Some remarks on even-hole-free graphs

Zi-Xia Song*

Department of Mathematics
University of Central Florida
Orlando, FL 32816, USA

Zixia.Song@ucf.edu

Submitted: Oct 4, 2021; Accepted: Jul 19, 2022; Published: Jul 29, 2022

Abstract

A vertex of a graph is *bisimplicial* if the set of its neighbors is the union of two cliques; a graph is *quasi-line* if every vertex is bisimplicial. A recent result of Chudnovsky and Seymour asserts that every non-empty even-hole-free graph has a bisimplicial vertex. Both Hadwiger's Conjecture and the Erdős-Lovász Tihany Conjecture have been shown to be true for quasi-line graphs, but are open for even-hole-free graphs. In this note, we prove that every even-hole-free graph G with $\omega(G) < \chi(G) = s + t - 1$ satisfies the Erdős-Lovász Tihany Conjecture provided that $t \geqslant s > \chi(G)/3$; every 9-chromatic graph G with $\omega(G) \leqslant 8$ has a $K_4 \cup K_6$ minor; and every even-hole-free graph with no K_k minor is $(2k-5)$-colorable for all $k \geqslant 7$. Our proofs rely heavily on the structural result of Chudnovsky and Seymour on even-hole-free graphs.

Mathematics Subject Classifications: 05C55, 05C35

1 Introduction

All graphs in this paper are finite and simple. For a graph G, we use $V(G)$ to denote the vertex set, $E(G)$ the edge set, $|G|$ the number of vertices, $e(G)$ the number of edges, $\delta(G)$ the minimum degree, $\Delta(G)$ the maximum degree, $\alpha(G)$ the independence number, and $\omega(G)$ the clique number. A graph G is *k-colorable* or has a *proper k-coloring* if there is a function $\tau : V(G) \to \{1, \dots, k\}$, such that for every edge uv of G, $\tau(u) \neq \tau(v)$. The chromatic number of G, denoted $\chi(G)$, is the minimum integer k for which G is k-colorable. We say that G is *k-chromatic* if $\chi(G) = k$. A graph H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges. We write $G \succcurlyeq H$ if

*Supported by the National Science Foundation under Grant No. DMS-1854903.

H is a minor of G. In those circumstances we also say that G has an H minor. Our work is motivated by the celebrated Hadwiger's Conjecture [12] and the Erdős-Lovász Tihany Conjecture [11].

Conjecture 1 (Hadwiger's Conjecture [12])**.** For every integer $k \geqslant 1$, every graph with no K_k minor is $(k-1)$-colorable.

Conjecture 1 is trivially true for $k \leqslant 3$, and reasonably easy for $k = 4$, as shown independently by Hadwiger [12] and Dirac [9]. However, for $k \geqslant 5$, Hadwiger's Conjecture implies the Four Color Theorem [1, 2]. Wagner [32] proved that the case $k = 5$ of Hadwiger's Conjecture is, in fact, equivalent to the Four Color Theorem, and the same was shown for $k = 6$ by Robertson, Seymour and Thomas [23]. Despite receiving considerable attention over the years, Hadwiger's Conjecture remains open for $k \geqslant 7$ and is widely considered among the most important problems in graph theory and has motivated numerous developments in graph coloring and graph minor theory. The best known upper bound on the chromatic number of graphs with no K_k minor is $O(k \log \log k)$ due to Delcourt and Postle [10], improving a recent breakthrough of Norin, Postle, and the present author [20] who improved a long-standing bound obtained independently by Kostochka [15, 16] and Thomason [30]. We refer the reader to recent surveys [5, 13, 24] for further background on Hadwiger's Conjecture.

Throughout the paper, let s and t be positive integers. A graph G is (s, t)*-splittable* if $V(G)$ can be partitioned into two sets S and T such that $\chi(G[S]) \geqslant s$ and $\chi(G[T]) \geqslant t$. In 1968, Erdős [11] published the following conjecture of Lovász, which has since been known as the Erdős-Lovász Tihany Conjecture.

