CL	not CL	not CL	not CL	not CL	
5	CL	CL	CL	not CL	not CL	not CL	not CL	not CL	
6	CL	CL	not CL	not CL	not CL	not CL	not CL	not CL	
$7 \sim 20$	CL	CL	not CL	not CL	not CL	not CL	not CL	not CL	
$\geqslant 21$	CL	CL							

Theorem 5. *$A_m^{\vee} \oplus A_n^{\vee}$ is CL if $\min\{m, n\} \leqslant 1$ or $m + n \leqslant 7$.*

It is not yet shown whether all the cases $m \geqslant 2$ and $n \geqslant 8$ (or vice versa) are not CL, though it is plausible that Theorem 5 is also necessary for $A_m^{\vee} \oplus A_n^{\vee}$ to be CL. By our computer calculation up to $n, m \leqslant 20$, no other CL parameters are found other than shown above.

4 CL-ness of $A_{n_1}^\vee \oplus A_{n_2}^\vee \oplus \cdots \oplus A_{n_r}^\vee$

In the following theorems, we have families of $A_{p_1}^\vee \oplus A_{p_2}^\vee \oplus \cdots \oplus A_{p_r}^\vee$ that are CL. In what follows, we denote $\underbrace{A_p^\vee \oplus A_p^\vee \oplus \cdots \oplus A_p^\vee}_{n}$ as $(A_p^\vee)^{\oplus n}$.

Proposition 6 ([9, Example 3.3]). *$(A_1^\vee)^{\oplus n}$ is CL for any n.*

Proof. This $A_1^{\vee\,\oplus n}$ is the n-dimensional cross polytope Cr_n, and is shown to be CL in [9, Example 3.3]. □

We can further show that $A_2^\vee \oplus (A_1^\vee)^{\oplus n}$ and $A_3^\vee \oplus (A_1^\vee)^{\oplus n}$ are also CL. For these, we use the following lemma. Here, R is the canonical line $\mathrm{Re}(z) = -1/2$, and two functions $f(x)$ and $g(x)$ with $\deg f = \deg g + 1$ are *R-interlacing* if all the zeros of $f(x)$ and $g(x)$ are on R and they appear alternatingly on R. That is, the zeros of f are $-1/2 + z_1 i, -1/2 + z_2 i, \ldots, -1/2 + z_d i$ and those of g are $-1/2 + w_1 i, -1/2 + w_2 i, \ldots, -1/2 + w_{d-1} i$, with $z_1 < w_1 < z_2 < w_2 < \cdots < w_{d-1} < z_d$, where $d = \deg f$.

Lemma 7 ([9, Lemma 2.5]). *Let f_1, f_2, and f_3 be real monic polynomials such that $\deg f_1 = \deg f_2 + 1 = \deg f_3 + 2$ and $f_1(x) = f_2(x) \cdot (x + \frac{1}{2}) + \beta f_3(x)$ for some $\beta > 0$. Then f_1 and f_2 are R-interlacing if and only if f_2 and f_3 are R-interlacing.*

Note that, when we use this lemma for three Ehrhart polynomials E_1, E_2, and E_3, the relation in the lemma should be

$$E_1(k) = \alpha E_2(k) \cdot (2k+1) + (1-\alpha) E_3(k) \quad \text{for some} \quad 0 \leqslant \alpha \leqslant 1.$$

See [9, Section 3].

Theorem 8. *$A_2^\vee \oplus (A_1^\vee)^{\oplus n}$ is CL for any n.*

Proof. We have the following equality:

$$E_{A_2^\vee \oplus (A_1^\vee)^{\oplus n}}(k) = \frac{3}{2n+4} E_{(A_1^\vee)^{\oplus (n+1)}}(k) \cdot (2k+1) + \frac{2n+1}{2n+4} E_{(A_1^\vee)^{\oplus n}}(k). \tag{2}$$

This follows from the relation of the Ehrhart series:

$$\mathrm{Ehr}_{A_2^\vee \oplus (A_1^\vee)^{\oplus n}}(t) = \frac{3}{2n+4}\left(2t\frac{d}{dt}\mathrm{Ehr}_{(A_1^\vee)^{\oplus(n+1)}}(t) + \mathrm{Ehr}_{(A_1^\vee)^{\oplus(n+1)}}(t)\right) + \frac{2n+1}{2n+4}\mathrm{Ehr}_{(A_1^\vee)^{\oplus n}}(t). \tag{3}$$

