A Type B Analogue of the Category of Finite Sets with Surjections

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Abstract

We define a type B analogue of the category of finite sets with surjections, and we study the representation theory of this category. We show that the opposite category is quasi-Gröbner, which implies that submodules of finitely generated modules are again finitely generated. We prove that the generating functions of finitely generated modules have certain prescribed poles, and we obtain restrictions on the representations of type B Coxeter groups that can appear in such modules. Our main example is a module that categorifies the degree i Kazhdan–Lusztig coefficients of type B Coxeter arrangements.

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1 Introduction

Let FS_A be the category whose objects are nonempty finite sets and whose morphisms are surjective maps. The A in the subscript is there to call attention to the fact that this is a "type A" structure. More concretely, for any positive integer n, the automorphism group of the object $[n] = \{1, \ldots, n\}$ is the Coxeter group type A_{n-1}, and the set of equivalence classes of morphisms with source $[n]$ may be identified with the set of flats of the Coxeter hyperplane arrangement of type A_n (Example 23). Our aim is to define and study a "type B" analogue of this category, which we call FS_B.

We begin with the definition. An object of FS_B is a pair (E, σ), where E is a finite set and $\sigma : E \to E$ is an involution with a unique fixed point. A morphism from (E_1, σ_1) to (E_2, σ_2) is a surjective map $\varphi : E_1 \to E_2$ with $\varphi \circ \sigma_1 = \sigma_2 \circ \varphi$. For any natural number

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n, we write $[-n, n]$ to denote the object given by the set of integers between $-n$ and n (inclusive) and the involution $k \mapsto -k$; every object of FS_B is isomorphic to $[-n, n]$ for some $n \in \mathbb{N}$. The automorphism group W_n of the object $[-n, n]$ is the Coxeter group of type B_n, and the set of equivalence classes of morphisms with source $[-n, n]$ may be identified with the set of flats of the Coxeter hyperplane arrangement of type B_n (Example 23).

Remark 1. A more naive definition of FS_B would be to take finite sets with free involutions and equivariant maps. This category would have the right automorphism groups, but it would not have the same relationship with flats of the Coxeter hyperplane arrangements of type B. This distinction is not relevant when one studies the type B analogue of finite sets with *injections* [11], since any equivariant injection would have to preserve the fixed point. The same comment applies to the category FS_{S_2} studied in [9] and [10].

Remark 2. It is natural to ask why we do not also introduce and study a "type D" analogue of this category. The brief answer is that the classes of Coxeter arrangements of types A and B are closed under contraction (Examples 23 and 24), but the analogous statement is false in type D. This property is crucial to the examples that we consider in this paper.

For the remainder of the introduction, we describe the results for FS_A and FS_B in parallel for comparison. All results that we state for FS_A appear in either [9] or [8].

1.1 Finiteness

The first half of this paper is devoted to applying the Sam–Snowden Gröbner theory of combinatorial categories [9] to the opposite category $\mathrm{FS}_B^{\mathrm{op}}$. More concretely, we fix a left Noetherian ring k and an essentially small category C (which will always be either FS_A or FS_B) and study the category $\mathrm{Rep}_k(\mathrm{C}^{\mathrm{op}})$ of contravariant functors from C to the category of left k-modules. Such a functor is called an C^{op}-module over k. Given an object x, the **principal projective** $P_x \in \mathrm{Rep}_k(\mathrm{C}^{\mathrm{op}})$ is the module that assigns to an object y the free k-module with basis $\mathrm{Hom}_{\mathrm{C}}(y, x)$, with maps defined on basis elements by composition. A module M is called **finitely generated** if there exists a finite set of objects $x_1, \ldots, x_r$ and a surjective map from $\oplus_i P_{x_i}$ to M. The following theorem of Sam and Snowden says that finitely generated $\mathrm{FS}_A^{\mathrm{op}}$-modules form an Abelian category [9, Theorem 8.1.2].

Theorem 3. *Any submodule of a finitely generated* $\mathrm{FS}_A^{\mathrm{op}}$*-module over* k *is itself finitely generated.*

We prove here the analogous theorem for FS_B.

Theorem 4. *Any submodule of a finitely generated* $\mathrm{FS}_B^{\mathrm{op}}$*-module over* k *is itself finitely generated.*

An $\mathrm{FS}_A^{\mathrm{op}}$-module M is called **finitely generated in degree $\leqslant d$** if the generating objects can all be taken to be sets of cardinality at most d. Similarly, an FS_B-module N is called **finitely generated in degree $\leqslant d$** if the generating objects can all be taken to have at most d free orbits; equivalently, they can all be taken to be objects of the form $[-n, n]$ with $n \leqslant d$. A module over either category is called **d-small** if it is isomorphic to a subquotient of a module that is finitely generated in degree $\leqslant d$. Theorems 3 and 4 immediately implies that a d-small object is itself finitely generated, though the degree of generation might be much larger than d.

Borrowing terminology from [5] and [6], we call a module **d-smallish** if it admits a filtration whose associated graded module is d-small. The motivation for this definition is that, if we have a spectral sequence converging to N for which the modules on the E_1-page are all d-small, the same will necessarily be true for the E_∞-page, which is isomorphic to the associated graded module of N with respect to some filtration, and N is therefore d-smallish. It is easy to prove that a d-smallish module is finitely generated [5, Proposition 2.14]. We do not know whether or not a d-smallish module must be d-small.

1.2 Growth

Fix a field k of characteristic zero. If $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ is a partition of n, we write V_λ to denote the corresponding irreducible representation of S_n over k. If λ and μ are partitions with $|\lambda| + |\mu| = n$, we write $V_{\lambda,\mu}$ to denote the corresponding irreducible representation of W_n over k.

For an $\mathrm{FS}_A^{\mathrm{op}}$-module M and a positive integer n, we write $M[n]$ to denote the S_n-representation $M([n])$, and we define the generating function

$$H_A(M; t) := \sum_{n=1}^{\infty} t^n \dim M[n].$$

If M is d-smallish, we define the limit

$$r_A^d(M) := \lim_{n \to \infty} \frac{\dim M[n]}{d^n},$$

which we will show always exists. The following theorem was proved in [8, Theorem 4.1].

Theorem 5. *Let M be a d-smallish $\mathrm{FS}_A^{\mathrm{op}}$-module.*

1. *The generating function $H_A(M; t)$ is a rational function whose poles are contained in the set $\{1/j \mid 1 \leqslant j \leqslant d\}$.*
2. *The limit $r_A^d(M)$ exists. Equivalently, $H_A(M; t)$ has at worst a simple pole at $1/d$, and $r_A^d(M)$ is the residue.*
3. *If $|\lambda| = n$ and $\mathrm{Hom}_{S_n}\big(V_\lambda, M[n]\big) \neq 0$, then $\ell(\lambda) \leqslant d$.*

We now state the type B analogue of Theorem 5. For an $\mathrm{FS}_B^{\mathrm{op}}$-module N and a nonnegative integer n, we write $N[-n, n]$ to denote the W_n-representation $N([-n, n])$, and we define the generating function

$$H_B(N; t) := \sum_{n=0}^{\infty} t^n \dim N[-n, n].$$

If N is d-smallish, we define the limit

$$r_B^d(N) := \lim_{n\to\infty} \frac{\dim N[-n,n]}{(2d+1)^n},$$

which we will show always exists.

