On a conjecture concerning shuffle-compatible permutation statistics

Lihong Yang
Department of Mathematics
Zhejiang Normal University
Jinhua 321004, P.R. China
yanglihong954@163.com

Sherry H.F. Yan *
Department of Mathematics
Zhejiang Normal University
Jinhua 321004, P.R. China
hfy@zjnu.cn

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Abstract

The notion of shuffle-compatible permutation statistics was implicit in Stanley's work on P-partitions and was first explicitly studied by Gessel and Zhuang. The aim of this paper is to prove that the triple $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})$ is shuffle-compatible as conjectured by Gessel and Zhuang, where udr denotes the number of up-down runs, pk denotes the peak number, and des denotes the descent number. This is accomplished by establishing an $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})$-preserving bijection in the spirit of Baker-Jarvis and Sagan's bijective proofs of the shuffle compatibility property of permutation statistics.

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1 Introduction

Let $\mathbb{P}$ denote the set of all positive integers. To denote the cardinality of a set U, we use $|U|$. For $U \subset \mathbb{P}$ with $|U| = n$, a permutation of U is a linear order $\pi = \pi_1\pi_2\ldots\pi_n$ of the elements of U. Denote by $L(U)$ the set of all permutations of U. The *length* of a permutation π is the cardinality of its underlying set, i.e. $|U|$, which is denoted by $|\pi|$. Permutations have been extensively studied over the last decades. For an introduction to research topics in permutation enumerations, see Bóna's book [2].

Three classical examples of permutation statistics are the descent set Des, the descent number des, and the major index maj. For $\pi \in L(U)$ with $|U| = n$, define

$$\mathrm{Des}(\pi) = \{i : \pi_i > \pi_{i+1}, 1 \leq i \leq n-1\},$$

*Corresponding author.

$$\mathrm{des}(\pi) = |\mathrm{Des}(\pi)|,$$

and

$$\mathrm{maj}(\pi) = \sum_{i \in \mathrm{Des}(\pi)} i.$$

A statistic st is said to be a *descent statistic* if $\mathrm{Des}(\pi) = \mathrm{Des}(\sigma)$ and $|\pi| = |\sigma|$ imply that $\mathrm{st}(\pi) = \mathrm{st}(\sigma)$ for any two permutations π and σ. Clearly, the statistics Des, des and maj are descent statistics. For $\pi \in L(U)$ with $|U| = n$, the *peak set* of π, denoted by $\mathrm{Pk}(\pi)$, is defined to be

$$\mathrm{Pk}(\pi) = \{i : \pi_{i-1} < \pi_i > \pi_{i+1}, 2 \leq i \leq n-1\}.$$

The *peak number* of π, denoted by $\mathrm{pk}(\pi)$, is defined to be the cardinality of $\mathrm{Pk}(\pi)$. The *exterior peak number* of π, denoted by $\mathrm{epk}(\pi)$, is defined to be the peak number of $0\pi0$. A *monotone factor* of a permutation is a factor that is either strictly increasing or strictly decreasing. A *birun* of π is a maximal monotone factor of π. An *updown run* of π is a birun of 0π. Let $\mathrm{udr}(\pi)$ denote the number of updown runs of π.

For any two permutations $\pi \in L(U)$ and $\sigma \in L(V)$ with $U \bigcap V = \varnothing$, we say that the permutation $\tau \in L(U \bigcup V)$ is a *shuffle* of π and σ if both π and σ are subsequences of τ. Denote by $S(\pi, \sigma)$ the set of shuffles of π and σ. For example, $S(31, 24) = \{3124, 3241, 2431, 3214, 2341, 2314\}$. A permutation statistic st is said to be *shuffle-compatible* if for any permutations π and σ with disjoint underlying sets, the multiset $\{\mathrm{st}(\tau) : \tau \in S(\pi, \sigma)\}$, which encodes the distribution of the statistic st over shuffles of π and σ, depends only on $\mathrm{st}(\pi)$, $\mathrm{st}(\sigma)$, $|\pi|$ and $|\sigma|$. For our convenience, we simply write $\mathrm{st}(S(\pi, \sigma))$ for the multiset $\{\mathrm{st}(\tau) : \tau \in S(\pi, \sigma)\}$. For instance, $\mathrm{des}(S(31, 24)) = \{1^3, 2^3\}$. We say that the permutation statistic st has shuffle compatibility property if st is shuffle-compatible.

