A Unique Characterization of Spectral Extrema for Friendship Graphs

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Abstract

Turán-type problem is one of central problems in extremal graph theory. Erdős et al. [J. Combin. Theory Ser. B 64 (1995), 89-100] obtained the exact Turán number of the friendship graph F_k for $n \geqslant 50k^2$, and characterized all its extremal graphs. Cioabă et al. [Electron. J. Combin. 27(4) (2020), #P4.22] initially introduced Triangle Removal Lemma into a spectral Turán-type problem, then showed that $SPEX(n, F_k) \subseteq EX(n, F_k)$ for n large enough, where $EX(n, F_k)$ and $SPEX(n, F_k)$ are the families of n-vertex F_k-free graphs with maximum size and maximum spectral radius, respectively. In this paper, the family $SPEX(n, F_k)$ is uniquely determined for sufficiently large n. Our key approach is to find various alternating cycles or closed trails in nearly regular graphs. Some typical spectral techniques are also used. This presents a probable way to characterize the uniqueness of extremal graphs for some of other spectral extremal problems. In the end, we mention several related conjectures.

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1 Introduction

A graph is said to be *H-free*, if it does not contain H as a subgraph. A classic problem in extremal graph theory asks what is the maximum number of edges in an H-free graph of order n, where the maximum number of edges is called the *Turán number* of H and denoted by $ex(n,H)$. The set of extremal H-free graphs with $ex(n,H)$ edges is denoted by $EX(n,H)$. An important motivation of investigating Turán numbers is that they are very useful for Ramsey theory. The original statements can be found in [9]. Turán-type problem can be at least traced back to Mantel's theorem in 1907, which says that $ex(n,C_3)=\lfloor \frac{n^2}{4}\rfloor$. In 1941, Turán [27] showed that $ex(n,K_{k+1})=|E(T_{n,k})|$ for $n\geqslant k+1\geqslant 3$, and the k-partite Turán graph $T_{n,k}$ is the unique extremal graph.

Let $A(G)$ be the adjacency matrix of a graph G and $\rho(G)$ be its spectral radius. By Perron-Frobenius theorem, every connected graph G exists a positive unit eigenvector corresponding to $\rho(G)$, which is called the *Perron vector* of G. In 1986, Wilf [29] showed that $\rho(G)\leqslant n(1-\frac{1}{k})$ for every n-vertex K_{k+1}-free graph G. Wilf's result was sharpened by Nikiforov [18], who proved that $\rho(G)\leqslant\rho(T_{n,k})$ for every n-vertex K_{k+1}-free graph G, with equality if and only if $G\cong T_{n,k}$. Babai and Guiduli [2] established asymptotically a Kővári-Sós-Turán bound $\rho(G)\leqslant((s-1)^{\frac{1}{t}}+o(1))n^{1-\frac{1}{t}}$ for every n-vertex $K_{s,t}$-free graph G (where $s\geqslant t\geqslant 2$), then Nikiforov [22] obtained a new upper bound with main term $(s-t+1)^{\frac{1}{t}}n^{1-\frac{1}{t}}$. Nikiforov [21] formally posed a spectral version of Turán-type problem, and established some interesting results and conjectures (see for example, [1, 19, 20, 21]). Moreover, Nikiforov also develops some methods for spectral Turán-type problems such as deleting vertices, removing edges and counting the number of walks (see [23]).

Problem 1 ([21])**.** What is the maximum spectral radius of an H-free graph of order n?

In the past decades, much attention has been paid to Problem 1 and its variations (see surveys [3, 23, 13] and some recent results [4, 5, 6, 7, 12, 15, 25, 26, 30]). For convenience, we denote by $SPEX(n,H)$ the family of extremal H-free graphs with maximum spectral radius in Problem 1.

A graph is called a *friendship graph*, and denoted by F_k, if it consists of k triangles which intersect in exactly one common vertex. In 1995, Erdős, Füredi, Gould and Gunderson proved the following classic result in extremal graph theory.

Theorem 2 ([10])**.** *For every $k\geqslant 1$, and for every $n\geqslant 50k^2$, $ex(n,F_k)=\lfloor\frac{n^2}{4}\rfloor+k^2-k$ if k is odd; and $ex(n,F_k)=\lfloor\frac{n^2}{4}\rfloor+k^2-\frac{3}{2}k$ if k is even.*

Moreover, the extremal graphs were completely determined in [10]. A *nearly $(k-1)$-regular graph* is a graph with one vertex of degree $k-2$ and all other vertices of degrees $k-1$. It was showed in [10] that $EX(n,F_k)$ consists of all graphs obtained by taking $T_{n,2}$ and embedding a subgraph H in one part, where $H\cong K_k\cup K_k$ for odd k and H is a nearly $(k-1)$-regular graph of order $2k-1$ for even k. A nearly $(k-1)$-regular graph of order $2k-1$ does exist for every even $k\geqslant 2$, and there are many if k is large.

We can first construct a graph with vertex set $\{w_0\}\cup A\cup B$ such that $N(w_0)=A$ and $|B|=|A|+2=k$. Then, we partition A into $A_1\cup A_2$ and B into $\{u_0\}\cup B^\star\cup B^{\star\star}$

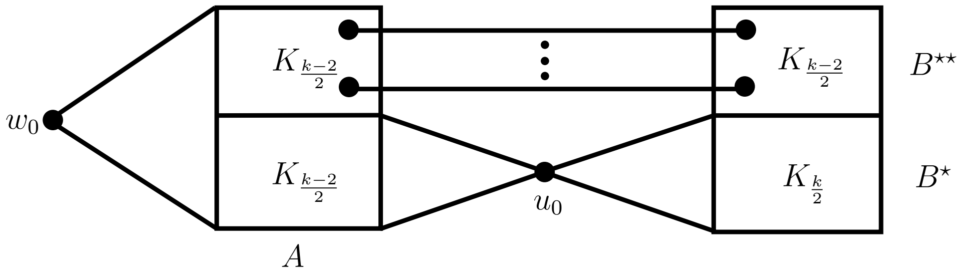

Figure 1: Graph $H^{\star}$ with $B = \{u_0\} \cup B^{\star} \cup B^{\star\star}$.

such that $|A_1| = |A_2| = |B^{\star\star}| = \frac{k-2}{2}$. Finally, we join $k-1$ edges from u_0 to $A_1 \cup B^{\star}$, $\frac{k-2}{2}$ independent edges from $B^{\star\star}$ to A_2, and some additional edges such that both A and $B \setminus \{u_0\}$ are cliques. The resulting graph $H^{\star}$ is a nearly $(k-1)$-regular graph of order $2k-1$ (see Fig. 1).