Theorem 6. *Let N be a d-smallish $\mathrm{FS}_B^{\mathrm{op}}$-module.*

1. *The generating function $H_B(M;t)$ is a rational function whose poles are contained in the set $\{1/j \mid 1 \leqslant j \leqslant 2d+1\}$.*
2. *The limit $r_B^d(N)$ exists. Equivalently, $H_B(N;t)$ has at worst a simple pole at $1/(2d+1)$, and $r_B^d(N)$ is the residue.*
3. *If $|\lambda|+|\mu| = n$ and $\mathrm{Hom}_{W_n}\big(V_{\lambda,\mu}, N[-n,n]\big) \neq 0$, then $\ell(\lambda) \leqslant d+1$ and $\ell(\mu) \leqslant d$.*

1.3 Examples

For any nonempty finite set E, we define in Example 21 a hyperplane arrangement $\mathcal{A}_E$ with the property that $\mathcal{A}_{[n]}$ is the Coxeter arrangement of type A_n. Similarly, for any object (E,σ) of FS_B, we define in Example 22 a hyperplane arrangement $\mathcal{A}_{(E,\sigma)}$ with the property that $\mathcal{A}_{[-n,n]}$ is the Coxeter arrangement of type B_n.

In Section 5, we define an FS_A-module S_A^i that takes E to the degree i part of the Orlik–Solomon algebra of $\mathcal{A}_E$; by taking the linear dual, we obtain an $\mathrm{FS}_A^{\mathrm{op}}$-module $(S_A^i)^*$. Similarly, we define an FS_B-module S_B^i that takes (E,σ) to the degree i part of the Orlik–Solomon algebra of $\mathcal{A}_{(E,\sigma)}$ and the dual $\mathrm{FS}_B^{\mathrm{op}}$-module $(S_B^i)^*$. The following proposition was proved in [8, Proposition 5.1].

Proposition 7. *The $\mathrm{FS}_A^{\mathrm{op}}$-module $(S_A^0)^*$ is 1-small. For all $i>0$, the $\mathrm{FS}_A^{\mathrm{op}}$-module $(S_A^i)^*$ is $2i$-small, and*

$$r_A^{2i}\big((S_A^i)^*\big) = 0.$$

Here we prove the following type B analogue of Proposition 7.

Proposition 8. *The $\mathrm{FS}_B^{\mathrm{op}}$-module $(S_B^0)^*$ is 0-small. For all $i>0$, the $\mathrm{FS}_B^{\mathrm{op}}$-module$(S_B^i)^*$ is $(2i-1)$-small, and*

$$r_B^{2i-1}\big((S_B^i)^*\big) = 0.$$

Remark 9. The smallness shift between Propositions 7 and 8 (which we will see again in Theorems 10 and 11) can be blamed on the fact that the object $[n]$ of FS_A corresponds to the Coxeter group and Coxeter arrangement of type A_{n-1}, while the objet $[-n,n]$ of FS_B corresponds to the Coxeter group and Coxeter arrangement of type B_n. It is also related to the fact that $[1]$ is the terminal object of FS_A while $[0,0]$ is the terminal object of FS_B.

For any hyperplane arrangement $\mathcal{A}$, one may define a singular algebraic variety $X_{\mathcal{A}}$ called the **reciprocal plane** of $\mathcal{A}$. This variety has vanishing intersection cohomology in odd degree, and the even degree intersection cohomology Poincaré polynomial coincides with the **Kazhdan–Lusztig polynomial** of the associated matroid [1, Proposition 3.12]. In Section 7, we define an FS_A-module D^i_A that takes a nonempty finite set E to $IH^{2i}(X_{\mathcal{A}_E})$ and an FS_B-module D^i_B that takes an object (E, σ) to $IH^{2i}\big(X_{\mathcal{A}_{(E,\sigma)}}\big)$. One can think of D^i_A and D^i_B as categorifications of the degree i Kazhdan–Lusztig coefficients of Coxeter arrangements in types A and B, respectively. The following theorem was proved in [8, Theorem 6.1].

Theorem 10. *For any $i > 0$, the $\mathrm{FS}_A^{\mathrm{op}}$-module $(D^i_A)^*$ is $2i$-smallish,[1] and we have*

$$r_{2i}\big((D^i_A)^*\big) = \frac{\dim D^{i-1}_A[2i]}{|S_{2i}|} = \frac{\dim D^{i-1}_A[2i]}{(2i)!}.$$

Here we prove the following type B analogue of Theorem 10.

Theorem 11. *For any $i > 0$, the $\mathrm{FS}_B^{\mathrm{op}}$-module $(D^i_B)^*$ is $(2i-1)$-smallish, and we have*

$$r_{2i-1}\big((D^i_B)^*\big) = \frac{\dim D^{i-1}_B[1-2i, 2i-1]}{|W_{2i-1}|} = \frac{\dim D^{i-1}_B[1-2i, 2i-1]}{2^{2i-1}(2i-1)!}.$$

2 Gröbner and O-lingual categories

We begin by reviewing the relevant machinery from [9] that we will need to prove Theorems 4 and 6. Let C be an essentially small category. Given morphisms $\varphi : x \to y$ and $\varphi' : x \to y'$, we say $\varphi \leqslant \varphi'$ if there exists a morphism $\psi : y \to y'$ with $\varphi' = \psi \circ \varphi$. If $\varphi \leqslant \varphi' \leqslant \varphi$, then φ and φ' are said to be **equivalent**. The poset of equivalence classes of morphisms out of x is denoted $|\,\mathrm{C}_x\,|$.

We say that C is **directed** if it has no endomorphisms other than the identity maps. We say that C has **property (G1)** if, for every object x, there exists a well order $\prec$ on C_x that with the property that $\varphi \prec \varphi' \Rightarrow \psi \circ \varphi \prec \psi \circ \varphi'$ whenever both compositions make sense. We say that C has **property (G2)** if, for every object x, the poset $|\,\mathrm{C}_x\,|$ is **Noetherian**, meaning that every ideal (upwardly closed subset) has only finitely many minimal elements. A directed category with properties (G1) and (G2) is called **Gröbner**.

A functor $\Phi : \mathrm{C} \to \mathrm{C}'$ has **property (F)** if, for any object x of C', there exist finitely many objects $y_1, \ldots, y_s$ of C and morphisms $\varphi_i : x \to \Phi(y_i)$ such that for any object y of C and any morphism $\varphi : x \to \Phi(y)$ in C, there exists a morphism $\psi : y_i \to y$ in C with $\varphi = \Phi(\psi) \circ \varphi_i$. This definition is engineered precisely so that the following result will hold [9, Propositions 3.2.3].

Proposition 12. *Suppose that $\Phi : \mathrm{C} \to \mathrm{C}'$ has property (F). Suppose that $N \in \mathrm{Rep}_k(\mathrm{C}')$ is finitely generated, with generating objects $x_1, \ldots, x_r$. For each $1 \leqslant i \leqslant r$, choose objects*

[1] In the published version of the paper, we claimed that the module was $2i$-small, but we only proved that it is $2i$-smallish. This mistake was corrected in the arXiv version.

$y_{i1}, \ldots, y_{is_i}$ of C *corresponding to x_i as in the definition of property (F). Then the module $\Phi^* N \in \mathrm{Rep}_k(\mathrm{C})$ is finitely generated, with generating objects $\{y_{ij} \mid 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s_i\}$.*

The category C′ is called **quasi-Gröbner** if there exists a Gröbner category C and an essentially surjective functor $\Phi : \mathrm{C} \to \mathrm{C}'$ with property (F). In this case, the category C is said to be a **Gröbner cover** of C′. Sam and Snowden use Proposition 12 to prove the following result [9, Theorem 4.3.2].

Theorem 13. *If* C′ *is quasi-Gröbner and k is a left Noetherian ring, then any submodule of a finitely generated* C′*-module over k is itself finitely generated.*

Given a finite set Σ, we denote the set of words (finite sequences) in Σ by $\Sigma^\star$. A **language** on Σ is a subset of $\Sigma^\star$. Given two languages $\mathcal{L}_1$ and $\mathcal{L}_2$ on Σ, their **concatenation** is the set of sequences formed by concatenating a word in $\mathcal{L}_1$ and a word in $\mathcal{L}_2$. The set of **ordered languages** on Σ is the smallest collection of languages on Σ that contains singleton languages and languages of the form $\Pi^\star$ for $\Pi \subset \Sigma$ and is closed under finite unions and concatenations.