The equation (2) is derived by comparing the coefficients of t^k in (3). The equation (3) can be verified using $\mathrm{Ehr}_{A_2^\vee \oplus (A_1^\vee)^{\oplus n}}(t) = \frac{(1+4t+t^2)(t+1)^n}{(1-t)^{n+3}}$ and $\mathrm{Ehr}_{(A_1^\vee)^{\oplus n}}(t) = \frac{(1+t)^n}{(1-t)^{n+1}}$ as follows:

$$\begin{aligned}\text{RHS of (3)} =& \frac{3}{2n+4}\left(2t\frac{d}{dt}\frac{(1+t)^{n+1}}{(1-t)^{n+2}} + \frac{(1+t)^{n+1}}{(1-t)^{n+2}}\right) + \frac{2n+1}{2n+4}\frac{(1+t)^n}{(1-t)^{n+1}} \\ =& \frac{(1+t)^n}{(1-t)^{n+3}}\Bigg(2t\frac{3}{2n+4}\big((n+1)(1-t) + (1+t)(n+2)\big)\end{aligned}$$

$$+\frac{3(1+t)(1-t)}{2n+4}+\frac{(2n+1)(1-t)^2}{2n+4}\Bigg)$$

$$=\frac{(1+t)^n(1+4t+t^2)}{(1-t)^{n+3}}=\mathrm{Ehr}_{A_2^\vee\oplus(A_1^\vee)^{\oplus n}}(t).$$

Since the Ehrhart polynomials of $(A_1^\vee)^{\oplus(n+1)}$ and $(A_1^\vee)^{\oplus n}$ (i.e., the cross polytopes Cr_{n+1} and Cr_n) are R-interlacing as shown in [9, Corollary 5.4], $A_2^\vee\oplus(A_1^\vee)^{\oplus n}$ and $(A_1^\vee)^{\oplus(n+1)}$ are R-interlacing by Lemma 7. Hence, we conclude that $A_2^\vee\oplus(A_1^\vee)^{\oplus n}$ is CL.

□

Theorem 9. *$A_3^\vee\oplus(A_1^\vee)^{\oplus n}$ is CL for any n.*

Proof. We have the following equality:

$$E_{A_3^\vee\oplus(A_1^\vee)^{\oplus n}}(k)=\frac{3}{n+3}E_{(A_1^\vee)^{\oplus(n+2)}}(k)\cdot(2k+1)+\frac{n}{n+3}E_{(A_1^\vee)^{\oplus(n+1)}}(k). \tag{4}$$

This equation follows from

$$\mathrm{Ehr}_{A_3^\vee\oplus(A_1^\vee)^{\oplus n}}(t)=\frac{3}{n+3}\left(2t\frac{d}{dt}\mathrm{Ehr}_{(A_1^\vee)^{\oplus(n+2)}}(t)+\mathrm{Ehr}_{(A_1^\vee)^{\oplus(n+2)}}(t)\right)+\frac{n}{n+3}\mathrm{Ehr}_{(A_1^\vee)^{\oplus(n+1)}}(t). \tag{5}$$

as in Theorem 8, and then the statement follows from Lemma 7.

The equation (5) is verified as follows:

$$\begin{aligned}\text{RHS of (5)}=&\frac{3}{n+3}\left(2t\frac{d}{dt}\frac{(1+t)^{n+2}}{(1-t)^{n+3}}+\frac{(1+t)^{n+2}}{(1-t)^{n+3}}\right)+\frac{n}{n+4}\frac{(1+t)^{n+1}}{(1-t)^{n+2}}\\ =&\frac{(1+t)^{n+1}}{(1-t)^{n+3}}\Bigg(2t\frac{3}{n+3}\big((n+2)(1-t)+(1+t)(n+3)\big)\\ &+\frac{3(1+t)(1-t)}{n+3}+\frac{n(1-t)^2}{n+3}\Bigg)\\ =&\frac{(1+t)^{n+1}(1+10t+t^2)}{(1-t)^{n+4}}=\frac{(1+t)^n(1+11t+11t^2+t^3)}{(1-t)^{n+4}}=\mathrm{Ehr}_{A_3^\vee\oplus(A_1^\vee)^{\oplus n}}(t).\end{aligned}$$

□

Remark 10. Other than Proposition 6, Theorems 8 and 9, $A_4^\vee\oplus(A_1^\vee)^{\oplus n}$ and $A_5^\vee\oplus(A_1^\vee)^{\oplus n}$ also seem to be CL by computer calculations for small n's. On the other hand, also from observation by computer calculation for small n's, $A_m^\vee\oplus(A_1^\vee)^{\oplus n}$ is not CL for $m\geqslant 7$ and $n\geqslant 2$. The behavior of $A_6^\vee\oplus(A_1^\vee)^{\oplus n}$ is somewhat strange so that it is CL for odd n's and not CL for even n's.