Conjecture 2 (The Erdős-Lovász Tihany Conjecture)**.** Let G be a graph with $\omega(G) < \chi(G) = s + t - 1$, where $t \geqslant s \geqslant 2$ are integers. Then G is (s, t)-splittable.

Conjecture 2 is hard, and few related results are known. The case $(2, 2)$ of Conjecture 2 is trivial; the cases $(2, 3)$ and $(3, 3)$ were shown by Brown and Jung [3] in 1969; Mozhan [19] and Stiebitz [26] independently proved the case $(2, 4)$ in 1987; the cases $(3, 4)$ and $(3, 5)$ were settled by Stiebitz [27] in 1988. A relaxed version of Conjecture 2 was proved in [29].

Recent work on both Conjecture 1 and Conjecture 2 have also focused on proving the conjectures for certain classes of graphs. A vertex of a graph is *bisimplicial* if the set of its neighbors is the union of two cliques; a graph is *quasi-line* if every vertex is bisimplicial. Note that every line graph is quasi-line and every quasi-line graph is claw-free [7]. A *hole* in a graph is an induced cycle of length at least four; a hole is *even* if it has an even length. A graph is *even-hole-free* if it contains no even hole. Hadwiger's Conjecture has been shown to be true for line graphs by Reed and Seymour [21]; quasi-line graphs by Chudnovsky and Ovetsky Fradkin [6]; graphs G with $\alpha(G) \geqslant 3$ and no hole of length between 4 and $2\alpha(G)-1$ by Thomas and the present author [31]. Meanwhile, the Erdős-Lovász Tihany Conjecture has also been verified to be true for line graphs by Kostochka and Stiebitz [17]; quasi-line graphs, and graphs G with $\alpha(G)=2$ by Balogh, Kostochka, Prince and Stiebitz [4]; graphs G with $\alpha(G)\geqslant 3$ and no hole of length between 4 and $2\alpha(G)-1$ by the present author [25].

Chudnovsky and Seymour [8] recently proved a structural result on even-hole-free graphs.

Theorem 3 (Chudnovsky and Seymour [8])**.** *Let G be a non-empty even-hole-free graph. Then G has a bisimplicial vertex and $\chi(G)\leqslant 2\omega(G)-1$.*

It is unknown whether Conjecture 1 and Conjecture 2 hold for even-hole-free graphs. Using Theorem 3, we prove in Section 2 that for all $k\geqslant 7$, every even-hole-free graph with no K_k minor is $(2k-5)$-colorable; and every even-hole-free graph G with $\omega(G)<\chi(G)=s+t-1$ satisfies Conjecture 2 provided that $t\geqslant s>\chi(G)/3$. It is worth noting that Kawarabayashi, Pedersen and Toft [14] observed that if Hadwiger's Conjecture holds, then the following conjecture might be easier to settle than the Erdős-Lovász Tihany Conjecture.

Conjecture 4 (Kawarabayashi, Pedersen, Toft [14])**.** Every graph G satisfying $\omega(G)<\chi(G)=s+t-1$ has two vertex-disjoint subgraphs G_1 and G_2 such that $G_1\succcurlyeq K_s$ and $G_2\succcurlyeq K_t$, where $t\geqslant s\geqslant 2$ are integers.

In the same paper [14], they settled Conjecture 4 for the additional values of $(s,t)\in\{(2,6),(3,6),(4,4),(4,5)\}$. We end Section 2 by proving the $(4,6)$ case for Conjecture 4, that is, we prove that every graph G with $\chi(G)=9>\omega(G)$ has a $K_4\cup K_6$ minor. Here $K_4\cup K_6$ denotes the disjoint union of K_4 and K_6.