A **norm** on C is a function ν from the set of isomorphism classes of objects of C to the natural numbers. The normed category C is said to be **O-lingual** if, for every object x of C, there exists a finite set Σ_x and an inclusion $\iota_x : |\,\mathrm{C}_x\,| \to \Sigma_x^\star$ satisfying the following two properties:

- For any $\varphi : x \to y$, $\iota_x(\varphi)$ is a word of length $\nu(y)$.
- For any ideal $I \subset |\,\mathrm{C}_x\,|$, $\iota_x(I) \subset \Sigma_x^\star$ is an ordered language.

The final result that we will need is the following, which is proved in [9, Corollary 5.3.8 and Theorem 6.3.2] (see also Corollary 8.1.4).

Theorem 14. *Suppose that* C *is endowed with a norm and an* O*-lingual structure, k is a field, and N is an* C*-module over k that is generated by the objects $x_1, \ldots, x_r$. Let $m := \max\{|\Sigma_{x_i}|\}$ and*

$$H_{\mathrm{C}}(N;t) := \sum_x t^{\nu(x)} \dim N(x),$$

where the sum is over isomorphism classes of objects. Then $H_{\mathrm{C}}(N;t)$ is a rational function whose poles are contained in the set $\{1/j \mid 1 \leqslant j \leqslant m\}$.

3 Ordered surjections

The purpose of this section is to prove theorems 4 and 6. We proceed by constructing a category OS_B such that $\mathrm{OS}_B^{\mathrm{op}}$ is an O-lingual Gröbner cover of $\mathrm{FS}_B^{\mathrm{op}}$. The objects of OS_B will be pairs (E, σ), where E is a totally ordered finite set and σ is an order-reversing involution with a unique fixed point. We will denote the fixed point by 0, and we will write $-e := \sigma(e)$ for any $e \in E$. Let

$$E^+ := \{e \in E \mid e > 0\} E^- := \{e \in E \mid e < 0\},$$

so that

$$E = E^- \sqcup \{0\} \sqcup E^+.$$

For any element $e \in E$, we will write $|e| := \max\{\pm e\}$. For any subset $D \subset E$, we will write $\operatorname{init} D := \min\{|e| \mid e \in S\}$. A morphism from (E_1, σ_1) to (E_2, σ_2) in OS_B will be a surjective map $\varphi : E_1 \to E_2$ with $\varphi \circ \sigma_1 = \sigma_2 \circ \varphi$ along with the following two additional properties:

(i) For all $e \in E_2^+$, $\operatorname{init} \varphi^{-1}(e) \in \varphi^{-1}(e)$.

(ii) For all $e < f \in E_2^+$, $\operatorname{init} \varphi^{-1}(e) < \operatorname{init} \varphi^{-1}(f)$.

The following lemma says that composition in OS_B is well defined.

Lemma 15. *If the maps $\varphi : (E_1, \sigma_1) \to (E_2, \sigma_2)$ and $\psi : (E_2, \sigma_2) \to (E_3, \sigma_3)$ each have properties (i) and (ii), then so does the composition $\psi \circ \varphi : (E_1, \sigma_1) \to (E_3, \sigma_3)$.*

Proof. It will suffice to check that, for all $e_3 \in E_3^+$, the elements

$$e_1 := \operatorname{init} \varphi^{-1}\big(\operatorname{init} \psi^{-1}(e_3)\big) \; f_1 := \operatorname{init}(\psi \circ \varphi)^{-1}(e_3)$$

coincide. Let $e_2 := \varphi(e_1)$ and $f_2 := \varphi(f_1)$. Property (i) for φ tells us that $e_2 = \operatorname{init} \psi^{-1}(e_3)$ and property (i) for ψ tells us that $\psi(e_2) = e_3$. Thus $(\psi \circ \varphi)(e_1) = e_3$, and therefore

$$f_1 = \operatorname{init}(\psi \circ \varphi)^{-1}(e_3) \leqslant e_1.$$

We have $\psi(f_2) = (\psi \circ \varphi)(f_1) \in \{\pm e_3\}$, therefore

$$e_2 = \operatorname{init} \psi^{-1}(e_3) = \operatorname{init} \psi^{-1}(\pm e_3) \leqslant |f_2|.$$

Applying property (ii) for φ, we find that

$$e_1 = \operatorname{init} \varphi^{-1}(e_2) \leqslant \operatorname{init} \varphi^{-1}(|f_2|) = \operatorname{init} \varphi^{-1}(f_2) \leqslant f_1.$$

This completes the proof that $e_1 = f_1$. □

Every object of OS_B is isomorphic to $[-n, n]$ for some natural number n, and that there are no nontrivial endomorphisms. In particular, OS_B is essentially small and directed. We define a norm ν on OS_B by taking $\nu(E, \sigma)$ to be equal to the number of free orbits of σ on E, so that $\nu([-n, n]) = n$. Let $\Phi : \mathrm{OS}_B^{\mathrm{op}} \to \mathrm{FS}_B^{\mathrm{op}}$ be the forgetful functor.

Lemma 16. *The functor $\Phi : \mathrm{OS}_B^{\mathrm{op}} \to \mathrm{FS}_B^{\mathrm{op}}$ has property (F).*

Proof. Unpacking the definition of property (F), we see that is is sufficient to show that, for any morphism $\varphi : (E_1, \sigma_1) \to (E_2, \sigma_2)$ in FS_B and any total order of E_2 compatible with σ_2, there is a total order of E_1 compatible with σ_1 such that φ is a morphism in OS_B. Indeed, it is clear that we can choose a total order on E_1, compatible with σ_1, with the even stronger condition that φ is weakly order preserving. □

For each object (E, σ) of OS_B, we define a poset structure on $E^\star$ by putting $e_1 \cdots e_m \leqslant f_1 \cdots f_n$ if there is a strictly increasing map $\theta : [m] \to [n]$ satisfying the following two conditions:

- For all $i \in [m]$, $e_i = f_{\theta(i)}$.
- For all $j \in [n]$, there exists $i \in [m]$ such that $\theta(i) \leqslant j$ and $f_{\theta(i)} \in \{\pm f_j\}$.

In plain English, we require that $e_1 \cdots e_m$ is a subword of $f_1 \cdots f_n$, and that this subword contains the first occurrence of every σ orbit appearing in $f_1 \cdots f_n$.

Proposition 17. *For any object, (E, σ) of OS_B, the poset $E^\star$ is Noetherian.*

Proof. Suppose not, and choose a sequence $w_1, w_2, w_3 \ldots$ of words such that $i < j \Rightarrow w_i \nleqslant w_j$. We may assume that our sequence is **minimal** in the sense that, for each i, the length of w_i is minimal among all such sequences that begin $w_1, \ldots, w_{i-1}$. Given a word w and an element $e \in E$, we say that e is **exceptional** in w if either e or $-e$ appears exactly once in w (and the other, if different, does not appear at all). If w has a non-exceptional element, we define $m(w)$ to be the number of letters appearing to the right of the last non-exceptional element.

There are only finitely many words of each length, thus we may choose a natural number i_0 such that, for all $i \geqslant i_0$, the length of w_i is strictly greater than $\nu(E, \sigma) + 1$. It follows that, for all $i \geqslant i_0$, w_i has a non-exceptional element. There are only finitely many possible values for $m(w_i)$ and only finitely many elements in E, so we may find a natural number m and an element $e \in E$ and pass to a subsequence $w_{i_1}, w_{i_2}, w_{i_3}, \ldots$ such that $m(w_{i_j}) = m$ for all j and the last non-exceptional element appearing in w_{i_j} is e for all j.