In [6], Cioabă, Feng, Tait and Zhang studied the spectral counterpart of Theorem 2. If $k = 1$, then Theorem 2 is just the well-known Mantel's theorem, and the spectral version is known from Nikiforov's spectral Turán theorem (see [18]). Both extremal graphs are the only graph $T_{n,2}$. For $k \geqslant 2$, the authors [6] obtained the following theorem by combining Triangle Removal Lemma (see [10, 11, 24]) and spectral techniques. This develops a new tool for Problem 1. Subsequently, Triangle Removal Lemma was also used efficiently to other spectral Turán-type problems (see [7, 8, 14]).

Theorem 3 ([6])**.** *For every fixed $k \geqslant 2$ and n large enough, $SPEX(n, F_k) \subseteq EX(n, F_k)$.*

In this paper, our goal is a further characterization of $SPEX(n, F_k)$. We obtain the following result, which gives the uniqueness of the graphs in $SPEX(n, F_k)$.

Theorem 4. *For every fixed $k \geqslant 2$ and n large enough, the only graph in $SPEX(n, F_k)$ is obtained from $T_{n,2}$ by embedding a graph H in a part of size $\lfloor \frac{n}{2} \rfloor$, where $H \cong K_k \cup K_k$ for odd k and $H \cong H^{\star}$ for even k (see Fig. 1).*

The rest of the paper is organized as follows. In Section 2, some structural and spectral propositions are obtained for graphs in $EX(n, F_k)$. In Section 3, the nearly $(k-1)$-regular graph H is characterized for graphs in $SPEX(n, F_k)$. In Section 4, it is showed that H must be embedded in a part of size $\lfloor \frac{n}{2} \rfloor$. In Section 5, some related conjectures are mentioned for further research.

2 Propositions for graphs in $EX(n, F_k)$

By the result due to Erdős et al. [10], we know that each graph in $EX(n, F_k)$ is obtained from $T_{n,2}$ with two parts (say S, T) by embedding a subgraph H in one part, where $H \cong K_k \cup K_k$ for odd k and H is a nearly $(k-1)$-regular graph of order $2k-1$ for even k. Without loss of generality we may assume that H is embedded in S.

For odd k, H has been uniquely determined up to isomorphism. In this section, we focus on the case that k is even and try to obtain some useful propositions for a graph $G \in EX(n, F_k)$ and its subgraph H. For even k, since H is a nearly $(k-1)$-regular graph of order $2k-1$, there exists a partition $V(H) = \{w_0\} \cup A \cup B$ such that $N_H(w_0) = A$, $|B| = |A| + 2 = k$, and $d_H(v) = k-1$ for every vertex $v \in A \cup B$.

A trail is a walk whose edges are distinct. In particular, a trail is closed if its original and terminal vertices are the same. Now, by coloring all the edges of $H[A \cup B]$ red and all its non-edges blue, we obtain a copy of $K_{|A\cup B|}$ with red and blue colours. If there exists a closed trail C in $K_{|A\cup B|}$ such that any two consecutive edges of C have distinct colours, then we call C an *alternating closed trail* of $H[A \cup B]$. In the following, we present a structural proposition of the nearly regular graph H.

Proposition 5. *If A is neither an empty set nor a clique, then there exists an alternating closed trail C such that $C = v_0v_1u_1u_2v_0$ or $C = v_0v_1u_1u_2v_2u_3v_0$, where each $v_i \in A$, each $u_i \in B$ and v_0v_1 is a blue edge (non-edge).*

Proof. Let $v_0 \in A$ with $d_B(v_0) = \max_{v\in A} d_B(v)$, and let $\overline{N}_A(v_0) = A \setminus (N_A(v_0) \cup \{v_0\})$. Note that $d_{A\cup B}(v) = k-2$ for each $v \in A$. Then $d_A(v_0) = \min_{v\in A} d_A(v)$. Moreover, since A is not a clique, we have $d_A(v_0) \leqslant |A| - 2 = k-4$, which implies that $\overline{N}_A(v_0) \neq \varnothing$ and $N_B(v_0) \neq \varnothing$. Now define $B' = \{u \in B : \exists\ v \in \overline{N}_A(v_0) \text{ with } v \sim u\}$, where $v \sim u$ denotes vertices v and u are adjacent. Since $d_B(v) = d_{A\cup B}(v) - d_A(v) \geqslant (k-2) - |A \setminus \{v, v_0\}| = 2$ for each $v \in \overline{N}_A(v_0)$, we also have $B' \neq \varnothing$. Now we give the following claim.

Claim 6. If there exists a blue edge with one endpoint in B' and the other in $N_B(v_0)$, then we have an alternating 4-cycle.

Proof. Let u_1u_2 be a blue edge, where $u_1 \in B'$ and $u_2 \in N_B(v_0) \setminus \{u_1\}$. By the definition of B', we can find a vertex $v_1 \in \overline{N}_A(v_0)$ with $v_1 \sim u_1$. Now it is easy to see that $C = v_0v_1u_1u_2v_0$ is an alternating 4-cycle and v_0v_1 is a blue edge. □

Next, we may assume that each edge $u'u''$ is red for any two distinct vertices $u' \in B'$ and $u'' \in N_B(v_0)$. Then we have the following claim.

Claim 7. If there exists a blue edge within B', then we have an alternating closed trail of length six.

Proof. Let u_1u_2 be a blue edge within B', that is, $u_1 \nsim u_2$ in H. By the above assumption, both u_1 and u_2 are not in $N_B(v_0)$. By the definition of B', we can find a vertex $v_i \in \overline{N}_A(v_0)$ with $v_i \sim u_i$ for $i \in \{1, 2\}$, where possibly $v_1 = v_2$. Note that $d_B(v_0) = \max_{v\in A} d_B(v) \geqslant d_B(v_2)$ and $u_2 \in N_B(v_2) \setminus N_B(v_0)$, then $N_B(v_0) \nsubseteq N_B(v_2)$. Thus, we can find a vertex $u_3 \in N_B(v_0)$ with $u_3 \nsim v_2$. Now $C = v_0v_1u_1u_2v_2u_3v_0$ is an alternating closed trail of length six and v_0v_1 is a blue edge. □

Now by Claims 6 and 7, we may assume that each edge $u'u''$ is red for any two distinct vertices $u' \in B'$ and $u'' \in B' \cup N_B(v_0)$.