For a nonnegative integer n, let

$$[n]_q = 1 + q + q^2 + \ldots + q^{n-1}$$

and

$$[n]_q! = [1]_q [2]_q \ldots [n]_q$$

where q is a variable. For $0 \leq k \leq n$, let

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

By utilizing P-partitions, Stanley [12] proved that for any two permutations π and σ with disjoint underlying sets,

$$\begin{aligned} \sum_{\tau \in S_k(\pi,\sigma)} q^{\mathrm{maj}(\tau)} &= q^{\mathrm{maj}(\pi)+\mathrm{maj}(\sigma)+(k-\mathrm{des}(\pi))(k-\mathrm{des}(\sigma))} \binom{|\pi|-\mathrm{des}(\pi)+\mathrm{des}(\sigma)}{k-\mathrm{des}(\pi)}_q \\ &\quad \times \binom{|\sigma|-\mathrm{des}(\sigma)+\mathrm{des}(\pi)}{k-\mathrm{des}(\sigma)}_q \end{aligned} \tag{1}$$

where $S_k(\pi, \sigma) = \{\tau : \tau \in S(\pi, \sigma), \mathrm{des}(\tau) = k\}$. The bijective proofs of (1) have been given by Goulden [5], Stadler [11], Ji and Zhang [7], respectively. Novick [9] provided a bijective proof of the following formula due to Garsia and Gessel [3]:

$$\sum_{\tau \in S(\pi,\sigma)} q^{\mathrm{maj}(\tau)} = q^{\mathrm{maj}(\pi)+\mathrm{maj}(\sigma)} \binom{|\pi| + |\sigma|}{|\pi|}_q \tag{2}$$

where π and σ are permutations with disjoint underlying sets. Very recently, Ji and Zhang [8] derived a cyclic analogue of (1). Formulae (1) and (2) imply that the statistics maj and $(\mathrm{maj}, \mathrm{des})$ are shuffle-compatible.

By using noncommutative symmetric functions and quasisymmetric functions, Gessel and Zhuang [4] further investigated the shuffle compatibility property of permutation statistics and proved that many permutation statistics do have this property. They also posed several conjectures concerning the shuffle compatibility of permutation statistics. Some of these conjectures were then confirmed by Grinberg [6] and disproved by Oğuz [10]. Recently, Baker-Jarvis and Sagan [1] presented a bijective approach to deal with the shuffle compatibility of permutation statistics. As an application, Baker-Jarvis and Sagan [1] proved that the pair $(\mathrm{udr}, \mathrm{pk})$ is shuffle-compatible as conjectured by Gessel and Zhuang [4].

The main objective of this paper is to prove the following conjecture posed by Gessel and Zhuang [4].

Conjecture 1. (See [4], conjecture 6.7) The triple $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})$ is shuffle-compatible.

In [1], Baker-Jarvis and Sagan remarked that their bijection for proving the shuffle compatibility of the statistic $(\mathrm{udr}, \mathrm{pk})$ does not preserve the statistic des and posed an open problem of finding a bijective proof of the the shuffle compatibility of the statistic $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})$ (see [1], Question 7.1). In this paper, we aim to provide such a bijective proof in the spirit of Baker-Jarvis and Sagan's bijective proofs of shuffle compatibility property of permutation statistics.

2 Proof of Conjecture 1

This section is devoted to the bijective proof of Conjecture 1. To this end, we need to recall the following two lemmas due to Baker-Jarvis and Sagan [1].

Lemma 2. (See [1], Theorem 4.2) *The statistic* Des *is shuffle-compatible.*

For $m, n \geq 1$, let $[n] = \{1, 2, \ldots, n\}$ and $[n] + m = \{n + i : 1 \leq i \leq m\}$.

Lemma 3. (See [1], Corollary 3.2) *Suppose that* st *is a descent statistic. The following are equivalent.*

(a) The statistic st *is shuffle-compatible.*

(b) If $\mathrm{st}(\pi) = \mathrm{st}(\pi')$ *where* $\pi, \pi' \in L([n])$*, and* $\sigma \in L([n] + m)$ *for some* $m, n \geq 1$*, then* $\mathrm{st}(S(\pi, \sigma)) = \mathrm{st}(S(\pi', \sigma))$.

For a permutation $\pi \in L(U)$ with k biruns, the *type* of π, denoted by $\mathrm{type}(\pi)$, is defined to be $(t_1, t_2, \ldots, t_k)$, where t_i denotes the length of the i-th birun (counting from left to right). For example, $\mathrm{type}(6534792) = (3,4,2)$. For a permutation $\pi = \pi_1\pi_2\ldots\pi_n$ with $n \geq 2$, define $\chi^+(\pi)$ to be 1 if $\pi_1 > \pi_2$ and to be 0 otherwise. Similarly, we define $\chi^-(\pi)$ to be 1 if $\pi_{n-1} < \pi_n$ and to be 0 otherwise. Notice that the peaks of a permutation occur precisely at the the end of every increasing birun except the rightmost increasing birun. Hence, one can easily check that for $n \geq 2$,

$$\mathrm{udr}(\pi) = \begin{cases} 2\mathrm{pk}(\pi) & \text{if } \chi^+(\pi) = \chi^-(\pi) = 0, \\ 2\mathrm{pk}(\pi) + 1 & \text{if } \chi^+(\pi) = 0, \chi^-(\pi) = 1, \\ 2\mathrm{pk}(\pi) + 2 & \text{if } \chi^+(\pi) = 1, \chi^-(\pi) = 0, \\ 2\mathrm{pk}(\pi) + 3 & \text{if } \chi^+(\pi) = \chi^-(\pi) = 1. \end{cases} \tag{3}$$