We need to introduce more notation. Let G be a graph. For a vertex $x\in V(G)$, we will use $N(x)$ to denote the set of vertices in G that are adjacent to x. We define $N[x]=N(x)\cup\{x\}$ and $d(x)=|N(x)|$. If $A,B\subseteq V(G)$ are disjoint, we say that A is *complete* to B if each vertex in A is adjacent to all vertices in B, and A is *anti-complete* to B if no vertex in A is adjacent to any vertex in B. If $A=\{a\}$, we simply say a is complete to B or a is anti-complete to B. The subgraph of G induced by A, denoted $G[A]$, is the graph with vertex set A and edge set $\{xy\in E(G): x,y\in A\}$. We denote by $B\setminus A$ the set $B-A$, and $G\setminus A$ the subgraph of G induced on $V(G)\setminus A$, respectively. If $A=\{a\}$, we simply write $B\setminus a$ and $G\setminus a$, respectively. An *(s,t)-graph* is a connected $(s+t-1)$-chromatic graph which does not contain two vertex-disjoint subgraphs with chromatic number s and t, respectively. We use the convention "A :=" to mean that A is defined to be the right-hand side of the relation.

Finally, we shall make use of the following results of Stiebitz [27, 28] and Mader [18].

Theorem 5 (Stiebitz [27])**.** *Suppose G is an (s,t)-graph with $t \geqslant s \geqslant 2$. If $\omega(G) \geqslant t$, then $\omega(G) \geqslant s+t-1$.*

Theorem 6 (Stiebitz [28])**.** *Every graph G satisfying $\delta(G) \geqslant s+t+1$ has two vertex-disjoint subgraphs G_1 and G_2 such that $\delta(G_1) \geqslant s$ and $\delta(G_2) \geqslant t$.*

Theorem 7 (Mader [18])**.** *For every integer $p \leqslant 7$, every graph on $n \geqslant p$ vertices and at least $(p-2)n - \binom{p-1}{2} + 1$ edges has a K_p minor.*

2 Main results

We begin with a lemma which plays a key role in the proof of Theorem 9 and Theorem 10.

Lemma 8. *Let G be a graph and $x \in V(G)$ with $p := \chi(G[N(x)]) \geqslant 2$. Let $V_1, \dots, V_p$ be the color classes of a proper p-coloring of $G[N(x)]$ with $|V_1| \geqslant \cdots \geqslant |V_p| \geqslant 1$. If $|V_r \cup \cdots \cup V_p| \leqslant \chi(G) - r - 1$ for some $r \in [p]$ with $2 \leqslant r \leqslant p$, then $p \leqslant \chi(G) - 2$ and G is $(r, \chi(G)+1-r)$-splittable.*

Proof. Let $G, p, r, V_1, \dots, V_p$ be as given in the statement. Note that $p-r+1 \leqslant |V_r \cup \cdots \cup V_p| \leqslant \chi(G)-r-1$ and so $p \leqslant \chi(G)-2$ and $V(G) \setminus N[x] \neq \emptyset$. Let $W := V_1 \cup \cdots \cup V_{r-1}$. Then $\chi(G[\{x\} \cup W]) = r$ and $\chi(G \setminus W) \geqslant \chi(G) - (r-1) = \chi(G)+1-r$. It suffices to show that $\chi(G \setminus (\{x\} \cup W)) \geqslant \chi(G \setminus W)$. Let $q := \chi(G \setminus (\{x\} \cup W)) \geqslant \chi(G \setminus W) - 1 \geqslant \chi(G) - r \geqslant 2$ and let $U_1, \dots, U_q$ be the color classes of a proper q-coloring of $G \setminus (\{x\} \cup W)$. Since x is adjacent to $|V_r \cup \cdots \cup V_p| \leqslant \chi(G) - r - 1 \leqslant q-1$ vertices in $G \setminus W$, we see that x is anti-complete to U_i for some $i \in [q]$. We may assume that $i = 1$. Then $U_1 \cup \{x\}, U_2, \dots, U_q$ form the color classes of a proper q-coloring of $G \setminus W$. Therefore, $\chi(G \setminus (\{x\} \cup W)) = q \geqslant \chi(G \setminus W) \geqslant \chi(G) - r + 1$, as desired. □

We now prove that the Erdős-Lovász Tihany Conjecture holds for even-hole-free graphs G with $\omega(G) < \chi(G) = s+t-1$ if $t \geqslant s > \chi(G)/3$. It would be nice if one can prove the same holds for all even-hole free graphs.