Let v_j be the word obtained from w_{i_j} by deleting the last appearance of e, and note that $v_j < w_{i_j}$ for all j. Consider the sequence $w_1, w_2, \ldots, w_{i_1-1}, v_1, v_2, \ldots$. By minimality of our original sequence, this sequence must contain a pair of elements with the first less than or equal to the second. We know that this cannot happen in the first $i_1 - 1$ terms, and we also cannot have $w_k \leqslant v_j$ for some $k < i_1$ and $j \geqslant 1$, because this would imply that $w_k < w_{i_j}$. Finally, there cannot exist $j < k$ such that $v_j \leqslant v_k$, because this would imply that $w_{i_j} < w_{i_k}$. Thus we have arrived at a contradiction. □

Corollary 18. *For any object, (E, σ) of OS_B, every ideal in the poset $E^\star$ is an ordered language.*

Proof. If $w = e_1 \cdots e_n \in E^\star$, we define I_w to be the principal ideal consisting of all words greater than or equal to w. By Proposition 17, every ideal in $E^\star$ is a finite union of principal ideals, so it is sufficient to show that I_w is an ordered language. For all $i \in [n]$, let $\Pi_i = \{\pm e_1, \ldots, \pm e_i\}$. Then

$$I_w = e_1 \Pi_1^\star e_2 \Pi_2^\star \cdots e_n \Pi_n^\star$$

is a concatenation of singleton languages and languages of the form $\Pi_i^\star$, so it is ordered. □

Consider the norm on $\mathrm{OS}_B^{\mathrm{op}}$ that takes (E,σ) to the number of free orbits in E; in other words, the object $[-n,n]$ has norm n. Given a morphism $\varphi : [-n,n] \to (E,\sigma)$ in OS_B, let

$$\iota_{(E,\sigma)}(\varphi) := \varphi(1)\cdots\varphi(n) \in E^\star.$$

Since every object of OS_B is uniquely isomorphic to $[-n,n]$ for some n, this defines a map

$$\iota_{(E,\sigma)} : |(\mathrm{OS}_B^{\mathrm{op}})_{(E,\sigma)}| \to E^\star.$$

Lemma 19. *Let (E,σ) be an object of OS_B.*

1. *The map $\iota_{(E,\sigma)}$ is strictly order preserving. That is, $\varphi < \varphi' \in |(\mathrm{OS}_B^{\mathrm{op}})_{(E,\sigma)}|$ if and only if $\iota_{(E,\sigma)}(\varphi) < \iota_{(E,\sigma)}(\varphi') \in E^\star$.*

2. *The image of an ideal in $|(\mathrm{OS}_B^{\mathrm{op}})_{(E,\sigma)}|$ is an ideal in $E^\star$.*

Proof. We begin with statement (1). Suppose that $\varphi : [-m,m] \to (E,\sigma)$, $\varphi' : [-n,n] \to (E,\sigma)$, and $\varphi < \varphi'$. Then there exists $\psi : [-n,n] \to [-m,m]$ such that $\varphi' = \varphi \circ \psi$. Define a map $\theta : [m] \to [n]$ by $\theta(i) := \operatorname{init} \psi^{-1}(i)$. Then θ exhibits the inequality $\iota_{(E,\sigma)}(\varphi) < \iota_{(E,\sigma)}(\varphi') \in E^\star$.

Conversely, suppose that $\iota_{(E,\sigma)}(\varphi) < \iota_{(E,\sigma)}(\varphi') \in E^\star$, and let $\theta : [m] \to [n]$ be the map that exhibits this inequality. By definition, for each $j \in [n]$, there exists an element $i \in [m]$ such that $\theta(i) \leqslant j$ and $\varphi'(\theta(i)) \in \{\pm\varphi'(j)\}$. Let i be the minimal such element. Define $\psi(j) = i$ if $\varphi'(\theta(i)) = \varphi'(j)$ and $-i$ if $\varphi'(\theta(i)) = -\varphi'(j)$. This extends uniquely to an OS_B morphism $\psi : [-n,n] \to [m,m]$ with $\varphi' = \varphi \circ \psi$, so $\varphi < \varphi'$.

For statement (2), we first observe that the image of $\iota_{[-n,n]}$ is equal to the ideal $I_{12\cdots n} \subset [-n,n]^\star$. Suppose that $I \subset |(\mathrm{OS}_B^{\mathrm{op}})_{[-n,n]}|$ is an ideal, $\varphi \in I$, and $w \geqslant \iota_{[-n,n]}(\varphi)$. Since the image of $\iota_{[-n,n]}$ is an ideal, we have $w = \iota_{[-n,n]}(\varphi')$ for some φ'. Statement (1) terlls us that $\varphi < \varphi'$, so $\varphi' \in I$ and $w \in \iota_{[-n,n]}(I)$. □

Proposition 20. *The category $\mathrm{OS}_B^{\mathrm{op}}$ is Gröbner, and* O*-lingual with respect to the maps $\iota_{(E,\sigma)}$.*

Proof. Property (G2) follows from Proposition 17 and Lemma 19(1). Property (G1) is proved by pulling back the lexicographic order from $E^\star$ to $|(\mathrm{OS}_B^{\mathrm{op}})_{(E,\sigma)}|$. This shows that $\mathrm{OS}_B^{\mathrm{op}}$ is Gröbner. The statement that $\mathrm{OS}_B^{\mathrm{op}}$ is O-lingual follows from Corollary 18 and Lemma 19(2). □

Proof of Theorem 4. This follows from Theorem 13, Lemma 16, and Proposition 20. □

Proof of Theorem 6. We begin by proving statement (1) for an $\mathrm{FS}_B^{\mathrm{op}}$-module N that is generated in degrees $\leqslant d$. By Proposition 12 and Lemma 16, the $\mathrm{OS}_B^{\mathrm{op}}$-module $\Phi^* N$ is also generated by objects of norm $\leqslant d$. Then Theorem 14 and Proposition 20 tell us that

$$H_B(N;t) = H_{\mathrm{FS}_B^{\mathrm{op}}}(N;t) = H_{\mathrm{OS}_B^{\mathrm{op}}}(\Phi^* N;t)$$

is a rational function with poles contained in the set $\{1/j \mid 1 \leqslant j \leqslant 2d+1\}$. Now suppose that N is d-small. By Theorem 4, there is some d' such that N is finitely generated in degrees $\leqslant d'$, so $H_B(N;t)$ is a rational function with poles contained in the set $\{1,\ldots,2d'+1\}$. However, the fact that N is d-small means that the dimension $\dim N[-n,n]$ can only grow as fast as the dimension of a module that is finitely generated in degree $\leqslant d$, therefore $H_B(N;t)$ cannot have a pole at $1/j$ when $j > 2d+1$. Finally, since passing to the associated graded of a filtration does not change the Hilbert series of a module, this proves statement (1) when N is d-smallish.