Let $\pi \in L([n])$ be a permutation with $\mathrm{type}(\pi) = (t_1, t_2, \ldots, t_k)$ such that $t_\ell \geq 3$ for some $\ell \geq 3$. Define $\Omega_\ell(\pi)$ to be the set of permutations $\pi' \in L([n])$ with $\chi^+(\pi') = \chi^+(\pi)$ and $\mathrm{type}(\pi') = (t'_1, t'_2, \ldots t'_k)$ where

$$t'_i = \begin{cases} t_i + 1 & \text{if } i = \ell - 2, \\ t_i - 1 & \text{if } i = \ell, \\ t_i & \text{otherwise.} \end{cases}$$

One can easily check that for any $\pi' \in \Omega_\ell(\pi)$, we have $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi'$ as the peaks of a permutation occur precisely at the the end of every increasing birun except the rightmost increasing birun.

Let $n \geq 2$. Notice that every permutation of length n either: (1) begins with an ascent and has an even number of biruns, (2) begins with an ascent and has an odd number of biruns, (3) begins with a descent and has an odd number of biruns, or (4) begins with a descent and has an even number of biruns. Clearly, these four types of permutations correspond directly to the four cases in Formula (3). In order to prove Conjecture 1, we define four canonical sets $\Pi^{(1)}_{n,k,d}$ through $\Pi^{(4)}_{n,k,d}$ of permutations corresponding to the four types of permutations mentioned above, where n is the length of the permutation, k is the number of peaks, and d is the number of descents.

Define

$$\Pi^{(1)}_{n,k,d} = \{\pi \in L([n]) : \chi^+(\pi) = 0, \mathrm{type}(\pi) = (t_1, t_2, \ldots, t_{2k})\}$$

where $t_1 = n - d - k + 1$, $t_2 = d - k + 2$, and $t_i = 2$ for $2 < i \leq 2k$. For example, we have $\pi = 25796431(10)8 \in \Pi^{(1)}_{10,2,5}$ with $\mathrm{type}(\pi) = (4,5,2,2)$

Define

$$\Pi^{(2)}_{n,k,d} = \{\pi \in L([n]) : \chi^+(\pi) = 0, \mathrm{type}(\pi) = (t_1, t_2, \ldots, t_{2k+1})\}$$

where $t_1 = n$ when $k = 0$ and $t_1 = n - d - k$, $t_2 = d - k + 2$, and $t_i = 2$ for $2 < i \leq 2k + 1$ otherwise. For example, we have $\pi = 2796431(10)58 \in \Pi^{(2)}_{10,2,5}$ with $\mathrm{type}(\pi) = (3,5,2,2,2)$

Define

$$\Pi^{(3)}_{n,k,d} = \{\pi \in L([n]) : \chi^+(\pi) = 1, \mathrm{type}(\pi) = (t_1, t_2, \ldots, t_{2k+1})\}$$

where $t_1 = n$ when $k = 0$ and $t_1 = d - k + 1$, $t_2 = n - d - k + 1$, and $t_i = 2$ for $2 < i \leq 2k + 1$ otherwise. For example, we have $\pi = 964123(10)785 \in \Pi^{(3)}_{10,2,5}$ with $\mathrm{type}(\pi) = (4,4,2,2,2)$

Define

$$\Pi^{(4)}_{n,k,d} = \{\pi \in L([n]) : \chi^+(\pi) = 1, \mathrm{type}(\pi) = (t_1, t_2, \dots, t_{2k+2})\}$$

where $t_1 = d - k + 1$, $t_2 = n - d - k$, and $t_i = 2$ for $2 < i \leq 2k + 2$. For example, we have $\pi = 96412(10)3857 \in \Pi^{(4)}_{10,2,5}$ with $\mathrm{type}(\pi) = (4, 3, 2, 2, 2, 2)$.

By (3), one can deduce the following result.

Lemma 4. *Let $n \geq 2$. We have*

$$(\mathrm{udr}, \mathrm{pk})\pi = \begin{cases} (2k, k) & \text{if } \pi \in \Pi^{(1)}_{n,k,d}, \\ (2k+1, k) & \text{if } \pi \in \Pi^{(2)}_{n,k,d}, \\ (2k+2, k) & \text{if } \pi \in \Pi^{(3)}_{n,k,d}, \\ (2k+3, k) & \text{if } \pi \in \Pi^{(4)}_{n,k,d}. \end{cases}$$

The following theorem will play an essential role in the proof of Conjecture 1.

