What Convex Geometries Tell About Shattering-Extremal Systems

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Abstract

We give a characterization of shattering-extremal set systems in terms of forbidden projections, in the spirit of Dietrich's characterization of antimatroids. Apart from that, we prove several metric and topological properties of such systems, which, however, do not amount to a characterization. The ideas for all these results come from the similar characterizations of antimatroids and convex geometries and due to the fact that both of them are special cases of shattering-extremal systems.

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1 Introduction

This paper is about shattering-extremal systems, that is, about set families that shatter as many sets as they have, thus achieving an exact bound in Sauer-Shelah-Perles inequality.

Shattering-extremal systems have been studied, for example, in connection with the unlabeled sample compression scheme conjecture of Littlestone and Warmuth [17], see also [4] and [19] for the recent progress in this direction. On the other hand, the study of such systems naturally falls under the category of extremal problems about traces of sets, see [13], Chapter 8, for a survey of recent results. One of the first systematic studies of shattering-extremal systems was done by Bollobás and Radcliffe [3]; they, in particular, have given several alternative characterizations of such systems. These descriptions, however, are not very descriptive and can hardly be used to construct nontrivial examples of such systems or to classify some special subclasses; see, for example, [21] for an example of such classification. There is, thus, a certain lack of a good characterization of shattering-extremal systems, although some attempts have been made in [20] and [16].

Convex geometries are a particular class of shattering-extremal systems, namely, they are precisely shattering-extremal closure systems [5]. In contrast with shattering-extremal systems in general, they are famous for admitting multiple different yet equivalent characterizations. It is thus natural to try and extend some of these characterizations from convex geometries to shattering-extremal systems. As it turns out, in at least one case it can be done. Namely, in Section 3 we prove Theorem 5, which is a characterization of shattering-extremal families in terms of forbidden projections. It is a direct counterpart of Dietrich's characterization of convex geometries in terms of implications [8], which we discuss in Section 2.

What about extending other characterizations? Unfortunately, there are certain obstructions to that. Some of those characterizations are formulated in terms of lattices; for example, $\mathcal{F}$ is a convex geometry iff it is meet-distributive as a lattice [9], which is equivalent to it being join semidistributive and lower semimodular [2]. As shattering-extremal systems lack the lattice structure, adapting these results for them is problematic. Moreover, shattering-extremal systems also lack the order in general: even though sets are ordered by inclusion, it can be shown that shattering-extremal systems are stable under bit flips, that is, the system remains shattering-extremal after taking a symmetric difference of every its set with some fixed set X. Due to the simple nature of this operation, it makes sense to distinguish shattering-extremal systems only up to a bit-flip. However, bit flips obviously do not preserve the inclusion. This makes it hard to generalize any characterizations relying on this order, like, for example, the canonic anti-exchange property, which is formulated in terms of a closure operator.

What is preserved under bit-flips is the Hamming distance between the sets. Thus, in Section 4, we reformulate the characterization of convex geometries in terms of maximal chains (Theorem 11) for shattering-extremal systems (Lemma 13). The resulting property then states that in a shattering-extremal system $\mathcal{F}$ any two sets at distance d can be connected by a path of length d in $\mathcal{F}$, that is, that the internal metric on $\mathcal{F}$ coincides with the one induced by the Hamming distance. However, unlike Theorem 11, which is a characterization, Lemma 13 turns out to be just a property that is far from being sufficient. Going further into this direction, we prove, in Theorem 16, that shattering-extremal systems are, in a way, simply connected. In particular, Theorem 16 states that any two maximal paths between the same endpoints can be deformed into each other, which, to an extent, is reminiscent of Theorem 2 in [14]. However, a rather simple example given at the end of Section 4 is sufficient to show that even together, these two properties do not give a sufficient condition.

Finally, in Section 5, we outline some open problems and related questions.