To prove statements (2) and (3), it is sufficient to check them for the principal projective $P_{[-d,d]}$. The dimension of $P_{[-d,d]}[-n,n]$ is equal to the number of equivariant surjections from $[-n,n]$ to $[-d,d]$. The total number of equivariant maps is n^{2d+1}, and when n is large, almost all equivariant maps are surjective, hence we have $r_B^d(P_{[-d,d]}) = 1$. Let φ be a morphism from $[-n,n]$ to $[-d,d]$, and consider the subgroup

$$W_\varphi \cong W_{|\varphi^{-1}(0)|} \times S_{|\varphi-1(1)|} \times \cdots \times S_{|\varphi^{-1}(d)|} \subset W_n$$

that stabilizes φ. Then the W_n representation $P_{[-d,d]}[-n,n]$ is isomorphic to

$$\bigoplus_{\varphi} \operatorname{Ind}_{W_\varphi}^{W_n}(\mathrm{triv}),$$

where the sum is over one representative of each W_n orbit in $\operatorname{Hom}_{\mathrm{FS}}\big([-n,n],[-d,d]\big)$. The fact that each one of these summands is a sum of representations of the form $V_{\lambda,\mu}$ with $\ell(\lambda) \leqslant d+1$ and $\ell(\mu) \leqslant d$ follows from induction on d using the type B Pieri rule [2, Section 6.1.9]. $\square$

4 Hyperplane arrangements

Let V be a finite dimensional vector space. A **hyperplane arrangement** in V is a finite set of codimension 1 linear subspaces of V. The following pair of examples will appear many times throughout this section.

Example 21. Given a nonempty finite set E and any element $e \in E$, let x_e be the e^{th} coordinate function on $\mathbb{C}^E$, and let $V_E \subset \mathbb{C}^E$ be the codimension 1 subspace consisting of vectors whose coordinates add to zero. For any unordered pair of distinct elements $e \neq f \in E$, consider the hyperplane

$$H_{ef} := \{v \in V_E \mid x_e(v) = x_f(v)\}.$$

Let

$$\mathcal{A}_E := \{H_{ef} \mid e \neq f \in E\}$$

be the corresponding hyperplane arrangement in V_E. When $E = [n]$, $\mathcal{A}_E$ can be identified with the Coxeter arrangement of type A_{n-1}, or equivalently the set of reflection hyperplanes for the Coxeter group S_n.

Example 22. For any object (E, σ) of FS_B, consider the vector space

$$V_{(E,\sigma)} := \left\{ v \in \mathbb{C}^E \mid x_e(v) + x_{\sigma(e)}(v) = 0 \text{ for all } e \in E \right\} \subset V_E \subset \mathbb{C}^E.$$

For each unordered pair $e \neq f \in E$, let

$$J_{ef} := V_{(E,\sigma)} \cap H_{ef} \subset V_{(E,\sigma)}.$$

Note that we have $J_{\sigma(e)\sigma(f)} = J_{ef}$ for all $e \neq f \in E$, and if $0 \in E$ is the unique fixed point, then $J_{e\sigma(e)} = J_{e0}$ for all $e \neq 0$. Let

$$\mathcal{A}_{(E,\sigma)} := \{ J_{ef} \mid e \neq f \in E \}$$

be the corresponding hyperplane arrangement in $V_{(E,\sigma)}$. When $(E, \sigma) = [-n, n]$, $\mathcal{A}_{(E,\sigma)}$ can be identified with the Coxeter arrangement of type B_n, or equivalently the set of reflection hyperplanes for the Coxeter group W_n.

Given a hyperplane arrangement $\mathcal{A}$ in V, a **flat** of $\mathcal{A}$ is a linear subspace $F \subset V$ obtained by intersecting some subset of the hyperplanes. The **contraction** of $\mathcal{A}$ at F is the hyperplane arrangement

$$\mathcal{A}^F := \{ F \cap H \mid F \not\subset H \in \mathcal{A} \}$$

in the vector space F. The **localization** of $\mathcal{A}$ at F is the hyperplane arrangement

$$\mathcal{A}_F := \{ H/F \mid F \subset H \in \mathcal{A} \}$$

in the vector space V/F. If $\mathcal{A}_1$ is a hyperplane arrangement in V_1 and $\mathcal{A}_2$ is a hyperplane arrangement in V_2, the **product** $\mathcal{A}_1 \times \mathcal{A}_2$ is defined to be the hyperplane arrangement in $V_1 \oplus V_2$ with hyperplanes

$$\{ H_1 \oplus V_2 \mid H_1 \in \mathcal{A}_1 \} \cup \{ V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2 \}.$$

Example 23. For any surjective map $\varphi : E_1 \to E_2$ of finite sets, we may define a flat

$$F_\varphi := \bigcap_{\substack{e \neq f \in E_1 \\ \varphi(e) = \varphi(f)}} H_{ef} \subset V_E$$

of the arrangement $\mathcal{A}_E$. Every flat of $\mathcal{A}_{E_1}$ is of this form, and if we have two surjections $\varphi : E_1 \to E_2$ and $\varphi' : E_1 \to E_2'$, then $F_\varphi = F_{\varphi'}$ if and only if there is a bijection $\psi : E_2 \to E_2'$ such that $\varphi' = \psi \circ \varphi$. The contraction of $\mathcal{A}_{E_1}$ at F_φ can be canonically identified with $\mathcal{A}_{E_2}$, and the localization of $\mathcal{A}_{E_1}$ at the flat F_φ can be canonically identified with the product

$$\prod_{e \in E_2} \mathcal{A}_{\varphi^{-1}(e)}.$$

Example 24. Given a morphism $\varphi : (E_1, \sigma_1) \to (E_2, \sigma_2)$ in FS_B, we may define a flat

$$G_\varphi := \bigcap_{\substack{e \neq f \in E_1 \\ \varphi(e) = \varphi(f)}} J_{ef} \subset V_{(E_1, \sigma_1)}$$

of the arrangement $\mathcal{A}_{(E_1,\sigma_1)}$. Every flat of $\mathcal{A}_{(E_1,\sigma_1)}$ is of this form, and if we have two morphisms $\varphi : (E_1, \sigma_1) \to (E_2, \sigma_2)$ and $\varphi' : (E_1, \sigma_1') \to (E_2, \sigma_2')$, then $G_\varphi = G_{\varphi'}$ if and only if there is an isomorphism $\psi : (E_2, \sigma_2) \to (E_2', \sigma_2')$ such that $\varphi' = \psi \circ \varphi$. The contraction of $\mathcal{A}_{(E_1,\sigma_1)}$ at G_φ can be canonically identified with $\mathcal{A}_{(E_2,\sigma_2)}$. To understand the localization, we first choose a decomposition

$$E_2 = P_2 \sqcup \{0\} \sqcup \sigma_2(P_2),$$

where $0 \in E_2$ is the unique fixed point. Then the localization of $\mathcal{A}_{(E_1,\sigma_1)}$ at the flat G_φ can be canonically identified with the product

$$\mathcal{A}_{(\varphi^{-1}(0),\sigma_1)} \times \prod_{e \in P_2} \mathcal{A}_{\varphi^{-1}(e)}.$$

Remark 25. If we want to avoid choosing a decomposition of E_2, we can replace the product over P_2 with a product over non-fixed σ_2-orbits, and replace the preimage of $e \in P_2$ with the set of σ_1-orbits in the preimage of the σ_2-orbit. This would be more canonical, but also more unwieldy to notate.

5 Orlik–Solomon algebras

Let $\mathcal{A}$ be a hyperplane arrangement. A set $\mathcal{D} \subset \mathcal{A}$ is called **dependent** if the codimension of its intersection is smaller than its cardinality (equivalently, if the corresponding set of normal vectors is linearly dependent). For any dependent set $\mathcal{D} = \{H_1, \ldots, H_k\} \subset \mathcal{A}$ of cardinality k, we define a class

$$\partial u_{\mathcal{D}} := \sum_{i=1}^{k} (-1)^i \prod_{j \neq i} u_{H_j}$$

in the exterior algebra $\Lambda_{\mathbb{C}}[u_H \mid H \in \mathcal{A}]$. Note that the element u_S as we have defined it depends on the ordering of the elements of S, but only up to sign. The **Orlik–Solomon algebra** $S(\mathcal{A})$[2] is defined as the quotient of $\Lambda_{\mathbb{C}}[u_H \mid H \in \mathcal{A}]$ by the ideal generated by ∂u_S for every dependent set $\mathcal{D}$. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two hyperplane arrangements, then

$$S(\mathcal{A}_1 \times \mathcal{A}_2) \cong S(\mathcal{A}_1) \otimes S(\mathcal{A}_2). \tag{1}$$

[2]It is typical to denote the Orlik–Solomon algebra either $OS(\mathcal{A})$ or $A(\mathcal{A})$, but we wish to avoid conflict with the notation for the category OS_B and with the use of the letter A for type A structures. So, with apologies to Peter Orlik, we are just using the letter S.