Theorem 5. *Let $\pi \in L([n])$ be a permutation with $(\mathrm{pk}, \mathrm{des})\pi = (k, d)$ and let $\sigma \in L([n] + m)$ for some $n \geq 2$, $m \geq 1$ and $k, d \geq 0$. The following statements hold.*

(i) *If $\mathrm{type}(\pi) = (t_1, t_2, \dots, t_{2k})$, $\chi^+(\pi) = 0$, and $\pi \notin \Pi^{(1)}_{n,k,d}$, then there exists a permutation $\pi' \in \Pi^{(1)}_{n,k,d}$ such that*

$$(\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi'$$

and

$$(\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi, \sigma) = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi', \sigma).$$

(ii) *If $\mathrm{type}(\pi) = (t_1, t_2, \dots, t_{2k+1})$, $\chi^+(\pi) = 0$, and $\pi \notin \Pi^{(2)}_{n,k,d}$, then there exists a permutation $\pi' \in \Pi^{(2)}_{n,k,d}$ such that*

$$(\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi'$$

and

$$(\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi, \sigma) = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi', \sigma).$$

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(iii) *If $\mathrm{type}(\pi) = (t_1, t_2, \dots, t_{2k+1})$, $\chi^+(\pi) = 1$, and $\pi \notin \Pi^{(3)}_{n,k,d}$, then there exists a permutation $\pi' \in \Pi^{(3)}_{n,k,d}$ such that*

$$(\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi'$$

and

$$(\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi, \sigma) = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi', \sigma).$$

(iv) *If* $\mathrm{type}(\pi) = (t_1, t_2, \ldots, t_{2k+2})$*,* $\chi^+(\pi) = 1$*, and* $\pi \notin \Pi^{(4)}_{n,k,d}$*, then there exists a permutation* $\pi' \in \Pi^{(4)}_{n,k,d}$ *such that*

$$(\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi'$$

and

$$(\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi, \sigma) = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi', \sigma).$$

Before we prove Theorem 5, we need the following lemma.

Lemma 6. *Let* $\pi \in L([n])$ *be a permutation with* $\mathrm{type}(\pi) = (t_1, t_2, \ldots, t_k)$ *such that* $t_\ell \geq 3$ *for some* $\ell \geq 3$ *and let* $\sigma \in L([n] + m)$ *for some* $n \geq 2$ *and* $m \geq 1$*. Then there exists an* $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})$*-preserving bijection* $\phi_\ell : S(\pi, \sigma) \longrightarrow S(\pi', \sigma)$ *for any permutation* $\pi' \in \Omega_\ell(\pi)$*.*

Proof. Let $\tau = \tau_1\tau_2 \ldots \tau_{n+m} \in S(\pi, \sigma)$. If the ℓ-th birun of π is increasing (resp. decreasing), then let π_j and π_{j+1} be the first (resp. last) two entries of the ℓ-th birun of π and let π_i be the first (resp. last) entry of the $(\ell-2)$-th birun of π. Then τ can be uniquely factored as $\tau^a\tau^b\tau^c$, where τ^b is the subsequence of τ between π_i and π_{j+1} including π_i and π_{j+1}. Then τ^b can be further decomposed as

$$\pi_i\sigma^{(1)}\pi_{i+1}\sigma^{(2)} \ldots \pi_j\sigma^{(j-i+1)}\pi_{j+1},$$

where $\sigma^{(s)}$ is a (possibly empty) subsequence of τ and all the entries of $\sigma^{(s)}$ belong to σ for all $1 \leq s \leq j - i + 1$. Now we proceed to construct $\phi_\ell(\tau)$ by distinguishing the following two cases.

Case 1. $\sigma^{(j-i+1)} = \varnothing$.

Define $\phi_\ell(\tau)$ to be the permutation $\theta^a\theta^b\theta^c$, where θ^a (resp. θ^c) is the permutation obtained from τ^a (resp. τ^c) by replacing each element π_k by π'_k for $1 \leq k < i$ (resp. $j + 1 < k \leq n$) and

$$\theta^b = \pi'_i\pi'_{i+1}\sigma^{(1)}\pi'_{i+2}\sigma^{(2)} \ldots \pi'_j\sigma^{(j-i)}\pi'_{j+1}.$$

For example, let $\ell = 4$, $\pi = 6351274 \in L([7])$ and $\sigma = (11)89(10) \in L([7] + 4)$. Also let $\tau = 63(\mathbf{11})\mathbf{8}5\mathbf{9}127(\mathbf{10})4 \in S(\pi, \sigma)$ and $\pi' = 6145273 \in \Omega_4(\pi)$. Then τ can be decomposed as $\tau^a\tau^b\tau^c$ as illustrated in Figure 1. Clearly, τ^b can be further decomposed as $3\sigma^{(1)}5\sigma^{(2)}1\sigma^{(3)}2$ where $\sigma^{(1)} = (\mathbf{11})\mathbf{8}$, $\sigma^{(2)} = \mathbf{9}$ and $\sigma^{(3)} = \varnothing$. By applying the map ϕ_4 to τ, we obtain $\phi_4(\tau) = \theta^a\theta^b\theta^c$ as shown in Figure 1, where $\theta^a = 6$, $\theta^b = 14(\mathbf{11})\mathbf{8}5\mathbf{9}2$ and $\theta^c = 7(\mathbf{10})3$.

