On the Minimal Sum of Weights on the Edges in a Signed Edge-Dominated Graph

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Abstract

Let G be a simple graph with n vertices and ± 1-weights on edges. Suppose that for every edge e the sum of edges adjacent to e (including e itself) is positive. Then the sum of weights over edges of G is at least $-\frac{n^2}{25}$. Also we provide an example of a weighted graph with described properties and the sum of weights $-(1+o(1))\frac{n^2}{8(1+\sqrt{2})^2}$.

The previous best known bounds were $-\frac{n^2}{16}$ and $-(1+o(1))\frac{n^2}{54}$ respectively. We show that the constant $-1/54$ is optimal under some additional conditions.

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1 Introduction

A graph (finite, simple, undirected) is a pair (V, E), where V stands for a set of vertices, and E denotes a set of unordered pairs of vertices, whose elements are called edges. Let G be a graph; for a given edge $e = (u, v)$ define its *closed edge-neighborhood* as an edge subset $N[e]$ formed by e and all edges of G adjacent to e. A weight function $f : E \to \{+1; -1\}$ is called a *signed edge domination function* of G if

$$\sum_{e' \in N[e]} f(e') \geqslant 1$$

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for every $e \in E$; in this case we say that (G, f) is an *SED-pair* of order $|V|$. Let $s[(G, f)]$ be the sum of weights over all edges of a graph G equipped by a weight function f.

Denote by E_+ the set $\{(u, v) \in E \mid f(u, v) = 1\}$ and by E_- the set $\{(u, v) \in E|f(u, v) = -1\}$. Define

$$s_v = \sum_{e \in N(v)} f(e)$$

for each $v \in V$, where $N(v)$ stands for the set of edges containing v. Let V_+ be $\{v \in V|s_v \geqslant 0\}$ and V_- be $\{v \in V|s_v < 0\}$.

The following problem was posed by Xu in [5, 6].

Problem 1. What is

$$g(n) := \min\{s[(G, f)] \mid (G, f) \text{ is an SED-pair of order } n\}$$

for each positive integer n?

Note that for every $g(n) \leqslant 0$ since an empty graph provides an SED-pair. The only known result was provided by the following theorem.

Theorem 2 (Akbari–Bolouki–Hatami–Siami [2])**.**

(i) For every n

$$g(n) \geqslant -\frac{n^2}{16}.$$

(ii) There is a sequence of SED-pairs of order n that satisfies[1]

$$s[G, f] \leqslant -(1 + o(1))\frac{n^2}{54}.$$

We refine both items as follows.

Theorem 3.

(i) For every n, $g(n) \geqslant -\frac{n^2}{25}$.

(ii) For every n there is an SED-pair of order n that satisfies

$$s[G, f] < -(1 + o(1))\frac{n^2}{8(1 + \sqrt{2})^2}.$$

Moreover, if $n = 4(p + q)p$, where $p > 1$ and $q > 1$ are positive integers satisfying $p^2 = 2q^2 - 1$, then

$$s[G, f] = \left\lfloor -\frac{n^2}{8(1 + \sqrt{2})^2} + \frac{3\sqrt{2} - 4}{4}n \right\rfloor.$$

[1] In fact the authors claim the bound $-\frac{n^2}{72}$ but the provided example gives the bound $-(1 + o(1))\frac{n^2}{54}$.

Note that there are infinitely many p and q satisfying the condition $p^2 = 2q^2 - 1$, since it is a special case of Pell's equation; it is well known that the positive solutions are

$$p = \frac{\sqrt{2}-1}{2}(3+2\sqrt{2})^k - \frac{1+\sqrt{2}}{2}(3-2\sqrt{2})^k, q = \frac{\sqrt{2}-1}{2\sqrt{2}}(3+2\sqrt{2})^k + \frac{1+\sqrt{2}}{2\sqrt{2}}(3-2\sqrt{2})^k,$$

for $k \in \mathbb{N}$.

We show that Theorem 2(ii) is optimal under additional assumptions.

Theorem 4. *Let (G, f) be an SED-pair of order n. Suppose that every $e \in E_-$ connects a vertex from V_+ and a vertex from V_-; and every $e \in E_+$ connects some vertices from V_+. Then*

$$s[(G, f)] \geqslant -\frac{1}{54}n^2.$$

1.1 Graphons

A *graphon* (also known as a graph limit) is a symmetric measurable function $W : [0,1]^2 \to [0,1]$. Define a *signed graphon* as a symmetric measurable function $W : [0,1]^2 \to [-1,1]$. A signed graphon is *edge-dominated* if $W(x,y) \neq 0$ implies

$$\int_0^1 (W(x,t) + W(y,t))dt \geqslant 0.$$

Here we consider a continuous analogue of Problem 1. Denote

$$\kappa := \inf \frac{1}{2} \int_0^1 \int_0^1 W(x,y)dxdy \tag{1}$$

where the infimum is taken over all edge-dominated graphons W.

The following theorem is a standard result in the theory of graph limits [3], we include the proof in Appendix A for completeness.

Theorem 5.

(i) $g(n) \geqslant \kappa n^2$, in other words $s(G, f) \geqslant \kappa n^2$ for any SED-pair (G, f) of order n;

(ii) $g(n) = (\kappa + o(1))n^2$ for large n.

Theorems 3 and 4 also have natural continuous analogues.

Structure of the paper. Theorem 3(ii) is proved in Section 2. Section 3 is devoted to the proof of Theorem 3(i). Section 4 cites a result, determining the maximal sum of squares of vertex degrees among all graphs with n vertices and e edges; we use it in Section 5, containing the proof of Theorem 4. Appendix A contains the proof of Theorem 5, Appendices B-D contain auxiliary calculations.

2 Examples

In this section we provide a sequence of SED-pairs that achieves the upper bound

$$-(1+o(1))\frac{n^2}{8(1+\sqrt{2})^2}.$$

2.1 A graphon example

The following signed graphon realizes an example for Theorem 3(ii). Put $[0,1]=A\sqcup B\sqcup C$, where $|A|=1-\frac{1}{\sqrt{2}}$, $|B|=\frac{1}{\sqrt{2}}-\frac{1}{2}$, $|C|=\frac{1}{2}$. The function W is defined in Fig. 1.