If F is a flat of $\mathcal{A}$, then there is a canonical map

$$S(\mathcal{A}) \to S(\mathcal{A}^F)$$

defined by sending u_H to $u_{F\cap H}$ if $F \not\subset H$ and to zero otherwise.

Remark 26. If V is a vector space over $\mathbb{C}$, then $S(\mathcal{A})$ is canonically isomorphic to the cohomology of the complement of $\mathcal{A}$ [4]. In this case, Equation (1) can be regarded as an application of the Künneth theorem. For a topological interpretation of the map from $S(\mathcal{A})$ to $S(\mathcal{A}^F)$, see [8, Section 3].

Fix a natural number i. By Example 23, we have an FS_A-module that assigns to a finite set E the vector space $S^i(\mathcal{A}_E)$, and to a surjection $\varphi : E_1 \to E_2$ the map

$$S^i(\mathcal{A}_{E_1}) \to S^i\big((\mathcal{A}_{E_1})^{F_\varphi}\big) \cong S^i(\mathcal{A}_{E_2})\,.$$

We denote this module by S^i_A, and we denote the dual $\mathrm{FS}_A^{\mathrm{op}}$-module by $(S^i_A)^*$. Similarly, by Example 24, we have an FS_B-module that assigns to an object (E,σ) the vector space $S^i\big(\mathcal{A}_{(E,\sigma)}\big)$, and to a morphism $\varphi : (E_1,\sigma_1) \to (E_2,\sigma_2)$ the map

$$S^i\big(\mathcal{A}_{(E_1,\sigma_1)}\big) \to S^i\big((\mathcal{A}_{(E_1,\sigma_1)})^{G_\varphi}\big) \cong S^i\big(\mathcal{A}_{(E_2,\sigma_2)}\big)\,.$$

We denote this module by S^i_B, and we denote the dual $\mathrm{FS}_B^{\mathrm{op}}$-module by $(S^i_B)^*$.

Proof of Proposition 8. We have $(S^0_B)^* \cong P_{[0,0]}$, so the first statement is trivial, and we may assume that $i > 0$. Since the Orlik–Solomon algebra is generated in degree 1, S^i_B is a quotient of $(S^1_B)^{\otimes i}$, and therefore $(S^i_B)^*$ is a submodule of $\big((S^1_B)^*\big)^{\otimes i}$. Thus it will suffice to show that, for any object of FS_B with at least $2i$ free orbits, every element of $(S^1_B)^*(E,\sigma)^{\otimes i}$ is a linear combination of pullbacks of classes along various maps to smaller objects.

Let $0 \in E$ denote the unique fixed point. The vector space $S^1_B(E,\sigma)$ is spanned by the elements u_{ef} for unordered pairs $e \neq f$ that are distinct from 0 (recall that we have $u_{e0} = u_{e\sigma(e)}$ for any $e \neq 0$). For such an unordered pair, let $v_{ef} \in S^1_B(E,\sigma)^*$ be the element that evaluates to 1 on $u_{ef} = u_{\sigma(e)\sigma(f)}$ and to 0 on all other generators. Then $(S^1_B)^*(E,\sigma)^{\otimes i}$ is spanned by classes of the form $v_{e_1f_1} \otimes \cdots \otimes v_{e_if_i}$.

Let

$$F := \{e_1, \sigma(e_1), f_1, \sigma(f_1), \ldots, e_i, \sigma(e_i), f_i, \sigma(f_i), 0\} \subset E,$$

so that (F,σ) is an object of FS_B with at most $2i$ free orbits. Define a morphism $\varphi : (E,\sigma) \to (F,\sigma)$ by fixing $F \subset E$ and sending $E \smallsetminus F$ to 0. Our hypothesis implies that the class $v_{e_1f_1} \otimes \cdots \otimes v_{e_if_i}$ is sent to itself by the map

$$\varphi^* : (S^1_B)^*(F,\sigma)^{\otimes i} \to (S^1_B)^*(E,\sigma)^{\otimes i}.$$

If the cardinality of F is strictly smaller than $2i+1$, then we are done. If not, then the classes appearing in the definition of F are all distinct, so we may assume for ease of notation that $(F,\sigma) = [-2i, 2i]$, with $e_j = (2j-1)$ and $f_i = 2j$ for all j.

We will consider three morphisms ψ_1, ψ_2, ψ_3 from $[-2i, 2i]$ to $[1-2i, 2i-1]$, and we will prove that the class

$$v_{12} \otimes \cdots \otimes v_{2i-3,2i-2} \otimes v_{2i-1,2i} \in (S_B^1)^*[-2i, 2i]^{\otimes i}$$

is in the span of the images of the pullbacks along these three morphisms. Each of these morphisms will fix $[2-2i, 2i-2]$, and they will be defined on the elements $\{2i-1, 2i\}$ as follows:[3]

- $\psi_1(2i) = 2i-1$ and $\psi_1(2i-1) = 1-2i$
- $\psi_2(2i) = 2i-1$ and $\psi_2(2i-1) = 0$
- $\psi_3(2i) = 0$ and $\psi_3(2i-1) = 2i-1$.

For each positive integer $j < i$, all three of these maps send the class $v_{2j-1,2j}$ to itself. Furthermore, we have

$$\begin{aligned}
\psi_1^*(v_{2i-1,1-2i}) &= v_{2i-1,1-2i} + v_{2i,-2i} + v_{2i-1,2i} \\
\psi_2^*(v_{2i-1,1-2i}) &= v_{2i-1,1-2i} + v_{2i-1,-2i} + v_{2i-1,2i} \\
\psi_3^*(v_{2i-1,1-2i}) &= v_{2i,-2i} + v_{2i-1,-2i} + v_{2i-1,2i}
\end{aligned}$$

and therefore

$$\psi_1^*(v_{2i-1,1-2i}) - \psi_2^*(v_{2i-1,1-2i}) + \psi_3^*(v_{2i-1,1-2i}) = v_{2i-1,2i}.$$

It follows that we have

$$\begin{aligned}
v_{12} \otimes \cdots \otimes v_{2i-3,2i-2} \otimes v_{2i-1,2i} &= \psi_1^*(v_{12} \otimes \cdots \otimes v_{2i-3,2i-2} \otimes v_{2i-1,1-2i}) \\
&\quad -\psi_2^*(v_{12} \otimes \cdots \otimes v_{2i-3,2i-2} \otimes v_{2i-1,1-2i}) \\
&\quad +\psi_2^*(v_{12} \otimes \cdots \otimes v_{2i-3,2i-2} \otimes v_{2i-1,1-2i}).
\end{aligned}$$

This completes the proof of smallness. For the final statement, we note that $\dim S_B^1[-n,n] = n^2$, therefore $\dim S_B^i[-n,n] \leqslant \binom{n^2}{i}$, and

$$\lim_{n\to\infty} \frac{\binom{n^2}{i}}{(2i-1)^n} = 0.$$

Thus $r_{2i-1}((S_B^i)^*) = 0$. □

[3] Note that this determines what the morphisms do to the elements $\{-2i, 1-2i\}$.

6 Combining small modules

We begin with the following analogue of [8, Lemma 4.2], which mixes modules over $\mathrm{FS}_A^{\mathrm{op}}$ and $\mathrm{FS}_B^{\mathrm{op}}$.