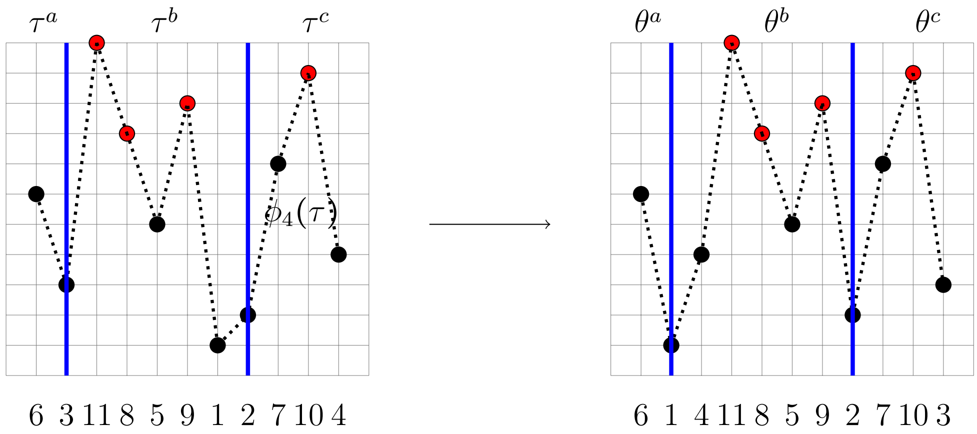

Figure 1: An example of Case 1.

Case 2. $\sigma^{(j-i+1)} \neq \varnothing$.
Suppose that $\sigma^{(s)} \neq \varnothing$ if and only if $s \in \{s_1, s_2, \ldots, s_p\}$ with $1 \le s_1 < s_2 < \ldots < s_p = j-i+1$. Define $\phi_\ell(\tau)$ to be the permutation $\theta^a\theta^b\theta^c$, where θ^a (resp. θ^c) is the permutation obtained from τ^a (resp. τ^c) by replacing each element π_k with π'_k for $1 \le k < i$ (resp. $j+1 < k \le n$) and θ^b is obtained from τ^b by replacing each π_k with π'_{k+1} for $i \le k \le j$, replacing each $\sigma^{(s_q)}$ by $\sigma^{(s_{q+1})}$ for $1 \le q \le p-1$, and inserting the subsequence $\pi'_i\sigma^{(s_1)}$ immediately to the left of π'_{i+1}.

For example, let $\ell = 3$, $\pi = 7426315 \in L([7])$ and $\sigma = (11)8(10)9(12) \in L([7]+5)$. Also let $\tau = \mathbf{(11)}74\mathbf{8}2\mathbf{(10)}63\mathbf{9(12)}15 \in S(\pi,\sigma)$ and $\pi' = 7432615 \in \Omega_3(\pi)$. Figure 2 illustrates the decomposition of τ, where $\tau^a = \mathbf{(11)}74\mathbf{8}$, $\tau^b = 2\mathbf{(10)}63\mathbf{9(12)}1$ and $\tau^c = 5$. Clearly, τ^b can be further decomposed as $2\sigma^{(1)}6\sigma^{(2)}3\sigma^{(3)}1$ where $\sigma^{(1)} = \mathbf{(10)}$, $\sigma^{(2)} = \varnothing$, and $\sigma^{(3)} = \mathbf{9(12)}$. By applying the map ϕ_3 to τ, we obtain $\phi_3(\tau) = \theta^a\theta^b\theta^c$ as shown in Figure 2, where $\theta^a = \mathbf{(11)}74\mathbf{8}$, $\theta^b = 3\mathbf{(10)}2\mathbf{9(12)}61$ and $\theta^c = 5$.

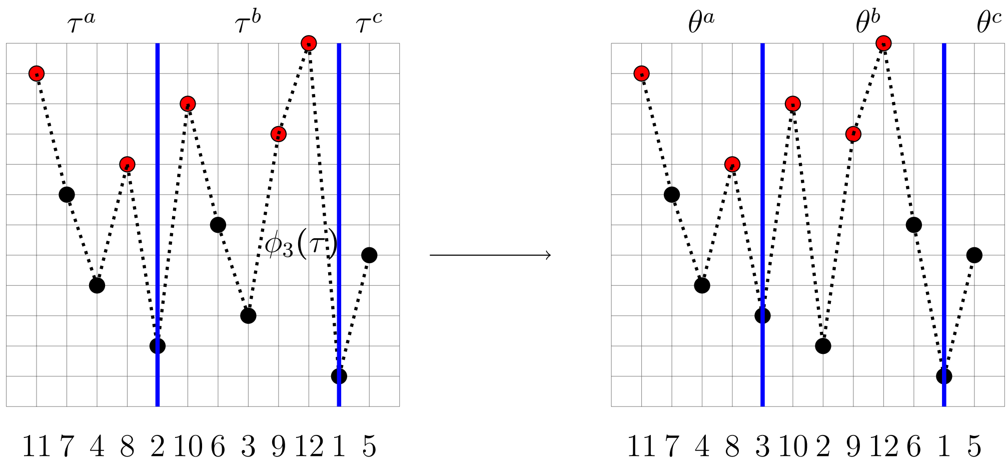

Figure 2: An example of Case 2.

