On the Minimal Sum of Weights on the Edges in a Signed Edge-Dominated Graph

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Abstract
Let \( G \) be a simple graph with \( n \) vertices and \( \pm 1 \)-weights on edges. Suppose that for every edge \( e \) the sum of edges adjacent to \( e \) (including \( e \) itself) is positive. Then the sum of weights over edges of \( G \) is at least \( -\frac{n^2}{25} \). Also we provide an example of a weighted graph with described properties and the sum of weights \( -(1 + o(1)) \frac{n^2}{8(1+\sqrt{2})^2} \).

The previous best known bounds were \( -\frac{n^2}{16} \) and \( -(1 + o(1)) \frac{n^2}{54} \) respectively. We show that the constant \(-1/54\) is optimal under some additional conditions.

Mathematics Subject Classifications: 05C07, 05C22

1 Introduction
A graph (finite, simple, undirected) is a pair \((V,E)\), where \( V \) stands for a set of vertices, and \( E \) denotes a set of unordered pairs of vertices, whose elements are called edges. Let \( G \) be a graph; for a given edge \( e = (u,v) \) define its closed edge-neighborhood as an edge subset \( N[e] \) formed by \( e \) and all edges of \( G \) adjacent to \( e \). A weight function \( f : E \to \{+1; -1\} \) is called a signed edge domination function of \( G \) if

\[
\sum_{e' \in N[e]} f(e') \geq 1
\]

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for every \( e \in E \); in this case we say that \((G, f)\) is an SED-pair of order \(|V|\). Let \( s[(G, f)] \) be the sum of weights over all edges of a graph \( G \) equipped by a weight function \( f \).

Denote by \( E_+ \) the set \( \{(u, v) \in E \mid f(u, v) = 1\} \) and by \( E_- \) the set \( \{(u, v) \in E \mid f(u, v) = -1\} \). Define

\[
 s_v = \sum_{e \in N(v)} f(e)
\]

for each \( v \in V \), where \( N(v) \) stands for the set of edges containing \( v \). Let \( V_+ \) be \( \{v \in V \mid s_v \geq 0\} \) and \( V_- \) be \( \{v \in V \mid s_v < 0\} \).

The following problem was posed by Xu in [5, 6].

**Problem 1.** What is

\[
 g(n) := \min\{s[(G, f)] \mid (G, f) \text{ is an SED-pair of order } n\}
\]

for each positive integer \( n \)?

Note that for every \( g(n) \leq 0 \) since an empty graph provides an SED-pair. The only known result was provided by the following theorem.

**Theorem 2** (Akbari–Bolouki–Hatami–Siami [2]).

(i) For every \( n \)

\[
 g(n) \geq \frac{n^2}{16}.
\]

(ii) There is a sequence of SED-pairs of order \( n \) that satisfies

\[
 s[(G, f)] \leq -(1 + o(1))\frac{n^2}{54}.
\]

We refine both items as follows.

**Theorem 3.**

(i) For every \( n \), \( g(n) \geq -\frac{n^2}{25} \).

(ii) For every \( n \) there is an SED-pair of order \( n \) that satisfies

\[
 s[(G, f)] < -(1 + o(1))\frac{n^2}{8(1 + \sqrt{2})^2}.
\]

Moreover, if \( n = 4(p + q)p \), where \( p > 1 \) and \( q > 1 \) are positive integers satisfying \( p^2 = 2q^2 - 1 \), then

\[
 s[(G, f)] = \left[ \frac{n^2}{8(1 + \sqrt{2})^2} + \frac{3\sqrt{2} - 4}{4}n \right].
\]

\(^1\)In fact the authors claim the bound \(-\frac{n^2}{25}\) but the provided example gives the bound \(-(1 + o(1))\frac{n^2}{54}\).
Note that there are infinitely many \( p \) and \( q \) satisfying the condition \( p^2 = 2q^2 - 1 \), since it is a special case of Pell’s equation; it is well known that the positive solutions are

\[
p = \frac{\sqrt{2} - 1}{2}(3 + 2\sqrt{2})^k - \frac{1 + \sqrt{2}}{2}(3 - 2\sqrt{2})^k, \quad q = \frac{\sqrt{2} - 1}{2\sqrt{2}}(3 + 2\sqrt{2})^k + \frac{1 + \sqrt{2}}{2\sqrt{2}}(3 - 2\sqrt{2})^k,
\]

for \( k \in \mathbb{N} \).

We show that Theorem 2(ii) is optimal under additional assumptions.

**Theorem 4.** Let \((G, f)\) be an SED-pair of order \( n \). Suppose that every \( e \in E_− \) connects a vertex from \( V_+ \) and a vertex from \( V_− \); and every \( e \in E_+ \) connects some vertices from \( V_+ \). Then

\[
\sigma[(G, f)] \geq -\frac{1}{54}n^2.
\]

### 1.1 Graphons

A graphon (also known as a graph limit) is a symmetric measurable function \( W : [0, 1]^2 \to [0, 1] \). Define a signed graphon as a symmetric measurable function \( W : [0, 1]^2 \to [-1, 1] \). A signed graphon is edge-dominated if \( W(x, y) \neq 0 \) implies

\[
\int_0^1 (W(x, t) + W(y, t))dt \geq 0.
\]

Here we consider a continuous analogue of Problem 1. Denote

\[
\kappa := \inf \frac{1}{2} \int_0^1 \int_0^1 W(x, y)dxdy
\]

where the infimum is taken over all edge-dominated graphons \( W \).

The following theorem is a standard result in the theory of graph limits [3], we include the proof in Appendix A for completeness.

**Theorem 5.**

(i) \( g(n) \geq \kappa n^2 \), in other words \( s(G, f) \geq \kappa n^2 \) for any SED-pair \((G, f)\) of order \( n \);

(ii) \( g(n) = (\kappa + o(1))n^2 \) for large \( n \).