Remark 11. In the proof of Theorems 8 and 9, the keys are the equations (2) and (4). Analogously, there are other relations among Ehrhart polynomials of $A_d^\vee$'s. We have found the following equations, though we do not currently find any application.

$$(a)\quad E_{A_3^\vee\oplus(A_2^\vee)^{\oplus n}}(k) = \frac{2}{2n+3}E_{(A_2^\vee)^{\oplus(n+1)}}(k)\cdot(2k+1) + \frac{2n+1}{2n+3}E_{A_1^\vee\oplus(A_2^\vee)^{\oplus n}}(k)$$

$$(b)\quad E_{A_3^\vee\oplus(A_1^\vee)^{\oplus n}}(k) = \frac{2}{n+3}E_{A_2^\vee\oplus(A_1^\vee)^{\oplus n}}(k)\cdot(2k+1) + \frac{2n+1}{n+3}E_{(A_1^\vee)^{\oplus(n+1)}}(k) - \frac{n}{n+3}E_{(A_1^\vee)^{\oplus(n-1)}}(k)$$

$$(c)\quad E_{A_4^\vee\oplus(A_1^\vee)^{\oplus n}}(k) = \frac{5}{2n+8}E_{A_3^\vee\oplus(A_1^\vee)^{\oplus n}}(k)\cdot(2k+1) + \frac{5(4n+2)}{3(2n+8)}E_{A_2^\vee\oplus(A_1^\vee)^{\oplus n}}(k) - \frac{14n+1}{3(2n+8)}E_{(A_1^\vee)^{\oplus n}}(k)$$

$$(d)\quad E_{(A_2^\vee)^{\oplus 2}\oplus(A_1^\vee)^{\oplus n}}(k) = \frac{3}{2n+8}E_{A_2^\vee\oplus(A_1^\vee)^{\oplus(n+1)}}(k)\cdot(2k+1) + \frac{2n+3}{2n+8}E_{A_2^\vee\oplus(A_1^\vee)^{\oplus n}}(k) + \frac{2}{2n+8}E_{(A_1^\vee)^{\oplus n}}(k)$$

$$(e)\quad E_{A_3^\vee\oplus(A_1^\vee)^{\oplus n}}(k) = \frac{2}{n+3}E_{A_2^\vee\oplus(A_1^\vee)^{\oplus n}}(k)\cdot(2k+1) + \frac{n+1}{n+3}\left(\frac{2n+1}{n+1}E_{(A_1^\vee)^{\oplus(n+1)}}(k) - \frac{n}{n+1}E_{(A_1^\vee)^{\oplus(n-1)}}(k)\right)$$

$$\frac{2n+1}{n+1}E_{(A_1^\vee)^{\oplus(n+1)}}(k) - \frac{n}{n+1}E_{(A_1^\vee)^{\oplus(n-1)}}(k) = \frac{2n+1}{(n+1)^2}E_{(A_1^\vee)^{\oplus n}}(k)\cdot(2k+1) + \frac{n^2}{(n+1)^2}E_{(A_1^\vee)^{\oplus(n-1)}}(k)$$

$$(f)\quad E_{A_4^\vee\oplus(A_1^\vee)^{\oplus n}}(k) = \frac{5}{2n+8}E_{A_3^\vee\oplus(A_1^\vee)^{\oplus n}}(k)\cdot(2k+1) + \frac{2n+3}{2n+8}\left(\frac{5(4n+2)}{3(2n+3)}E_{A_2^\vee\oplus(A_1^\vee)^{\oplus n}}(k) - \frac{14n+1}{3(2n+3)}E_{(A_1^\vee)^{\oplus n}}(k)\right)$$

$$\frac{5(4n+2)}{3(2n+3)}E_{(A_2^\vee)\oplus(A_1^\vee)^{\oplus n}}(k) - \frac{14n+1}{3(2n+3)}E_{(A_1^\vee)^{\oplus n}}(k) = \frac{5(2n+1)}{(2n+3)(n+2)}E_{(A_1^\vee)^{\oplus(n+1)}}(k)\cdot(2k+1) + \frac{(n-1)(2n-1)}{(2n+3)(n+2)}E_{(A_1^\vee)^{\oplus n}}(k)$$

$$(f')\quad E_{A_4^\vee\oplus(A_1^\vee)^{\oplus n}}(k) = \frac{15}{(2n+8)(n+3)}E_{(A_1^\vee)^{\oplus(n+2)}}(k)\cdot(2k+1)^2 + \frac{15(n^2+3n+1)}{2(n+2)(n+3)(n+4)}E_{(A_1^\vee)^{\oplus(n+1)}}(k)\cdot(2k+1) + \frac{(2n-1)(n-1)}{2(n+2)(n+4)}E_{(A_1^\vee)^{\oplus n}}(k)$$

5 Free sums of A_d's

In the previous sections, we have studied the root distributions of the Ehrhart polynomials of the free sums of $A_d^\vee$'s. It is also of interest in studying the free sums of other reflexive polytopes. For example, how about the free sums of the classical root polytopes A_d's? Note that since $A_1 = A_1^\vee$, the CL-ness and the R-interlacing property for $A_1^{\oplus n} = \mathrm{Cr}_n$ also hold.