From the construction of $\phi_\ell(\tau)$, it is easily seen that $\phi_\ell(\tau)$ still contains σ as a subsequence. Hence, we have $\phi_\ell(\tau) \in S(\pi', \sigma)$, that is, the map ϕ_ℓ is well-defined.

Conversely, given any $\tau' \in S(\pi', \sigma)$, we can recover the permutation $\tau \in S(\pi, \sigma)$ as follows. If the ℓ-th birun of π' is increasing (resp. decreasing), then let π'_i be the first (resp. last) entry of the $(\ell-2)$-th birun of π'. Let k be a positive integer such that $\tau'_k = \pi'_i$. Then we can recover a permutation $\tau \in S(\pi, \sigma)$ by reversing the procedure in Case 1 when the ℓ-th birun of π' is increasing (resp. decreasing) and $\tau'_{k+1} = \pi'_{i+1}$ (resp. $\tau'_{k-1} = \pi'_{i-1}$). Otherwise, we can recover a permutation $\tau \in S(\pi, \sigma)$ by reversing the procedure in Case 2. So the construction of the map ϕ_ℓ is reversible and hence it is a bijection.

In the following, we aim to show that $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})\tau = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})\phi_\ell(\tau)$. We have four cases: (i) the ℓ-th birun is increasing and $\sigma^{(j-i+1)} = \varnothing$, (ii) the ℓ-th birun is increasing and $\sigma^{(j-i+1)} \neq \varnothing$, (iii) the ℓ-th birun is decreasing and $\sigma^{(j-i+1)} = \varnothing$, and (iv) the ℓ-th birun is decreasing and $\sigma^{(j-i+1)} \neq \varnothing$. Here we only prove the assertion for cases (i) and (iv). All the other cases can be verified by similar arguments.

(i) The ℓ-th birun is increasing and $\sigma^{(j-i+1)} = \varnothing$.
It is easy to verify that

$$\mathrm{des}(\tau) = \mathrm{des}(\tau^a \pi_i) + \mathrm{des}(\pi_{j+1}\tau^c) + t_{\ell-1} - 1 + \sum_{s=1}^{j-i+1} \mathrm{des}(\sigma^{(\mathrm{s})}) + \sum_{s=1}^{t_{\ell-2}-1} \delta(|\sigma^{(s)}| > 0)$$

and

$$\mathrm{pk}(\tau) = \mathrm{pk}(\tau^a \pi_i) + \mathrm{pk}(\pi_{j+1}\tau^c) + \sum_{s=1}^{j-i+1} \mathrm{epk}(\sigma^{(\mathrm{s})}) + \delta(|\sigma^{(t_{\ell-2}-1)}| = |\sigma^{(t_{\ell-2})}| = 0).$$

Here $\delta(S) = 1$ if the statement S is true, and $\delta(S) = 0$ otherwise. Similarly, we have

$$\mathrm{des}(\phi_\ell(\tau)) = \mathrm{des}(\theta^a \pi'_i) + \mathrm{des}(\pi'_{j+1}\theta^c) + t'_{\ell-1} - 1 + \sum_{s=1}^{j-i+1} \mathrm{des}(\sigma^{(\mathrm{s})}) + \sum_{s=1}^{t_{\ell-2}-1} \delta(|\sigma^{(s)}| > 0),$$

and

$$\mathrm{pk}(\phi_\ell(\tau)) = \mathrm{pk}(\theta^a \pi'_i) + \mathrm{pk}(\pi'_{j+1}\theta^c) + \sum_{s=1}^{j-i+1} \mathrm{epk}(\sigma^{(\mathrm{s})}) + \delta(|\sigma^{(t_{\ell-2}-1)}| = |\sigma^{(t_{\ell-2})}| = 0).$$

As $\mathrm{Des}(\pi_1\pi_2 \ldots \pi_i) = \mathrm{Des}(\pi'_1\pi'_2 \ldots \pi'_i)$ and $\mathrm{Des}(\pi_{j+1}\pi_{j+2} \ldots \pi_n) = \mathrm{Des}(\pi'_{j+1}\pi'_{j+2} \ldots \pi'_n)$, we have $\mathrm{Des}(\tau^a\pi_i) = \mathrm{Des}(\theta^a\pi'_i)$ and $\mathrm{Des}(\pi_{j+1}\tau^c) = \mathrm{Des}(\pi'_{j+1}\theta^c)$. This yields that $\mathrm{des}(\phi_\ell(\tau)) = \mathrm{des}(\tau)$ and $\mathrm{pk}(\phi_\ell(\tau)) = \mathrm{pk}(\tau)$ as $t_{\ell-1} = t'_{\ell-1}$.