Theorem 9. *Let G be an even-hole-free graph with $\omega(G) < \chi(G) = s+t-1$, where $t \geqslant s \geqslant 2$. If $s > \chi(G)/3$, then G is (s,t)-splittable.*

Proof. Suppose the assertion is false. Let G be a counterexample with $|G|$ minimum. Then G is vertex-critical; in addition, G is an (s,t)-graph. Thus $\delta(G) \geqslant \chi(G)-1 = s+t-2$. By Theorem 5, $\omega(G) \leqslant t-1$. Since G is even-hole-free, by Theorem 3, G has a bisimplicial vertex v. Then $N(v)$ is the union of two cliques. Thus $\alpha(G[N(v)]) \leqslant 2$, $\omega(G[N(v)]) \leqslant t-2$ and

$$s+t-2 = \chi(G) - 1 \leqslant \delta(G) \leqslant d(v) \leqslant 2\omega(G[N(v)]) \leqslant 2t-4.$$

It follows that $t \geqslant s+2 \geqslant 4$ and $\chi(G) = s+t-1 \geqslant 2s+1$. We next claim that $\Delta(G) \leqslant |G|-2$. Suppose there exists $x \in V(G)$ such that $d(x) = |G|-1$. Then

$$\chi(G\setminus x) = \chi(G)-1 = s+(t-1)-1 > \omega(G)-1 = \omega(G\setminus x) \text{ and } t-1 > s > \chi(G\setminus x)/3.$$

By the minimality of G, $G \setminus x$ is $(s, t-1)$-splittable and thus G is (s,t)-splittable, a contradiction. Thus $\Delta(G) \leqslant |G|-2$, as claimed. It follows that $V(G) \setminus N[v] \neq \emptyset$ and so $\chi(G[N[v]]) \leqslant \chi(G)-1$. Let $p := \chi(N(v))$. Then $p = \chi(G[N[v]])-1 \leqslant \chi(G)-2$. Note that

$$p \geqslant \omega(G[N(v)]) \geqslant d(v)/2 \geqslant (\chi(G)-1)/2 \geqslant ((2s+1)-1)/2 = s \geqslant 2.$$

Let $V_1, \ldots, V_p$ be the color classes of a proper p-coloring of $G[N(v)]$ with $2 \geqslant |V_1| \geqslant \cdots \geqslant |V_p| \geqslant 1$. Suppose $p \geqslant t-1$. Then $|V_{t-2}| = 1$ because $d(v) \leqslant 2t-4$. Therefore,

$$|V_t \cup \cdots \cup V_p| = p-t+1 \leqslant (\chi(G)-2)-t+1 = \chi(G)-t-1.$$

By Lemma 8 applied to G and v with $r = t$, we see that G is (s,t)-splittable, a contradiction. Thus $s \leqslant p \leqslant t-2$. Next, if $|V_s \cup \cdots \cup V_p| \leqslant \chi(G)-s-1$, then G is (s,t)-splittable by applying Lemma 8 to G and v with $r = s$, a contradiction. Hence, $|V_s \cup \cdots \cup V_p| \geqslant \chi(G)-s = t-1 \geqslant 3$. Note that $p-s+1 \leqslant (t-2)-2+1 = t-3$, and so $|V_s| = 2$ and

$$d(v) = (|V_1|+\cdots+|V_{s-1}|) + |V_s \cup \cdots \cup V_p| \geqslant 2(s-1)+t-1 = 2s+t-3.$$

It follows that $t-2 \geqslant \omega(G[N(v)]) \geqslant d(v)/2 \geqslant (2s+t-3)/2$, which implies that $t \geqslant 2s+1$. Thus $\chi(G) = s+t-1 \geqslant 3s$, contrary to the assumption that $3s > \chi(G)$. $\square$

We next prove that Conjecture 4 is true when $(s,t) = (4,6)$.