Lemma 27. *Let N be an $\mathrm{FS}_B^{\mathrm{op}}$-module and let $M_1, \ldots, M_p$ be $\mathrm{FS}_A^{\mathrm{op}}$-modules, with N d-small and M_i c_i-small for all i. Consider the $\mathrm{FS}_B^{\mathrm{op}}$-module R defined on objects by the formula*

$$R(E,\sigma) = \bigoplus_{\varphi:(E,\sigma)\to[-p,p]} N(\varphi^{-1}(0),\sigma) \otimes M_1(\varphi^{-1}(1)) \otimes \cdots \otimes M_p(\varphi^{-1}(p)),$$

where the sum is over all morphisms in FS_B from (E,σ) to $[-p,p]$, and maps are defined in the natural way. The module R is $(d + c_1 + \cdots + c_p)$-small.

Proof. Since smallness is preserved by taking direct sums and passing to subquotients, we may immediately reduce to the case where N is the principal projective $P_{[-n,n]}$ for some $n \leqslant d$ and for each i, M_i is the principal projective $P_{[m_i]}$ for some $m_i \leqslant c_i$. Then

$$\begin{aligned} R(E,\sigma) &\cong \bigoplus_{\varphi:(E,\sigma)\to[-p,p]} N(\varphi^{-1}(0),\sigma) \otimes M_1(\varphi^{-1}(1)) \otimes \cdots \otimes M_p(\varphi^{-1}(p)) \\ &\cong \bigoplus_{\varphi:(E,\sigma)\to[-p,p]} \mathbb{C}\left\{ \mathrm{Hom}_{\mathrm{FS}_B}\big((\varphi^{-1}(0),\sigma),[-n,n]\big) \times \prod_{i=1}^{p} \mathrm{Hom}_{\mathrm{FS}_A}\big(\varphi^{-1}(i),[m_i]\big) \right\} \\ &\cong \mathbb{C}\Big\{ \mathrm{Hom}_{\mathrm{FS}_B}\big((E,\sigma),[-(n+m_1+\cdots+m_p),(n+m_1+\cdots+m_p)]\big) \Big\} \\ &\cong P_{[-(n+m_1+\cdots+m_p),(n+m_1+\cdots+m_p)]}(E,\sigma). \end{aligned}$$

Thus R is $(n + m_1 + \cdots + m_p)$-small, and therefore $(d + c_1 + \cdots + c_p)$-small. □

For any natural numbers p and i and any object (E,σ) of FS_B, let

$$\begin{aligned} C_{p,i}(E,\sigma) &:= \bigoplus_{\varphi:(E,\sigma)\to[-p,p]} S^i\big((\mathcal{A}_{(E,\sigma)})_{G_\varphi}\big) \\ &\cong \bigoplus_{\varphi:(E,\sigma)\to[-p,p]} S^i\big(\mathcal{A}_{(\varphi^{-1}(0),\sigma)} \times \mathcal{A}_{\varphi^{-1}(1)} \times \cdots \times \mathcal{A}_{\varphi^{-1}(p)}\big) \\ &\cong \bigoplus_{\varphi:(E,\sigma)\to[-p,p]} \Big(S\big(\mathcal{A}_{(\varphi^{-1}(0),\sigma)}\big) \otimes S\big(\mathcal{A}_{\varphi^{-1}(1)}\big) \otimes \cdots \otimes S\big(\mathcal{A}_{\varphi^{-1}(p)}\big)\Big)^i \\ &= \bigoplus_{\substack{\varphi:(E,\sigma)\to[-p,p] \\ i_0+i_1+\cdots+i_p=i}} S_B^{i_0}\big(\varphi^{-1}(0),\sigma\big) \otimes S_A^{i_1}\big(\varphi^{-1}(1)\big) \otimes \cdots \otimes S_A^{i_p}\big(\varphi^{-1}(p)\big). \end{aligned}$$

Then $C_{p,i}$ is naturally an FS_B-module, and its dual $C_{p,i}^*$ is an $\mathrm{FS}_B^{\mathrm{op}}$-module. The following proposition is the type B analogue of [8, Proposition 5.3], and will be needed in the next section for the proof of Theorem 11.

Proposition 28. *If $i > 0$, the $\mathrm{FS}_B^{\mathrm{op}}$-module $C^*_{p,i}$ is $(2i-1+p)$-small. If $i = 0$, it is p-small.*

Proof. By Propositions 7 and 8 and Lemma 27, the direct summand of $C^*_{p,i}$ corresponding to the tuple $(i_0, i_1, \ldots, i_p)$ is $(2i-1+d)$-small, where d is the number of $k \in \{0, 1, \ldots, p\}$ such that $i_k = 0$. If $i > 0$, the maximum possible value of d is p, so the entire sum is $(2i-1+p)$-small. If $i = 0$, then $d = p+1$, and the sum is p-small. □

7 Kazhdan–Lusztig coefficients

Let V be a vector space over $\mathbb{C}$ and $\mathcal{A}$ a hyperplane arrangement in V with $\bigcap_{H\in\mathcal{A}} H = \{0\}$. We have an inclusion

$$V \to \prod_{H\in\mathcal{A}} V/H \cong \prod_{H\in\mathcal{A}} \mathbb{A}^1 \subset \prod_{H\in\mathcal{A}} \mathbb{P}^1.$$

Let $Y_\mathcal{A}$ be the closure of V inside of the product of projective lines, and let $X_\mathcal{A} \subset Y_\mathcal{A}$ be the open subset consisting of points where no coordinate is equal to zero. The affine variety $X_\mathcal{A}$ was introduced in [7], and is called the **reciprocal plane** of $\mathcal{A}$. We will be interested in the intersection cohomology of $X_\mathcal{A}$ with coefficients in $\mathbb{C}$, which vanishes in odd degree, and has the property that its Poincaré polynomial

$$\sum_{i\geqslant 0} t^i \dim IH^{2i}(X_\mathcal{A})$$

is equal to the **Kazhdan–Lusztig polynomial** of $\mathcal{A}$ [1, Proposition 3.12]. For this reason, we may regard the vector space $IH^{2i}(X_\mathcal{A})$ as a catigorification of the i^{th} Kazhdan–Lusztig coefficient of $\mathcal{A}$.

If F is a flat of $\mathcal{A}$, there is a (noncanonical) inclusion of varieties $X_{\mathcal{A}^F} \to X_\mathcal{A}$, which induces a (canonical) map of intersection cohomology groups $IH^{2i}(X_\mathcal{A}) \to IH^{2i}(X_{\mathcal{A}^F})$. These maps are functorial [8, Theorem 3.3]; in particular, we have an FS_A-module D^i_A that takes a finite set E to the vector space $IH^{2i}(X_{\mathcal{A}_E})$ and a morphism $\varphi : E_1 \to E_2$ to the map

$$IH^{2i}\left(X_{\mathcal{A}_{E_1}}\right) \to IH^{2i}\left(X_{(\mathcal{A}_{E_1})^{F_\varphi}}\right) \cong IH^{2i}\left(X_{\mathcal{A}_{E_2}}\right),$$

and we have an FS_B-module D^i_B that takes an object (E, σ) to the vector space $IH^{2i}\left(X_{\mathcal{A}_{(E,\sigma)}}\right)$ and a morphism $\varphi : (E_1, \sigma_1) \to (E_2, \sigma_2)$ to the map

$$IH^{2i}\left(X_{\mathcal{A}_{E_1}}\right) \to IH^{2i}\left(X_{(\mathcal{A}_{E_1})^{F_\varphi}}\right) \cong IH^{2i}\left(X_{\mathcal{A}_{E_2}}\right).$$