Let v_1u_1 be a red edge, where $v_1 \in \overline{N}_A(v_0)$ and $u_1 \in B'$. Then $d_B(u_1) = d_H(u_1) - d_A(u_1) = (k-1) - d_A(u_1) \leqslant k-2 = |B| - 2$, and so we can find a vertex $u_2 \in B \setminus \{u_1\}$

such that u_1u_2 is blue. Furthermore, it follows from the above assumption that $u_2 \in B \setminus (B' \cup N_B(v_0))$. This implies that $N_A(u_2) \subseteq N_A(v_0)$. Since $d_A(u_2) = d_H(u_2) - d_B(u_2) \geqslant (k-1) - |B \setminus \{u_1, u_2\}| = 1$, then we can find a vertex $v_2 \in N_A(v_0)$ with $v_2 \sim u_2$. Now by the definition of v_0, we also have $N_B(v_0) \nsubseteq N_B(v_2)$. Subsequently, we can find a vertex $u_3 \in N_B(v_0)$ with $u_3 \nsim v_2$ (possibly $u_3 = u_1$). Therefore, $C = v_0v_1u_1u_2v_2u_3v_0$ is an alternating closed trail of length six and v_0v_1 is a blue edge. This completes the proof. $\square$

Recall that G contains a spanning subgraph $K_{|S|,|T|}$ with H being embedded in S, where $|S| + |T| = n$ and $||S| - |T|| \leqslant 1$. In the following, let $\rho := \rho(G)$ and $X = (x_1, \ldots, x_n)^T$ be the Perron vector of G. For convenience, we write $x_{\max V(H)} = \max_{v \in V(H)} x_v$ and $x_{\min V(H)} = \min_{v \in V(H)} x_v$. Then we can obtain some propositions for sufficiently large n.

Proposition 8. *Let $X_T = \sum_{v \in T} x_v$. Then*

$$x_{\max V(H)} \leqslant \frac{X_T}{\rho - k + 1}, \quad x_{\min V(H)} \geqslant \frac{X_T}{\rho - k + 2}. \tag{1}$$

Proof. Note that $V(H) \subseteq S$. Then $T \subseteq N_G(v)$ for each $v \in V(H)$. Moreover, we know that $\rho x_v = \sum_{u \in N_G(v)} x_u$ and $k - 2 \leqslant d_H(v) \leqslant k - 1$ for each $v \in V(H)$. Thus,

$$\rho x_{\max V(H)} \leqslant X_T + (k-1)x_{\max V(H)}, \qquad \rho x_{\min V(H)} \geqslant X_T + (k-2)x_{\min V(H)}.$$

Consequently, $x_{\max V(H)} \leqslant \frac{X_T}{\rho-k+1}$ and $x_{\min V(H)} \geqslant \frac{X_T}{\rho-k+2}$. $\square$

We now evaluate Perron components for vertices in A and B. This will be frequently used in the subsequent sections.

Proposition 9. *For each vertex $v \in A$, we have*

$$\Big(\rho + k - 1 + \frac{k(k-2)}{\rho - k + 2}\Big) X_T \leqslant \rho^2 x_v \leqslant \Big(\rho + k - 1 + \frac{k(k-2)}{\rho - k + 1}\Big) X_T.$$

Proof. For each $v \in A$, we have $\rho x_v = X_T + \sum\limits_{u \in N_H(v)} x_u$ and $\rho^2 x_v = \rho X_T + \sum\limits_{u \in N_H(v)} \rho x_u$, where

$$\rho x_u \leqslant \begin{cases} X_T + (k-2)x_{\max V(H)}, & u = w_0, \\ X_T + (k-1)x_{\max V(H)}, & u \in N_H(v) \setminus \{w_0\}. \end{cases}$$

Since $d_H(v) = k-1$, we have $\sum\limits_{u \in N_H(v)} \rho x_u \leqslant (k-1)X_T + k(k-2)x_{\max V(H)}$, and hence

$$\rho^2 x_v \leqslant (\rho + k - 1)X_T + k(k-2)x_{\max V(H)}.$$

Similarly, we can obtain

$$\rho^2 x_v \geqslant (\rho + k - 1)X_T + k(k-2)x_{\min V(H)}.$$

By Proposition 8, the result follows. $\square$

Proposition 10. *For each vertex $v \in B$, we have*

$$\Big(\rho + k - 1 + \frac{(k-1)^2}{\rho - k + 2}\Big) X_T \leqslant \rho^2 x_v \leqslant \Big(\rho + k - 1 + \frac{(k-1)^2}{\rho - k + 1}\Big) X_T.$$

Proof. For each $v \in B$, we know that $N_H(v) \subseteq A \cup B$. Hence, $d_H(u) = k - 1$ for each vertex $u \in N_H(v)$. Moreover, $\rho x_v = X_T + \sum_{u \in N_H(v)} x_u$ and $\rho^2 x_v = \rho X_T + \sum_{u \in N_H(v)} \rho x_u$, where

$$X_T + (k-1) x_{\min V(H)} \leqslant \rho x_u \leqslant X_T + (k-1) x_{\max V(H)}.$$

Since $|N_H(v)| = k - 1$, we have

$$(\rho + k - 1) X_T + (k-1)^2 x_{\min V(H)} \leqslant \rho^2 x_v \leqslant (\rho + k - 1) X_T + (k-1)^2 x_{\max V(H)}.$$

Therefore, the result holds from (1). □

Proposition 11. *For any two vertices $u_1, u_2 \in B$, if $d_B(u_1) > d_B(u_2)$, then $x_{u_1} > x_{u_2}$.*

Proof. Since $u_1, u_2 \in B$, we have $d_{A \cup B}(u_1) = d_{A \cup B}(u_2) = k - 1$. Now assume that $d_B(u_1) = d_B(u_2) + a$, where $a \geqslant 1$. Then $d_A(u_1) = d_A(u_2) - a$. Write $b = (\rho - k + 1)(\rho - k + 2)$. By Propositions 9-10, we obtain

$$\rho^2 (x_{\max A} - x_{\min A}) \leqslant \frac{k(k-2)}{b} X_T, \qquad \rho^2 (x_{\max B} - x_{\min B}) \leqslant \frac{(k-1)^2}{b} X_T; \tag{2}$$

while

$$\rho^2 (x_{\min B} - x_{\min A}) \geqslant \rho^2 (x_{\min B} - x_{\max A}) \geqslant \frac{\rho - k^2 + k + 1}{b} X_T. \tag{3}$$

Since $T \subseteq N_G(u_1) \cap N_G(u_2)$, we can see that

$$\begin{aligned}
\rho x_{u_1} - \rho x_{u_2} &= \sum_{v \in N_H(u_1)} x_v - \sum_{v \in N_H(u_2)} x_v \\
&\geqslant d_B(u_1) x_{\min B} + d_A(u_1) x_{\min A} - d_B(u_2) x_{\max B} - d_A(u_2) x_{\max A} \\
&= a(x_{\min B} - x_{\min A}) - d_A(u_2)(x_{\max A} - x_{\min A}) - d_B(u_2)(x_{\max B} - x_{\min B}).
\end{aligned}$$