Theorems 3 and 4 also have natural continuous analogues.

**Structure of the paper.** Theorem 3(ii) is proved in Section 2. Section 3 is devoted to the proof of Theorem 3(i). Section 4 cites a result, determining the maximal sum of squares of vertex degrees among all graphs with \( n \) vertices and \( e \) edges; we use it in Section 5, containing the proof of Theorem 4. Appendix A contains the proof of Theorem 5, Appendices B-D contain auxiliary calculations.
2 Examples

In this section we provide a sequence of SED-pairs that achieves the upper bound
\[-(1 + o(1)) \frac{n^2}{8(1 + \sqrt{2})^2}.\]

2.1 A graphon example

The following signed graphon realizes an example for Theorem 3(ii). Put $[0, 1] = A \sqcup B \sqcup C$, where $|A| = 1 - \frac{1}{\sqrt{2}}$, $|B| = \frac{1}{\sqrt{2}} - \frac{1}{2}$, and $|C| = \frac{1}{2}$. The function $W$ is defined in Fig. 1.

\[
\begin{array}{|c|c|c|}
\hline
C & 0 & -\frac{1}{\sqrt{2}} & 0 \\
\hline
B & 1 & 1 & -\frac{1}{\sqrt{2}} \\
\hline
A & -\frac{1}{\sqrt{2}} & 1 & 0 \\
\hline
\end{array}
\]

Figure 1: A graphon example for Theorem 3(ii).

Note that $W$ is edge-dominated: indeed, for $(x, y) \in A \times A$

\[
\int_0^1 (W(x, t) + W(t, y)) dt = 2 \left( -\frac{1}{\sqrt{2}} |A| + |B| \right) = 0,
\]

for $(x, y) \in A \times B$

\[
\int_0^1 (W(x, t) + W(t, y)) dt = -\frac{1}{\sqrt{2}} |A| + |B| + |A| + |B| - \frac{1}{\sqrt{2}} |C| = \frac{1}{2} - \frac{1}{2\sqrt{2}} > 0,
\]

for $(x, y) \in B \times B$

\[
\int_0^1 (W(x, t) + W(t, y)) dt = 2 \left( |A| + |B| - \frac{1}{\sqrt{2}} |C| \right) = 1 - \frac{1}{\sqrt{2}} > 0,
\]

and for $(x, y) \in B \times C$

\[
\int_0^1 (W(x, t) + W(t, y)) dt = |A| + |B| - \frac{1}{\sqrt{2}} |C| - \frac{1}{\sqrt{2}} |B| = 0.
\]

Finally,

\[
\frac{1}{2} \int_0^1 \int_0^1 W(x, y) dxdy = \frac{1}{2} \left( -\frac{|A|^2}{\sqrt{2}} + 2|A| \cdot |B| + |B|^2 - \frac{2|B| \cdot |C|}{\sqrt{2}} \right) = -\frac{1}{8(1 + \sqrt{2})^2}.
\]
2.2 An explicit graph approximation

Here we provide the best approximation we can do. Fix $p$ and $q$ such that $p^2 = 2q^2 - 1$, and $p, q > 1$.

We need several auxiliary definitions. Define a graph $K_{X,Y,\frac{k}{l}} = (X \cup Y, E_{X,Y,\frac{k}{l}})$ for $|X| = al, |Y| = bl$ and integers $a, b, k \leq l$. Split $X$ into $a$ disjoint sets of size $l$: $X = X_1 \cup X_2 \cup \cdots \cup X_a$ with $|X_i| = l$; also split $Y$ into $b$ disjoint sets of the same size: $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_b$ with $|Y_i| = l$. For each pair $1 \leq i \leq a, 1 \leq j \leq b$ consider the following bipartite graph $G_{ij} = (X_i \cup Y_j, E_{ij})$ with parts $X_i$ and $Y_j$ (all graphs $G_{ij}$ are isomorphic). Enumerate vertices as follows $X_i = \{v_1, v_2, \ldots, v_l\}$, $Y_j = \{u_1, u_2, \ldots, u_l\}$. Define $E_{ij}$ as the set of all pairs $(v_a, u_b)$, for which $g - h \mod l$ lies in $\{1, 2, \ldots, k\}$. Put

$$E_{X,Y,\frac{k}{l}} = \bigcup_{1 \leq i \leq a, 1 \leq j \leq b} E_{ij}.$$ 

Obviously the degree of every vertex in $G_{ij}$ is equal to $k$, so the degree of a vertex in $K_{X,Y,\frac{k}{l}}$ is $bk = |Y| \frac{k}{l}$ for vertices in $X$, and $ak = |X| \frac{k}{l}$ for vertices in $Y$.

Now define graph $K_{X,\frac{k}{l}} = (X, E_{X,\frac{k}{l}})$ for $|X| = 2al$ and integer $a, k < l$. Split $X$ into $2l$ disjoint sets of size $a$: $X = X_1 \cup X_2 \cup \cdots \cup X_{2l}$. The edge between vertices $u$ and $v$ exists if and only if $i - j \mod 2l$ lies in

$$\{-k, -(k - 1), \ldots, -2, -1, 1, 2, \ldots, k - 1, k\},$$

where $v \in X_i, u \in X_j$. Then the degree of every vertex in $K_{X,\frac{k}{l}}$ is equal to $2ak = |X| \frac{k}{l}$.

Let $K_X = (X, E_X)$ be the complete graph (i.e. every pair of vertices forms an edge) on the vertex set $X$. Degree of each vertex in $K_X$ is equal to $|X| - 1$.

