A Combinatorial Approach to the Groebner Bases for Ideals Generated by Elementary Symmetric Functions

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Abstract

Previous work by Mora and Sala provides the reduced Groebner basis of the ideal formed by the elementary symmetric polynomials in n variables of degrees $k = 1, \dots, n$, $\langle e_{1,n}(x), \dots, e_{n,n}(x)\rangle$. Haglund, Rhoades, and Shimonozo expand upon this, finding the reduced Groebner basis of the ideal of elementary symmetric polynomials in n variables of degree d for $d = n - k + 1, \dots, n$ for $k \leqslant n$. In this paper, we further generalize their findings by using symbolic computation and experimentation to conjecture the reduced Groebner basis for the ideal generated by the elementary symmetric polynomials in n variables of arbitrary degrees and prove that it is a basis of the ideal.

Mathematics Subject Classifications: 05E05

1 Introduction

In their paper [6], Mora and Sala use computational and algebraic means to find the reduced Groebner basis of the ideal generated by the elementary symmetric polynomials in n variables of degrees $d = 1, \dots, n$. Haglund, Rhoades, and Shimonozo expand upon this, finding the reduced Groebner basis of the ideal of elementary symmetric polynomials in n variables of degree d for $d = n - k + 1, \dots, n$ for $k \leqslant n$ [4]. In this paper, we further generalize their findings by using symbolic computation and experimentation to conjecture the reduced Groebner basis for the ideal generated by the elementary symmetric polynomials in n variables of arbitrary degrees and prove that it is in fact a basis of the ideal.

Definition 1. Let k and n be natural numbers. The *elementary symmetric polynomial* of degree k in n variables $x_1, \ldots, x_n$ is

$$e_{k,n}(x) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} x_{i_1} \ldots x_{i_k}.$$

Definition 2. The *homogeneous symmetric polynomial* of degree k in n variables $x_1, \ldots, x_n$ is

$$h_{k,n}(x) = \sum_{1 \leqslant i_1 \leqslant \ldots \leqslant i_k \leqslant n} x_{i_1} \ldots x_{i_k}.$$

Given a set or multiset S with elements in $\{1, \ldots, n\}$, define the weight of S to be

$$wt(S) = \prod_{s \in S} x_s^{m(s)},$$

where $m(s)$ is the multiplicity of s in S. For example,

$$wt(\{1,2,5\}) = x_1 x_2 x_5, \text{ and } wt(\{1,1,3,4\}) = x_1^2 x_3 x_4.$$

Then, $e_{k,n}(x)$ (respectively, $h_{k,n}(x)$) is the weight enumerator of the sets (respectively, multisets) with cardinality k whose elements are in $\{1, \ldots, n\}$. Moreover, considering subsets of $\{1, \ldots, n\}$ which do and do not contain n separately, we have the following recursive definition

$$e_{k,n}(x) = \begin{cases} 0, & \text{if } n < k \\ 1, & \text{if } k = 0 \\ e_{k,n-1}(x) + x_n e_{k-1,n-1}(x), & \text{otherwise.} \end{cases}$$

Similarly, when looking at multisets, we get

$$h_{k,n}(x) = \begin{cases} 0, & \text{if } n = 0 \text{ and } k > 0 \\ 1, & \text{if } k = 0 \\ h_{k,n-1}(x) + x_n h_{k-1,n}(x), & \text{otherwise.} \end{cases}$$

We use the recursive definitions to write Maple functions `eknS(x,k,n)` and `hknS(x,k,n)`, which output $e_{k,n}(x)$ and $h_{k,n}(x)$, respectively. These functions – along with others used to investigate the Groebner basis of ideals generated by elementary symmetric polynomials – can be found in the accompanying Maple package `Solomon.txt`, written by AJ Bu and Doron Zeilberger.