Theorem 10. *Every 9-chromatic graph G with $\omega(G) \leqslant 8$ has a $K_4 \cup K_6$ minor.*

Proof. Suppose for a contradiction that G is a counterexample to the statement with minimum number of vertices. Then G is vertex-critical, and so $\delta(G) \geqslant 8$ and G is connected. Suppose G contains two vertex-disjoint subgraphs G_1 and G_2 such that $\chi(G_1) \geqslant 4$ and $\chi(G_2) \geqslant 6$. Since Hadwiger's Conjecture holds for k-chromatic graphs with $k \leqslant 6$, we see that $G_1 \succcurlyeq K_4$ and $G_2 \succcurlyeq K_6$, a contradiction. Thus G is a $(4,6)$-graph, and so $\omega(G) \leqslant 5$ by Theorem 5. Note that G is not necessarily contraction-critical, as a proper minor of G may have clique number 9. We claim that

Claim 1. $2 \leqslant \alpha(G[N(x)]) \leqslant d(x)-7$ for each $x \in V(G)$.

Proof. Let $x \in V(G)$. Since $\omega(G) \leqslant 5$ and $\delta(G) \geqslant 8$, we see that $\alpha(G[N(x)]) \geqslant 2$. Suppose $\alpha(G[N(x)]) \geqslant d(x)-6$. Let A be a maximum independent set of $G[N(x)]$. Let G^* be obtained from G by contracting $G[A \cup \{x\}]$ into a single vertex, say w. Note that

$\omega(G^*) < 8$ and G^* has no $K_4 \cup K_6$ minor. By the minimality of G, $\chi(G^*) \leqslant 8$. Let $c : V(G^*) \to [8]$ be a proper 8-coloring of G^*. Since $|N(x) \setminus A| = d(x) - |A| \leqslant 6$, we may assume that $c(N(x) \setminus A) \subseteq [6]$ and $c(w) = 7$. But then we obtain a proper 8-coloring of G from c by coloring all the vertices in A with color 7 and the vertex x with color 8, a contradiction. Thus $2 \leqslant \alpha(G[N(x)]) \leqslant d(x) - 7$, as claimed. □

By Claim 1, $\delta(G) \geqslant 9$. Suppose $\delta(G) \geqslant 13$. By Theorem 6, G contains two vertex-disjoint subgraphs G_1 and G_2 such that $\delta(G_1) \geqslant 4$ and $\delta(G_2) \geqslant 8$. By Theorem 7, we see that $G_1 \succcurlyeq K_4$ and $G_2 \succcurlyeq K_6$, a contradiction. Thus $9 \leqslant \delta(G) \leqslant 12$. We next claim that

Claim 2. $G[N(x)]$ is even-hole-free and $\chi(G[N(x)]) \leqslant 2\omega(G[N(x)]) - 1$ for each $x \in V(G)$.

Proof. Let $x \in V(G)$. Suppose $G[N(x)]$ contains an even hole C. Then $\chi(G[V(C) \cup \{x\}]) = 3$ and so $\chi(G \backslash (V(C) \cup \{x\})) \geqslant \chi(G) - 3 = 6$. It is easy to see that $G[V(C) \cup \{x\}] \succcurlyeq K_4$. Since Hadwiger's Conjecture holds for 6-chromatic graphs, we see that $G \setminus (V(C) \cup \{x\})$ has a K_6 minor, and so G has a $K_4 \cup K_6$ minor, a contradiction. Thus $G[N(x)]$ is even-hole-free. By Theorem 3, $\chi(G[N(x)]) \leqslant 2\omega(G[N(x)]) - 1$. □