Note that $d_A(u_2) + d_B(u_2) = k - 1$ and $\rho \geqslant \sqrt{|S||T|} = \sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil} >> k^3$. Combining (2) and (3), we have $\rho x_{u_1} - \rho x_{u_2} > 0$. Therefore, $x_{u_1} > x_{u_2}$. □

Since $\rho \geqslant \sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil} >> k^3$, by (3) we can also conclude that

$$x_{\min B} > x_{\max A}. \tag{4}$$

3 Characterization of H for graphs in $SPEX(n, F_k)$

Let $G \in SPEX(n, F_k)$, where $k \geqslant 2$ and n is large enough. By Theorem 3 we know that $G \in EX(n, F_k)$, that is, G is obtained from $K_{|S|,|T|}$ with $|S| + |T| = n$ and $||S| - |T|| \leqslant 1$ by embedding a subgraph H in one part (say S). In this section, we focus on the characterization of H. For odd k, H has already been uniquely determined.

Theorem 12 ([6]). *If k is odd and n is sufficiently large, then $H \cong K_k \cup K_k$.*

In the following, we assume that k is even. Recall that H is a nearly $(k-1)$-regular graph of order $2k-1$ with $V(H) = \{w_0\} \cup A \cup B$, where $N_H(w_0) = A$, $|B| = |A| + 2 = k$, and $d_H(v) = k-1$ for each $v \in A \cup B$. To characterize H for even k, it suffices to determine the structure of $H[A \cup B]$.

Lemma 13. *If $k \geqslant 4$ and n is sufficiently large, then A must be a clique.*

Proof. Suppose, to the contrary, that A is not a clique. By Proposition 5, $H[A \cup B]$ contains an alternating closed trail C. Next we consider the following two cases.

Case 1. $C = v_0v_1u_1u_2v_0$, where $v_0, v_1 \in A$; $u_1, u_2 \in B$ and v_0v_1 is a blue edge.

Now let G' be the graph obtained from G by exchanging the color of each edge in C, that is, let $G' = G - \{u_1v_1, u_2v_0\} + \{v_0v_1, u_1u_2\}$. Then $G'[\{w_0\} \cup A \cup B]$ and $G[\{w_0\} \cup A \cup B]$ have the same degree sequence, and thus $G'[\{w_0\} \cup A \cup B]$ is also a nearly $(k-1)$-regular graph. This implies that $G' \in EX(n, F_k)$ too.

On the other hand, write $\rho = \rho(G)$ and $\rho' = \rho(G')$. Then

$$\begin{aligned} \rho' - \rho \geqslant X^T(A(G') - A(G))X &= 2(x_{v_0}x_{v_1} + x_{u_1}x_{u_2}) - 2(x_{u_1}x_{v_1} + x_{u_2}x_{v_0}) \\ &= 2(x_{u_1} - x_{v_0})(x_{u_2} - x_{v_1}). \end{aligned} \tag{5}$$

It follows from (4) that $x_u > x_v$ for any $v \in A$ and $u \in B$. Therefore $\rho' > \rho$ by (5), which contradicts the fact that $G \in SPEX(n, F_k)$.

Case 2. $C = v_0v_1u_1u_2v_2u_3v_0$, where $v_0, v_1, v_2 \in A$; $u_1, u_2, u_3 \in B$ and v_0v_1 is a blue edge.

We also define G' to be the graph obtained from G by exchanging the color of each edge in C. Similarly, G' is also F_k-free. Moreover,

$$\begin{aligned} \rho' - \rho &\geqslant 2(x_{v_0}x_{v_1} + x_{u_1}x_{u_2} + x_{v_2}x_{u_3}) - 2(x_{u_1}x_{v_1} + x_{u_2}x_{v_2} + x_{u_3}x_{v_0}) \\ &= 2(x_{u_1} - x_{v_0})(x_{u_2} - x_{v_1}) - 2(x_{u_2} - x_{u_3})(x_{v_2} - x_{v_0}). \end{aligned} \tag{6}$$

Combining (2), (3) and (6), we obtain

$$\rho' - \rho \geqslant \Big(\frac{(\rho - k^2 + k + 1)^2}{b^2} - \frac{k(k-2)(k-1)^2}{b^2}\Big)\frac{2X_T^2}{\rho^4} > 0$$

for sufficiently large n, a contradiction.

By Cases 1 and 2, we can conclude that A is a clique. □

By Lemma 13, $A \cup \{w_0\}$ is a clique. For each vertex $v \in A$, since $d_H(v) = k-1$, we have $d_B(v) = (k-1) - |A| = 1$. Let $e(A, B)$ be the number of edges with one endpoint in A and the other in B. Then

$$e(A, B) = |A| = k - 2. \tag{7}$$

Now assume that $u_0 \in B$ with $x_{u_0} = \min_{u \in B} x_u$. Moreover, let $B^* = \{u \in B : d_B(u) = k-1\}$ and $B^{**} = B \setminus (B^* \cup \{u_0\})$. Then $e(B^*, A) = 0$, as $d_H(u) = k-1$ for each $u \in B$. Now we are ready to give a complete characterization of H for even k.

Theorem 14. *If k is even and n is sufficiently large, then $H \cong H^\star$ (see Fig. 1).*

Proof. If $k = 2$, then $|A| = k-2 = 0$ and $|B| = k = 2$. Since H is a nearly $(k-1)$-regular graph of order $2k-1$, it is easy to see that $H \cong K_1 \cup K_2$, and hence $H \cong H^\star$. In the following, we assume that $k \geqslant 4$. We first give five claims.

Claim 15. *B^* is a clique and $x_{u^*} = \max_{v \in B} x_v > x_u$ for each $u^* \in B^*$ and each $u \in B \backslash B^*$.*

Proof. We first show that $B^* \neq \varnothing$. If not, then $B^* = \varnothing$, that is, for each $u \in B$, $d_B(u) \leqslant k-2$, and hence $d_A(u) \geqslant 1$. It follows that $e(B, A) \geqslant |B| = k$, which contradicts (7). Therefore, $B^* \neq \varnothing$.