In [6], Mora and Sala proved that $\{h_{1,n}(x), h_{2,n-1}(x), \ldots, h_{n,1}(x)\}$ is a Groebner basis of the ideal $\langle e_{1,n}(x), \ldots, e_{n,n}(x) \rangle$. Using the accompanying package to efficiently generate the reduced Groebner bases of many *specific* ideals, we can extend their findings. We first use experimental methods to deduce a pattern for the reduced Groebner bases of the ideals $\langle e_{1,n}(x), \ldots, e_{k,n}(x) \rangle$ and $\langle e_{1,n}(x), e_{k,n}(x) \rangle$ for arbitrary $k \leqslant n$, and prove them by combinatorial means. We then investigate other cases to expand upon our results to the ideal $\langle e_{k_1,n}(x), \ldots, e_{k_m,n}(x) \rangle$. We find a basis for this general case, proving that it generates the ideal, and show empirically that it is a Groebner basis.

1.1 Groebner Bases

Let the monomial $x_1^{a_1} \dots x_n^{a_n} \in k[x_1, \dots, x_n]$ be denoted by x^α where $\alpha = (a_1, \dots, a_n)$.

Definition 3. A *monomial order* on $k[x_1, \dots, x_n]$ is any relation $>$ on the set of monomials $x^\alpha \in k[x_1, \dots, x_n]$ such that

1. $>$ is a total linear ordering relation
2. If $x^\alpha > x^\beta$ and x^γ is any monomial, then

$$x^\alpha x^\gamma = x^{\alpha+\gamma} > x^{\beta+\gamma} = x^\beta x^\gamma$$

3. $>$ is a well ordering.

Definition 4. The *leading term* of a nonzero polynomial

$$f = \sum_\alpha a_\alpha x^\alpha$$

is the term $LT(f) = a_\alpha x^\alpha$, such that x^α is the largest monomial appearing in f in the ordering $>$. The *leading monomial* of f is $LM(f) = x^\alpha$, and the *leading coefficient* of f is a_α.

In this paper, we use lexicographical order, where $x_n > x_{n-1} > \dots > x_1$. More precisely, if x^α and x^β are monomials in $k[x_1, \dots, x_n]$, then

$$x^\alpha > x^\beta$$

if the rightmost nonzero entry of the difference vector $\alpha - \beta \in \mathbb{Z}^n$ is positive.

Definition 5. A *Groebner basis* of an ideal $I \subset k[x_1, \dots, x_n]$ (with respect to a given monomial order) is a finite subset $G = \{g_1, \dots, g_t\}$ of I such that for that every nonzero polynomial f in I, the leading term of f is divisible by the leading term of g_i for some i.

Moreover, it is *reduced* if, for all distinct elements $g, p \in G$, the leading coefficient of g is 1 and no monomial appearing in g is a multiple of $LT(p)$. Any ideal $I \subset k[x_1, \dots, x_n]$ has a unique reduced Groebner basis for a given monomial order.

Theorem 6 (The Division Algorithm in $\boldsymbol{k[x_1, \dots, x_n]}$)**.** *Let $>$ be a fixed monomial order in $k[x_1, \dots, x_n]$. Let $F := (f_1, \dots, f_m)$ be an ordered list of polynomials in $k[x_1, \dots, x_n]$. Then for any $f \in k[x_1, \dots, x_n]$, there exists $a_1, \dots, a_m, r \in k[x_1, \dots, x_n]$ such that*

1. $f = a_1 f_1 + \dots + a_m f_m + r$,

2. for all i, either $a_i f_i = 0$ or $LT(f) \geqslant LT(a_i f_i)$, and

3. r is a sum of monomials, none of which are divisible by any $LT(f_i)$.

We call r the remainder *of f on r.*

If the monomial order and the order on F are fixed, the polynomials $a_1, \dots, a_m, r$ are unique. These polynomials can change, however, if different monomial orders or orders on F are selected.