For the root polytope of type A, the Ehrhart polynomial and the δ-polynomial known to be as follows ([2, Theorem 1], [1, Theorem 2]):

$$E_{A_d}(k) = \sum_{j=0}^{d} \binom{d}{j}^2 \binom{k+d-j}{d}, \quad \delta_{A_d}(t) = \sum_{j=0}^{d} \binom{d}{j}^2 t^j.$$

We have the following several analogous results.

Theorem 12. *$A_1 \oplus A_n$ is CL for any $n \geqslant 1$.*

Proof. We have the following equality:

$$E_{A_1 \oplus A_n}(k) = \frac{1}{n+1} E_{A_n}(k) \cdot (2k+1) + \frac{n}{n+1} E_{A_{n-1}}(k). \tag{6}$$

This relation follows from the following relation of the Ehrhart series:

$$\mathrm{Ehr}_{A_1 \oplus A_n}(t) = \frac{1}{n+1} \left(2 \frac{d}{dt} \mathrm{Ehr}_{A_n}(t) + \mathrm{Ehr}_{A_n}(t) \right) + \frac{n}{n+1} \mathrm{Ehr}_{A_{n-1}}(t), \tag{7}$$

which is verified as follows. Since we have

$$\mathrm{Ehr}_{A_n} = \frac{\sum_{j=0}^{n} \binom{n}{j}^2 t^j}{(1-t)^{n+1}}, \quad \mathrm{Ehr}_{A_1 \oplus A_n} = \frac{(1+t) \sum_{j=0}^{n} \binom{n}{j}^2 t^j}{(1-t)^{n+2}},$$

the equation (7) is equivalent to

$$\frac{(1+t) \sum_{j=0}^{n} \binom{n}{j}^2 t^j}{(1-t)^{n+2}} = \frac{1}{n+1} \left(2 \frac{d}{dt} \frac{\sum_{j=0}^{n} \binom{n}{j}^2 t^j}{(1-t)^{n+1}} + \frac{\sum_{j=0}^{n} \binom{n}{j}^2 t^j}{(1-t)^{n+1}} \right) + \frac{n}{n+1} \frac{\sum_{j=0}^{n-1} \binom{n-1}{j}^2 t^j}{(1-t)^{n}},$$

and we have

$$\begin{aligned} (1+t) \sum_{j=0}^{n} \binom{n}{j}^2 t^j =& \frac{2t}{n+1} (1-t) \sum_{j=1}^{n} j \binom{n}{j}^2 t^{j-1} + 2t \sum_{j=0}^{n} \binom{n}{j}^2 t^j \\ &+ \frac{1}{n+1} (1-t) \sum_{j=0}^{n} \binom{n}{j}^2 t^j + \frac{n}{n+1} (1-t)^2 \sum_{j=0}^{n-1} \binom{n-1}{j}^2 t^j. \end{aligned}$$

By comparing the coefficients of t^i, what we have to show is

$$\binom{n}{i}^2+\binom{n}{i-1}^2=\frac{2}{n+1}(i\binom{n}{i}^2-(i-1)\binom{n}{i-1}^2)+2\binom{n}{i-1}^2$$
$$+\frac{1}{n+1}(\binom{n}{i}^2-\binom{n}{i-1}^2)+\frac{n}{n+1}(\binom{n-1}{i}^2-2\binom{n-1}{i-1}^2+\binom{n-1}{i-2}^2), \quad (8)$$

where $\binom{n}{i}$ is assumed to be 0 when $i<0$ or $i>n$. This is verified by

$$\binom{n}{i}^2+\binom{n}{i-1}^2=\binom{n}{i}^2+\frac{i^2}{(n-i+1)^2}\binom{n}{i}^2=\frac{n^2-2in+2n+2i^2-2i+1}{(n-i+1)^2}\binom{n}{i}^2$$

and

$$\begin{aligned}\text{RHS of (8)} =&\frac{2}{n+1}(i\binom{n}{i}^2-(i-1)\binom{n}{i-1}^2)+2\binom{n}{i-1}^2+\frac{1}{n+1}(\binom{n}{i}^2-\binom{n}{i-1}^2)\\ &+\frac{n}{n+1}(\frac{(n-i)^2}{n^2}\binom{n}{i}^2-2\frac{(n-i+1)^2}{n^2}\binom{n}{i-1}^2+\frac{(i-1)^2}{n^2}\binom{n}{i-1}^2)\\ =&\frac{n^2+n+i^2}{n(n+1)}\binom{n}{i}^2+\frac{2ni-n-i^2+2i-1}{n(n+1)}\binom{n}{i-1}^2\\ =&\frac{n^2-2in+2n+2i^2-2i+1}{(n-i+1)^2}\binom{n}{i}^2.\end{aligned}$$