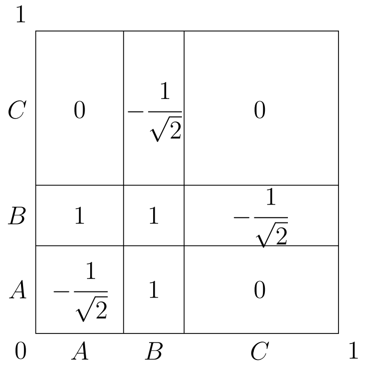

Figure 1: A graphon example for Theorem 3(ii).

Note that W is edge-dominated: indeed, for $(x,y)\in A\times A$

$$\int_0^1 (W(x,t)+W(t,y))dt = 2\left(-\frac{1}{\sqrt{2}}|A|+|B|\right)=0,$$

for $(x,y)\in A\times B$

$$\int_0^1 (W(x,t)+W(t,y))dt = -\frac{1}{\sqrt{2}}|A|+|B|+|A|+|B|-\frac{1}{\sqrt{2}}|C| = \frac{1}{2}-\frac{1}{2\sqrt{2}}>0,$$

for $(x,y)\in B\times B$

$$\int_0^1 (W(x,t)+W(t,y))dt = 2\left(|A|+|B|-\frac{1}{\sqrt{2}}|C|\right)=1-\frac{1}{\sqrt{2}}>0,$$

and for $(x,y)\in B\times C$

$$\int_0^1 (W(x,t)+W(t,y))dt = |A|+|B|-\frac{1}{\sqrt{2}}|C|-\frac{1}{\sqrt{2}}|B|=0.$$

Finally,

$$\frac{1}{2}\int_0^1\int_0^1 W(x,y)dxdy = \frac{1}{2}\left(-\frac{|A|^2}{\sqrt{2}}+2|A|\cdot|B|+|B|^2-\frac{2|B|\cdot|C|}{\sqrt{2}}\right)=-\frac{1}{8(1+\sqrt{2})^2}.$$

2.2 An explicit graph approximation

Here we provide the best approximation we can do. Fix p and q such that $p^2 = 2q^2 - 1$, and $p, q > 1$.

We need several auxiliary definitions. Define a graph $K_{X,Y,\frac{k}{l}} = (X \cup Y, E_{X,Y,\frac{k}{l}})$ for $|X| = al, |Y| = bl$ and integers $a, b, k \leqslant l$. Split X into a disjoint sets of size l: $X = X_1 \cup X_2 \cup \cdots \cup X_a$ with $|X_i| = l$; also split Y into b disjoint sets of the same size: $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_b$ with $|Y_i| = l$. For each pair $1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b$ consider the following bipartite graph $G_{ij} = (X_i \cup Y_j, E_{ij})$ with parts X_i and Y_j (all graphs G_{ij} are isomorphic). Enumerate vertices as follows $X_i = \{v_1, v_2, \dots, v_l\}$, $Y_j = \{u_1, u_2, \dots, u_l\}$. Define E_{ij} as the set of all pairs (v_g, u_h), for which $g - h \mod l$ lies in $\{1, 2, \dots, k\}$. Put

$$E_{X,Y,\frac{k}{l}} = \bigcup_{1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b} E_{ij}.$$

Obviously the degree of every vertex in G_{ij} is equal to k, so the degree of a vertex in $K_{X,Y,\frac{k}{l}}$ is $bk = |Y|\frac{k}{l}$ for vertices in X, and $ak = |X|\frac{k}{l}$ for vertices in Y.

Now define graph $K_{X,\frac{k}{l}} = (X, E_{X,\frac{k}{l}})$ for $|X| = 2al$ and integer $a, k < l$. Split X into $2l$ disjoint sets of size a: $X = X_1 \cup X_2 \cup \cdots \cup X_{2l}$. The edge between vertices u and v exists if and only if $i - j \mod 2l$ lies in

$$\{-k, -(k-1), \dots, -2, -1, 1, 2, \dots, k-1, k\},$$

where $v \in X_i$, $u \in X_j$. Then the degree of every vertex in $K_{X,\frac{k}{l}}$ is equal to $2ak = |X|\frac{k}{l}$.

Let $K_X = (X, E_X)$ be the complete graph (i.e. every pair of vertices forms an edge) on the vertex set X. Degree of each vertex in K_X is equal to $|X| - 1$.

Now we are ready to provide the desired construction. Let p and q be a positive solution of $p^2 = 2q^2 - 1$. Put

$$A = \{a_1, a_2, \dots, a_{2p^2}\}, \quad B_1 = \{b_1, b_2, \dots, b_{2p(p-q)}\}, \quad B_2 = \{b_{2p(p-q)+1}, b_{2p(p-q)+2}, \dots, b_{2pq}\},$$

$$C_1 = \{c_1, c_2, \dots, c_{6p(p-q)}\}, \quad C_2 = \{c_{6p(p-q)+1}, c_{6p(p-q)+2}, \dots, c_{2(p+q)p}\}.$$

Define the vertex set

$$V = A \cup B_1 \cup B_2 \cup C_1 \cup C_2$$

(so $n = 4p^2 + 4pq$). The edge set E and weight function f are defined by explicit expressions for E_+ and E_-:

$$E_+ = E_{A,B_1 \cup B_2,\frac{1}{1}} \cup E_{B_1,\frac{p^2-pq-1}{p(p-q)}} \cup E_{B_1,B_2,\frac{1}{1}} \cup E_{B_2};$$

$$E_- = E_{A,\frac{q}{p}} \cup E_{B_1,C_2,\frac{q}{p}} \cup E_{B_2,C_1,\frac{q}{p}}, \cup E_{B_1,C_1,\frac{2pq-2q^2-1}{2p(p-q)}} \cup E_{B_2,C_2,\frac{4q^2-2pq-1}{2p(2q-p)}}.$$

Since p divides all of the cardinalities $|A|, |B_1|, |B_2|, |C_1|, |C_2|$; $2p(p-q)$ divides $|B_1|$, $|C_1|$, and $2p(2q-p)$ divides $|B_2|$, $|C_2|$, the definition of f is correct.

Some annoying calculation gives

$$s_{a_i} = 0, \quad s_{b_i} = p^2, \quad s_{c_i} = -p^2$$

for every i.

Note that there is no edge between A and C or inside C. Also all edges inside A of between B and C are negative, so our construction is an SED-pair.