2 Preliminaries

If not mentioned otherwise, all objects that we consider in this paper are finite. In a poset $\mathcal{L}$, an *antichain* $\mathcal{A}$ over $\mathcal{L}$ is a subset $\mathcal{A} \subseteq \mathcal{L}$ such that no two distinct elements from $\mathcal{A}$ are comparable in $\mathcal{L}$; moreover, $\mathcal{A}$ is *maximal* if $\mathcal{A} \cup \{x\}$ is not an antichain for any $x \in \mathcal{L} - \mathcal{A}$. A subset $\mathcal{S} \subseteq \mathcal{L}$ is *hereditary* (*upward closed*) if for any $x \in \mathcal{S}$ and any $y \leqslant x$ ($y \geqslant x$) it follows that $y \in \mathcal{S}$. There is a natural bijection between antichains and upward closed sets in $\mathcal{L}$: for an antichain $\mathcal{A}$ its *upward closure* $\mathcal{A}^u = \{x \in \mathcal{L} \mid x \geqslant a \text{ for some } a \in \mathcal{A}\}$ is, as expected, upward closed. Conversely, for an upward closed set $\mathcal{I}$, the set of its minimal elements $\mathcal{I}_m = \{x \in \mathcal{I} \mid y \not< x \text{ for every } y \in \mathcal{I}\}$ is an antichain. Moreover, these two operations are mutually inverse. We denote an operation of taking a minimal antichain by $\mathrm{Min}\colon 2^{\mathcal{L}} \to 2^{\mathcal{L}}$.

For two subsets $Q, R \subseteq \mathcal{L}$ we say that Q *refines* R, denoted $Q \ll R$, if for any $q \in Q$

there is $r \in R$ such that $q \leqslant r$; and we say that Q *dually refines* R denoted $Q \ll_d R$, if for any $r \in R$ there is $q \in Q$ such that $q \leqslant r$. In this paper, we will deal exclusively with the dual refinement, and the former definition is given simply for compliance with the standard lattice theory terminology.

Dual refinement relation is a preorder on $2^{\mathcal{L}}$, however, when restricted to antichains in $\mathcal{L}$, it becomes a partial order. It is easy to see that $Q \ll_d R$ iff $Q^u \supseteq R^u$, in particular, an antichain of minimal elements of a set dually refines this set. Similarly, $Q \ll R$ iff $Q^l \subseteq R^l$, where Q^l and R^l are the downward closures of Q and R.

For $A, B \subseteq U$, we denote the set difference of A and B by $A - B$ and the symmetric difference by $A \triangle B$. For $A \subseteq U$ and $x \in U$ we write $A - x$ for $A - \{x\}$ and $A + x$ for $A \cup \{x\}$. For $\varphi\colon X \to Y$ and $A \subseteq X$, $\varphi[A]$ denotes the image of X under φ.

Our primary object of study will be a set family $\mathcal{F}$ (which we also call a *system*) over a fixed *base set* U, that is, $\mathcal{F} \subseteq 2^U$; we consider 2^U to be a poset ordered by set inclusion. A system $\mathcal{F}$ *shatters* a set $X \subseteq U$ (alternatively, X *is shattered* by $\mathcal{F}$) if for any $Y \subseteq X$ there is $F \in \mathcal{F}$ such that $F \cap X = Y$. We denote the family of sets shattered by $\mathcal{F}$ by $\operatorname{Str}(\mathcal{F})$. Trivially, for any $\mathcal{F}$, $\operatorname{Str}(\mathcal{F})$ is hereditary. Also, by the Sauer-Shelah-Perles (SSP) lemma [23–25], it holds:

$$|\mathcal{F}| \leqslant |\operatorname{Str}(\mathcal{F})|, \tag{1}$$

and we say that $\mathcal{F}$ is *shattering-extremal* if it attains equality in (1), that is, if $|\mathcal{F}| = |\operatorname{Str}(\mathcal{F})|$. Every hereditary system $\mathcal{H}$ is shattering-extremal with $\operatorname{Str}(\mathcal{H}) = \mathcal{H}$.