By (3), in order to prove that $\mathrm{udr}(\tau) = \mathrm{udr}(\phi_\ell(\tau))$, it suffices to show that $\chi^+(\tau) = \chi^+(\phi_\ell(\tau))$ and $\chi^-(\tau) = \chi^-(\phi_\ell(\tau))$. Let x and y be positive integers such that $\tau_x = \pi_i$ and $\tau_y = \pi_{j+1}$. If $x = 1$, then we have $\chi^+(\tau) = 0 = \chi^+(\phi_\ell(\tau))$ since $\pi_i < \pi_{i+1}$ and $\pi'_i < \pi'_{i+1}$ guarantee that $1 \notin \mathrm{Des}(\tau)$ and $1 \notin \mathrm{Des}(\phi_\ell(\tau))$. If $x > 1$, then $\mathrm{Des}(\tau^a\pi_i) = \mathrm{Des}(\theta^a\pi'_i)$ implies that $\chi^+(\tau) = \chi^+(\phi_\ell(\tau))$. Notice that π_{j+1} (resp. π'_{j+1}) is not the last entry of the ℓ-th birun of π (resp. π'). This implies that $y < n + m$. Then $\mathrm{Des}(\pi_{j+1}\tau^c) = \mathrm{Des}(\pi'_{j+1}\theta^c)$ implies that $\chi^-(\tau) = \chi^-(\phi_\ell(\tau))$. So far, we have concluded that $\chi^+(\tau) = \chi^+(\phi_\ell(\tau))$ and $\chi^-(\tau) = \chi^-(\phi_\ell(\tau))$. Thus, we have $\mathrm{udr}(\tau) = \mathrm{udr}(\phi_\ell(\tau))$ as desired.

(iv) The ℓ-th birun is decreasing and $\sigma^{(j-i+1)} \neq \emptyset$.
It is routine to check that

$$\mathrm{des}(\tau) = \mathrm{des}(\tau^a \pi_i) + \mathrm{des}(\pi_{j+1}\tau^c) + t_\ell - 1 + \sum_{s=1}^{j-i+1} \mathrm{des}(\sigma^{(s)}) + \sum_{s=1}^{t_{\ell-1}-1} \delta(|\sigma^{(s)}| > 0)$$

and

$$\mathrm{pk}(\tau) = \mathrm{pk}(\tau^a \pi_i) + \mathrm{pk}(\pi_{j+1}\tau^c) + \sum_{s=1}^{j-i+1} \mathrm{epk}(\sigma^{(s)}) + \delta(|\sigma^{(t_{\ell-1}-1)}| = |\sigma^{(t_{\ell-1})}| = 0).$$

Similarly, we have

$$\mathrm{des}(\phi_\ell(\tau)) = \mathrm{des}(\theta^a \pi'_i) + \mathrm{des}(\pi'_{j+1}\theta^c) + t'_\ell + \sum_{s=1}^{j-i+1} \mathrm{des}(\sigma^{(s)}) + \sum_{s=1}^{t_{\ell-1}-1} \delta(|\sigma^{(s)}| > 0),$$

and

$$\mathrm{pk}(\phi_\ell(\tau)) = \mathrm{pk}(\theta^a \pi'_i) + \mathrm{pk}(\pi'_{j+1}\theta^c) + \sum_{s=1}^{j-i+1} \mathrm{epk}(\sigma^{(s)}) + \delta(|\sigma^{(t_{\ell-1}-1)}| = |\sigma^{(t_{\ell-1})}| = 0).$$

As $\mathrm{Des}(\pi_1\pi_2\ldots\pi_i) = \mathrm{Des}(\pi'_1\pi'_2\ldots\pi'_i)$ and $\mathrm{Des}(\pi_{j+1}\pi_{j+2}\ldots\pi_n) = \mathrm{Des}(\pi'_{j+1}\pi'_{j+2}\ldots\pi'_n)$, we have $\mathrm{Des}(\tau^a\pi_i) = \mathrm{Des}(\theta^a\pi'_i)$ and $\mathrm{Des}(\pi_{j+1}\tau^c) = \mathrm{Des}(\pi'_{j+1}\theta^c)$. This yields that $\mathrm{des}(\phi_\ell(\tau)) = \mathrm{des}(\tau)$ and $\mathrm{pk}(\phi_\ell(\tau)) = \mathrm{pk}(\tau)$ since $t'_\ell = t_\ell - 1$.