Finally we count

$$s[G,f] = \frac{1}{2}\sum_{v\in V} s_v = \frac{p^2(|B_1|+|B_2|-|C_1|-|C_2|)}{2} = -p^4.$$

Recall that $p^2 = 2q^2 - 1$ and $n = 4p^2 + 4pq = 2p(2p + \sqrt{2}\sqrt{1+p^2})$. So

$$\frac{s[G,f]}{n^2} = \frac{-p^4}{(2p(2p+\sqrt{2}\sqrt{1+p^2}))^2} = -\frac{1}{8(1+\sqrt{2})^2} + \frac{5\sqrt{2}-7}{8p^2} + \frac{31\sqrt{2}-44}{32p^4} + O(p^{-5}).$$

Since $n = (4+2\sqrt{2})p^2 + \sqrt{2} - \frac{1}{2\sqrt{2}p^2} + O(p^{-3})$

$$s[G,f] = -\frac{n^2}{8(1+\sqrt{2})^2} + \frac{3\sqrt{2}-4}{4}n - \frac{1}{2(2+\sqrt{2})} + o(1).$$

One can also derive

$$s[G,f] = \left\lfloor -\frac{n^2}{8(1+\sqrt{2})^2} + \frac{3\sqrt{2}-4}{4}n \right\rfloor.$$

3 The lower bound of Theorem 3

Consider an arbitrary SED-pair (G,f), where $G=(V,E)$.

It is known that for each $v,u\in V$ if $(v,u)\in E_-\cup E_+$, then $s_v + s_u \geqslant 0$ (check it by hands or see Lemma 1 in [2]). If V_- is empty, then $s[G,f]\geqslant 0$. Let x be

$$-\min_{v\in V_-} s_v$$

and consider an arbitrary vertex a such that $s_a = -x$. Let $N_-(a)$ be $\{v\in V | (a,v)\in E_-\}$. Then $|N_-(a)|\geqslant x$ and $s_v \geqslant x$ for each $v\in N_-(a)$, so $N_-(a)\subset V_+$. Then

$$x^2 \leqslant \sum_{v\in N_-(a)} s_v \leqslant \sum_{v\in V_+} s_v.$$

Clearly, V_- is an independent set (i.e. has no edges inside) so

$$\sum_{v\in V_+} s_v = \sum_{v\in V_-} s_v + 2\left(\sum_{(u,v)\in E_+|u,v\in V_+} 1 - \sum_{(u,v)\in E_-|u,v\in V_+} 1\right)$$

$$\leqslant \sum_{v\in V_-} s_v + 2\frac{|V_+|\cdot(|V_+|-1)}{2} \leqslant \sum_{v\in V_-} s_v + |V_+|^2.$$

So

$$\sum_{v\in V_-} s_v \geqslant x^2 - |V_+|^2;$$

recall that

$$\sum_{v\in V_+} s_v \geqslant x|N_-(a)| \geqslant x^2.$$

On the other hand

$$s[(G,f)] = \sum_{(x,y)\in E_+} 1 - \sum_{(x,y)\in E_-} 1 = \frac{\sum_{v\in V} s_v}{2},$$

and

$$\sum_{v\in V} s_v = \sum_{v\in V_+} s_v + \sum_{v\in V_-} s_v \geqslant 2x^2 - |V_+|^2.$$

Also

$$\sum_{v\in V} s_v = \sum_{v\in V_+} s_v + \sum_{v\in V_-} s_v \geqslant x^2 - x|V_-| = -x(|V_-|-x) = -x(|V|-|V_+|-x).$$

Put $y = \frac{x}{|V|}$, $k = \frac{|V_+|}{|V|}$. Then we have the following system of inequalities:

$$\begin{cases} s[(G,f)] \geqslant (y^2 - \frac{k^2}{2})|V|^2 \\ s[(G,f)] \geqslant \frac{-y(1-k-y)}{2}|V|^2. \end{cases}$$

So

$$g(n) \geqslant \min_{0\leqslant y\leqslant 1, 0\leqslant k\leqslant 1} \left(\max\left(y^2 - \frac{k^2}{2}, -\frac{y(1-k-y)}{2}\right)\right) n^2.$$

One may check by computer (or read explicit calculus in Appendix B) that the minimum is $-\frac{1}{25}$ and is reached at $y = \frac{1}{5}$, $k = \frac{2}{5}$.

4 Degree sequences of a graph

Here we display the results from [1], which are required in the proof of Theorem 4; for a survey see [4].

Definition 6. Let n, $e \leqslant \binom{n}{2}$ be integer numbers. Consider the unique representation

$$e = \binom{a}{2} + b, \quad 0 \leqslant b < a.$$

The quasi-complete graph C_n^e with e edges and n vertices $v_1, \ldots, v_n$ has edges (v_i, v_j) for $i, j \leqslant a$ and $i = a+1$, $j \in \{1, \ldots, b\}$.

Definition 7. Let n, $e \leqslant \binom{n}{2}$ be integer numbers. Consider the unique representation

$$\binom{n}{2} - e = \binom{c}{2} + d, \quad 0 \leqslant d < c.$$

The quasi-star graph S_n^e is the graph with e edges and n vertices $v_1, \ldots, v_n$, such that vertices $v_1, \ldots, v_{n-c-1}$ are connected with all vertices and vertex v_{n-c} is connected with vertices $v_1, \ldots, v_{n-d}$.

Let $F(n, e)$ be the maximal value of

$$\sum_{v \in V} (\deg v)^2$$

among the graphs $G = (V, E)$ with n vertices and e edges. We use the following result.