Proof of Theorem 11. For any hyperplane arrangement $\mathcal{A}$, there a spectral sequence $N(i, \mathcal{A})$ converging to $IH^{2i}(X_\mathcal{A})$ with

$$N(i,\mathcal{A})_1^{p,q} = \bigoplus_{\dim F = p} S^{2i-p-q}(\mathcal{A}_F) \otimes IH^{2(i-q)}(X_{\mathcal{A}^F}),$$

where the direct sum is over flats F of $\mathcal{A}$ [8, Theorem 3.1]. For any object (E,σ) of FS_B, let $N(i,E,\sigma) = N\big(i,\mathcal{A}_{(E,\sigma)}\big)$. Then $N(i,E,\sigma)$ converges to $D^i_B(E,\sigma)$, and Example 24 tells us that

$$N(i,E,\sigma)^{p,q}_1 \cong \big(C_{p,2i-p-q}(E,\sigma)\otimes D^{i-q}_B(p)\big)^{W_p}.$$

This construction is functorial [8, Theorem 3.3], meaning that we have a spectral sequence $N(i)$ in the category of FS_B-modules converging to D^i_B with

$$N(i)^{p,q}_1 = \big(C_{p,2i-p-q}\otimes D^{i-q}_B(p)\big)^{W_p}.$$

Dualizing, we obtain a spectral sequence $N^*(i)$ in the category of $\mathrm{FS}_B^{\mathrm{op}}$-modules converging to $(D^i_B)^*$. Since $N(i)^{p,q}_1$ is a submodule of $C_{p,2i-p-q}\otimes D^{i-q}_B(p)$, $N^*(i)^{p,q}_1$ is a quotient of $C^*_{p,2i-p-q}\otimes D^{i-q}_B(p)^*$, and Proposition 28 implies that it is $(2(2i-p-q)-1+p)$-small unless $p+q=2i$, in which case it is p-small. Furthermore, we have $D^{i-q}_B(p)=0$ unless either $(p,q)=(0,i)$ or $p>2(i-q)$ [1, Proposition 3.4].

Let us consider first the case where $p+q=2i$. Since $i>0$, we cannot have $(p,q)=(0,i)$, so we must have $p>2(i-q)$ for $N^*(i)^{p,q}_1$ to be nonzero. This means that p cannot be equal to $2i$, so we have $p\leqslant 2i$, which implies that $N^*(i)^{p,q}_1$ is $(2i-1)$-small. Even better, it tells us that $N^*(i)^{p,q}_1$ is $(2i-2)$-small unless $p=2i-1$ and $q=1$.

Now let us consider the case where $p+q<2i$. If $(p,q)=(0,i)$, then $(2(2i-p-q)-1+p)=2i-1$, so $N^*(i)^{0,i}_1$ is $(2i-1)$-small. If $p>2(i-q)$, then $2(2i-p-q)-1+p = 2(i-q)-p+2i-1<2i-1$, so $N^*(i)^{p,q}_1$ is $(2i-2)$-small.

Since $N^*(i)$ converges to $(D^i_B)^*$ and the entries of the E_1-page of $N^*(i)$ are all $(2i-1)$-small, we can conclude that $(D^i_B)^*$ is $(2i-1)$-smallish. Furthermore, the E_∞ page of $N^*(i)$ is concentrated on the diagonal $p+q=2i$, hence

$$\begin{aligned} r_{2i-1}\big((D^i_B)^*\big) &= \sum_{p,q} r_{2i-1}\Big(N^*(i)^{p,q}_\infty\Big)\\ &= \sum_{p,q}(-1)^{p+q} r_{2i-1}\Big(N^*(i)^{p,q}_\infty\Big)\\ &= \sum_{p,q}(-1)^{p+q} r_{2i-1}\Big(N^*(i)^{p,q}_1\Big). \end{aligned}$$

Since r_{2i-1} vanishes on any $\mathrm{FS}_B^{\mathrm{op}}$-module that is $(2i-2)$-small, this equation simplifies to

$$r_{2i-1}\big((D^i_B)^*\big) = r_{2i-1}\big(N^*(i)^{2i-1,1}_1\big) + (-1)^i r_{2i-1}\big(N^*(i)^{0,i}_1\big).$$

We have $N^*(i)^{0,i}_1 = C^*_{0,i} = (S^i_B)^*$, thus Proposition 8 says that $r_{2i-1}\big(N^*(i)^{0,i}_1\big)=0$. Finally, we have

$$\begin{aligned} N(i)^{2i-1,1}_1 &= \big(C_{2i-1,0}\otimes D^{i-1}_B[1-2i,2i-1]\big)^{W_{2i-1}}\\ &\cong \big(P_{[1-2i,2i-1]}[-n,n]\otimes D^{i-1}_B[1-2i,2i-1]\big)^{W_{2i-1}}, \end{aligned}$$

so

$$\dim N^*(i)_1^{2i-1,1}[-n,n] = \dim N(i)_1^{2i-1,1}[-n,n] \\ = \frac{\dim P_{[1-2i,2i-1]}[-n,n]\cdot \dim D_B^{i-1}[1-2i,2i-1]}{|W_{2i-1}|},$$

where the last equality follows from the fact that the group W_{2i-1} acts freely on a basis for $P_{[1-2i,2i-1]}[-n,n]$. We therefore have

$$r_{2i-1}\big((D_B^i)^*\big) = r_{2i-1}\big(P_{[1-2i,2i-1]}\big)\cdot \frac{\dim D_B^{i-1}[1-2i,2i-1]}{|W_{2i-1}|} = \frac{\dim D_B^{i-1}[1-2i,2i-1]}{|W_{2i-1}|}.$$

This completes the proof. □

Example 29. We illustrate Theorems 6 and 11 when $i=1$. The coefficient of t in the Kazhdan–Lusztig polynomial of a hyperplane arrangement $\mathcal{A}$ is equal to the number of flats of dimension 1 minus the number of hyperplanes [1, Proposition 2.12], thus Example 24 tells us that

$$\dim D_B^1[-n,n] = \Big|\operatorname{Hom}_{\mathrm{FS}_B}\big([-n,n],[-1,1]\big)/W_1\Big| - n^2 = \frac{3^n-1}{2} - n^2.$$

This means that

$$H_B\big((D_B^1)^*,t\big) = \sum_{n=0}^{\infty}\left(\frac{3^n-1}{2}-n^2\right)t^n = \frac{1}{2(1-3t)} - \frac{1}{2(1-t)} - \frac{t}{(1-t)^2} - \frac{2t^2}{(1-t)^3}.$$

This is a rational function with a poles at 1 and 1/3. The pole at 1/3 is simple, with residue

$$\frac{1}{2} = \frac{\dim D_B^0[-1,1]}{|W_1|}.$$

As a representation of W_n, $D_B^1[-n,n]^* \cong D_B^1[-n,n]$ is isomorphic to the permutation representation with basis given by the flats of dimension 1 modulo the permutation representation with basis given by the hyperplanes [3, Corollary 2.10]. If $n<3$, then $D_B^1[-n,n]=0$, while if $n\geqslant 3$, using the branching rule in [2, Lemma 6.1.3] allows us to compute

$$D_B^1[-n,n] = \bigoplus_{\substack{|\lambda|\leqslant n\\ \ell(\lambda)\leqslant 2}} V_{\lambda,[n-|\lambda|]}^{\oplus c_\lambda},$$

where

$$c_\lambda = \begin{cases} \lfloor \lambda_1/2\rfloor - 1 & \text{if } \lambda=[n] \text{ or } \lambda=[n-1,1]\\ \lfloor \lambda_1/2\rfloor & \text{if } \lambda=[n-2,2] \text{ or } \lambda=[n-2]\\ \lfloor \lambda_1/2\rfloor + 1 & \text{otherwise.}\end{cases}$$

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