Note that $|B| = k$ and $d_B(u^*) = k-1$ for each $u^* \in B^*$. Then every vertex of B^* is a dominating vertex of B, and so B^* is a clique. Since $e(B^*, A) = 0$, by symmetry we have $x_{u_1^*} = x_{u_2^*}$ for any two vertices $u_1^*, u_2^* \in B^*$. Moreover, since $d_B(u^*) > d_B(u)$ for each $u^* \in B^*$ and each $u \in B \setminus B^*$, by Proposition 11, we have $x_{u^*} = \max_{v \in B} x_v > x_u$. ☐

Claim 16. *$u_0 \notin B^*$ and $B^{**} \neq \varnothing$.*

Proof. Note that $x_{u_0} = \min_{u \in B} x_u$. If $u_0 \in B^*$, then by the definition of u_0 and Claim 15, we know that $x_u = x_v$ for all distinct vertices $u, v \in B$. That is to say, $x_u = \max_{v \in B} x_v$ for each $u \in B$. It follows that $B = B^*$ and $e(B, A) = e(B^*, A) = 0$, which contradicts (7). Therefore, $u_0 \notin B^*$.

Since every vertex of B^* is a dominating vertex of B, we have $d_B(u_0) \geqslant |B^*|$. On the other hand, $u_0 \notin B^*$, then $d_B(u_0) \leqslant k-2$, and hence $|B^*| \leqslant k-2$. Therefore, $|B^{**}| = |B \setminus (B^* \cup \{u_0\})| \geqslant 1$. ☐

Claim 17. *$e(\{u_0\}, B^{**}) = 0$.*

Proof. Suppose, to the contrary, that there exists a vertex $u_1 \in B^{**}$ with $u_0 \sim u_1$. By the definition of B^{**}, we have $d_B(u_1) \leqslant k-2$, and hence there exists a vertex $u_2 \in B$ with $u_2 \nsim u_1$. Now since $d_B(u_2) \leqslant |B \setminus \{u_1, u_2\}| = k-2$, we have $d_A(u_2) \geqslant 1$. Thus we can find a vertex $v_1 \in A$ such that $v_1 \sim u_2$. Recall that $d_B(v) = 1$ for each $v \in A$. Then $v_1 \nsim u_0$. Therefore, we can obtain an alternating 4-cycle $u_0u_1u_2v_1u_0$.

Let $G' = G - \{u_0u_1, u_2v_1\} + \{u_1u_2, u_0v_1\}$. Similar to the analysis of Lemma 13, we have that G' is F_k-free. Moreover,

$$\begin{aligned} \rho(G') - \rho(G) \geqslant X^T(A(G') - A(G))X &= 2(x_{u_1}x_{u_2} + x_{u_0}x_{v_1}) - 2(x_{u_0}x_{u_1} + x_{u_2}x_{v_1}) \\ &= 2(x_{u_1} - x_{v_1})(x_{u_2} - x_{u_0}). \end{aligned}$$

By (4) we know that $x_{u_1} > x_{v_1}$, and by the choice of u_0 we have $x_{u_2} \geqslant x_{u_0}$. Thus $\rho(G') \geqslant \rho(G)$. On the other hand, since $G \in SPEX(n, F_k)$, we have $\rho(G') = \rho(G)$, and hence X is also the Perron vector of G'. Observe that $N_G(u_2) \setminus N_{G'}(u_2) = \{v_1\}$ and $N_{G'}(u_2) \setminus N_G(u_2) = \{u_1\}$. Hence,

$$0 = \rho(G')x_{u_2} - \rho(G)x_{u_2} = x_{u_1} - x_{v_1} > 0,$$

a contradiction. Therefore, $e(\{u_0\}, B^{**}) = 0$. □

Claim 18. $|B^*| = \bar{e}(B^{**}) + \frac{k}{2}$, *where* $\bar{e}(B^{**})$ *denotes the number of non-edges in* $G[B^{**}]$.

Proof. Note that $e(\{u_0\}, B^{**}) = 0$. Then $\bar{e}(B) = \bar{e}(B^{**}) + |B^{**}|$. Recall that $d_H(u) = k-1$ for each $u \in B$ and $|B| = k$. Then we have $k(k-1) = \sum_{u \in B} d_H(u) = e(B, A) + 2e(B) = e(B, A) + k(k-1) - 2\bar{e}(B)$, and hence $e(B, A) = 2\bar{e}(B)$. Note that $|B^{**}| = |B \setminus (B^* \cup \{u_0\})| = k - 1 - |B^*|$. Then $e(B, A) = 2\bar{e}(B) = 2\bar{e}(B^{**}) + 2k - 2 - 2|B^*|$. Combining (7), we have $|B^*| = \bar{e}(B^{**}) + \frac{k}{2}$. □

Claim 19. *If* B^{**} *is a clique, then* $H \cong H^\star$.

Proof. By Claim 17, $d_{B^{**}}(u_0) = 0$, and so $d_B(u_0) = |B^*|$. Note that $d_H(u_0) = k - 1 = |B^*| + |B^{**}|$. Thus $d_A(u_0) = d_H(u_0) - d_B(u_0) = |B^{**}|$. Now if B^{**} is a clique, then $d_B(u) = |B \setminus \{u, u_0\}| = k - 2$ and $d_A(u) = 1$ for each $u \in B^{**}$. Recall that $e(B^*, A) = 0$. Therefore,

$$e(B, A) = \sum_{u \in B^{**}} d_A(u) + d_A(u_0) = 2|B^{**}|.$$

Combining (7), we have $|B^{**}| = \frac{e(B,A)}{2} = \frac{k-2}{2}$, and so $|B^*| = (k-1) - |B^{**}| = \frac{k}{2}$. Now $d_B(u_0) = |B^*| = \frac{k}{2}$, hence $d_A(u_0) = \frac{k-2}{2}$. Combining Lemma 13 and $d_B(v) = 1$ for each $v \in A$, we have each vertex of $A \setminus N_A(u_0)$ has exactly one neighbor in B^{**} and vice versa. Note that $|B^{**}| = \frac{k-2}{2}$. It follows that there are $\frac{k-2}{2}$ independent edges between B^{**} and $A \setminus N_A(u_0)$, and hence $H \cong H^\star$ (see Fig. 1). □

By Claim 19, it suffices to show that B^{**} is a clique in the following. The case $|B^{**}| = 1$ is trivial. We may assume that $|B^{**}| \geqslant 2$. Suppose, to the contrary, that B^{**} is not a clique. Then we can find two vertices $u_1, u_2 \in B^{**}$ with $u_1 \nsim u_2$. Without loss of generality, we assume that $x_{u_2} \geqslant x_{u_1}$. For any vertex u^* of B^*, by Claim 15 and the choice of u_0, we have

$$x_{u^*} > x_{u_2} \geqslant x_{u_1} \geqslant x_{u_0}. \tag{8}$$

Moreover, $d_B(u_i) \leqslant k - 2$ implies that $d_A(u_i) \geqslant 1$ for $i \in \{1, 2\}$. Thus we can find $v_1, v_2 \in A$ such that $v_1 \sim u_1$ and $v_2 \sim u_2$. Recall that A is a clique and $d_B(v) = 1$ for each $v \in A$. Then $v_1 \neq v_2$, and by symmetry, we have

$$\sum_{v \in N_A(u_2)} x_v = d_A(u_2)x_{v_2}. \tag{9}$$

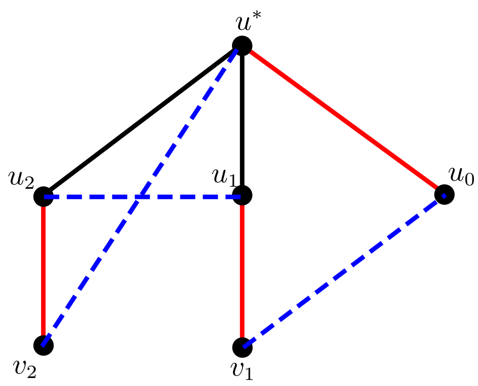

Figure 2: An alternating 6-cycle.