In order to efficiently determine whether or not a given basis is a Groebner basis, we use Buchberger's Criterion. The *S-polynomial* of two polynomials f and g is

$$S(f,g) = \frac{LCM(LM(f), LM(g))}{LT(f)} f - \frac{LCM(LM(f), LM(g))}{LT(g)} g.$$

Theorem 7 (Buchberger's Criterion)**.** *$G = \{g_1, \dots, g_t\}$ is a Groebner basis of I with respect to a given monomial order if and only if G generates I and, for any distinct g_i and g_j in G,*

$$\overline{S(g_i, g_j)}^G = 0,$$

where $\overline{S(f_i, f_j)}^G$ denotes the remainder of the S-Polynomial of f_i and f_j upon division by G.

2 The Ideal $\langle e_{1,n}(x), \dots, e_{k,n}(x)\rangle$

The procedure `Gkn(k,n,x)` in `Solomon.txt` outputs the reduced Groebner basis (with respect to lexicographical order where $x_n > x_{n-1} > \cdots > x_1$) for the ideal $\langle e_{1,n}(x), \dots, e_{k,n}(x)\rangle$. After running the procedure for multiple values k and n, we can conjecture that the reduced Groebner basis is $\{h_{i,n-i+1}(x) | i = 1 \dots k\}$. Indeed, for the case $k = n$, this agrees with the Groebner basis that Mora and Sala proved in their paper [6]. In order to prove our conjecture, we use the following two relations between the elementary and homogeneous symmetric polynomials. This is essentially a well-known classical identity that can be found in [5], Eq. (2.6'). It has a very quick proof using generating functions, which is left to the reader. Nevertheless, we prefer the following somewhat longer, but more insightful combinatorial proof, inspired by Zeilberger's proof [8].

Lemma 8. *Let k and n be natural numbers. Then*

$$h_{k,n-k+1}(x) = \sum_{i=1}^{k} (-1)^{i+1} e_{i,n}(x) h_{k-i,n-k+1}(x)$$

Proof. This is equivalent to proving

$$\sum_{i=0}^{k} (-1)^{i} e_{i,n}(x) h_{k-i,n-k+1}(x) = 0.$$

This is trivial when $k > n$ because $h_{k-i,n-k+1}(x) = 0$ when $0 \leqslant i \leqslant k-1$, and $e_{k,n} = 0$. So, assume $k \leqslant n$. Then, the left-hand side is the weight enumerator of the set $\mathcal{S}_{k,n}$ of pairs (A, B), where

- A is a subset of $\{1, \ldots, n\}$ of order $|A|$,
- B is a multiset with cardinality $k - |A|$ whose elements are in $\{1, \ldots, n-k+1\}$,

and the weight of (A, B) is

$$w(A, B) = (-1)^{|A|} wt(A) wt(B).$$

Let $f : \mathcal{S}_{k,n} \to \mathcal{S}_{k,n}$ be defined as

$$f(A, B) = \begin{cases} (A \cup \{\min(B)\}, B - \{\min(B)\}), & \text{if } \min(B) < \min(A) \\ (A \setminus \{\min(A)\}, B + \{\min(A)\}), & \text{otherwise.} \end{cases}$$

Note that this mapping is defined for all possible pairs of sequences, and it changes sign since the size of the first subset is either increasing or decreasing by 1. Moreover, if $\min(A) > \min(B)$ then

$$\begin{aligned} f(A, B) &= (A \cup \{\min(B)\}, B - \{\min(B)\}) =: (A', B'), \text{ and} \\ f(A', B') &= (A, B), \end{aligned}$$

since clearly $\min(A') = \min(B) \leqslant \min(B')$. If $\min(A) \leqslant \min(B)$ then

$$\begin{aligned} f(A, B) &= (A \setminus \{\min(A)\}, B + \{\min(A)\}) =: (A', B'), \text{ and} \\ f(A', B') &= (A, B), \end{aligned}$$

since $\min(B') = \min(A) < \min(A')$. Thus, all elements of $\mathcal{S}_{k,n}$ can be paired up into mutually cancelling pairs, concluding our proof. ☐