Theorem 8 (Alshwede–Katona, [1]). *For every n and $0 \leqslant e \leqslant \binom{n}{2}$ the value $F(n, e)$ is achieved on C_n^e or S_n^e.*

For $G = C_n^e$ the sum of squares of degrees equals to

$$\begin{aligned} ba^2 + (a-b)(a-1)^2 + b^2 &= a^3 - 2a^2 + 2ab + b^2 + a - b \\ &= (1 + o(1)) \left(\frac{a}{n}\right)^3 n^3 \\ &= (1 + o(1)) \left(\frac{2e}{n^2}\right)^{3/2} n^3, \end{aligned}$$

because $a = (1 + o(1))\sqrt{2e}$. For $G = S_n^e$ this sum equals to

$$\begin{aligned} &(n-c-1)(n-1)^2 + (n-d-1)^2 + (c-d)(n-c)^2 + d(n-c-1)^2 \\ &= (1+o(1))((n-c)n^2 + (c-d)(n-c)^2 + d(n-c)^2) \\ &= (1+o(1))(n-c)(n^2 + (n-c)c) = (1+o(1))n^3 \left(1 - \frac{c}{n}\right)\left(1 + \frac{c}{n} - \frac{c^2}{n^2}\right) \\ &= (1+o(1)) \left(1 - \sqrt{1 - \frac{2e}{n^2}}\right)\left(1 + \sqrt{1 - \frac{2e}{n^2}} - \left(1 - \frac{2e}{n^2}\right)\right) n^3 \\ &= (1+o(1)) \left(1 - \sqrt{1 - \frac{2e}{n^2}}\right)\left(\sqrt{1 - \frac{2e}{n^2}} + \frac{2e}{n^2}\right) n^3, \end{aligned}$$

because $c = (1 + o(1))(\sqrt{n^2 - 2e})$.

Corollary 9. *Put $\alpha = \frac{2e}{n^2}$. Then*

$$F(n, e) = (1 + o(1)) \max\left(\alpha^{\frac{3}{2}}, (1 - \sqrt{1-\alpha})(\sqrt{1-\alpha} + \alpha)\right) n^3.$$

Define

$$R(\alpha) := \alpha^{\frac{3}{2}}, \qquad T(\alpha) = (1-\sqrt{1-\alpha})(\sqrt{1-\alpha}+\alpha).$$

We show that $R(\alpha) < T(\alpha)$ for $\alpha \in (0, 1/2)$ and $R(\alpha) > T(\alpha)$ for $\alpha \in (1/2, 1)$. Define $t = \sqrt{1-\alpha}$. Note that

$$R^2(\alpha) - T^2(\alpha) = (1-t)^2(1+t-t^2)^2 - (1-t^2)^3 = t^2(1-t)^2(2t^2-1).$$

For $\alpha \in (1/2, 1)$ one has $t \in \left(0, \frac{\sqrt{2}}{2}\right)$ and $R(\alpha) > T(\alpha)$. For $\alpha \in (0, 1/2)$ one has $t \in \left(\frac{\sqrt{2}}{2}, 1\right)$ and $R(\alpha) < T(\alpha)$.

There are several weaker and better-looking bounds on $F(n, e)$, but they do not meet our aims.

5 Proof of Theorem 3

Put $k = |V_+|$. Let the degrees of vertices in $G[V_+]$ be equal to $a_1, \dots, a_k$; the degrees of vertices in $G[V_+, V_-]$ be equal to $b_1, \dots, b_k$ for $v_i \in V_+$ and $c_1, \dots, c_{n-k}$ for $v_j \in V_-$. Define

$$a = \frac{1}{k}\sum_{1 \leqslant i \leqslant k} a_i; \qquad b = \frac{1}{k}\sum_{1 \leqslant i \leqslant k} b_i; \qquad c = \frac{1}{n-k}\sum_{1 \leqslant j \leqslant n-k} c_j;$$

by double-counting in the graph $G[V_+, V_-]$ we have $kb = (n-k)c$.

By the main condition, if we have an edge (v_i^+, v_j^-) then

$$a_i - b_i \geqslant c_j.$$

Sum up all these inequalities; then every vertex v_i^+ is counted b_i times, and every vertex v_j^- is counted c_j times. Hence

$$\sum_{1 \leqslant i \leqslant k} (a_i - b_i) b_i \geqslant \sum_{1 \leqslant j \leqslant n-k} c_j^2.$$

Applying Cauchy–Bunyakovsky–Schwarz inequality, we get

$$\sqrt{\sum_{1 \leqslant i \leqslant k} a_i^2 \sum_{1 \leqslant i \leqslant k} b_i^2} - \sum_{1 \leqslant i \leqslant k} b_i^2 \geqslant \sum_{1 \leqslant i \leqslant k} (a_i - b_i) b_i.$$

The AM-GM inequality implies

$$\sum_{1 \leqslant j \leqslant n-k} c_j^2 \geqslant (n-k)c^2 = \frac{k^2}{n-k} b^2.$$

Consider the following "dimensionless" quantities

$$\alpha = \frac{a}{k} = \frac{1}{k^2}\sum_{1 \leqslant i \leqslant k} a_i; \qquad \beta = \frac{b}{k} = \frac{1}{k^2}\sum_{1 \leqslant i \leqslant k} b_i; \qquad B = \sqrt{\frac{1}{k^3}\sum_{1 \leqslant i \leqslant k} b_i^2}; \qquad K = \frac{k}{n}.$$

Then applying Corollary 9 to V_+ and using obtained inequalities we have

$$U(\alpha)k^{3/2}Bk^{3/2} - B^2k^3 \geqslant \sqrt{\sum_{1\leqslant i\leqslant k} a_i^2 \sum_{1\leqslant i\leqslant k} b_i^2} - \sum_{1\leqslant i\leqslant k} b_i^2 \geqslant \sum_{1\leqslant i\leqslant k} (a_i - b_i)b_i \geqslant$$

$$\sum_{1\leqslant j\leqslant n-k} c_j^2 \geqslant \frac{k^2}{n-k}b^2 = \beta^2\frac{k^4}{n-k},$$

where $U(\alpha) = \max\left(\sqrt{R(\alpha)}, \sqrt{T(\alpha)}\right)$. By the AM-GM inequality $\beta \leqslant B$. Also,

$$s[(G,f)] = \frac{1}{2}\sum_{1\leqslant i\leqslant k} a_i - \sum_{1\leqslant i\leqslant k} b_i = \left(\frac{\alpha}{2}K^2 - \beta K^2\right)n^2.$$

Thus we have reduced our problem to the following optimization problem:

$$\begin{cases} U(\alpha)B - B^2 \geqslant \beta^2\frac{K}{1-K}; \\ \text{minimize} \quad \frac{\alpha}{2}K^2 - \beta K^2; \\ 0 \leqslant \alpha \leqslant 1, \quad 0 \leqslant K \leqslant 1, \quad 0 \leqslant \beta \leqslant B \leqslant \frac{1-K}{K}, \end{cases} \tag{2}$$

where the last inequality follows from the fact that every b_i is at most $n-k$.