Since the Ehrhart polynomials of A_n and A_{n-1} are R-interlacing as shown in [9], the statement follows from Lemma 7 and (6). □

Proposition 13. *$A_2\oplus A_1^{\oplus n}$ are CL for any $n\geqslant 1$.*

Proof. This follows from Theorem 8, since we have $E_{A_1}(k)=E_{A_1^\vee}(k)$ and $E_{A_2}(k)=E_{A_2^\vee}(k)$. □

Theorem 14. *$A_3\oplus A_1^{\oplus n}$ is CL for any $n\geqslant 1$.*

Proof. We have the following relation:

$$E_{A_3\oplus A_1^{\oplus n}}(k)=\frac{5}{2(n+3)}E_{\mathrm{Cr}_{n+2}}(k)\cdot(2k+1)+\frac{2n+1}{2(n+3)}E_{\mathrm{Cr}_{n+1}}(k). \quad (9)$$

This follows from the relation of the Ehrhart series:

$$\mathrm{Ehr}_{A_3\oplus A_1^{\oplus n}}=\frac{5}{2(n+3)}\left(2t\frac{d}{dt}\mathrm{Ehr}_{\mathrm{Cr}_{n+2}}(t)+\mathrm{Ehr}_{\mathrm{Cr}_{n+2}}(t)\right)+\frac{2n+1}{2(n+3)}\mathrm{Ehr}_{\mathrm{Cr}_{n+1}}(t). \quad (10)$$

The equation (9) is derived by comparing the coefficients of t^k in (10). The equation (10) can be verified using $\mathrm{Ehr}_{A_3\oplus A_1^{\oplus n}}(t)=\frac{(1+9t+9t^2+t^3)(1+t)^n}{(1-t)^{n+4}}$ and $\mathrm{Ehr}_{\mathrm{Cr}_n}(t)=\frac{(1+t)^n}{(1-t)^{n+1}}$ as follows:

$$\text{RHS of (10)} =\frac{5}{2(n+3)}\left(2t\frac{d}{dt}\frac{(1+t)^{n+2}}{(1-t)^{n+3}}+\frac{(1+t)^{n+2}}{(1-t)^{n+3}}\right)+\frac{2n+1}{2(n+3)}\frac{(1+t)^{n+1}}{(1-t)^{n+2}}$$

$$
\begin{aligned}
=&\frac{(1+t)^n}{(1-t)^{n+4}}\Bigg(2t\frac{5}{2(n+3)}\big((n+2)(1+t)(1-t)+(1+t)^2(n+3)\big)\\
&\qquad\qquad\qquad\qquad+\frac{5(1-t)(1+t)^2}{2(n+3)}+\frac{(2n+1)(1+t)(1-t)^2}{2(n+3)}\Bigg)\\
=&\frac{(1+t)^n(1+9t+9t^2+t^3)}{(1-t)^{n+4}}=\mathrm{Ehr}_{A_3\oplus A_1^{\oplus n}}(t).
\end{aligned}
$$

The R-interlacing property follows from Lemma 7, since the Ehrhart polynomials of the cross polytopes Cr_{n+1} and Cr_n are R-interlacing. □

Table 2 shows the CL-ness of $A_m \oplus A_n$, calculated by computer using Pari/GP. Comparing with that of $A_m^\vee \oplus A_n^\vee$, the behavior is somewhat complex. (Here, "C" means CL, and "n" means not CL.) Similar to the case of $A_m \oplus A_n$, it is CL for small m and n. On the other hand, the behavior looks different when m and n are large. Those around the diagonal tend to be CL and it is plausible that $A_n \oplus A_n$ are CL for all n, for example, but we currently do not have any proof. By a computation using Pari/GP, $A_n \oplus A_n$ and $A_n \oplus A_{n+1}$ are CL for all n up to 100, while $A_n \oplus A_{n+2}$ are CL up to $n = 54$ but are not CL from $n = 55$ up to 100.