Now we are ready to provide the desired construction. Let $p$ and $q$ be a positive solution of $p^2 = 2q^2 - 1$. Put

$$A = \{a_1, a_2, \ldots, a_{2p}\}, \quad B_1 = \{b_1, b_2, \ldots, b_{2p(p-q)}\}, \quad B_2 = \{b_{2p(p-q)+1}, b_{2p(p-q)+2}, \ldots, b_{2pq}\},$$

$$C_1 = \{c_1, c_2, \ldots, c_{6p(p-q)}\}, \quad C_2 = \{c_{6p(p-q)+1}, c_{6p(p-q)+2}, \ldots, c_{2(p+q)p}\}.$$ 

Define the vertex set

$$V = A \cup B_1 \cup B_2 \cup C_1 \cup C_2$$

(so $n = 4p^2 + 4pq$). The edge set $E$ and weight function $f$ are defined by explicit expressions for $E_+$ and $E_-:

$$E_+ = E_{A,B_1 \cup B_2,1} \cup E_{B_1,\frac{p^2-pq-1}{2(p-q)}} \cup E_{B_1,B_2,1} \cup E_{B_2};$$

$$E_- = E_{A,\frac{2}{p}} \cup E_{B_1,C_2,\frac{2}{p}} \cup E_{B_2,C_1,\frac{2}{p}} \cup E_{B_1,C_1,\frac{2p^2-2q^2-1}{2(p-q)}} \cup E_{B_2,C_2,\frac{4p^2-2pq-1}{2(p-q)}}.$$ 

Since $p$ divides all of the cardinalities $|A|, |B_1|, |B_2|, |C_1|, |C_2|; 2p(p-q)$ divides $|B_1|, |C_1|, \text{ and } 2p(2q-p)$ divides $|B_2|, |C_2|$, the definition of $f$ is correct.

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Some annoying calculation gives

\[ s_{a_i} = 0, \quad s_{b_i} = p^2, \quad s_{c_i} = -p^2 \]

for every \( i \).

Note that there is no edge between \( A \) and \( C \) or inside \( C \). Also all edges inside \( A \) of between \( B \) and \( C \) are negative, so our construction is an SED-pair.

Finally we count

\[ s[G, f] = \frac{1}{2} \sum_{v \in V} s_v = \frac{p^2(|B_1| + |B_2| - |C_1| - |C_2|)}{2} = -p^4. \]

Recall that \( p^2 = 2q^2 - 1 \) and \( n = 4p^2 + 4pq = 2p(2p + \sqrt{2}/1 + p^2) \). So

\[ \frac{s[G, f]}{n^2} = \frac{-p^4}{(2p(2p + \sqrt{2}/1 + p^2))^2} = -\frac{1}{8(1 + \sqrt{2})^2} + \frac{5\sqrt{2} - 7}{8p^2} + \frac{31\sqrt{2} - 44}{32p^4} + O(p^{-5}). \]

Since \( n = (4 + 2\sqrt{2})p^2 + \sqrt{2} - \frac{1}{2\sqrt{2}p^2} + O(p^{-3}) \)

\[ s[G, f] = -\frac{n^2}{8(1 + \sqrt{2})^2} + \frac{3\sqrt{2} - 4}{4}n - \frac{1}{2(2 + \sqrt{2})} + o(1). \]

One can also derive

\[ s[G, f] = \left\lfloor -\frac{n^2}{8(1 + \sqrt{2})^2} + \frac{3\sqrt{2} - 4}{4}n \right\rfloor. \]

### 3 The lower bound of Theorem 3

Consider an arbitrary SED-pair \((G, f)\), where \( G = (V, E) \).

It is known that for each \( v, u \in V \) if \((v, u) \in E_- \cup E_+\), then \( s_v + s_u \geq 0 \) (check it by hands or see Lemma 1 in [2]). If \( V_- \) is empty, then \( s[G, f] \geq 0 \). Let \( x \) be

\[ -\min_{v \in V_-} s_v \]

and consider an arbitrary vertex \( a \) such that \( s_a = -x \). Let \( N_-(a) \) be \( \{v \in V | (a, v) \in E_-\} \). Then \( |N_-(a)| \geq x \) and \( s_v \geq x \) for each \( v \in N_-\), so \( N_- \subseteq V_+ \). Then

\[ x^2 \leq \sum_{v \in N_-(a)} s_v \leq \sum_{v \in V_+} s_v. \]

Clearly, \( V_- \) is an independent set (i.e. has no edges inside) so

\[ \sum_{v \in V_+} s_v = \sum_{v \in V_-} s_v + 2 \left( \sum_{(u,v) \in E_+ | u,v \in V_+} 1 - \sum_{(u,v) \in E_- | u,v \in V_+} 1 \right) \]
\[ \sum_{v \in V_-} s_v + 2|V_+| \cdot (|V_+| - 1) \leq \sum_{v \in V_-} s_v + |V_+|^2. \]

So
\[ \sum_{v \in V_-} s_v \geq x^2 - |V_+|^2; \]
recalling that
\[ \sum_{v \in V_+} s_v \geq x|N_-(a)| \geq x^2. \]

On the other hand
\[ s[(G, f)] = \sum_{(x, y) \in E_+} 1 - \sum_{(x, y) \in E_-} 1 = \frac{\sum_{v \in V} s_v}{2}, \]
and
\[ \sum_{v \in V} s_v = \sum_{v \in V_+} s_v + \sum_{v \in V_-} s_v \geq 2x^2 - |V_+|^2. \]

Also
\[ \sum_{v \in V} s_v = \sum_{v \in V_+} s_v + \sum_{v \in V_-} s_v \geq x^2 - x|V_-| = -x(|V_+| - x) = -x(|V| - |V_+| - x). \]

Put \( y = \frac{x}{|V|}, \ k = \frac{|V_+|}{|V|}. \) Then we have the following system of inequalities:
\[
\begin{cases}
  s[(G, f)] \geq (y^2 - \frac{k^2}{2})|V|^2 \\
  s[(G, f)] \geq -y(1 - k - y)|V|^2.
\end{cases}
\]

So
\[ g(n) \geq \min_{0 \leq y \leq 1, 0 \leq k \leq 1} \left( \max \left( y^2 - \frac{k^2}{2}, -y(1 - k - y) \right) \right) n^2. \]

One may check by computer (or read explicit calculus in Appendix B) that the minimum is \( -\frac{1}{25} \) and is reached at \( y = \frac{3}{5}, \ k = \frac{2}{5}. \)

4 Degree sequences of a graph

Here we display the results from [1], which are required in the proof of Theorem 4; for a survey see [4].