We say that $\mathcal{F}$ is a *closure system* if it is intersection closed and contains U; in this case, for any $X \subseteq U$ there is the smallest set in $\mathcal{F}$, containing U, which we denote by $\overline{X}$. It is typical, for example, in formal concept analysis, see [12], to study closure systems using *implications*. Formally, an implication is a tuple (A, a), for $A \subseteq U$ and $a \in U - A$, denoted by $A \to a$. We assume that the implications are partially ordered by $A \to a \leqslant B \to b$ if $a = b$ and $A \subseteq B$. A set X *satisfies* an implication $A \to a$ if $A \nsubseteq X$ or if $a \in X$. We note that then $A \to a \leqslant B \to b$ iff $B \to b$ satisfies all sets satisfied by $A \to a$. For example, for $U = \{1, 2, 3\}$, $\varnothing \to 3 \leqslant 12 \to 3$, and the former satisfies all subsets of U containing 3, while the latter satisfies all sets except for 23. Here and further on we drop the curly brackets in the notation for the subsets of U, that is, 12 and 23 stand for $\{1, 2\}$ and $\{2, 3\}$ respectively. A family of all subsets of U satisfying a fixed set of implications is always a closure system. Conversely, any closure system can be defined by a set of implications that it satisfies.

A system $\mathcal{F}$ is a *convex geometry* if it is a closure system and satisfies an *anti-exchange property*:

$$\text{if } x \in \overline{F + y} \text{ then } y \notin \overline{F + x},$$

for all $F \in \mathcal{F}$ and $x, y \notin F$, $x \neq y$. Alternatively, a closure system is a convex geometry iff it is shattering-extremal [5]. A system $\mathcal{F}$ is an *antimatroid* if the family of its complements $\{U - F \mid F \in \mathcal{F}\}$ is a convex geometry; this is trivially equivalent to $\mathcal{F}$ being a shattering-extremal union closed system containing $\varnothing$.

Let us now recall a well-known characterization of convex geometries by Dietrich [8, Theorem 7]; the original theorem is about antimatroids, its reformulation for convex geometries is straightforward.

Theorem 1 (Dietrich)**.** *Let $\mathcal{P}$ be a set of pairwise incomparable implications (that is, as implications, they form an antichain). Additionally, suppose $\mathcal{P}$ satisfies:*

$$\begin{array}{l}\textit{for every } A \to a \textit{ and } B \to b \textit{ in } \mathcal{P} \textit{ such that } a \neq b \textit{ and} \\ a \in B, \textit{ there is } C \to b \in \mathcal{P} \textit{ such that } C \subseteq A \cup B - a.\end{array} \tag{2}$$

Then the system $\mathcal{F} = \mathcal{F}(\mathcal{P})$ of all sets satisfying $\mathcal{P}$ is a convex geometry. Moreover, if $\mathcal{F}$ satisfies an implication $D \to d$, then there is $A \to a \in \mathcal{P}$ such that $A \to a \leqslant D \to d$.

Conversely, for a convex geometry $\mathcal{F}$ let $\mathcal{P} = \mathcal{P}(\mathcal{F})$ be the set of minimal implications satisfied by $\mathcal{F}$, then $\mathcal{P}$ satisfies (2)*.*

As we have mentioned, our goal is to generalize Theorem 1 from the characterization of shattering-extremal closure systems to arbitrary shattering-extremal families. As the first step in this direction, we will now introduce *forbidden projections*, which generalize implications.

We call a tuple (P, h_P), where $P \subseteq U$ and $h_P \colon P \to \{0, 1\}$ a *projection*, where P is called the *support* and h_P the *pattern* of the projection. Alternatively, a projection can be given by a tuple (P, H_P), $H_P \subseteq P \subseteq U$. Two definitions are obtained from each other by identifying H_P with its characteristic function $h_P(x)$. If no confusion arises, we will drop h_P (or H_P) when mentioning a projection. For a function $h \colon P \to \{0, 1\}$ and $Q \subseteq P$ we define a function $h|_Q \colon Q \to \{0, 1\}$ as $h|_Q(x) = h(x)$, for $x \in Q$. We define PRJ to be a poset of all projections with the following partial order: for projections (P, h_P) and (Q, h_Q), $(P, h_P) \leqslant (Q, h_Q)$ if $P \subseteq Q$ and $h_P = h_Q|_P$; empty projection $(\varnothing, \varnothing)$ is a minimal element of PRJ.