Table 2: CL-ness of $A_m \oplus A_n$ with $m, n \leqslant 20$

$n\backslash m$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C
1	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C
2	C	C	C	C	C	C	C	C	C	C	C	C	C	C	n	n	n	n	n	n	n
3	C	C	C	C	C	C	C	C	C	n	n	n	n	n	n	n	n	n	n	n	C
4	C	C	C	C	C	C	C	C	C	n	n	n	n	n	n	C	C	C	C	n	n
5	C	C	C	C	C	C	C	C	C	n	n	n	n	n	C	C	C	n	n	n	n
6	C	C	C	C	C	C	C	C	C	C	n	n	n	n	C	C	C	n	n	n	C
7	C	C	C	C	C	C	C	C	C	C	C	n	n	n	C	C	C	n	n	C	n
8	C	C	C	C	C	C	C	C	C	C	C	C	n	n	C	C	C	n	n	n	n
9	C	C	C	n	n	n	C	C	C	C	C	C	n	n	n	C	C	C	n	n	n
10	C	C	C	n	n	n	n	C	C	C	C	C	C	n	n	n	C	C	n	n	n
11	C	C	C	n	n	n	n	n	C	C	C	C	C	C	n	n	C	C	C	n	n
12	C	C	C	n	n	n	n	n	n	n	C	C	C	C	C	n	n	C	C	C	n
13	C	C	C	n	n	n	n	n	n	n	n	C	C	C	C	C	n	n	C	C	n
14	C	C	n	n	n	C	C	C	C	n	n	n	C	C	C	C	C	n	n	C	C
15	C	C	n	n	C	C	C	C	C	C	n	n	n	C	C	C	C	C	n	n	C
16	C	C	n	n	C	C	C	C	C	C	C	C	n	n	C	C	C	C	C	n	n
17	C	C	n	n	C	n	n	n	n	C	C	C	C	n	n	C	C	C	C	C	n
18	C	C	n	n	C	n	n	n	n	n	n	C	C	C	n	n	C	C	C	C	C
19	C	C	n	n	n	n	n	C	n	n	n	n	C	C	C	n	n	C	C	C	C
20	C	C	n	C	n	n	C	n	n	n	n	n	n	n	C	C	n	n	C	C	C

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Appendix: Some results by computer calculation

As mentioned in Section 3, we investigated the CL-ness for $A_m^\vee \oplus A_n^\vee$ with $m, n \geqslant 2$ by computer calculations. The δ-vectors, Ehrhart polynomials, and the CL-ness are listed in Table 3. The computation is done by using `Pari/GP`. The results are summarized in Theorem 5.

We also calculated the case of the free sums of three or four $A_d^\vee$'s with small parameters (up to 20) by using `Pari/GP`. For the case of the free sums of three $A_d^\vee$, our computer calculations are as follows.
$A_1^\vee \oplus A_1^\vee \oplus A_n^\vee$: CL for $n = 1, \dots, 5$, not CL for $n = 6, \dots, 20$
$A_1^\vee \oplus A_2^\vee \oplus A_n^\vee$: CL for $n = 2, \dots, 4$, not CL for $n = 5, \dots, 20$
$A_1^\vee \oplus A_3^\vee \oplus A_n^\vee$: CL for $n = 3, 4$, not CL for $n = 5, \dots, 20$
$A_1^\vee \oplus A_4^\vee \oplus A_n^\vee$: not CL for $n = 4$, CL for $n = 5$, not CL for $n = 6, \dots, 20$
$A_2^\vee \oplus A_2^\vee \oplus A_n^\vee$: CL for $n = 2, 3$, not CL for $n = 4, \dots, 20$
$A_2^\vee \oplus A_3^\vee \oplus A_n^\vee$: CL for $n = 3, \dots, 6$, not CL for $n = 7, \dots, 20$

Other parameters (each from 1 to 20) not listed here are not CL, up to permutation of the parameters.

Similarly to the case of two parameters, roughly speaking, we can observe that CL-ness hods for small parameters and CL-ness does not hold for large parameters. However, the boundary is sometimes complexified such that the CL/nonCL is not monotone: $A_1^\vee \oplus A_4^\vee \oplus A_n^\vee$ is CL for small $n \leqslant 3$, not CL for $n = 4$, CL for $n = 5$, and not CL for $n = 6, \dots, 20$. The same can be observed for $A_1^\vee \oplus A_m^\vee \oplus A_5^\vee$. It is not CL for $m = 2, 3$, CL for $m = 4$, and not CL for $m \geqslant 5$.

