DP-Colorings of Uniform Hypergraphs and Splittings of Boolean Hypercube into Faces

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Abstract

We develop a connection between DP-colorings of k-uniform hypergraphs of order n and coverings of n-dimensional Boolean hypercube by pairs of antipodal $(n-k)$-dimensional faces. Bernshteyn and Kostochka established a lower bound on the number of edges in a non-2-DP-colorable k-uniform hypergraph namely, 2^{k-1} for odd k and $2^{k-1}+1$ for even k. They proved that these bounds are tight for $k = 3, 4$. In this paper, we prove that the bound is achieved for all odd $k \geqslant 3$.

Mathematics Subject Classifications: 05C15, 05C65, 05C35, 51E05

1 Introduction

Let Q_2^n be the n-dimensional Boolean hypercube. We consider coverings and splittings of Q_2^n into faces. A k-*covering* of Q_2^n is a set of $(n-k)$-dimensional axis-aligned planes or $(n-k)$-faces such that the union of the faces is equal to Q_2^n. Two m-faces are called *parallel* if they have the same directions and a pair of parallel faces is called *antipodal* if for each vertex from one face there exists an antipodal vertex in another face.

We denote an $(n-k)$-face of Q_2^n by an n-tuple $(a_1, a_2, \dots, a_n)$ of symbols $0, 1, *$ where the symbol $*$ is used $n-k$ times. In more detail, $(a_1, \dots, a_n) = \{(x_1, \dots, x_n) : x_i = a_i \text{ if } a_i = 0 \text{ or } a_i = 1\}$. Parallel faces have symbols $*$ in the same coordinates. In the case of antipodal faces symbols in other coordinates are different. Notice that two faces are disjoint if and only if the corresponding tuples are different from each other and from $*$ in one of the coordinates.

It is clear that each k-covering of Q_2^n consists of 2^k or more $(n-k)$-faces. If a k-covering C of Q_2^n consists of exactly 2^k $(n-k)$-faces then C is a k-*splitting* of Q_2^n into

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$(n-k)$-faces. If $n-k=1$ then such splitting is equivalent to a perfect matching in the Boolean hypercube. A k-covering of Q_2^n is called an *antipodal* k-splitting if it consists of exactly 2^k $(n-k)$-faces and it does not contain pairs of parallel non-antipodal faces. An antipodal k-splitting contains two or zero faces of any direction (see Proposition 4).

The concept of DP-coloring was introduced by Dvořák and Postle [3] for graphs in order to generalize the notion of a proper coloring. In [1] Bernshteyn and Kostochka extended the definition of DP-colorings to the hypergraph case.

Definition 1. Let G be an r-uniform hypergraph on n vertices. For each $e \in E(G)$ we consider two antipodal 2-colorings $\varphi_e : e \to \{0,1\}$ and $\overline{\varphi_e} = \varphi_e \oplus 1$. Let Φ be a collection of φ_e, $e \in E(G)$. We say that a 2-coloring $f : V(G) \to \{0,1\}$ avoids Φ if $f|_e \neq \varphi_e$ and $f|_e \neq \overline{\varphi}_e$ for each $e \in E(G)$. A hypergraph G is called *2-DP-colorable* if for every Φ there exists a 2-coloring f avoiding Φ.

Note that a hypergraph G is a proper 2-colorable if there exists a 2-coloring f avoiding Φ_0, where Φ_0 consists of constant maps.

Bernshteyn and Kostochka ([1]) considered the problem to estimate the minimum number of edges in non-2-DP-colorable k-uniform hypergraphs. The existence of a non-2-DP-colorable k-uniform hypergraph with e edges and n vertices is equivalent to the existence of a covering of Q_2^n by e pairs of antipodal $(n-k)$-faces. If the hypergraph has no multiple edges then the definition of DP-coloring implies that this covering does not contain pairs of parallel non-antipodal faces. If $e = 2^{k-1}$ then a non-2-DP-colorable k-uniform hypergraph with e edges generates an antipodal k-splitting and vice versa. The connection between 2-colorings of hypergraphs and coverings of the hypercube will be stated in more detail in Section 3.