Definition 6. Let \( n, e \leq \binom{n}{2} \) be integer numbers. Consider the unique representation
\[ e = \binom{a}{2} + b, \quad 0 \leq b < a. \]

The quasi-complete graph \( C_n^e \) with \( e \) edges and \( n \) vertices \( v_1, \ldots, v_n \) has edges \( (v_i, v_j) \) for \( i, j \leq a \) and \( i = a + 1, \ j \in \{1, \ldots, b\} \).
Definition 7. Let \( n, e \leq \binom{n}{2} \) be integer numbers. Consider the unique representation
\[
\binom{n}{2} - e = \binom{c}{2} + d, \quad 0 \leq d < c.
\]
The quasi-star graph \( S_e^n \) is the graph with \( e \) edges and \( n \) vertices \( v_1, \ldots, v_n \), such that vertices \( v_1, \ldots, v_{n-c-1} \) are connected with all vertices and vertex \( v_{n-c} \) is connected with vertices \( v_1, \ldots, v_{n-d} \).

Let \( F(n, e) \) be the maximal value of
\[
\sum_{v \in V} (\deg v)^2
\]
among the graphs \( G = (V, E) \) with \( n \) vertices and \( e \) edges. We use the following result.

Theorem 8 (Alshwede–Katona, [1]). For every \( n \) and \( 0 \leq e \leq \binom{n}{2} \) the value \( F(n, e) \) is achieved on \( C_e^n \) or \( S_e^n \).

For \( G = C_e^n \) the sum of squares of degrees equals to
\[
ba^2 + (a-b)(a-1)^2 + b^2 = a^3 - 2a^2 + 2ab + b^2 + a - b
= (1 + o(1)) \left( \frac{a}{n} \right)^3 n^3
= (1 + o(1)) \left( \frac{2e}{n^2} \right)^{3/2} n^3,
\]
because \( a = (1 + o(1)) \sqrt{2e} \). For \( G = S_e^n \) this sum equals to
\[
(n-c-1)(n-1)^2 + (n-d-1)^2 + (c-d)(n-c)^2 + d(n-c-1)^2
= (1 + o(1))((n-c)n^2 + (c-d)(n-c)^2 + d(n-c)^2)
= (1 + o(1))(n-c)(n^2 + (n-c)c) = (1 + o(1))n^3 \left( 1 - \frac{c}{n} \right) \left( 1 + \frac{c}{n} - \frac{c^2}{n^2} \right)
= (1 + o(1)) \left( 1 - \sqrt{1 - \frac{2e}{n^2}} \right) \left( 1 + \sqrt{1 - \frac{2e}{n^2}} - \left( 1 - \frac{2e}{n^2} \right) \right) n^3
= (1 + o(1)) \left( 1 - \sqrt{1 - \frac{2e}{n^2}} \right) \left( \sqrt{1 - \frac{2e}{n^2} + \frac{2e}{n^2}} \right) n^3,
\]
because \( c = (1 + o(1)) \sqrt{n^2 - 2e} \).

Corollary 9. Put \( \alpha = \frac{2e}{n^2} \). Then
\[
F(n, e) = (1 + o(1)) \max \left( \alpha^3, (1 - \sqrt{1 - \alpha})(\sqrt{1 - \alpha} + \alpha) \right) n^3.
\]
Define
\[ R(\alpha) := \alpha^3, \quad T(\alpha) = (1 - \sqrt{1 - \alpha})(\sqrt{1 - \alpha} + \alpha). \]
We show that \( R(\alpha) < T(\alpha) \) for \( \alpha \in (0, 1/2) \) and \( R(\alpha) > T(\alpha) \) for \( \alpha \in (1/2, 1) \). Define \( t = \sqrt{1 - \alpha} \). Note that

\[ R^2(\alpha) - T^2(\alpha) = (1 - t)^2(1 + t - t^2)^2 - (1 - t^2)^3 = t^2(1 - t)^2(2t^2 - 1). \]

For \( \alpha \in (1/2, 1) \) one has \( t \in \left(0, \frac{\sqrt{2}}{2}\right) \) and \( R(\alpha) > T(\alpha) \). For \( \alpha \in (0, 1/2) \) one has \( t \in \left(\frac{\sqrt{2}}{2}, 1\right) \) and \( R(\alpha) < T(\alpha) \).

There are several weaker and better-looking bounds on \( F(n,e) \), but they do not meet our aims.

5 Proof of Theorem 3

Put \( k = |V_+| \). Let the degrees of vertices in \( G[V_+] \) be equal to \( a_1, \ldots, a_k \); the degrees of vertices in \( G[V_+, V_-] \) be equal to \( b_1, \ldots, b_k \) for \( v_i \in V_+ \) and \( c_1, \ldots, c_{n-k} \) for \( v_j \in V_- \). Define

\[ a = \frac{1}{k} \sum_{1 \leq i \leq k} a_i; \quad b = \frac{1}{k} \sum_{1 \leq i \leq k} b_i; \quad c = \frac{1}{n-k} \sum_{1 \leq j \leq n-k} c_j; \]

by double-counting in the graph \( G[V_+, V_-] \) we have \( kb = (n-k)c \).