For the case of four $A_d^\vee$'s, our computer calculations are shown as follows.
$A_1^\vee \oplus A_1^\vee \oplus A_1^\vee \oplus A_n^\vee$: CL for $n = 1, \dots, 4$, not CL for $n = 5$, CL for $n = 6$, not CL for $n = 7, \dots, 20$
$A_1^\vee \oplus A_1^\vee \oplus A_2^\vee \oplus A_n^\vee$: CL for $n = 2, \dots, 5$, not CL for $n = 6, \dots, 20$
$A_1^\vee \oplus A_1^\vee \oplus A_3^\vee \oplus A_n^\vee$: CL for $n = 3, \dots, 5$, not CL for $n = 6, \dots, 20$
$A_1^\vee \oplus A_1^\vee \oplus A_4^\vee \oplus A_n^\vee$: CL for $n = 4$, not CL for $n = 5, \dots, 20$
$A_1^\vee \oplus A_2^\vee \oplus A_2^\vee \oplus A_n^\vee$: CL for $n = 2, \dots, 6$, not CL for $n = 7, \dots, 20$
$A_1^\vee \oplus A_2^\vee \oplus A_3^\vee \oplus A_n^\vee$: CL for $n = 3, \dots, 5$, not CL for $n = 6, \dots, 20$
$A_1^\vee \oplus A_2^\vee \oplus A_4^\vee \oplus A_n^\vee$: CL for $n = 4$, not CL for $n = 5$, CL for $n = 6$, not CL for $n = 7, \dots, 20$
$A_1^\vee \oplus A_3^\vee \oplus A_3^\vee \oplus A_n^\vee$: not CL for $n = 3$, CL for $n = 4$, not CL for $n = 5, \dots, 20$
$A_1^\vee \oplus A_3^\vee \oplus A_4^\vee \oplus A_n^\vee$: not CL for $n = 4$, CL for $n = 5$, not CL for $n = 6, \dots, 20$
$A_2^\vee \oplus A_2^\vee \oplus A_3^\vee \oplus A_n^\vee$: CL for $n = 3$, not CL for $n = 4, \dots, 20$
$A_2^\vee \oplus A_3^\vee \oplus A_3^\vee \oplus A_n^\vee$: not CL for $n = 3$, CL for $n = 4$, not CL for $n = 5, \dots, 20$

Other parameters (each from 1 to 20) not listed here are not CL, up to permutation of the parameters.

Table 3: δ-vectors, Ehrhart polynomials, and CL-ness of $A_m^\vee \oplus A_n^\vee$ with $2 \leqslant m, n \leqslant 7$