Almost exclusively we talk about projections in the context of them being forbidden, in particular, we say that $F \subseteq U$ *satisfies* a projection P if $H_P \neq P \cap F$, otherwise F *invalidates* P. Similarly, a system $\mathcal{F}$ satisfies a set of projections $\mathcal{P}$ if F satisfies P, for all $F \in \mathcal{F}$, $P \in \mathcal{P}$. Note that for all $P, Q \in \mathsf{PRJ}$, $P \leqslant Q$ iff Q satisfies all sets satisfied by P; alternatively, for any $F \subseteq U$, if Q invalidates F then so does P. It is now easy to see that forbidden projections are a generalization of implications. Indeed, by associating $A \to a$ with a forbidden projection $(A + a, A)$ it can be seen that the order and the notion of satisfiability for implications coinsides with the one for their forbidden projections counterparts. Moreover, implications correspond precisely to forbidden projections of the form (P, H_P), for $|P - H_P| = 1$.

For an arbitrary system $\mathcal{F}$, we define $\mathcal{P}^u_{\mathcal{F}}$ as a set of all projections satisfied by $\mathcal{F}$, that is, $\mathcal{P}^u_{\mathcal{F}} = \{P \in \mathsf{PRJ} \mid H_P \neq P \cap F \text{ for all } F \in \mathcal{F}\}$, and define $\mathcal{P}_{\mathcal{F}}$ as $\mathcal{P}_{\mathcal{F}} = \mathrm{Min}(\mathcal{P}^u_{\mathcal{F}})$. The latter is called *the set of forbidden projections* of $\mathcal{F}$. Note that, in general, $\mathcal{P}^u_{\mathcal{F}}$ and $\mathcal{P}_{\mathcal{F}}$ can have projections with the same supports (but with different patterns). In fact, save for some degenerate cases, $\mathcal{P}^u_{\mathcal{F}}$ will have a lot of them. The situation with $\mathcal{P}_{\mathcal{F}}$ is, to an extent, similar: although in some cases, which are of particular interest to us, supports of $\mathcal{P}_{\mathcal{F}}$ will all be different and form an antichain (in 2^U), in general, supports of $\mathcal{P}_{\mathcal{F}}$ can coincide or be comparable to each other.

In the opposite direction, we define PRJ^* to be the poset of all antichains over PRJ, ordered by dual refinement; its maximal element is an empty set of projections and its minimal element is a one-element antichain $\{(\varnothing, \varnothing)\}$. For $\mathcal{P} \in \mathsf{PRJ}^*$ we define $\mathcal{F}_{\mathcal{P}}$ as a system of sets satisfying $\mathcal{P}$, that is, $\mathcal{F}_{\mathcal{P}} = \{F \subseteq U \mid H_P \neq P \cap F \text{ for all } (P, H_P) \in \mathcal{P}\}$. Sometimes we will write $\mathcal{F}(\mathcal{P})$ and $\mathcal{P}(\mathcal{F})$ instead of $\mathcal{F}_{\mathcal{P}}$ and $\mathcal{P}_{\mathcal{F}}$. Note that the definition of $\mathcal{F}_{\mathcal{P}}$ does not require $\mathcal{P}$ to be an antichain. However, including other projection sets does not add to the expressiveness of this definition. We justify this in Proposition 2 below, together with some other basic properties of operations $\mathcal{P}$ and $\mathcal{F}$.

Proposition 2.

1. *For any set of projections $\mathcal{P}$, $\mathcal{F}(\mathcal{P}) = \mathcal{F}(\mathrm{Min}(\mathcal{P})) = \mathcal{F}(\mathcal{P}^u)$;*

2. *Both $\mathcal{P}$ and $\mathcal{F}$ operations are monotone, that is, for $\mathcal{F}, \mathcal{G} \subseteq 2^U$ and $\mathcal{P}, \mathcal{Q} \subseteq 2^{\mathsf{PRJ}}$, $\mathcal{F} \subseteq \mathcal{G}$ implies $\mathcal{P}(\mathcal{F}) \ll_d \mathcal{P}(\mathcal{G})$ and $\mathcal{P} \ll_d \mathcal{Q}$ implies $\mathcal{F}(\mathcal{P}) \subseteq \mathcal{F}(\mathcal{Q})$;*