Lemma 9. *For any $n, k \in \mathbb{N}$,*

$$e_{k,n}(x) = \sum_{i=1}^{k} (-1)^{i+1} h_{i,n-i+1}(x) e_{k-i,n-i}(x).$$

Proof. Note that this is equivalent to

$$\sum_{i=0}^{k} (-1)^{i} h_{i,n-i+1}(x) e_{k-i,n-i}(x) = 0.$$

Again, this is trivial for $k > n$, so assume $k \leqslant n$. The left-hand side is the weight enumerator of the set $\mathcal{S}_{k,n}$ of ordered pairs (A, B), where

- A is a multiset with elements in $\{1, \ldots, n - |A| + 1\}$, where $|A|$ is the cardinality of A,
- B is a subset of $\{1, \ldots, n - |A|\}$ of order $|B| := k - |A|$,

and the weight of (A, B) is

$$w(A, B) = (-1)^{|A|} wt(A) wt(B).$$

Let $f : \mathcal{S}_{k,n} \to \mathcal{S}_{k,n}$ be defined as

$$f(A, B) = \begin{cases} (A + \{\min(B)\}, B \setminus \{\min(B)\}), & \text{if } \min(B) < \min(A) \\ (A - \{\min(A)\}, B \cup \{\min(A)\}), & \text{otherwise.} \end{cases}$$

As in the previous proof, this involution pairs all elements of $\mathcal{S}_{k,n}$ into mutually cancelling pairs. □

We use the polynomial identities in the preceding lemmas to construct the Groebner basis of the ideal $\langle e_{1,n}(x), \ldots, e_{k,n}(x) \rangle$ generated by the elementary symmetric polynomials of low degree. To start, we determine a basis of this ideal.

Lemma 10. *Let k and n be natural numbers such that $k \leqslant n$.*

$$\langle e_{1,n}(x), \ldots, e_{k,n}(x) \rangle = \langle h_{1,n}(x), h_{2,n-1}(x), \ldots, h_{k,n-k+1}(x) \rangle.$$

Proof. For $i = 1, \ldots, k$, we have

$$h_{i,n-i+1}(x) \in \langle e_{1,n}(x), \ldots, e_{k,n}(x) \rangle, \quad \text{and} \quad e_{i,n}(x) \in \langle h_{1,n}(x), h_{2,n-1}(x), \ldots, h_{k,n-k+1}(x) \rangle$$

by Lemmas 8 and 9, respectively. It immediately follows that

$$\langle e_{1,n}(x), \ldots, e_{k,n}(x) \rangle = \langle h_{1,n}(x), h_{2,n-1}(x), \ldots, h_{k,n-k+1}(x) \rangle.$$ □

Proposition 11. *Let k and n be natural numbers. The set $G := \{h_{i,n-i+1}(x) \mid 1 \leqslant i \leqslant k\}$ is the reduced Groebner basis of the ideal $\langle e_{1,n}(x), \ldots, e_{k,n}(x) \rangle$ with respect to lexicographical order, where $x_n > x_{n-1} > \cdots > x_1$.*

Proof. By Lemma 10, the set G generates the ideal $I := \langle e_{1,n}(x), \ldots, e_{k,n}(x) \rangle$. The S-polynomial of any two distinct elements $h_{i,n-i+1}(x)$ and $h_{j,n-j+1}(x)$ in G is

$$\begin{aligned} S(h_{i,n-i+1}(x), h_{j,n-j+1}(x)) &= x_{n-j+1}^{j} h_{i,n-i+1}(x) - x_{n-i+1}^{i} h_{j,n-j+1}(x) \\ &= h_{j,n-j+1}(x) \sum_{\ell=0}^{i-1} x_{n-i+1}^{\ell} h_{i-\ell,n-i}(x) \\ &\quad - h_{i,n-i+1}(x) \sum_{\ell=0}^{j-1} x_{n-j+1}^{\ell} h_{j-\ell,n-j}(x). \end{aligned}$$