m	n	$\delta(A_m^\vee \oplus A_n^\vee),\ E_{A_m^\vee \oplus A_n^\vee}(k)$	
2	2	(1,8,18,8,1) $\frac{3}{2}x^4 + 3x^3 + \frac{9}{2}x^2 + 3x + 1$	CL
2	3	(1,15,56,56,15,1) $\frac{6}{5}x^5 + 3x^4 + 6x^3 + 6x^2 + \frac{19}{5}x + 1$	CL
2	4	(1,30,171,316,171,30,1) $x^6 + 3x^5 + \frac{15}{2}x^4 + 10x^3 + \frac{19}{2}x^2 + 5x + 1$	CL
2	5	(1,61,531,1567,1567,531,61,1) $\frac{6}{7}x^7 + 3x^6 + 9x^5 + 15x^4 + 19x^3 + 15x^2 + \frac{43}{7}x + 1$	CL
2	6	(1,124,1672,7300,12046,7300,1672,124,1) $\frac{3}{4}x^8 + 3x^7 + \frac{21}{2}x^6 + 21x^5 + \frac{133}{4}x^4 + 35x^3 + \frac{43}{2}x^2 + 7x + 1$	not CL
2	7	(1,251,5282,33038,82388,82388,33038,5282,251,1) $\frac{2}{3}x^9 + 3x^8 + 12x^7 + 28x^6 + \frac{266}{5}x^5 + 70x^4 + \frac{172}{3}x^3 + 28x^2 + \frac{39}{5}x + 1$	not CL
3	3	(1,22,143,244,143,22,1) $\frac{4}{5}x^6 + \frac{12}{5}x^5 + 6x^4 + 8x^3 + \frac{36}{5}x^2 + \frac{18}{5}x + 1$	CL
3	4	(1,37,363,1039,1039,363,37,1) $\frac{4}{7}x^7 + 2x^6 + 6x^5 + 10x^4 + 12x^3 + 9x^2 + \frac{31}{7}x + 1$	CL
3	5	(1,68,940,4252,6758,4252,940,68,1) $\frac{3}{7}x^8 + \frac{12}{7}x^7 + 6x^6 + 12x^5 + 18x^4 + 18x^3 + \frac{95}{7}x^2 + \frac{44}{7}x + 1$	not CL
3	6	(1,131,2522,16838,40988,40988,16838,2522,131,1) $\frac{1}{3}x^9 + \frac{3}{2}x^8 + 6x^7 + 14x^6 + \frac{126}{5}x^5 + \frac{63}{2}x^4 + \frac{95}{3}x^3 + 22x^2 + \frac{39}{5}x + 1$	not CL
3	7	(1,258,7021,65560,234898,352204,234898,65560,7021,258,1) $\frac{4}{15}x^{10} + \frac{4}{3}x^9 + 6x^8 + 16x^7 + \frac{168}{5}x^6 + \frac{252}{5}x^5 + \frac{190}{3}x^4 + \frac{176}{3}x^3 + \frac{154}{5}x^2 + \frac{38}{5}x + 1$	not CL
4	4	(1,52,808,3484,5710,3484,808,52,1) $\frac{5}{14}x^8 + \frac{10}{7}x^7 + 5x^6 + 10x^5 + 15x^4 + 15x^3 + \frac{135}{14}x^2 + \frac{25}{7}x + 1$	not CL
4	5	(1,83,1850,11942,29324,29324,11942,1850,83,1) $\frac{5}{21}x^9 + \frac{15}{14}x^8 + \frac{30}{7}x^7 + 10x^6 + 18x^5 + \frac{45}{2}x^4 + \frac{415}{21}x^3 + \frac{80}{7}x^2 + \frac{33}{7}x + 1$	not CL
4	6	(1,146,4377,41328,145734,221628,145734,41328,4377,146,1) $\frac{1}{6}x^{10} + \frac{5}{6}x^9 + \frac{15}{4}x^8 + 10x^7 + 21x^6 + \frac{63}{2}x^5 + \frac{415}{12}x^4 + \frac{80}{3}x^3 + \frac{37}{2}x^2 + 9x + 1$	not CL
4	7	(1,273,10781,143565,711474,1553106,1553106,711474,143565,10781,273,1) $\frac{4}{33}x^{11} + \frac{2}{3}x^{10} + \frac{10}{3}x^9 + 10x^8 + 24x^7 + 42x^6 + \frac{166}{3}x^5 + \frac{160}{3}x^4 + \frac{146}{3}x^3 + 35x^2 + \frac{127}{11}x + 1$	not CL
5	5	(1,114,3853,35032,125746,188908,125746,35032,3853,114,1) $\frac{1}{7}x^{10} + \frac{5}{7}x^9 + \frac{45}{14}x^8 + \frac{60}{7}x^7 + 18x^6 + 27x^5 + \frac{425}{14}x^4 + \frac{170}{7}x^3 + \frac{72}{7}x^2 + \frac{10}{7}x + 1$	not CL
5	6	(1,177,8333,106845,534882,1164162,1164162,534882,106845,8333,177,1) $\frac{1}{11}x^{11} + \frac{1}{2}x^{10} + \frac{5}{2}x^9 + \frac{15}{2}x^8 + 18x^7 + \frac{63}{2}x^6 + \frac{85}{2}x^5 + \frac{85}{2}x^4 + 28x^3 + 11x^2 + \frac{43}{11}x + 1$	not CL
5	7	(1,304,18674,335216,2277039,6922080,9923772,6922080,2277039,335216,18674,304,1) $\frac{2}{33}x^{12} + \frac{4}{11}x^{11} + 2x^{10} + \frac{20}{3}x^9 + 18x^8 + 36x^7 + \frac{170}{3}x^6 + 68x^5 + 55x^4 + \frac{82}{3}x^3 + \frac{289}{11}x^2 + \frac{216}{11}x + 1$	not CL
6	6	(1,240,16782,290672,2000703,6040992,8702820,6040992,2000703,290672,16782,240,1) $\frac{7}{132}x^{12} + \frac{7}{22}x^{11} + \frac{7}{4}x^{10} + \frac{35}{6}x^9 + \frac{63}{4}x^8 + \frac{63}{2}x^7 + \frac{595}{12}x^6 + \frac{119}{2}x^5 + 56x^4 + \frac{119}{3}x^3 + \frac{63}{22}x^2 - \frac{119}{11}x + 1$	not CL
6	7	(1,367,35124,827372,7600805,31146987,61995744,61995744,31146987,7600805,827372,35124,367,1) $\frac{14}{429}x^{13} + \frac{7}{33}x^{12} + \frac{14}{11}x^{11} + \frac{14}{3}x^{10} + 14x^9 + \frac{63}{2}x^8 + \frac{170}{3}x^7 + \frac{238}{3}x^6 + \frac{441}{5}x^5 + \frac{455}{6}x^4 + \frac{357}{11}x^3$ $-\frac{28}{11}x^2 - \frac{1163}{715}x + 1$	not CL
7	7	(1,494,69595,2151980,26176873,141829106,380179131,524888040,380179131,141829106,26176873, 2151980,69595,494,1) $\frac{8}{429}x^{14} + \frac{56}{429}x^{13} + \frac{28}{33}x^{12} + \frac{112}{33}x^{11} + \frac{56}{5}x^{10} + 28x^9 + \frac{170}{3}x^8 + \frac{272}{3}x^7 + \frac{1736}{15}x^6 + \frac{1736}{15}x^5 + \frac{1260}{11}x^4$ $+\frac{1120}{11}x^3 - \frac{32184}{715}x^2 - \frac{61306}{715}x + 1$	not CL