3. *For any system $\mathcal{F}$, $\mathcal{F}(\mathcal{P}(\mathcal{F})) = \mathcal{F}$;*

4. *For any $\mathcal{P} \in \mathsf{PRJ}^*$, $\mathcal{P}(\mathcal{F}(\mathcal{P})) \ll_d \mathcal{P}$, and there are antichains of projections for which this inequality is strict;*

5. *For any $\mathcal{P} \in PRJ^*$, $\mathcal{P}(\mathcal{F}(\mathcal{P})) = \mathcal{P}(\mathcal{F}(\mathcal{P}(\mathcal{F}(\mathcal{P}))))$.*

Proof. 1. As $\mathrm{Min}(\mathcal{P}) \subseteq \mathcal{P}$, $\mathcal{F}(\mathrm{Min}(\mathcal{P})) \supseteq \mathcal{F}(\mathcal{P})$, because the former is the family of sets satisfying a smaller family of projections than the latter. And if $F \notin \mathcal{F}(\mathcal{P})$, then there is $P \in \mathcal{P}$, invalidating F. Take $Q \in \mathrm{Min}(\mathcal{P})$, $Q \leqslant P$. Then Q also invalidates F, hence $F \notin \mathcal{F}(\mathrm{Min}(\mathcal{P}))$. So, $\mathcal{F}(\mathcal{P}) = \mathcal{F}(\mathrm{Min}(\mathcal{P}))$; the second equality follows from the fact that $\mathrm{Min}(\mathcal{P}) = \mathrm{Min}(\mathcal{P}^u)$.

2. Let $\mathcal{F}$ and $\mathcal{G}$ be families of subsets such that $\mathcal{F} \subseteq \mathcal{G}$. Then $\mathcal{P}^u(\mathcal{F}) \supseteq \mathcal{P}^u(\mathcal{G})$. By the definition of the dual refinement, this implies $\mathcal{P}(\mathcal{F}) \ll_d \mathcal{P}(\mathcal{G})$. Conversely, $\mathcal{P} \ll_d \mathcal{Q}$ is equivalent to $\mathcal{P}^u \supseteq \mathcal{Q}^u$, which implies $\mathcal{F}(\mathcal{P}) = \mathcal{F}(\mathcal{P}^u) \subseteq \mathcal{F}(\mathcal{Q}^u) = \mathcal{F}(\mathcal{Q})$.

3. Let us define $\mathcal{F}' = \mathcal{F}(\mathcal{P}(\mathcal{F}))$. Thus, $\mathcal{F}'$ is a family of all subsets satisfying $\mathcal{P}_{\mathcal{F}}$. As all elements of $\mathcal{F}$ satisfy $\mathcal{P}_{\mathcal{F}}$, $\mathcal{F}' \supseteq \mathcal{F}$. On the other hand, for any $N \notin \mathcal{F}$, (U, N) is satisfied by $\mathcal{F}$ and hence $(U, N) \in \mathcal{P}^u_{\mathcal{F}}$. Then there is $P \in \mathcal{P}_{\mathcal{F}}$ such that $P \leqslant (U, N)$. As (U, N) invalidates N then so does P. But then $N \notin \mathcal{F}'$.

4. Let $\mathcal{F} = \mathcal{F}(\mathcal{P})$, $\mathcal{P}' = \mathcal{P}(\mathcal{F})$, and $\mathcal{P}'^u$ and $\mathcal{P}^u$ be upward closures of $\mathcal{P}'$ and $\mathcal{P}$. Note that $\mathcal{F} = \mathcal{F}(\mathcal{P}) = \mathcal{F}(\mathcal{P}^u)$, and hence $\mathcal{P}^u$ is satisfied by $\mathcal{F}$. Also note that, by definition, $\mathcal{P}'^u = \mathcal{P}^u_{\mathcal{F}}$, and hence $\mathcal{P}'^u$ is the set of all projections satisfied by $\mathcal{F}$. But then $\mathcal{P}'^u \supseteq \mathcal{P}^u$, which holds iff $\mathcal{P}' \ll_d \mathcal{P}$.