It is known (see [1]) that for any even k each k-uniform hypergraph with 2^{k-1} edges has a 2-DP-coloring. Bernshteyn and Kostochka initially conjectured that for any odd $k \geqslant 3$ there exists a non-2-DP-colorable k-uniform hypergraph with 2^{k-1} edges. But they changed their hypothesis because I mistakenly claimed that each 5-uniform hypergraph with 16 edges is 2-DP-colorable (see [1]). The main result of this paper is a construction of antipodal k-splittings for any odd $k \geqslant 3$ and, consequently, a proof of the existence of non-2-DP-colorable k-uniform hypergraphs with 2^{k-1} edges. Thereby the initial conjecture is true.

It is not difficult to prove that any s-colorable hypergraph is s-DP-colorable (see [1]) and every k-uniform hypergraph with s^{k-1} or fewer edges is properly s-colorable. A better bound for the case of proper colorings is known. Cherkashin and Kozik [2], Radhakrishnan and Srinivasan [4] (for $s=2$) showed that any k-uniform hypergraph with $c(s)(\frac{k}{\ln k})^{\frac{s-1}{s}} s^{k-1}$ or fewer edges is properly s-colorable, where $c(s) > 0$ does not depend on k (k is large enough). A survey of results on proper colorings of hypergraphs and related problems can be found in [5]. A Brooks' type theorem for DP-colorings of hypergraphs is proved in [6].

2 Splittings of a hypercube

We denote a $(n-k)$-face of Q_2^n by a n-tuple $(a_1, a_2, \ldots, a_n)$ of symbols $0, 1, *$ where the symbol $*$ is used $n-k$ times. In more detail, $(a_1, \ldots, a_n) = \{(x_1, \ldots, x_n) : x_i = a_i \text{ if } a_i = 0 \text{ or } a_i = 1\}$. Isometries of the Boolean hypercube are generated by permutations of coordinates and vector additions over GF(2). Consider the action of isometries on k-splittings. If $A = \{(a_1^i, \ldots, a_n^i) : i = 1, \ldots, 2^k\}$ is an antipodal k-splitting, then $A_\tau = \{(a_{\tau 1}^i, \ldots, a_{\tau n}^i) : i = 1, \ldots, 2^k\}$ is an antipodal k-splitting for any permutation τ of the set $\{1, \ldots, n\}$. Let us agree that $* \oplus 0 = * \oplus 1 = *$. We define addition of n-tuples to act coordinate-wise. Then for any $(n-k)$-face a and any $b \in Q_2^n$ the sum $a \oplus b$ is an $(n-k)$-face of Q_2^n. It is clear that if A is an antipodal k-splitting, then $A \oplus b = \{a \oplus b : a \in A\}$ is an antipodal k-splitting for each $b \in Q_2^n$. A and A' are called *equivalent* antipodal k-splittings if A' is obtained from A by an isometry.

Proposition 2. *If there exists an antipodal k-splitting of Q_2^n, then there exists an antipodal k-splitting of Q_2^{n+1} with the same cardinality.*

Proof. If A is an antipodal k-splitting of Q_2^n, then $B = \{(a_1, \ldots, a_n, *) : (a_1, \ldots, a_n) \in A\}$ is an antipodal k-splitting of Q_2^{n+1}. □

Let a be an $(n-k)$-face in Q_2^n. A k-face $a^\perp$ is called *orthogonal (dual)* to a if positions of asterisks in a and $a^\perp$ are complementary and other positions are arbitrary. For example, $a = (0, 1, 1, 0, *, *)$ and $a^\perp = (*, *, *, *, 1, 0)$. The following statement can be found in [1]. Below we prove it in the notation of this article.

Proposition 3 ([1])**.** *If k is even then an antipodal k-splitting of Q_2^n does not exist.*

Proof. Let A be an antipodal k-splitting and k is even. Let us consider an $(n-k)$-face $a \in A$, the $(n-k)$-face $\overline{a} \in A$ antipodal to a. By the definitions, we obtain that $x = a \cap a^\perp$ and $\widetilde{x} = \overline{a} \cap a^\perp$ are antipodal vertices within the face $a^\perp$. For example, $a = (0, 1, 1, 0, *, *)$, $a^\perp = (*, *, *, *, 1, 0)$, $x = (0, 1, 1, 0, 1, 0)$, $\widetilde{x} = (1, 0, 0, 1, 1, 0)$. The number of units in the Boolean vector is called the weight of the vector. The parity of this weight is called the parity of the vector. The vertices x and $\widetilde{x}$ have the same parity because k is even. But for all other $b \in A$ we obtain that $b \cap a^\perp$ has the same number of even-weighted and odd-weighted vertices because the intersection of faces is a face or the empty set. Since A is a splitting, the set $\{b \cap a^\perp : b \in A\}$ is a splitting of $a^\perp$ as well. Because the numbers of even-weighted and odd-weighted vertices in $a^\perp$ are equal, we have a contradiction. □