We show that the desired minimum is $-\frac{1}{54}$; it can be reached by the example from Theorem 2(ii). Note that a possible (with respect to conditions of the system (2)) value of (α, β, B, K) may not correspond to an SED-pair.

Case 1. In this case $\alpha \geqslant \frac{1}{2}$, so $U(\alpha) = \sqrt{R(\alpha)}$. Then we have to solve the following system

$$\begin{cases} \alpha^{\frac{3}{4}}B - B^2 \geqslant \beta^2\frac{K}{1-K}; \\ \frac{1}{2} \leqslant \alpha \leqslant 1, \quad 0 < K < 1, \quad 0 \leqslant \beta \leqslant B \leqslant \frac{1-K}{K}; \\ \text{minimize} \quad \frac{\alpha}{2}K^2 - \beta K^2. \end{cases}$$

In Appendix C we show that the minimum is $-\frac{1}{54}$.

Case 2. In this case $\alpha \leqslant \frac{1}{2}$. Then $U(\alpha) = \sqrt{T(\alpha)}$. Then we have a deal with the following system

$$\begin{cases} \sqrt{(1-\sqrt{1-\alpha})(\sqrt{1-\alpha}+\alpha)}B - B^2 \geqslant \beta^2\frac{K}{1-K}; \\ 0 \leqslant \alpha \leqslant \frac{1}{2}, \quad 0 < K < 1, \quad 0 \leqslant \beta \leqslant B \leqslant \frac{1-K}{K}; \\ \text{minimize} \quad \frac{\alpha}{2}K^2 - \beta K^2. \end{cases}$$

This system is analyzed in Appendix D; the minimum is bigger than the desired value $-\frac{1}{54}$.

So we prove $s[(G,f)] \geqslant -(1+o(1))\frac{n^2}{54}$; Theorem 5 finishes the proof.

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References

[1] Rudolf Ahlswede and Gyula O. H. Katona. Graphs with maximal number of adjacent pairs of edges. *Acta Mathematica Hungarica*, 32(1-2):97–120, 1978.

[2] Saeed Akbari, Sadegh Bolouki, Pooya Hatami, and Milad Siami. On the signed edge domination number of graphs. *Discrete Mathematics*, 309(3):587–594, 2009.

[3] László Lovász. *Large networks and graph limits*, volume 60. American Mathematical Soc., 2012.

[4] Vladimir Nikiforov. The sum of the squares of degrees: Sharp asymptotics. *Discrete Mathematics*, 307(24):3187–3193, 2007.

[5] Baogen Xu. On signed edge domination numbers of graphs. *Discrete Mathematics*, 239(1-3):179–189, 2001.

[6] Baogen Xu. On edge domination numbers of graphs. *Discrete Mathematics*, 294(3):311–316, 2005.

Appendix A

Proof of Theorem 5. (i) Let $(G = (V, E), f)$ be an SED-pair of order n. We partition $[0, 1]$ into n disjoint sets of measure $1/n$ and identify these n sets with n vertices of G. For points $x, y \in [0, 1]$ denote by v, u the vertices which contain them, respectively, and put

$$W(x,y) = \begin{cases} f(v,u), & \text{if } (v,u) \in E \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\int_0^1 (W(x,t) + W(y,t))dt = s_v + s_u \geqslant 0$ whenever $x \in v$, $y \in u$ and $(v, u) \in E$. Thus signed graphon W is edge-dominated, and $\kappa \leqslant \frac{1}{2}\int_0^1 \int_0^1 W = \frac{1}{n^2} s(G, f)$ that proves (i).

(ii) Fix $\varepsilon \in (0, 1)$ and an edge-dominated signed graphon W such that $\frac{1}{2}\int_0^1 \int_0^1 W < \kappa + \varepsilon$. Let n be a (large) integer. Denote $k = \lfloor \varepsilon n \rfloor$, $m = n - k$. Since $\varepsilon > 0$ is arbitrary, and the lower bound $g(n) \geqslant \kappa n^2$ is already established in (i), for proving (ii) it suffices to prove that

$$g(n) \leqslant 2kn + m^2(\kappa + \varepsilon) \tag{3}$$

for all large enough n.

Choose m points $v_1, \dots, v_m \in [0, 1]$ uniformly and independently at random. Denote $V = \{1, 2, \dots, n\}$, and define the signed graph $G = (V, E)$ as follows:

1) if $i > m$, the vertex i is joined with all other vertices and $f(i, j) = 1$ for all $j \in V \setminus \{i\}$;

2) if $i, j \leqslant m$, we join i and j by an edge with probability $|W(v_i, v_j)|$ and put $f(i, j) = \operatorname{sign} W(v_i, v_j)$ if i and j become joined (the above events are independent).

If we define

$$\tilde{f}(i,j) = \begin{cases} f(i,j), & \text{if } (i,j) \in E \\ 0, & \text{otherwise,} \end{cases}$$

then the expectation of $\tilde{f}(i, j)$ equals $W(v_i, v_j)$. If $v_1, \dots, v_m$ are fixed, the Chernoff bound guarantees that:

a) the probability that $s_i - k = \sum_{j \leqslant m} \tilde{f}(i, j)$ *differs from* $\sum_j W(v_i, v_j)$ *by a value greater than* $k/5$ is exponentially small, and this holds true even if $v_1, \dots, v_m$ are fixed;

b) the probability that $\sum_j W(v_i, v_j)$ *differs from* $m \int_0^1 W(v_i, t)dt$ *by a value greater than* $k/5$ is also exponentially small, and this holds true even if v_i is fixed;

c) the probability that $\sum_i \int_0^1 W(v_i, t)dt$ *differs from* $m \int_0^1 \int_0^1 W(x, y)dxdy$ *by more than* $k/5$ is also exponentially small.