By the main condition, if we have an edge \((v_i^+, v_j^-)\) then

\[ a_i - b_i \geq c_j. \]

Sum up all these inequalities; then every vertex \( v_i^+ \) is counted \( b_i \) times, and every vertex \( v_j^- \) is counted \( c_j \) times. Hence

\[ \sum_{1 \leq i \leq k} (a_i - b_i)b_i \geq \sum_{1 \leq j \leq n-k} c_j^2. \]

Applying Cauchy–Bunyakovsky–Schwarz inequality, we get

\[ \sqrt{\sum_{1 \leq i \leq k} a_i^2 \sum_{1 \leq i \leq k} b_i^2 - \sum_{1 \leq i \leq k} b_i^2} \geq \sum_{1 \leq i \leq k} (a_i - b_i)b_i. \]

The AM-GM inequality implies

\[ \sum_{1 \leq j \leq n-k} c_j^2 \geq (n-k)c^2 = \frac{k^2}{n-k} b^2. \]

Consider the following “dimensionless” quantities

\[ \alpha = \frac{a}{k} = \frac{1}{k^2} \sum_{1 \leq i \leq k} a_i; \quad \beta = \frac{b}{k} = \frac{1}{k^2} \sum_{1 \leq i \leq k} b_i; \quad B = \frac{1}{k^3} \sum_{1 \leq i \leq k} b_i^2; \quad K = \frac{k}{n}. \]
Then applying Corollary 9 to $V_+$ and using obtained inequalities we have

$$U(\alpha)k^{3/2}Bk^{3/2} - B^2k^3 \geq \sqrt{\sum_{1 \leq i \leq k} a_i^2 \sum_{1 \leq i \leq k} b_i^2 - \sum_{1 \leq i \leq k} b_i^2} \geq \sum_{1 \leq i \leq k} (a_i - b_i)b_i \geq$$

$$\sum_{1 \leq j \leq n-k} c_j^2 \geq \frac{k^2}{n-k}b^2 = \beta^2 \frac{k^4}{n-k},$$

where $U(\alpha) = \max\left(\sqrt{R(\alpha)}, \sqrt{T(\alpha)}\right)$. By the AM-GM inequality $\beta \leq B$. Also,

$$s[(G, f)] = \frac{1}{2} \sum_{1 \leq i \leq k} a_i - \sum_{1 \leq i \leq k} b_i = \left(\frac{\alpha}{2}K^2 - \beta K^2\right)n^2.$$

Thus we have reduced our problem to the following optimization problem:

$$\begin{cases}
U(\alpha)B - B^2 \geq \beta^2 \frac{K}{1-K}; \\
\min \frac{\alpha}{2}K^2 - \beta K^2; \\
0 \leq \alpha \leq 1,
0 \leq K < 1,
0 \leq \beta \leq B \leq \frac{1-K}{K};
\end{cases}\quad (2)$$

where the last inequality follows from the fact that every $b_i$ is at most $n-k$.

We show that the desired minimum is $-\frac{1}{54}$; it can be reached by the example from Theorem 2(ii). Note that a possible (with respect to conditions of the system (2)) value of $(\alpha, \beta, B, K)$ may not correspond to an SED-pair.

**Case 1.** In this case $\alpha \geq \frac{1}{2}$, so $U(\alpha) = \sqrt{R(\alpha)}$. Then we have to solve the following system

$$\begin{cases}
\alpha^2B - B^2 \geq \beta^2 \frac{K}{1-K}; \\
\frac{1}{2} \leq \alpha \leq 1,
0 < K < 1,
0 \leq \beta \leq \frac{1-K}{K}; \\
\min \frac{\alpha}{2}K^2 - \beta K^2.
\end{cases}$$

In Appendix C we show that the minimum is $-\frac{1}{54}$.

**Case 2.** In this case $\alpha \leq \frac{1}{2}$. Then $U(\alpha) = \sqrt{T(\alpha)}$. Then we have a deal with the following system

$$\begin{cases}
(1 - \sqrt{1-\alpha})(\sqrt{1-\alpha} + \alpha)B - B^2 \geq \beta^2 \frac{K}{1-K}; \\
0 \leq \alpha \leq \frac{1}{2},
0 < K < 1,
0 \leq \beta \leq \frac{1-K}{K}; \\
\min \frac{\alpha}{2}K^2 - \beta K^2.
\end{cases}$$

This system is analyzed in Appendix D; the minimum is bigger than the desired value $-\frac{1}{54}$.

So we prove $s[(G, f)] \geq -(1 + o(1))\frac{n^2}{54}$. Theorem 5 finishes the proof.
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References


Appendix A

Proof of Theorem 5. (i) Let \((G = (V, E), f)\) be an SED-pair of order \(n\). We partition \([0,1]\) into \(n\) disjoint sets of measure \(1/n\) and identify these \(n\) sets with \(n\) vertices of \(G\). For points \(x, y \in [0,1]\) denote by \(v, u\) the vertices which contain them, respectively, and put

\[
W(x,y) = \begin{cases} f(v,u), & \text{if } (v,u) \in E \\ 0, & \text{otherwise.} \end{cases}
\]

It is easy to see that \(\int_0^1 (W(x,t) + W(y,t))dt = s_v + s_u \geq 0\) whenever \(x \in v, y \in u\) and \((v,u) \in E\). Thus signed graphon \(W\) is edge-dominated, and \(\kappa \leq \frac{1}{2} \int_0^1 \int_0^1 W = \frac{1}{n^2} s(G,f)\) that proves (i).

(ii) Fix \(\varepsilon \in (0,1)\) and an edge-dominated signed graphon \(W\) such that \(\frac{1}{2} \int_0^1 \int_0^1 W < \kappa + \varepsilon\). Let \(n\) be a \((\large\text{large})\) integer. Denote \(k = \lfloor \varepsilon n \rfloor, \ m = n - k\). Since \(\varepsilon > 0\) is arbitrary, and the lower bound \(\varepsilon(n) \geq \kappa n^2\) is already established in (i), for proving (ii) it suffices to prove that

\[
g(n) \leq 2kn + m^2(\kappa + \varepsilon)
\]

for all large enough \(n\).