Let $v \in V(G)$ with $d(v) = \delta(G)$, and let $p := \chi(G[N(v)])$. Since $9 \leqslant d(v) \leqslant 12$, we see that $p \geqslant |N(x)|/\alpha(G[N(x)]) \geqslant 3$ by Claim 1. Suppose $G[N(v)]$ is K_3-free. By Claim 2, $p \leqslant 2\omega(G[N(v)]) - 1 = 3$. Thus $\chi(G[N[v]]) = 4$ and $\chi(G \backslash N[v]) = \chi(G \backslash N(v)) \geqslant 9 - 3 = 6$, contrary to the fact that G is a $(4,6)$-graph. Thus $\omega(G[N(v)]) \geqslant 3$. Let $v_1, v_2, v_3 \in N(v)$ be pairwise adjacent in G and let $H := G \setminus \{v, v_1, v_2, v_3\}$. Then $G[\{v, v_1, v_2, v_3\}] = K_4$ and

$$
\begin{aligned}
2e(H) &\geqslant (d(v) - 3)(|G \setminus N[v]|) + (d(v) - 4) \cdot |N(v) \setminus \{v_1, v_2, v_3\}| \\
&= (d(v) - 3)(|G| - d(v) - 1) + (d(v) - 4)(d(v) - 3) \\
&= (d(v) - 3)(|H| - 1).
\end{aligned}
$$

Suppose $d(v) \in \{11, 12\}$. Then $2e(H) \geqslant 8(|H| - 1)$. By Theorem 7, $H \succcurlyeq K_6$, and so G has a $K_4 \cup K_6$ minor, a contradiction. This proves that $9 \leqslant d(v) \leqslant 10$. Then $p \geqslant 4$ by Claim 1. Since $\omega(G) \leqslant 5$, we see that $\omega(G[N(v)]) \leqslant 4$ and so $G[N(v)]$ contains the complement of a matching of size at least three. It follows that $4 \leqslant p \leqslant d(v) - 3$. Let $V_1, \ldots, V_p$ be the color classes of a proper p-coloring of $G[N(v)]$ with $|V_1| \geqslant \cdots \geqslant |V_p| \geqslant 1$. If $p \in \{4, 5\}$, then $|V_4| \leqslant 2$ because $d(v) \leqslant 10$. Thus $|V_4 \cup \cdots \cup V_p| \leqslant 4 = \chi(G) - 4 - 1$. By Lemma 8 applied to G and v with $r = 4$, we see that G is $(4,6)$-splittable, contrary to the fact that G is a $(4,6)$-graph. It remains to consider the case $6 \leqslant p \leqslant d(v) - 3$. Since $d(v) \leqslant 10$, we see that $|V_5| = 1$. Thus $|V_6 \cup \cdots \cup V_p| = p - 5 \leqslant (d(v) - 3) - 5 \leqslant 2 = \chi(G) - 6 - 1$. By Lemma 8 applied to G and v with $r = 6$, we see that G is $(4,6)$-splittable, a contradiction.

This completes the proof of Theorem 10. □

We end this section with an easy result (Theorem 11) on coloring even-hole-free graphs with no K_k minor, where $k \geqslant 7$. It seems non-trivial to improve the bound in Theorem 11 to $2k-6$. Rolek and the present author [22, Theorem 5.2] proved that if Mader's bound in Theorem 7 can be generalized to all values of p (as in [22, Conjecture 5.1]), then every graph with no K_p minor is $(2p-6)$-colorable for all $p \geqslant 7$.

Theorem 11. *For all $k \geqslant 7$, every even-hole-free graph with no K_k minor is $(2k-5)$-colorable.*

Proof. Suppose the assertion is false. Let G be an even-hole free graph with no K_k minor and $\chi(G) \geqslant 2k-4$. We choose G with $|G|$ minimum. Then G is vertex-critical and $\chi(G) = 2k-4$. Thus $\delta(G) \geqslant 2k-5$; in addition, G is connected and has no clique-cut. Suppose $\omega(G) \geqslant k-1$. Let K be a $(k-1)$-clique in G. Then $G \setminus K$ is connected because G has no clique-cut; by contracting $G \setminus K$ into a single vertex we obtain a K_k minor, a contradiction. Thus $\omega(G) \leqslant k-2$. Since G is even-hole-free, by Theorem 3, $\chi(G) \leqslant 2\omega(G) - 1 \leqslant 2(k-2) - 1 = 2k-5$, a contradiction. □

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