Proposition 4. *For any k-splitting A of Q_2^n $(n > k > 0)$ and for any direction of faces the number of $(n-k)$-faces of this direction in A is even.*

Proof. Suppose $a \in A$ and A contains m $(n-k)$-faces of the same direction as a. Consider a face $a^\perp$. If $b \in A$ has the same direction as a, then $|b \cap a^\perp| = 1$, otherwise the number $|b \cap a^\perp|$ is even. Since $|a^\perp| = 2^k = \sum_{b \in A} |b \cap a^\perp|$ and all terms except m are even, m is even. □

Let T be a subset of $\{1, \ldots, n\}$. An antipodal k-splitting A in Q_2^n is called *t-balanced on T* if every $a \in A$ has t elements 0 and 1 ($|T| - t$ asterisks) in coordinates from T.

Proposition 5. *If there exist an antipodal k_1-splitting of $Q_2^{n_1}$ and an antipodal k_2-splitting of $Q_2^{n_2}$ which is t-balanced on T, $|T| = n_3 \leqslant n_2$, then there exists an antipodal $(k_2 + (k_1 - 1)t)$-splitting of $Q_2^{n_2+(n_1-1)n_3}$.*

Proof. Let A be an antipodal k_2-splitting of $Q_2^{n_2}$ and $B = B_0 \cup B_1$ be an antipodal k_1-splitting of $Q_2^{n_1}$ where sets B_0 and B_1 do not contain parallel $(n_1 - k_1)$-faces. Consider an $(n_2 - k_2)$-face $(a_1, \ldots, a_{n_2}) \in A$. For $i \in T$, if $a_i = 0$ we replace a_i by every possible $b \in B_0$; if $a_i = 1$ then we replace a_i by every possible $b \in B_1$; if $a_i = *$ then we replace a_i by

$$\underbrace{(*, \ldots, *)}_{n_1}.$$

So, we obtain a set C of $|A|(|B|/2)^t = 2^{k_2} \cdot 2^{(k_1-1)t}$ tuples corresponding to m-faces in $Q_2^{n_2+(n_1-1)n_3}$, where

$$\begin{aligned} m &= n_2 - k_2 - (n_3 - t) + (n_3 - t)n_1 + t(n_1 - k_1) \\ &= n_2 + (n_1 - 1)n_3 - (k_2 + (k_1 - 1)t). \end{aligned}$$

It is not difficult to verify that:
1) all faces in C are disjoint because A and B consist of disjoint faces;
2) C is a covering of $Q_2^{n_2+(n_1-1)n_3}$ by counting cardinality of $\cup C$ and, consequently, C is a $(k_2 + (k_1 - 1)t)$-splitting;
3) C contains pairs of antipodal faces because A and B contain pairs of antipodal faces;
4) C does not contain parallel non-antipodal faces because A and B do not contain such faces. □

Corollary 6. *If there exist an antipodal k_1-splitting of $Q_2^{n_1}$ and an antipodal k_2-splitting of $Q_2^{n_2}$, then there exists an antipodal k_1k_2-splitting of $Q_2^{n_1n_2}$.*

Proof. Every k-splitting of Q_2^n is k-balanced on $\{1, \ldots, n\}$ by definition. We can define $n_3 = n_2$, $t = k_2$ and use Proposition 5. □

The following antipodal 3-splitting of Q_2^4 corresponds to the well-known antipodal perfect matching in Q_2^4. We will denote it by E_3.

$$\begin{array}{llll} * & 0 & 0 & 0, \\ 0 & * & 0 & 1, \\ 0 & 1 & * & 0, \\ 0 & 0 & 1 & *, \end{array} \qquad \begin{array}{llll} * & 1 & 1 & 1, \\ 1 & * & 1 & 0, \\ 1 & 0 & * & 1, \\ 1 & 1 & 0 & *. \end{array}$$