To prove the second equality, note that it is equivalent to

$$h_{i,n-i+1}(x) \sum_{\ell=0}^{j} x_{n-j+1}^{\ell} h_{j-\ell,n-j}(x) = h_{j,n-j+1}(x) \sum_{\ell=0}^{i} x_{n-i+1}^{\ell} h_{i-\ell,n-i}(x).$$

$x^{\ell}_{n-j+1}h_{j-\ell,n-j}(x)$ is the weight enumerator of all multisets of cardinality j with elements taken from $\{1,\dots,n-j+1\}$, where $n-j+1$ appears exactly ℓ times. Thus, it is clear that

$$\sum_{\ell=0}^{j} x^{\ell}_{n-j+1}h_{j-\ell,n-j}(x) = h_{j,n-j+1}(x).$$

It follows that

$$\begin{aligned} S(h_{i,n-i+1}(x), h_{j,n-j+1}(x)) = h_{j,n-j+1}(x) \sum_{\ell=0}^{i-1} x^{\ell}_{n-i+1}h_{i-\ell,n-i}(x) \\ - h_{i,n-i+1}(x) \sum_{\ell=0}^{j-1} x^{\ell}_{n-j+1}h_{j-\ell,n-j}(x). \end{aligned}$$

Moreover, for $i \neq j$

$$\begin{aligned} LT\left(h_{i,n-j+1}(x)\sum_{\ell=0}^{i-1} x^{\ell}_{n-i+1}h_{i-\ell,n-i}(x)\right) &= x^{j}_{n-j+1}x^{i-1}_{n-i+1}x_{n-i} \\ &\neq x^{i}_{n-i+1}x^{j-1}_{n-j+1}x_{n-j} \\ &= LT\left(h_{i,n-i+1}(x)\sum_{\ell=0}^{j-1} x^{\ell}_{n-j+1}h_{j-\ell,n-j}(x)\right). \end{aligned}$$

Hence,

$$LT(S(h_{i,n-i+1}(x), h_{j,n-j+1}(x))) = \max(x^{j}_{n-j+1}x^{i-1}_{n-i+1}x_{n-i}, x^{i}_{n-i+1}x^{j-1}_{n-j+1}x_{n-j})$$

and, by the division algorithm,

$$\overline{S(h_{i,n-i+1}(x), h_{j,n-j+1}(x))}^{G} = 0.$$

Therefore, G is a Groebner basis of I by Buchberger's Criterion. Furthermore, G is a reduced Groebner basis because, for any distinct i,j, $LT(h_{i,n-i+1}(x)) = x^{i}_{n-i+1}$ cannot divide the terms in $h_{j,n-j+1}(x)$. This follows from the fact that the terms of $h_{j,n-j+1}(x)$ have lower degree if $i > j$, and they cannot be multiples of x_{n-i+1} if $i < j$. □

3 Investigation into the General Case

The procedure `GSn(S,n,x)` in `Solomon.txt` inputs a set $S = \{k_1,\dots,k_m\}$, a non-negative integer n, and a variable x. It outputs the reduced Groebner basis (with respect to lexicographical order where $x_n > x_{n-1} > \cdots > x_1$) for the ideal $\langle e_{k_1,n}(x),\dots,e_{k_m,n}(x)\rangle$. Using this procedure to analyze the reduced Groebner bases for various ideals, we conjecture the following basis for arbitrary S and n.

Proposition 12. *Let $k_1, \ldots, k_m$, and n be positive integers such that $1 \leqslant k_1 < \cdots < k_m \leqslant n$. Let I be the ideal*

$$I := \langle e_{k_1,n}(x), \ldots, e_{k_m,n}(x) \rangle,$$

and let M be the set of matrices of the form

$$\begin{bmatrix} e_{k_m - i_{m-1}, n-i_{m-1}}(x) & \ldots & e_{k_m-i_1,n-i_1}(x) & e_{k_m,n}(x) \\ \vdots & \ldots & & \vdots \\ e_{k_1-i_{m-1},n-i_{m-1}}(x) & \ldots & e_{k_1-i_1,n-i_1}(x) & e_{k_1,n}(x) \end{bmatrix},$$

where $i_1 \in \{1, 2, \ldots, k_1, k_2\}$ and $i_j \in \{i_{j-1}+1, i_{j-1}+2, \ldots, k_j, k_{j+1}\}$ for $j > 1$. Then the set

$$G := \{det(m) \mid m \in M\}$$

is a basis of I.