Choose \(m\) points \(v_1, \ldots, v_m \in [0,1]\) uniformly and independently at random. Denote \(V = \{1,2,\ldots,n\}\), and define the signed graph \(G = (V, E)\) as follows:

1) if \(i > m\), the vertex \(i\) is joined with all other vertices and \(f(i,j) = 1\) for all \(j \in V \setminus \{i\}\);

2) if \(i,j \leq m\), we join \(i\) and \(j\) by an edge with probability \(|W(v_i,v_j)|\) and put \(f(i,j) = \sign W(v_i,v_j)\) if \(i\) and \(j\) become joined (the above events are independent).

If we define

\[
\tilde{f}(i,j) = \begin{cases} f(i,j), & \text{if } (i,j) \in E \\ 0, & \text{otherwise,} \end{cases}
\]

then the expectation of \(\tilde{f}(i,j)\) equals \(W(v_i,v_j)\). If \(v_1, \ldots, v_m\) are fixed, the Chernoff bound guarantees that:

a) the probability that \(s_i - k = \sum_{j \leq m} \tilde{f}(i,j)\) differs from \(\sum_j W(v_i,v_j)\) by a value greater than \(k/5\) is exponentially small, and this holds true even if \(v_1, \ldots, v_m\) are fixed;

b) the probability that \(\sum_j W(v_i,v_j)\) differs from \(m \int_0^1 W(v_i,t)dt\) by a value greater than \(k/5\) is also exponentially small, and this holds true even if \(v_i\) is fixed;

c) the probability that \(\sum_i \int_0^1 W(v_i,t)dt\) differs from \(m \int_0^1 \int_0^1 W(x,y)dxdy\) by more than \(k/5\) is also exponentially small.

Therefore with high probability none of the above \(2m+1\) events happens, and we get

\[
\left| s_i - k - \int_0^1 W(v_i,t)dt \right| \leq \frac{2k}{5}
\]

for all \(i = 1, \ldots, m\), and

\[
\sum_{i=1}^m s_i - km - m^2 \int_0^1 \int_0^1 W \leq \frac{3km}{5}.
\]
These bounds yield that \((G, f)\) is an SED-pair, and
\[
g(n) \leq s[G, f] = \frac{1}{2} \sum_{j=1}^{n} s_j \leq k(n - 1) + \frac{km}{2} + \frac{3km}{10} + \frac{1}{2} \int_{0}^{1} \int_{0}^{1} W \leq 2kn + m^{2}(\kappa + \varepsilon)
\]
that is (3).

Appendix B

We have to calculate
\[
\min \left( \max \left( y^2 - \frac{k^2}{2}, -\frac{y(1 - k - y)}{2}\right) \right) = -\frac{1}{2} \max \left( \min(k^2 - 2y^2, y - y^2 - ky) \right).
\]

Let \(k_1, y_1 \in [0, 1]\) be any values representing this maximum (the maximum is reached by compactness).

First, we show that
\[
y_1^2 - \frac{k_1^2}{2} = -\frac{y_1(1 - k_1 - y_1)}{2}.
\]

Indeed, this equality means that
\[
k_1 = -\frac{y_1 + \sqrt{5y_1^2 + 4y_1}}{2}.
\]

Suppose the contrary; if \(k_1 > -\frac{y_1 + \sqrt{5y_1^2 + 4y_1}}{2}\) then
\[
\min(k_1^2 - 2y_1^2, y_1 - y_1^2 - k_1y_1) \leq y_1 - y_1^2 - k_1y_1
\]
\[
< y_1 - y_1^2 - y_1^2 + \frac{\sqrt{5y_1^2 + 4y_1}}{2}
\]
\[
= \left( -\frac{y_1 + \sqrt{5y_1^2 + 4y_1}}{2} \right)^2 - 2y_1^2
\]
\[
= \min \left( y_1 - y_1^2 - y_1 - \frac{\sqrt{5y_1^2 + 4y_1}}{2} \right)^2, \left( -\frac{y_1 + \sqrt{5y_1^2 + 4y_1}}{2} \right)^2 - 2y_1^2
\]
and if \(k_1 < -\frac{y_1 + \sqrt{5y_1^2 + 4y_1}}{2}\) then
\[
\min(k_1^2 - 2y_1^2, y_1 - y_1^2 - k_1y_1) \leq k_1^2 - 2y_1^2
\]
\[
< \left( -\frac{y_1 + \sqrt{5y_1^2 + 4y_1}}{2} \right)^2 - 2y_1^2 = y_1 - y_1^2 - y_1 - \frac{\sqrt{5y_1^2 + 4y_1}}{2}
\]
\[
= \min \left( y_1 - y_1^2 - y_1 \right)^2, \left( -\frac{y_1 + \sqrt{5y_1^2 + 4y_1}}{2} \right)^2 - 2y_1^2
\]
In both cases
\[
\min(k_1^2 - 2y_1^2, y_1 - y_1^2 - k_1y_1)
\]
\[
< \min \left( y_1 - y_1^2 - y_1 - \frac{\sqrt{5y_1^2 + 4y_1}}{2} \right)^2, \left( -\frac{y_1 + \sqrt{5y_1^2 + 4y_1}}{2} \right)^2 - 2y_1^2,
\]
and \(0 < \frac{-y_1 + \sqrt{5y_1^2 + 4y_1}}{2} < 1\) (because \(y_1 < \sqrt{5y_1^2 + 4y_1} < y_1 + 2\)), so \((k_1, y_1)\) doesn’t represent the maximum, a contradiction.