Therefore with high probability none of the above $2m + 1$ events happens, and we get

$$\left| s_i - k - \int_0^1 W(v_i, t)dt \right| \leqslant \frac{2k}{5}$$

for all $i = 1, \dots, m$, and

$$\left| \sum_{i=1}^m s_i - km - m^2 \int_0^1 \int_0^1 W \right| \leqslant \frac{3km}{5}.$$

These bounds yield that (G,f) is an SED-pair, and

$$g(n) \leqslant s[G,f] = \frac{1}{2}\sum_{j=1}^{n} s_j \leqslant k(n-1) + \frac{km}{2} + \frac{3km}{10} + \frac{1}{2}m^2 \int_0^1 \int_0^1 W \leqslant 2kn + m^2(\kappa + \varepsilon)$$

that is (3). □

Appendix B

We have to calculate

$$\min\left(\max\left(y^2 - \frac{k^2}{2}, -\frac{y(1-k-y)}{2}\right)\right) = -\frac{1}{2}\max\left(\min(k^2 - 2y^2, y - y^2 - ky)\right).$$

Let $k_1, y_1 \in [0,1]$ be any values representing this maximum (the maximum is reached by compactness).

First, we show that $y_1^2 - \frac{k_1^2}{2} = -\frac{y_1(1-k_1-y_1)}{2}$. Indeed, this equality means that $k_1 = \frac{-y_1+\sqrt{5y_1^2+4y_1}}{2}$. Suppose the contrary; if $k_1 > \frac{-y_1+\sqrt{5y_1^2+4y_1}}{2}$ then

$$\begin{aligned}
&\min(k_1^2 - 2y_1^2, y_1 - y_1^2 - k_1y_1) \leqslant y_1 - y_1^2 - k_1y_1 \\
&< y_1 - y_1^2 - y_1\frac{-y_1 + \sqrt{5y_1^2 + 4y_1}}{2} \\
&= \left(\frac{-y_1 + \sqrt{5y_1^2 + 4y_1}}{2}\right)^2 - 2y_1^2 \\
&= \min\left(y_1 - y_1^2 - y_1\frac{-y_1 + \sqrt{5y_1^2 + 4y_1}}{2}, \left(\frac{-y_1 + \sqrt{5y_1^2 + 4y_1}}{2}\right)^2 - 2y_1^2\right)
\end{aligned}$$

and if $k_1 < \frac{-y_1+\sqrt{5y_1^2+4y_1}}{2}$ then

$$\begin{aligned}
&\min(k_1^2 - 2y_1^2, y_1 - y_1^2 - k_1y_1) \leqslant k_1^2 - 2y_1^2 \\
&< \left(\frac{-y_1 + \sqrt{5y_1^2 + 4y_1}}{2}\right)^2 - 2y_1^2 = y_1 - y_1^2 - y_1\frac{-y_1 + \sqrt{5y_1^2 + 4y_1}}{2} \\
&= \min\left(y_1 - y_1^2 - y_1\frac{-y_1 + \sqrt{5y_1^2 + 4y_1}}{2}, \left(\frac{-y_1 + \sqrt{5y_1^2 + 4y_1}}{2}\right)^2 - 2y_1^2\right).
\end{aligned}$$

In both cases

$$\begin{aligned}
&\min(k_1^2 - 2y_1^2, y_1 - y_1^2 - k_1y_1) \\
&< \min\left(y_1 - y_1^2 - y_1\frac{-y_1 + \sqrt{5y_1^2 + 4y_1}}{2}, \left(\frac{-y_1 + \sqrt{5y_1^2 + 4y_1}}{2}\right)^2 - 2y_1^2\right),
\end{aligned}$$

and $0 < \frac{-y_1+\sqrt{5y_1^2+4y_1}}{2} < 1$ (because $y_1 < \sqrt{5y_1^2+4y_1} < y_1+2$), so (k_1, y_1) doesn't represent the maximum, a contradiction.

Since $y_1^2 - \frac{k_1^2}{2} = -\frac{y_1(1-k_1-y_1)}{2}$ for $k_1 = \frac{-y_1+\sqrt{5y_1^2+4y_1}}{2}$, one may search for $\max S(y)$ with $0 \leqslant y \leqslant 1$, where

$$S(y) = y - y^2 - y\frac{-y+\sqrt{5y^2+4y}}{2} = y - y\frac{y+\sqrt{5y^2+4y}}{2}.$$

Consider the derivative of S

$$\begin{aligned} S'(y) &= \left(y - y\frac{y+\sqrt{5y^2+4y}}{2}\right)' \\ &= 1 - y - \frac{\sqrt{5y^2+4y}}{2} - y\frac{10y+4}{4\sqrt{5y^2+4y}} \\ &= -\frac{(y+\sqrt{5y^2+4y})(5y-1)(y+1)}{\sqrt{5y^2+4y}(\sqrt{5y^2+4y}+1)}. \end{aligned}$$

For $y > \frac{1}{5}$ one has $S'(y) < 0$, so $S(y) < S(\frac{1}{5})$ for each $y > \frac{1}{5}$. Analogously $y < \frac{1}{5}$ one has $S'(y) > 0$, so $S(y) < S(\frac{1}{5})$ for each $y < \frac{1}{5}$. Then $S(y) \leqslant S(\frac{1}{5}) = \frac{2}{25}$ for each $y \in [0, 1]$. So

$$\begin{aligned} \min\left(\max\left(y^2 - \frac{k^2}{2}, -\frac{y(1-k-y)}{2}\right)\right) &= -\frac{1}{2}\max\left(\min(k^2-2y^2, y-y^2-ky)\right) \\ &= -\frac{1}{2}\max S(y) = -\frac{1}{25}. \end{aligned}$$

Appendix C

Here we solve the system

$$\begin{cases} \alpha^{\frac{3}{4}}B - B^2 \geqslant \beta^2\frac{K}{1-K}; \\ \frac{1}{2} \leqslant \alpha \leqslant 1, \quad 0 < K < 1, \quad 0 \leqslant \beta \leqslant B \leqslant \frac{1-K}{K}; \\ \text{minimize} \quad \frac{\alpha}{2}K^2 - \beta K^2. \end{cases}$$

Case 1: $K > \frac{1}{2}$. Then by AM-GM inequality $\sqrt{R(\alpha)} \geqslant 2\beta\sqrt{\frac{K}{1-K}}$ and equality holds for $B = \beta\sqrt{\frac{K}{1-K}}$. Then

$$\beta \leqslant \frac{\sqrt{R(\alpha)}}{2\sqrt{\frac{K}{1-K}}} = \frac{\sqrt{R(\alpha)}}{2}\sqrt{\frac{1-K}{K}}.$$