Proof. Note that the entries of the last column of any matrix in M are the elementary symmetric polynomials

$$e_{k_1,n}(x), \ldots, e_{k_m,n}(x)$$

that generate I. It immediately follows that $\langle G \rangle \subseteq I$.

For the other containment, let m_1 be the matrix in M where $i_j = k_{j+1}$. Then,

$$\begin{aligned} \det(m_1) &= \det\left(\begin{bmatrix} 1 & e_{k_m-k_{m-1},n-k_{m-1}}(x) & \ldots & e_{k_m-k_2,n-k_2}(x) & e_{k_m,n}(x) \\ 0 & 1 & \ldots & e_{k_{m-1}-k_2,n-k_2}(x) & e_{k_{m-1},n}(x) \\ \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & e_{k_2,n}(x) \\ 0 & 0 & \ldots & 0 & e_{k_1,n}(x) \end{bmatrix}\right) \\ &= e_{k_1,n}(x). \end{aligned}$$

Therefore, $e_{k_1,n}(x) \in \langle G \rangle$. Now suppose that for $L > 1$, $e_{k_\ell,n}(x) \in \langle G \rangle$ for all $1 \leqslant \ell < L$. Let m_L denote the matrix in M such that $i_j = k_j$ for $j < L$ and $i_j = k_{j+1}$ for $j \geqslant L$. Then,

$$m_L = \begin{bmatrix} A_L & B_L \\ 0 & C_L \end{bmatrix},$$

where A is an $(m-L) \times (m-L)$ triangular matrix with whose diagonal entries are all 1, and 0 is an $L \times (m-L)$ zero matrix. Therefore, $\det m_L = \det C_L$, where C_L is the $L \times L$ matrix

$$\begin{bmatrix} e_{k_L-k_{L-1},n-k_{L-1}}(x) & e_{k_L-k_{L-2},n-k_{L-2}}(x) & \ldots & e_{k_L-k_1,n-k_1}(x) & e_{k_L,n}(x) \\ 1 & e_{k_{L-1}-k_{L-2},n-k_{L-2}}(x) & \ldots & e_{k_{L-1}-k_1,n-k_1}(x) & e_{k_{L-1},n}(x) \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & e_{k_2-k_1,n-k_1}(x) & e_{k_2,n}(x) \\ 0 & \ldots & 0 & 1 & e_{k_1,n}(x) \end{bmatrix}.$$

Define c_i to be the $(L-1)\times(L-1)$ matrix formed by removing the $i-th$ row and the last column from C_L. Then,

$$\det(C_L) = \sum_{i=1}^{L}(-1)^{i+1}e_{k_i,n}(x)\det(c_{L+1-i}).$$

Since $\det c_1 = 1$, it follows that

$$\begin{aligned}\det M &= \det C_L\\ &= (-1)^{L+1}e_{k_L,n}(x) + \sum_{i=1}^{L-1}(-1)^{i+1}e_{k_i,n}(x)\det(c_{L+1-i}).\end{aligned}$$

Since $\det M \in \langle G\rangle$ and, by our inductive hypothesis, $e_{k_i,n}(x) \in \langle G\rangle$ for $1 \leqslant i \leqslant L-1$, it follows $e_{k_L,n}(x)$ is in the ideal as well. Thus, $e_{k_i,n}(x) \in \langle G\rangle$ for $i = 1,\dots,m$, and $I = \langle G\rangle$. □

Since we found this basis by studying specifically the reduced Groebner bases of various ideals, we further conjecture that it is the reduced Groebner basis for arbitrary S and n. Indeed, the following proposition states that this conjecture holds for the ideal $\langle e_{1,n}(x), e_{k,n}(x)\rangle$.