Since \(y^2 - \frac{k_1^2}{2} = -y(1-k_1-y_1)\) for \(k_1 = \frac{-y_1 + \sqrt{5y_1^2 + 4y_1}}{2}\), one may search for \(\max S(y)\) with \(0 \leq y \leq 1\), where

\[
S(y) = y - y^2 - y - y + \frac{\sqrt{5y^2 + 4y}}{2} = y - y + \frac{\sqrt{5y^2 + 4y}}{2}.
\]

Consider the derivative of \(S\)

\[
S'(y) = \left(y - y + \frac{\sqrt{5y^2 + 4y}}{2}\right)'
= 1 - y - \frac{\sqrt{5y^2 + 4y}}{2} - y \frac{10y + 4}{4\sqrt{5y^2 + 4y}}
= -\frac{(y + \sqrt{5y^2 + 4y})(5y - 1)(y + 1)}{\sqrt{5y^2 + 4y}(\sqrt{5y^2 + 4y} + 1)}.
\]

For \(y > \frac{1}{5}\) one has \(S'(y) < 0\), so \(S(y) < S(\frac{1}{5})\) for each \(y > \frac{1}{5}\). Analogously for \(y < \frac{1}{5}\) one has \(S'(y) > 0\), so \(S(y) < S(\frac{1}{5})\) for each \(y < \frac{1}{5}\). Then \(S(y) \leq S(\frac{1}{5}) = \frac{2}{25}\) for each \(y \in [0, 1]\).

So

\[
\min \left(\max \left(y^2 - \frac{k^2}{2}, -\frac{y(1-k-y)}{2}\right)\right) = -\frac{1}{2} \max \left(\min (k^2 - 2y^2, y - y^2 - ky)\right)
= -\frac{1}{2} \max S(y) = -\frac{1}{25}.
\]

**Appendix C**

Here we solve the system

\[
\begin{align*}
\begin{cases}
\alpha^3 B - B^2 & \geq \beta^2 \frac{K}{1-K}; \\
\frac{1}{2} \leq \alpha \leq 1, \quad 0 < K < 1, \quad 0 \leq \beta \leq \frac{1-K}{K}; \\
\end{cases}
\end{align*}
\]

**Case 1:** \(K > \frac{1}{2}\). Then by AM-GM inequality \(\sqrt{R(\alpha)} \geq 2\beta \sqrt{\frac{K}{1-K}}\) and equality holds for \(B = \beta \sqrt{\frac{K}{1-K}}\). Then

\[
\beta \leq \frac{\sqrt{R(\alpha)}}{2\sqrt{\frac{K}{1-K}}} = \frac{\sqrt{R(\alpha)} \sqrt{1-K}}{K}.
\]

Hence

\[
\frac{\alpha}{2} K^2 - \beta K^2 \geq \frac{\alpha}{2} K^2 - \frac{\sqrt{R(\alpha)} \sqrt{1-K}}{2} K^{3/2} =: q(\alpha, K);
\]
we are going to minimize $q(\alpha, K)$. Derive with respect to $K$:

$$\frac{dq(\alpha, K)}{dK} = K\alpha - \frac{\sqrt{R(\alpha)}}{2} \frac{3 - 4K}{\sqrt{1-K}}.$$ 

Find the roots of the derivative. We may multiply by $\sqrt{\frac{1-K}{K}}$

$$\sqrt{(1-K)K}\alpha = \sqrt{R(\alpha)}\left(\frac{3}{4} - K\right).$$

Then $K < \frac{3}{4}$. Square the equation

$$(1-K)K\alpha^2 = R(\alpha)\left(\frac{3}{4} - K\right)^2.$$ 

It is quadratic in $K$

$$(\alpha^2 + R(\alpha))K^2 - \left(\frac{3}{2} R(\alpha) + \alpha^2\right)K + \frac{9}{16} R(\alpha) = 0.$$ 

Then $D = \frac{3}{4}R(\alpha)\alpha^2 + \alpha^4$ and the roots are

$$K_1 = \frac{(\frac{3}{2} R(\alpha) + \alpha^2) + \sqrt{\frac{3}{4}R(\alpha)\alpha^2 + \alpha^4}}{2(\alpha^2 + R(\alpha))}; \quad K_2 = \frac{(\frac{3}{2} R(\alpha) + \alpha^2) - \sqrt{\frac{3}{4}R(\alpha)\alpha^2 + \alpha^4}}{2(\alpha^2 + R(\alpha))}.$$ 

Obviously, the first root is always bigger than $3/4$. Note that

$$K_2 = \frac{1}{2} + \frac{R(\alpha)}{2} - \sqrt{\frac{3}{4}R(\alpha)\alpha^2 + \alpha^4}.$$ 

Easily

$$\sqrt{\frac{3}{4}R(\alpha)\alpha^2 + \alpha^4} > \alpha^2 > \frac{1}{2} \alpha^{1.5} = \frac{R(\alpha)}{2}$$

since $\alpha \geq \frac{1}{2}$ so the second root is smaller than $1/2$. Hence we should check only $K = 1/2$ and $K = 1$. Clearly $q(\alpha, 1)$ is non-negative; one may check (see Fig. 2) that $q(\alpha, \frac{1}{2})$ is bigger than $-\frac{1}{15}$. 
Figure 2: The plot of $q\left(\alpha, \frac{1}{2}\right) + \frac{1}{54}$.

Figure 3: the plot of $q(\alpha, K_0(\alpha)) + \frac{1}{54}$.