By (8) and Proposition 11, $d_B(u_2) \geqslant d_B(u_1) \geqslant d_B(u_0)$, and so $d_A(u_2) \leqslant d_A(u_1) \leqslant d_A(u_0)$. Combining (7), we can see that

$$d_A(u_2) \leqslant \left\lfloor \frac{e(A,B)}{3} \right\rfloor = \left\lfloor \frac{k-2}{3} \right\rfloor \leqslant \frac{k-4}{2}, \tag{10}$$

as $k \geqslant 4$.

Since A is a clique, we have $N_G(v_1) \setminus N_G(v_2) = \{v_2, u_1\}$ and $N_G(v_2) \setminus N_G(v_1) = \{v_1, u_2\}$. Hence $\rho(x_{v_2} - x_{v_1}) = (x_{v_1} - x_{v_2}) + (x_{u_2} - x_{u_1})$, which gives

$$x_{v_2} - x_{v_1} = \frac{x_{u_2} - x_{u_1}}{\rho + 1} \geqslant 0. \tag{11}$$

Observe that

$$N_G(u_2) \setminus N_G(u^*) = N_A(u_2) \cup \{u^*\}, \qquad N_G(u^*) \setminus N_G(u_2) = N_B(u^*) \setminus N_B(u_2), \tag{12}$$

where $|N_B(u^*) \setminus N_B(u_2)| = |N_A(u_2) \cup \{u^*\}| = d_A(u_2) + 1$. Thus

$$\sum_{u \in N_B(u^*) \setminus N_B(u_2)} x_u \leqslant d_A(u_2) x_{u^*} + x_{u_2}, \tag{13}$$

as $u_2 \in N_B(u^*) \setminus N_B(u_2)$ and $x_{u^*} = \max_{u \in B} x_u$. Combining (12), (13) and (9), we obtain

$$\rho(x_{u^*} - x_{u_2}) \leqslant (x_{u_2} - x_{u^*}) + d_A(u_2)(x_{u^*} - x_{v_2}) < d_A(u_2)(x_{u^*} - x_{v_2}),$$

as $x_{u^*} > x_{u_2}$ by (8). Furthermore, by (10) and (11), we conclude that

$$\rho(x_{u^*} - x_{u_2}) < \frac{k-4}{2}(x_{u^*} - x_{v_2}) \leqslant \frac{k-4}{2}(x_{u^*} - x_{v_1}). \tag{14}$$

Now note that $u_1 \nsim u_2$ and $v_1, v_2 \notin N_G(u_0) \cup N_G(u^*)$. Thus we can find an alternating 6-cycle $C_1 = u_1 v_1 u_0 u^* v_2 u_2 u_1$ (see Fig. 2).

Let

$$G_1 = G - \{u_1v_1, u_0u^*, v_2u_2\} + \{v_1u_0, u^*v_2, u_2u_1\}. \tag{15}$$

Then G_1 is F_k-free by previous analysis. If there still exist $u_1', u_2' \in B^{**}$ with $u_1' \nsim u_2'$, by Claim 4, we can also find a vertex $u^{*\prime} \in B^* \setminus \{u^*\}$ and an alternating 6-cycle C_2 with $u^{*\prime} \in V(C_2)$ and $E(C_1) \cap E(C_2) = \varnothing$, then obtain an F_k-free graph G_2 by a similar operation. Continue this operation until B^{**} is a clique, and denote by G' the resulting graph. Then $G' \in EX(n, F_k)$, as $G \in EX(n, F_k)$. Let $H' = G'[\{w_0\} \cup A \cup B]$, $\rho' = \rho(G')$ and $Y = (y_1, \ldots, y_n)^T$ be the Perron vector of G'. In order to differentiate with the above B^* and B^{**}, we write $B^\star = \{u \in B : d_{H'[B]}(u) = k-1\}$ and $B^{\star\star} = B \setminus (B^\star \cup \{u_0\})$ in H'. Note that $d_{H'[B^{\star\star}]}(u_0) = 0$ and $B^{\star\star}$ becomes a clique with $B^{**} \cup \{u^*\} \subseteq B^{\star\star}$. By Claim 19, $H' \cong H^\star$, which implies that $|B^{\star\star}| = d_{H'[A]}(u_0) = \frac{k-2}{2}$ and $N_{H'[B]}(u_0) = B^\star$ (see Fig. 1).

Since $N_{H'[B]}(u_0) = B^\star$, we have $\{u^*, u_1, u_2\} \subseteq B^{\star\star}$ in H'. By (15), $v_1 \sim u_0$ and $v_2 \sim u^*$ in H'. Thus we can observe from Fig. 1 that

$$N_{G'}(u^*) \setminus N_{G'}(u_0) = (B^{\star\star} \setminus \{u^*\}) \cup \{v_2\}, \qquad N_{G'}(u_0) \setminus N_{G'}(u^*) = N_{H'[A]}(u_0). \tag{16}$$

By symmetry,

$$\sum_{u \in B^{\star\star} \setminus \{u^*\}} y_u = (|B^{\star\star}| - 1) y_{u^*} = \frac{k-4}{2} y_{u^*}$$

and

$$\sum_{v \in N_{H'[A]}(u_0)} y_v = d_{H'[A]}(u_0) y_{v_1} = \frac{k-2}{2} y_{v_1}.$$

Combining (16), we have

$$\rho'(y_{u^*} - y_{u_0}) = (y_{v_2} - y_{v_1}) + \frac{k-4}{2}(y_{u^*} - y_{v_1}). \tag{17}$$