Hence

$$\frac{\alpha}{2}K^2 - \beta K^2 \geqslant \frac{\alpha}{2}K^2 - \frac{\sqrt{R(\alpha)}}{2}\sqrt{1-K}K^{3/2} =: q(\alpha, K);$$

we are going to minimize $q(\alpha, K)$. Derive with respect to K:

$$\frac{dq(\alpha, K)}{dK} = K\alpha - \frac{\sqrt{R(\alpha)}}{2} \frac{3-4K}{2\sqrt{\frac{1-K}{K}}}.$$

Find the roots of the derivative. We may multiply by $\sqrt{\frac{1-K}{K}}$

$$\sqrt{(1-K)K}\alpha = \sqrt{R(\alpha)} \left(\frac{3}{4} - K\right).$$

Then $K < \frac{3}{4}$. Square the equation

$$(1-K)K\alpha^2 = R(\alpha) \left(\frac{3}{4} - K\right)^2.$$

It is quadratic in K

$$(\alpha^2 + R(\alpha))K^2 - \left(\frac{3}{2}R(\alpha) + \alpha^2\right) K + \frac{9}{16}R(\alpha) = 0.$$

Then $D = \frac{3}{4}R(\alpha)\alpha^2 + \alpha^4$ and the roots are

$$K_1 = \frac{(\frac{3}{2}R(\alpha) + \alpha^2) + \sqrt{\frac{3}{4}R(\alpha)\alpha^2 + \alpha^4}}{2(\alpha^2 + R(\alpha))}; \qquad K_2 = \frac{(\frac{3}{2}R(\alpha) + \alpha^2) - \sqrt{\frac{3}{4}R(\alpha)\alpha^2 + \alpha^4}}{2(\alpha^2 + R(\alpha))}.$$

Obviously, the first root is always bigger than 3/4. Note that

$$K_2 = \frac{1}{2} + \frac{\frac{R(\alpha)}{2} - \sqrt{\frac{3}{4}R(\alpha)\alpha^2 + \alpha^4}}{2(\alpha^2 + R(\alpha))}.$$

Easily

$$\sqrt{\frac{3}{4}R(\alpha)\alpha^2 + \alpha^4} > \alpha^2 > \frac{1}{2}\alpha^{1.5} = \frac{R(\alpha)}{2}$$

since $\alpha \geqslant \frac{1}{2}$ so the second root is smaller than 1/2. Hence we should check only $K = 1/2$ and $K = 1$. Clearly $q(\alpha, 1)$ is non-negative; one may check (see Fig. 2) that $q\left(\alpha, \frac{1}{2}\right)$ is bigger than $-\frac{1}{54}$.

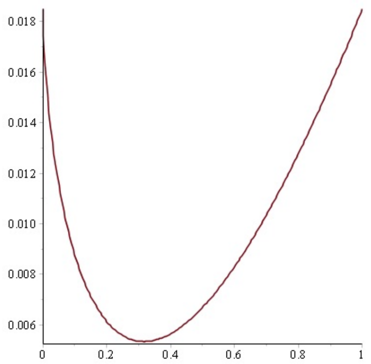

Figure 2: The plot of $q\left(\alpha, \frac{1}{2}\right) + \frac{1}{54}$.

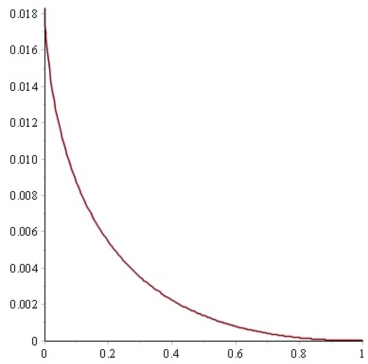

Figure 3: the plot of $q(\alpha, K_0(\alpha)) + \frac{1}{54}$.

Case 2: $K < \frac{1}{2}$. Consider

$$\sqrt{R(\alpha)} \geqslant B + \frac{\beta^2}{B}\frac{K}{1-K}.$$

It also implies that $B \geqslant \beta\sqrt{\frac{K}{1-K}}$ but the condition $B \geqslant \beta$ is stronger since $K < \frac{1}{2}$. Then the optimal B is equal to β and hence $\beta = \sqrt{R(\alpha)}(1-K)$ and we minimize

$$q(\alpha, K) := \frac{\alpha}{2}K^2 - \sqrt{R(\alpha)}(1-K)K^2.$$

The derivative with respect to K is

$$\alpha K - 2\sqrt{R(\alpha)}K + 3\sqrt{R(\alpha)}K^2.$$

It has zeros at 0 and $\frac{2\sqrt{R(\alpha)}-\alpha}{3\sqrt{R(\alpha)}}$. The derivative is negative on $\left(0, \frac{2\sqrt{R(\alpha)}-\alpha}{3\sqrt{R(\alpha)}}\right)$, so $q(\alpha, K)$ is a decreasing function. After $\frac{2\sqrt{R(\alpha)}-\alpha}{3\sqrt{R(\alpha)}}$ the derivative is positive, so the function increases. Hence $q(\alpha, K)$ has local minimum in K at

$$K_0(\alpha) = \frac{2\sqrt{R(\alpha)}-\alpha}{3\sqrt{R(\alpha)}} = \frac{2}{3} - \frac{\alpha}{3\sqrt{R(\alpha)}}.$$

Substitution gives

$$\begin{aligned} q(\alpha, K_0(\alpha)) &= \frac{\alpha}{2}\left(\frac{2}{3} - \frac{\alpha}{3\sqrt{R(\alpha)}}\right)^2 - \sqrt{R(\alpha)}\left(\frac{1}{3} + \frac{\alpha}{3\sqrt{R(\alpha)}}\right)\left(\frac{2}{3} - \frac{\alpha}{3\sqrt{R(\alpha)}}\right)^2 \\ &= \frac{\sqrt{R(\alpha)}}{54}\left(\frac{\alpha}{\sqrt{R(\alpha)}} - 2\right)^3. \end{aligned}$$

One may check (see Fig. 3) that $q(\alpha, K_0(\alpha)) > -\frac{1}{54}$.