Case 2: $K < \frac{1}{2}$. Consider

$$\sqrt{R(\alpha)} \geq B + \frac{\beta^2}{B} \frac{K}{1-K}.$$ 

It also implies that $B \geq \beta \sqrt{\frac{K}{1-K}}$ but the condition $B \geq \beta$ is stronger since $K < \frac{1}{2}$. Then the optimal $B$ is equal to $\beta$ and hence $\beta = \sqrt{R(\alpha)(1-K)}$ and we minimize

$$q(\alpha, K) := \frac{\alpha}{2} K^2 - \sqrt{R(\alpha)}(1-K)K^2.$$ 

The derivative with respect to $K$ is

$$\alpha K - 2\sqrt{R(\alpha)}K + 3\sqrt{R(\alpha)}K^2.$$ 

It has zeros at 0 and $\frac{2\sqrt{R(\alpha)} - \alpha}{3\sqrt{R(\alpha)}}$. The derivative is negative on $\left(0, \frac{2\sqrt{R(\alpha)} - \alpha}{3\sqrt{R(\alpha)}}\right)$, so $q(\alpha, K)$ is a decreasing function. After $\frac{2\sqrt{R(\alpha)} - \alpha}{3\sqrt{R(\alpha)}}$ the derivative is positive, so the function increases. Hence $q(\alpha, K)$ has local minimum in $K$ at

$$K_0(\alpha) = \frac{2\sqrt{R(\alpha)} - \alpha}{3\sqrt{R(\alpha)}} = \frac{2}{3} - \frac{\alpha}{3\sqrt{R(\alpha)}}.$$ 

Substitution gives

$$q(\alpha, K_0(\alpha)) = \frac{\alpha}{2} \left(\frac{2}{3} - \frac{\alpha}{3\sqrt{R(\alpha)}}\right)^2 - \sqrt{R(\alpha)} \left(\frac{1}{3} + \frac{\alpha}{3\sqrt{R(\alpha)}}\right) \left(\frac{2}{3} - \frac{\alpha}{3\sqrt{R(\alpha)}}\right)^2$$

$$= \frac{\sqrt{R(\alpha)}}{54} \left(\frac{\alpha}{\sqrt{R(\alpha)}} - 2\right)^3.$$ 

One may check (see Fig. 3) that $q(\alpha, K_0(\alpha)) > -\frac{1}{54}$. 

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Appendix D

Now we solve the system

\[
\begin{cases}
\sqrt{(1 - \sqrt{1 - \alpha})(\sqrt{1 - \alpha} + \alpha)} B - B^2 \geq \beta^2 \frac{K}{1-K}; \\
0 \leq \alpha \leq \frac{1}{2}, \quad 0 < K < 1, \quad 0 \leq \beta \leq B \leq \frac{1-K}{K}; \\
\text{minimize} \quad \frac{\alpha}{2} K^2 - \beta K^2.
\end{cases}
\]

First, consider \(T(\alpha)\). Since it is positive, \(\sqrt{T(\alpha)}\) and \(T(\alpha)\) have the same intervals of monotonicity. Change the variable \(t = \sqrt{1 - \alpha}\). Note that \(\alpha \in [0; 1/2)\) implies \(t \in \left(\frac{1}{\sqrt{2}}; 1\right]\).

Then \(T(\alpha) = (1 - t)(t + 1 - t^2) = t^3 - 2t^2 + 1\).

Since \(T'(t) = 3t^2 - 4t = 3t(t - \frac{4}{3}) < 0\) for all \(t\), \(T(t)\) is a decreasing function. Note that \(t(\alpha)\) is also decreasing, so \(\sqrt{T(\alpha)}\) and \(T(\alpha)\) are increasing functions.

Consider two cases.

Case 1: \(K > \frac{1}{2}\). Then by AM-GM inequality \(\sqrt{T(\alpha)} \geq 2\beta \sqrt{\frac{K}{1-K}}\) and equality holds for \(B = \beta \sqrt{\frac{K}{1-K}}\). Then

\[
\beta \leq \frac{\sqrt{T(\alpha)}}{2\sqrt{\frac{K}{1-K}}} = \frac{\sqrt{T(\alpha)}}{2} \sqrt{\frac{1-K}{K}}.
\]

Analogously to Appendix C we reduce to finding the minimum of

\[
q(\alpha, K) := \frac{\alpha}{2} K^2 - \frac{\sqrt{T(\alpha)}}{2} \sqrt{1-K} K^{3/2}.
\]

Again derive with respect to \(K\) and find the roots

\[
K_1 = \frac{\left(\frac{3}{2} T(\alpha) + \alpha^2\right) + \sqrt{\frac{9}{4} T(\alpha)^2 + \alpha^4}}{2(\alpha^2 + T(\alpha))}; \quad K_2 = \frac{\left(\frac{3}{2} T(\alpha) + \alpha^2\right) - \sqrt{\frac{9}{4} T(\alpha)^2 + \alpha^4}}{2(\alpha^2 + T(\alpha))}.
\]

Obviously, \(K_1 > 3/4\). So the only possible root is \(K_2\). We should examine \(K = K_2\) (in the case when it is bigger than \(1/2\), \(K = \frac{1}{2}\) and \(K = 1\). One can see (for example by compare the plots on Fig. 4 and Fig. 5) that \(K_2(\alpha) > \frac{1}{2}\) implies that \(q(\alpha, K_2(\alpha))\) is bigger than \(-\frac{1}{54}\).
Figure 4: The plot of \( q(\alpha, K_2(\alpha)) + \frac{1}{54} \).

Finally, note that for \( K = 1 \) function \( q \) is positive. For \( K = 1/2 \) one may see the plot on Fig. 6 to check that \( q(\alpha, \frac{1}{2}) > -\frac{1}{54} \).

Figure 6: The plot of \( q(\alpha, \frac{1}{2}) + \frac{1}{54} \).

Case 2: One can repeat step-by-step the second case of Appendix C. We minimize

\[
q(\alpha, K) := \frac{\alpha}{2} K^2 - \sqrt{T(\alpha)}(1 - K)K^2.
\]

Derivation and substitution gives

\[
q(\alpha, K_0(\alpha)) = \frac{\sqrt{T(\alpha)}}{54} \left( \frac{\alpha}{\sqrt{T(\alpha)}} - 2 \right)^3.
\]

One may check (see Fig. 7) that \( q(\alpha, K_0(\alpha)) > -\frac{1}{54} \).

Figure 5: The plot of \( K_2(\alpha) - \frac{1}{2} \).

Figure 7: The plot of \( q(\alpha, K_0(\alpha)) + \frac{1}{54} \).