Since $|B^{\star\star}| \geqslant |\{u^*, u_1, u_2\}| \geqslant 3$, we have $d_{H'[B]}(u^*) > d_{H'[B]}(u_0)$. Since $G' \in EX(n, F_k)$, by Proposition 11, we obtain $y_{u^*} > y_{u_0}$. Recall that $v_1 \sim u_0$ and $v_2 \sim u^*$ in H'. Similar to (11), we can obtain

$$y_{v_2} - y_{v_1} = \frac{y_{u^*} - y_{u_0}}{\rho' + 1} > 0.$$

Combining (17), we have

$$\rho'(y_{u^*} - y_{u_0}) > \frac{k-4}{2}(y_{u^*} - y_{v_1}) \geqslant \frac{k-4}{2}(y_{u^*} - y_{v_2}). \tag{18}$$

Now let $E = E(G)$, $E' = E(G')$ and $\overline{E}$ be the set of non-edges in $G[B^{**}]$. Then

$$\begin{aligned} Y^T(\rho' - \rho)X &= Y^T(A(G') - A(G))X \\ &= \sum_{uv \in E'} (x_u y_v + x_v y_u) - \sum_{uv \in E} (x_u y_v + x_v y_u) \\ &= \sum_{u_1u_2 \in \overline{E}} \gamma, \end{aligned}$$

where by (15)

$$\begin{aligned}\gamma &= (x_{u_0}y_{v_1}+x_{v_1}y_{u_0}+x_{u^*}y_{v_2}+x_{v_2}y_{u^*}+x_{u_1}y_{u_2}+x_{u_2}y_{u_1})\\ &\quad -(x_{u_1}y_{v_1}+x_{v_1}y_{u_1}+x_{u^*}y_{u_0}+x_{u_0}y_{u^*}+x_{u_2}y_{v_2}+x_{v_2}y_{u_2}).\end{aligned}$$

Since $u^*, u_1, u_2 \in B^{\star\star}$ in H', we have $y_{u^*} = y_{u_1} = y_{u_2}$ by symmetry, and thus $x_{v_2}y_{u^*} - x_{v_2}y_{u_2} = 0$. Moreover, $x_{u_1} \geqslant x_{u_0}$ by (8) and $y_{u^*} > y_{v_1}$ by (4). Then

$$(x_{u_0}y_{v_1}+x_{u_1}y_{u_2})-(x_{u_1}y_{v_1}+x_{u_0}y_{u^*}) = (x_{u_0}-x_{u_1})(y_{v_1}-y_{u^*}) \geqslant 0.$$

Hence we have

$$\begin{aligned}\gamma &\geqslant y_{u_0}(x_{v_1}-x_{u^*})+y_{v_2}(x_{u^*}-x_{u_2})+y_{u_1}(x_{u_2}-x_{v_1})\\ &= (y_{u^*}-y_{u_0})(x_{u^*}-x_{v_1})-(y_{u^*}-y_{v_2})(x_{u^*}-x_{u_2}),\end{aligned}$$

as $y_{u_1} = y_{u^*}$. By (4), $y_{u^*} > y_{v_2}$ and $x_{u^*} > x_{v_1}$. Combining (14) and (18), we obtain

$$\gamma > \frac{k-4}{2}(y_{u^*}-y_{v_2})(x_{u^*}-x_{v_1})\Big(\frac{1}{\rho'}-\frac{1}{\rho}\Big).$$

Note that $G \in SPEX(n, F_k)$ and $G' \in EX(n, F_k)$. Then $\rho' \leqslant \rho$, and so $\gamma > 0$. It follows that $Y^T(\rho'-\rho)X = \sum_{u_1u_2\in\overline{E}}\gamma > 0$, and thus $\rho' > \rho$, a contradiction. Therefore, B^{**} is a clique of H. This completes the proof. $\square$

4 The only graph in $SPEX(n, F_k)$

By Theorems 12 and 14, H is uniquely determined up to isomorphism. Let $e(H)$ be the number of edges of the subgraph H. Recall the known fact that $||S|-|T|| \leqslant 1$ and the assumption that H is embedded in S. To obtain the uniqueness of the graphs in $SPEX(n, F_k)$, we only need to prove $|S| \leqslant |T|$.

Lemma 20. *Regardless of parity of k, we have*

$$\rho^2 < |S||T| + 2e(H)\Big(\frac{|T|}{\rho}+1\Big).$$

Proof. For each $v \in T$, we have $\rho x_v = X_S$ and $\rho^2 x_v = \rho X_S = \sum\limits_{u\in S}\rho x_u$. Note that

$$\rho x_u = \begin{cases} X_T, & u \in S\setminus V(H),\\ X_T + \sum\limits_{w\in N_H(u)} x_w, & u \in V(H).\end{cases} \tag{19}$$

It follows that

$$\rho^2 x_v = \sum_{u\in S} X_T + \sum_{u\in V(H)}\sum_{w\in V_H(u)} x_w \leqslant |S|X_T + 2e(H)x_{\max V(H)}$$

for each $v \in T$. By (1), we know that $x_{\max V(H)} \leqslant \frac{X_T}{\rho-k+1} < \frac{X_T}{\rho-k}$, and hence $\rho^2 x_v \leqslant \Big(|S| + \frac{2e(H)}{\rho-k}\Big)X_T$. Summing this inequality for all $v \in T$, we have

$$\rho^2 X_T \leqslant \Big(|S||T| + 2e(H)\frac{|T|}{\rho - k}\Big)X_T. \tag{20}$$

Recall that $\rho^2 \geqslant |S||T| = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. Hence for sufficiently large n, we have

$$\rho^2 - k\rho - k|T| = \Big(\frac{\rho^2}{2} - k\rho\Big) + \Big(\frac{\rho^2}{2} - k|T|\Big) \geqslant \rho\Big(\frac{\rho}{2} - k\Big) + |T|\Big(\frac{|S|}{2} - k\Big) > 0.$$

Solving $\rho^2 - k\rho - k|T| > 0$, we obtain $\frac{|T|}{\rho-k} < \frac{|T|}{\rho} + 1$. Combining (20), we have $\rho^2 < |S||T| + 2e(H)\Big(\frac{|T|}{\rho} + 1\Big)$, as desired. □