By (3), in order to prove that $\mathrm{udr}(\tau) = \mathrm{udr}(\phi_\ell(\tau))$, it suffices to show that $\chi^+(\tau) = \chi^+(\phi_\ell(\tau))$ and $\chi^-(\tau) = \chi^-(\phi_\ell(\tau))$. Let x and y be positive integers such that $\tau_x = \pi_i$ and $\tau_y = \pi_{j+1}$. Clearly, we have $x > 1$. Then $\mathrm{Des}(\tau^a\pi_i) = \mathrm{Des}(\theta^a\pi'_i)$ implies that $\chi^+(\tau) = \chi^+(\phi_\ell(\tau))$. If $y < n+m$, $\mathrm{Des}(\pi_{j+1}\tau^c) = \mathrm{Des}(\pi'_{j+1}\theta^c)$ implies that $\chi^-(\tau) = \chi^-(\phi_\ell(\tau))$. If $y = n+m$, then we have $\chi^-(\tau) = 0 = \chi^-(\phi_\ell(\tau))$ since $\pi_{n-1} > \pi_n$ and $\pi'_{n-1} > \pi'_n$ guarantee that $n+m-1 \in \mathrm{Des}(\tau)$ and $n+m-1 \in \mathrm{Des}(\phi_\ell(\tau))$. So far, we have concluded that $\chi^+(\tau) = \chi^+(\phi_\ell(\tau))$ and $\chi^-(\tau) = \chi^-(\phi_\ell(\tau))$. Thus, we have $\mathrm{udr}(\tau) = \mathrm{udr}(\phi_\ell(\tau))$ as desired. Hence, the map ϕ_ℓ is an $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})$-preserving bijection between $S(\pi,\sigma)$ and $S(\pi',\sigma)$, completing the proof. □

Proof of Theorem 5. Take $i \in [4]$ to be arbitrary. As $\pi \notin \Pi^{(i)}_{n,k,d}$, we can find the largest integer $\ell^{(1)}$ with $\ell^{(1)} > 2$ such that $t_{\ell^{(1)}} \geq 3$. Let $\pi^{(1)}$ be a permutation in $\Omega_{\ell^{(1)}}(\pi)$. By Lemma 6, the map $\phi_{\ell^{(1)}}$ serves as an $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})$-preserving bijection between $S(\pi,\sigma)$ and $S(\pi^{(1)},\sigma)$. Thus we have

$$(\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi^{(1)}$$

and

$$(\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi,\sigma) = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi^{(1)},\sigma).$$

If $\pi^{(1)} \in \Pi^{(i)}_{n,k,d}$, then we stop and set $\pi' = \pi^{(1)}$. Otherwise, let t'_i denote the i-th birun of $\pi^{(1)}$. Then, find the largest integer $\ell^{(2)}$ with $\ell^{(2)} > 2$ such that $t'_{\ell^{(2)}} \geq 3$. Again by Lemma 6, the map $\phi_{\ell^{(2)}}$ serves as an $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})$-preserving bijection between $S(\pi^{(1)},\sigma)$ and $S(\pi^{(2)},\sigma)$

where $\pi^{(2)} \in \Omega_{\ell(2)}(\pi^{(1)})$. We continue this process until we get some $\pi^{(s)} \in \Pi^{(i)}_{n,k,d}$. Then we set $\pi' = \pi^{(s)}$. Clearly, we have $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi'$. By Lemma 6, we have

$$(\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi, \sigma) = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi', \sigma)$$

as desired, completing the proof. □

Now we are ready for the proof of Conjecture 1.

Proof of Conjecture 1. By Lemma 3, in order to prove Conjecture 1, it suffices to show that for any two permutations $\pi, \pi' \in L([n])$ with $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi'$ and $\sigma \in L([n]+m)$ for $n, m \geq 1$, we have $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi, \sigma) = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi', \sigma)$. It is easily seen that the assertion holds for $n = 1$. We now assume that $n \geq 2$.

Let $\pi, \pi' \in L([n])$ with $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi'$ and $(\mathrm{pk}, \mathrm{des})\pi = (\mathrm{pk}, \mathrm{des})\pi' = (k, d)$ and let $\sigma \in L([n]+m)$. Notice that $\mathrm{Des}(\pi) = \mathrm{Des}(\pi')$ for any permutations $\pi, \pi' \in \Pi^{(i)}_{n,k,d}$ for fixed $i \in [4]$. Then by Lemma 2, we have

$$(\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi, \sigma) = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi', \sigma)$$

when $\pi, \pi' \in \Pi^{(i)}_{n,k,d}$ for fixed $i \in [4]$. Otherwise, by Theorem 5, there exist two permutations $\tau, \tau' \in \Pi_{n,k,d}$ satisfying that

$$(\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})\tau,$$

$$(\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi, \sigma) = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\tau, \sigma),$$

$$(\mathrm{udr}, \mathrm{pk}, \mathrm{des})\pi' = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})\tau',$$

and

$$(\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi', \sigma) = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\tau', \sigma).$$

In order to show that $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi, \sigma) = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})S(\pi', \sigma)$, it remains to show that both τ and τ' are elements of $\Pi^{(i)}_{n,k,d}$ for some $i \in [4]$. This follows immediately from Lemma 4 and the equality $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})\tau = (\mathrm{udr}, \mathrm{pk}, \mathrm{des})\tau'$. This completes the proof. □

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