Proposition 13. *Let k and n be natural numbers such that $n \geqslant k$. Let I be the ideal $\langle e_{1,n}(x), e_{k,n}(x)\rangle$, and let M be the set of matrices*

$$M = \left\{\begin{bmatrix}1 & e_{k,n}(x)\\ 0 & e_{1,n}\end{bmatrix}, \begin{bmatrix}e_{k-1,n-1}(x) & e_{k,n}(x)\\ 1 & e_{1,n}(x)\end{bmatrix}\right\}.$$

Then the set

$$G := \{det(m) \mid m \in M\}$$

is the reduced Groebner basis of I with respect to lexicographical order, where $x_n > x_{n-1} > \cdots > x_1$.

Proof. By Proposition 12, G generates $I := \langle e_{1,n}(x), e_{k,n}(x)\rangle$. Note that by evaluating the determinants and then using the recursive properties of the elementary symmetric polynomials, we can rewrite G as

$$G = \{e_{1,n}(x), e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x)\}.$$

Taking the S-polynomial of the elements in G, we have

$$\begin{aligned}S(e_{1,n}(x),&e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x))\\ &= x_{n-1}^2x_{n-2}\dots x_{n-k+1}e_{1,n}(x) - x_n\big(e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x)\big)\\ &= \big(e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x)\big)e_{1,n-1}(x)\\ &\qquad - e_{1,n}(x)\big(e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x) - x_{n-1}^2x_{n-2}\dots x_{n-k+1}\big)\end{aligned}$$

Note that the second equality obviously holds since it can be rewritten as

$$e_{1,n}(x)\big(e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x)\big) = \big(e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x)\big)e_{1,n}(x).$$

It is also clear that

$$\begin{aligned} &LT\big((e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x))e_{1,n-1}(x)\big) \\ &\qquad < LT\big(e_{1,n}(x)(e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x) - x_{n-1}^2 x_{n-2}\dots x_{n-k+1})\big), \end{aligned}$$

since the latter is a multiple of x_n. Hence,

$$\overline{S(e_{1,n}(x), e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x))}^{G} = 0,$$

and G is a Groebner basis. It is clearly reduced since no term in $e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x)$ is divisible by x_n and no term in $e_{1,n}(x)$ is divisible by x_{n-1}^2. □

Upon further investigation, we can show that our basis for the general case also gives the reduced Groebner basis of the ideal $\langle e_{1,n}(x), \dots, e_{k,n}(x)\rangle$. We show that this is true in the following proposition, which is equivalent to Proposition 11.

Proposition 14. *Let k and n be positive integers such that $1 \leqslant k \leqslant n$. Let I be the ideal*

$$I := \langle e_{1,n}(x), \dots, e_{k,n}(x)\rangle,$$

and let M be the set of matrices of the form

$$\begin{bmatrix} e_{k-i_{k-1},n-i_{k-1}}(x) & \dots & e_{k-i_1,n-i_1}(x) & e_{k,n}(x) \\ \vdots & \dots & \vdots & \vdots \\ e_{1-i_{k-1},n-i_{k-1}}(x) & \dots & e_{1-i_1,n-i_1}(x) & e_{1,n}(x) \end{bmatrix},$$

where $1 \leqslant i_1 < \dots < i_{k-1} \leqslant k$. Then the set

$$G := \{det(m) \mid m \in M\}$$

is the reduced Groebner basis of I.

Proof. By Proposition 12, G is a basis of I. Moreover, by Proposition 11, the reduced Groebner basis of I is $G' := \{h_{i,n-i}(x) \mid 1 \leqslant i \leqslant k\}$. Thus, it suffices to prove that $G = G'$.