Theorem 21. *Regardless of parity of k, we have $|S| \leqslant |T|$.*

Proof. Recall that $|S| + |T| = n$ and $||S| - |T|| \leqslant 1$. Suppose, to the contrary, that $|S| \geqslant |T| + 1$. Then $|S| = |T| + 1 = \frac{n+1}{2}$. Select a vertex $v_0 \in S \setminus V(H)$, and define $G' = G - \{v_0 v : v \in T\} + \{v_0 u : u \in S \setminus \{v_0\}\}$. Then $G' \in EX(n, F_k)$, and so $\rho(G') \leqslant \rho(G)$. Let $\rho' = \rho(G')$ and $Y = (y_1, \ldots, y_n)^T$ be the Perron vector of G'. Then we have $\rho x_{v_0} = X_T$ and $y_v = y_{v_0}$ for each $v \in T$. Hence

$$x_{v_0} = \frac{X_T}{\rho}, \qquad y_{v_0} = \frac{Y_T}{|T|}. \tag{21}$$

Thus, $Y_T + y_{v_0} = \frac{|T|+1}{|T|} Y_T = \frac{|S|}{|T|} Y_T$. Note that

$$\rho' y_u = \begin{cases} Y_T + y_{v_0}, & u \in S \setminus (V(H) \cup \{v_0\}), \\ Y_T + y_{v_0} + \sum\limits_{w \in N_H(u)} y_w, & u \in V(H). \end{cases} \tag{22}$$

Moveover, for each $u \in V(H)$, we have

$$\rho \sum_{w \in N_H(u)} x_w = \sum_{w \in N_H(u)} \rho x_w \geqslant d_H(u) X_T, \tag{23}$$

and

$$\rho' \sum_{w \in N_H(u)} y_w = \sum_{w \in N_H(u)} \rho' y_w \geqslant \sum_{w \in N_H(u)} (Y_T + y_{v_0}) = d_H(u) \frac{|S|}{|T|} Y_T. \tag{24}$$

Combining (19) and (23), we obtain that

$$\sum_{u \in S \setminus \{v_0\}} \rho x_u = (|S| - 1) X_T + \sum_{u \in V(H)} \sum_{w \in N_H(u)} x_w \geqslant |T| X_T + \frac{2e(H)}{\rho} X_T. \tag{25}$$

Note that $(|S|-1)(Y_T+y_{v_0}) = |T|\frac{|S|}{|T|}Y_T = |S|Y_T$. By (22) and (24), we have

$$\sum_{u\in S\setminus\{v_0\}} \rho' y_u = (|S|-1)(Y_T+y_{v_0}) + \sum_{u\in V(H)}\sum_{w\in N_H(u)} y_w \geqslant |S|Y_T + \frac{2e(H)}{\rho'}\frac{|S|}{|T|}Y_T. \quad (26)$$

Furthermore, it follows from (25) and (26) that

$$\sum_{u\in S\setminus\{v_0\}} x_u \geqslant \Big(\frac{|T|}{\rho} + \frac{2e(H)}{\rho^2}\Big)X_T, \quad (27)$$

and

$$\sum_{u\in S\setminus\{v_0\}} y_u \geqslant \Big(\frac{|S|}{\rho'} + \frac{2e(H)}{\rho'^2}\Big)Y_T \geqslant \Big(\frac{|S|}{\rho} + \frac{2e(H)}{\rho^2}\Big)Y_T. \quad (28)$$

Observe that

$$\begin{aligned} Y^T(\rho'-\rho)X &= Y^T(A(G')-A(G))X \\ &= \sum_{uv\in E(G')}(x_uy_v+x_vy_u) - \sum_{uv\in E(G)}(x_uy_v+x_vy_u). \\ &= \sum_{u\in S\setminus\{v_0\}}(x_{v_0}y_u + x_uy_{v_0}) - \sum_{v\in T}(x_{v_0}y_v + x_vy_{v_0}). \end{aligned}$$

Clearly, $\sum_{v\in T}(x_{v_0}y_v + x_vy_{v_0}) = x_{v_0}Y_T + y_{v_0}X_T$; moreover, by (27) and (28), we have

$$\sum_{u\in S\setminus\{v_0\}}(x_{v_0}y_u + x_uy_{v_0}) \geqslant x_{v_0}\Big(\frac{|S|}{\rho} + \frac{2e(H)}{\rho^2}\Big)Y_T + y_{v_0}\Big(\frac{|T|}{\rho} + \frac{2e(H)}{\rho^2}\Big)X_T.$$

Note that $x_{v_0} = \frac{X_T}{\rho}$ and $y_{v_0} = \frac{Y_T}{|T|}$ by (21). Therefore,

$$\begin{aligned} Y^T(\rho'-\rho)X &\geqslant \frac{X_T}{\rho}\Big(\frac{|S|}{\rho} + \frac{2e(H)}{\rho^2} - 1\Big)Y_T + \frac{Y_T}{|T|}\Big(\frac{|T|}{\rho} + \frac{2e(H)}{\rho^2} - 1\Big)X_T. \\ &= \frac{X_TY_T}{\rho^2|T|}\Big(|S||T| + 2e(H)\Big(\frac{|T|}{\rho}+1\Big) - \rho^2\Big). \end{aligned}$$

By Lemma 20, $\rho' > \rho$, a contradiction. The proof is completed. □

Combining Theorem 21, Theorems 12 and 14, we complete the proof of Theorem 4. The only graph in $SPEX(n, F_k)$ is determined for every fixed positive integer $k \geqslant 2$ and sufficiently large n.

5 Concluding remarks

Let $\mathcal{G}_{n,k}$ be the family of connected irregular graphs on n vertices with given maximum degree k. Define $\lambda = \max\{\rho(G) : G \in \mathcal{G}_{n,k}\}$. Until now, the value of λ is still unknown. In [16], Liu and Li proposed the following conjecture.

Conjecture 22 ([16])**.** Let $3 \leqslant k \leqslant n-2$ and G be a graph attaining the maximum spectral radius among all connected non-regular graphs of order n with fixed maximum degree k. Then G is a nearly k-regular graph for odd nk, and a graph with degree sequence $(k, \dots, k, k-2)$ for even nk.

Liu [17] has just confirmed Conjecture 22 for $k \in \{3, 4\}$ by determining the unique extremal graph respectively. For general k, the conjecture is still open. We hope that our method to characterize a nearly $\frac{n-1}{2}$-regular graph will be helpful for studying the above conjecture. We also expect a characterization of extremal nearly k-regular graphs on n vertices with maximum spectral radius.

To end this paper, we would like to introduce a recent conjecture due to Cioabă, Desai and Tait [7]. This extends a spectral color critical edge theorem of Nikiforov ([20]).

Conjecture 23 ([7])**.** Let F be any graph such that the graphs in $EX(n, F)$ are Turán graphs plus $O(1)$ edges. Then $SPEX(n, F) \subseteq EX(n, F)$ for sufficiently large n.

We are happy to see that Conjecture 23 has just been solved by Wang, Kang and Xue (see [28]). We further wonder whether there exists only one graph in $SPEX(n, F)$ for every F satisfying the condition. Nikiforov's result (see Theorem 2 in [20]; see also a direct version in [31]) implies that the extremal graph is unique for every graph F with a color critical edge, when the $O(1)$ is replaced by 0. Theorems 3 and 4 give a new support.

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