Appendix D

Now we solve the system

$$\begin{cases} \sqrt{(1-\sqrt{1-\alpha})(\sqrt{1-\alpha}+\alpha)}B - B^2 \geqslant \beta^2 \frac{K}{1-K}; \\ 0 \leqslant \alpha \leqslant \frac{1}{2}, \quad 0 < K < 1, \quad 0 \leqslant \beta \leqslant B \leqslant \frac{1-K}{K}; \\ \text{minimize} \quad \frac{\alpha}{2}K^2 - \beta K^2. \end{cases}$$

First, consider $T(\alpha)$. Since it is positive, $\sqrt{T(\alpha)}$ and $T(\alpha)$ have the same intervals of monotonicity. Change the variable $t = \sqrt{1-\alpha}$. Note that $\alpha \in [0; 1/2)$ implies $t \in \left(\frac{1}{\sqrt{2}}; 1\right]$. Then

$$T(\alpha) = (1-t)(t+1-t^2) = t^3 - 2t^2 + 1.$$

Since $T'(t) = 3t^2 - 4t = 3t(t - \frac{4}{3}) < 0$ for all t, $T(t)$ is a decreasing function. Note that $t(\alpha)$ is also decreasing, so $\sqrt{T(\alpha)}$ and $T(\alpha)$ are increasing functions.

Consider two cases.

Case 1: $K > \frac{1}{2}$. Then by AM-GM inequality $\sqrt{T(\alpha)} \geqslant 2\beta\sqrt{\frac{K}{1-K}}$ and equality holds for $B = \beta\sqrt{\frac{K}{1-K}}$. Then

$$\beta \leqslant \frac{\sqrt{T(\alpha)}}{2\sqrt{\frac{K}{1-K}}} = \frac{\sqrt{T(\alpha)}}{2}\sqrt{\frac{1-K}{K}}.$$

Analogously to Appendix C we reduce to finding the minimum of

$$q(\alpha, K) := \frac{\alpha}{2}K^2 - \frac{\sqrt{T(\alpha)}}{2}\sqrt{1-K}K^{3/2}.$$

Again derive with respect to K and find the roots

$$K_1 = \frac{(\frac{3}{2}T(\alpha) + \alpha^2) + \sqrt{\frac{3}{4}T(\alpha)\alpha^2 + \alpha^4}}{2(\alpha^2 + T(\alpha))}; \qquad K_2 = \frac{(\frac{3}{2}T(\alpha) + \alpha^2) - \sqrt{\frac{3}{4}T(\alpha)\alpha^2 + \alpha^4}}{2(\alpha^2 + T(\alpha))}.$$

Obviously, $K_1 > 3/4$. So the only possible root is K_2. We should examine $K = K_2$ (in the case when it is bigger than 1/2), $K = \frac{1}{2}$ and $K = 1$. One can see (for example by compare the plots on Fig. 4 and Fig. 5) that $K_2(\alpha) > \frac{1}{2}$ implies that $q(\alpha, K_2(\alpha))$ is bigger than $-\frac{1}{54}$.

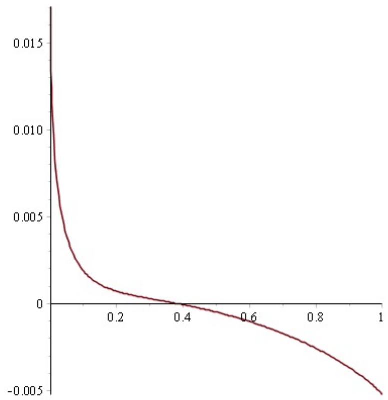

Figure 4: The plot of $q(\alpha, K_2(\alpha)) + \frac{1}{54}$.

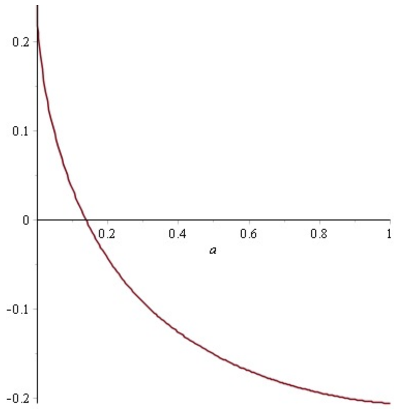

Figure 5: The plot of $K_2(\alpha) - \frac{1}{2}$.

Finally, note that for $K = 1$ function q is positive. For $K = 1/2$ one may see the plot on Fig. 6 to check that $q\left(\alpha, \frac{1}{2}\right) > -\frac{1}{54}$.

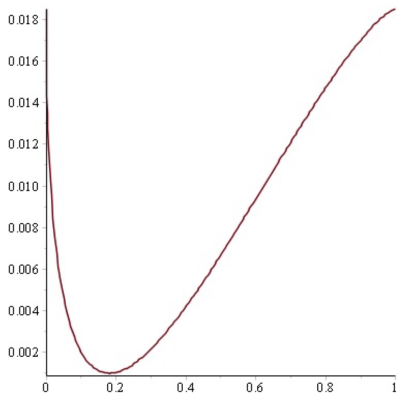

Figure 6: The plot of $q\left(\alpha, \frac{1}{2}\right) + \frac{1}{54}$.

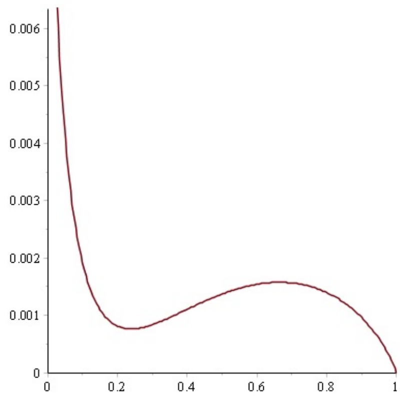

Figure 7: The plot of $q\left(\alpha, K_0(\alpha)\right) + \frac{1}{54}$.

Case 2: One can repeat step-by-step the second case of Appendix C. We minimize

$$q(\alpha, K) := \frac{\alpha}{2}K^2 - \sqrt{T(\alpha)}(1-K)K^2.$$

Derivation and substitution gives

$$q(\alpha, K_0(\alpha)) = \frac{\sqrt{T(\alpha)}}{54}\left(\frac{\alpha}{\sqrt{T(\alpha)}} - 2\right)^3.$$

One may check (see Fig. 7) that $q(\alpha, K_0(\alpha)) > -\frac{1}{54}$.