We find two antipodal 5-splittings of Q_2^8.

$$
\begin{array}{llll@{\qquad}llll}
00 & 1\ast & 00 & \ast\ast, & 11 & 0\ast & 11 & \ast\ast,\\
00 & \ast1 & 10 & \ast\ast, & 11 & \ast0 & 01 & \ast\ast,\\
0\ast & 01 & 00 & \ast\ast, & 1\ast & 10 & 11 & \ast\ast,\\
\ast0 & 10 & 01 & \ast\ast, & \ast1 & 01 & 10 & \ast\ast,\\
01 & 1\ast & 0\ast & 0\ast, & 10 & 0\ast & 1\ast & 1\ast,\\
00 & \ast0 & 1\ast & 1\ast, & 11 & \ast1 & 0\ast & 0\ast,\\
0\ast & 10 & 1\ast & 0\ast, & 1\ast & 01 & 0\ast & 1\ast,\\
\ast0 & 00 & 0\ast & 1\ast, & \ast1 & 11 & 1\ast & 0\ast,\\
01 & 0\ast & \ast1 & 0\ast, & 10 & 1\ast & \ast0 & 1\ast,\\
10 & \ast1 & \ast0 & 0\ast, & 01 & \ast0 & \ast1 & 1\ast,\\
1\ast & 10 & \ast0 & 0\ast, & 0\ast & 01 & \ast1 & 1\ast,\\
\ast1 & 00 & \ast0 & \ast0, & \ast0 & 11 & \ast1 & \ast1,\\
\ast0 & 00 & \ast\ast & 00, & \ast1 & 11 & \ast\ast & 11,\\
\ast0 & 0\ast & \ast1 & 01, & \ast1 & 1\ast & \ast0 & 10,\\
\ast0 & \ast1 & \ast1 & 00, & \ast1 & \ast0 & \ast0 & 11,\\
\ast\ast & 00 & \ast0 & 01, & \ast\ast & 11 & \ast1 & 10;
\end{array}
$$

$$
\begin{array}{llll@{\qquad}llll}
0\ast & \ast0 & \ast0 & 10, & 1\ast & \ast1 & \ast1 & 01,\\
0\ast & \ast1 & 1\ast & 10, & 1\ast & \ast0 & 0\ast & 01,\\
0\ast & \ast1 & 00 & \ast0, & 1\ast & \ast0 & 11 & \ast1,\\
0\ast & \ast0 & 00 & 0\ast, & 1\ast & \ast1 & 11 & 1\ast,\\
\ast1 & 0\ast & \ast1 & 00, & \ast0 & 1\ast & \ast0 & 11,\\
\ast1 & 1\ast & 1\ast & 00, & \ast0 & 0\ast & 0\ast & 11,\\
\ast0 & 0\ast & 01 & \ast0, & \ast1 & 1\ast & 10 & \ast1,\\
\ast1 & 0\ast & 10 & 0\ast, & \ast0 & 1\ast & 01 & 1\ast,\\
\ast1 & \ast0 & \ast1 & 10, & \ast0 & \ast1 & \ast0 & 01,\\
\ast0 & \ast1 & 1\ast & 00, & \ast1 & \ast0 & 0\ast & 11,\\
\ast0 & \ast0 & 11 & \ast0, & \ast1 & \ast1 & 00 & \ast1,\\
\ast0 & \ast0 & 10 & 0\ast, & \ast1 & \ast1 & 01 & 1\ast,\\
1\ast & 1\ast & \ast0 & 10, & 0\ast & 0\ast & \ast1 & 01,\\
1\ast & 1\ast & 0\ast & 00, & 0\ast & 0\ast & 1\ast & 11,\\
1\ast & 0\ast & 00 & \ast0, & 0\ast & 1\ast & 11 & \ast1,\\
0\ast & 1\ast & 01 & 0\ast, & 1\ast & 0\ast & 10 & 1\ast\,.
\end{array}
$$

By counting of asterisks in the columns we find that the first antipodal 5-splitting is not 1-balanced on any pairs of coordinates. But the second antipodal 5-splitting is 1-balanced on the sets $\{1,2\}$ and $\{3,4\}$. We will denote it by E_5. Note that any isometry of the hypercube exchanges only the order of columns and symbols 0 and 1 in any fixed column. Then any isometry preserves the property of splittings to be 1-balanced. Consequently, the above two antipodal 5-splittings are nonequivalent. We believe that all antipodal 5-splittings are reducible to these two ones.