For any positive integer L, let m_L denote the matrix such that no $i_j = L$. Then, as shown in the proof of Proposition 12,

$$\det m_L = \det C_{L,n},$$

where

$$C_{L,n} = \begin{bmatrix} e_{1,n-L+1}(x) & e_{2,n-L+2(x)} & \dots & e_{L-1,n-1}(x) & e_{L,n}(x) \\ 1 & e_{1,n-L+2}(x) & \dots & e_{L-2,n-1}(x) & e_{L-1,n}(x) \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & e_{1,n-1}(x) & e_{2,n}(x) \\ 0 & \dots & 0 & 1 & e_{1,n}(x) \end{bmatrix}.$$

For any positive integers L and n where $L \leqslant n$, we claim

$$\det C_{L,n} = h_{L,n-L+1}(x).$$

To prove this claim, we use induction over L, where for each $L \in \mathbb{N}$, we will show that the claim holds for all n. We begin with the base case; for any positive integer n, we have $C_{1,n} = \left[e_{1,n}(x)\right]$, so clearly

$$\begin{aligned}\det C_{1,n} &= e_{1,n}(x)\\ &= h_{1,n}(x).\end{aligned}$$

Now suppose that for any given $L \geqslant 2$, we have $\det C_{\ell,N} = h_{\ell,N-\ell+1}(x)$ for any $0 < \ell < L$ and $N > \ell$. Since

$$\det C_{L,n} = \sum_{i=1}^{L} (-1)^{i+1} e_{i,n}(x) \det(c_{L+1-i}),$$

where c_i is formed by removing the $i-th$ row and the last column from $C_{L,n}$, and we have shown that

$$h_{k,n-k+1}(x) = \sum_{i=1}^{k} (-1)^{i+1} e_{i,n}(x) h_{k-i,n-k+1}(x),$$

it is enough to show that $\det c_i = h_{i-1,n-L+1}$. Since c_1 is a triangular matrix whose diagonal entries are 1, it is obvious that

$$\begin{aligned}\det c_1 &= 1\\ &= h_{0,n-L+1}.\end{aligned}$$

For $i > 1$, c_i can be written as

$$c_i = \begin{bmatrix} a_i & b_i \\ 0 & d_i, \end{bmatrix}$$

where d_i is an $(L-i) \times (L-i)$ triangular matrix whose diagonal entries are all 1, and $a_i = C_{i-1,n-L+i-1}$. Therefore,

$$\begin{aligned}\det c_i &= \det C_{i-1,n-L+i-1}\\ &= h_{i-1,n-L+1},\end{aligned}$$

by our inductive hypothesis. Thus,

$$\begin{aligned}\det C_{L,n} &= \sum_{i=1}^{L} (-1)^{i+1} e_{i,n}(x) \det(c_{L+1-i})\\ &= \sum_{i=1}^{L} (-1)^{i+1} e_{i,n}(x) h_{L-i,n-L+1}(x)\\ &= h_{L,n-L+1}(x),\end{aligned}$$

as desired. □

4 Conclusion

We have given a formal proof for a basis of $\langle e_{k_1,n}(x), \ldots, e_{k_m,n}(x)\rangle$, the ideal generated by an arbitrary set of elementary symmetric functions of degree n, and we proved that it is the reduced Groebner basis for ideals of the form $\langle e_{1,n}(x), e_{k,n}(x)\rangle$ and $\langle e_{1,n}(x), \ldots, e_{k,n}(x)\rangle$. Using the procedure `CheckConjGSn(S,n,x)` in `Solomon.txt`, we can verify that the generalized basis given in Proposition 12 is the reduced Groebner basis of the ideal $\langle e_{k_1,n}(x), \ldots, e_{k_m}(x)\rangle$ for a given set $S = \{k_1, \ldots, k_m\}$ and positive integer n. Verifying that this is the case for many S and n, we can empirically show that this basis is the reduced Groebner basis for the ideal generated by an arbitrary set of elementary symmetric functions.

One direction for further research is to formally prove that the basis we have found for the general case is the reduced Groebner basis. We can also try to find similar identities for other ideals, such as those generated by various power sum symmetric polynomials or homogeneous symmetric polynomials of arbitrary degrees.

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