Theorem 7. *There exists an antipodal k-splitting for every odd $k \geqslant 3$.*

Proof. Let us use antipodal 5-splitting E_5 and 3-splitting E_3 in the construction from Proposition 5 with $k_1 = 3, k_2 = 5$, $n_1 = 4, n_2 = 8$, $T = \{1,2\}$, $t = 1$. We obtain an antipodal 7-splitting E_7. Replacing E_3 by E_5 in this construction, we obtain an antipodal 9-splitting E_9. Consider E_7 and E_9 as the base cases. Suppose that we can construct an antipodal $(2s-3)$-splitting E_{2s-3}. Then by Proposition 5 we obtain an antipodal $(2s+1)$-splitting E_{2s+1} from E_{2s-3} and 1-balanced 5-splitting E_5. Consequently, we prove the theorem by induction. $\square$

3 2-DP-colorings

Let G be an r-uniform hypergraph on n vertices. A 2-coloring f of a k-uniform hypergraph on n vertices is in a one-to-one correspondence to an n-tuple over alphabet $\{0,1\}$. Each k-hyperedge corresponds to $(n-k)$-faces of Q_2^n of some direction. For example, a k-hyperedge consisting of i_1th,…,i_kth vertices corresponds to faces

$$(*,\dots,*,a_{i_1},*,\dots,*,a_{i_2},*,\dots,a_{i_k},*\dots,*)$$

where $a_{i_j}\in\{0,1\}$. The set $\{a_{i_j}\}$ corresponds to some coloring of vertices from the hyperedge. Thereby f corresponds to an element of Q_2^n and 2-colorings of k-hyperedges correspond to $(n-k)$-faces.

Let us remember the definition of 2-DP-coloring. A 2-coloring f avoids a 2-coloring $\varphi_e=(*,\dots,1,\dots,*,\dots,0,\dots,*)$ of $e\in E(G)$ if and only if $f\notin\varphi_e$. A 2-coloring f avoids a collection Φ if $f\notin\varphi_e\cup\overline{\varphi}_e$ for each $\varphi_e\in\Phi$. A hypergraph G is a 2-DP-colorable if for every collection Φ of antipodal 2-colorings of hyperedges there exists a 2-coloring f avoiding Φ. Thereby G is a 2-DP-colorable if and only if any collection $\{\varphi_e,\overline{\varphi_e}: e\in E(G)\}$ is not a covering of Q_2^n.

Consider a table of size $n\times\ell$, where every column corresponds to a $(n-k)$-face of an antipodal covering of Q_2^n. Let us replace in the table symbols $*$ by 0 and other symbols by 1. By the definition, the resulting table is the incidence matrix of a non-2-DP-colorable k-uniform hypergraph with ℓ edges. Consequently, we have the following statement.

Proposition 8. *A k-uniform hypergraph with ℓ edges and n vertices is non-2-DP-colorable if and only if its incidence matrix corresponds to a k-covering of Q_2^n by ℓ pairs of antipodal $(n-k)$-faces.*

Moreover, Proposition 8 implies the following statement.

Corollary 9. *There exists a non-2-DP-colorable k-uniform hypergraph with 2^{k-1} edges if and only if there exists an antipodal k-splitting of Q_2^n.*

A non-2-DP-colorable 3-uniform hypergraph with 4 edges that corresponds to the antipodal 3-splitting E_3 is presented in [1]. By Theorem 7 and Corollary 9, we obtain the following statement.

Corollary 10. *If $k\geqslant 3$ is odd there exists a non-2-DP-colorable k-uniform hypergraph with 2^{k-1} edges.*

Since a union of at most 2ℓ $(n-k)$-faces contains $\ell 2^{n-k+1}$ vertices, we obtain the following corollary.

Corollary 11 ([1])**.** *Every k-uniform hypergraph with $\ell<2^{k-1}$ edges is 2-DP-colorable.*

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