For an example of the strict inequality, let U be a one element set, $U = \{1\}$, and let $\mathcal{P}$ be a two-element antichain of projections, $\mathcal{P} = \{(1, \varnothing), (1, 1)\}$. Then $\mathcal{F}(\mathcal{P})$ is empty and hence $\mathcal{P}(\mathcal{F}(\mathcal{P})) = \{(\varnothing, \varnothing)\} \neq \mathcal{P}$.

5. This is a consequence of part 3. $\square$

One way of looking at the connection between the families of sets and of forbidden projections is to treat the projections as statements and sets as models which satisfy or do not satisfy certain statements. Thus, projection families in PRJ^* correspond to theories and set families to families of models. Then the operation $\mathcal{F}$ constructs a collection of models, satisfying certain theory, and $\mathcal{P}$ constructs a theory for a collection of models.

Let us note that Proposition 2 implies that $\mathcal{P} \circ \mathcal{F} \colon \mathsf{PRJ}^* \to \mathsf{PRJ}^*$ is a closure operator, and we define $\mathsf{PRJ}^\diamond \subsetneq \mathsf{PRJ}^*$ as its image, that is, $\mathsf{PRJ}^\diamond = \mathcal{P} \circ \mathcal{F}\,[\mathsf{PRJ}^*]$. Note that by Proposition 2, part 5, $\mathcal{P} \in \mathsf{PRJ}^\diamond$ iff $\mathcal{P} = \mathcal{P}(\mathcal{F}(\mathcal{P}))$. Then the application of $\mathcal{P} \circ \mathcal{F}$ is somewhat similar to semantic inference, that is, we can say $\mathcal{P} \models P$ iff $P \in \mathcal{P}(\mathcal{F}(\mathcal{P}))$.

This parallel is justified by the fact that then P is satisfied by all models that satisfy $\mathcal{P}$. We emphasize that all parallels with logic are only referential.

We can now reformulate Dietrich's characterization using the newly developed terminology.

Theorem 3 (Dietrich, reformulated)**.** *Let $\mathcal{P} \in \mathsf{PRJ}^*$ be such that every projection in $\mathcal{P}$ is an implication, that is, has a form $(P + p, P)$, for $p \notin P$. Additionally, suppose $\mathcal{P}$ satisfies:*

$$\begin{array}{l}\textit{for every } (A{+}a, A) \textit{ and } (B{+}b, B) \textit{ in } \mathcal{P}, \textit{ such that } a \in B,\\ \textit{there is } (C + b, C) \in \mathcal{P} \textit{ such that } C \subseteq A \cup B - a.\end{array} \tag{$\dagger$}$$

Then $\mathcal{P} \in \mathsf{PRJ}^\diamond$ and the system $\mathcal{F}(\mathcal{P})$ is a convex geometry.

Conversely, for a convex geometry $\mathcal{F}$, every projection in $\mathcal{P}(\mathcal{F}) \in \mathsf{PRJ}^\diamond$ is an implication, and $\mathcal{P}(\mathcal{F})$ satisfies ($\dagger$).

Let us formulate the following remark on Theorem 3.

Proposition 4. *Let $\mathcal{P}$ be as in Theorem 3. Then all supports of the projections from $\mathcal{P}$ are distinct and form an antichain.*

Proof. Suppose not, and let $A' = (A + a, A)$, $B' = (B + b, B)$ be two distinct implications from $\mathcal{P}$ such that $A + a \subseteq B + b$. Notice that $a \neq b$ as otherwise $A' \leqslant B'$. But this implies $a \in B$ and, by ($\dagger$), there is $(C + b, C) \in \mathcal{P}$ such that $C \subseteq A \cup B - a$. Note that as $A + a \subseteq B + b$ and $b \notin C$, this implies $C \subseteq B - a \subsetneq B$. Then $(C + b, C) < (B + b, B)$, a contradiction. $\square$

Finally, let us note that Theorem 3 does not rule out the possibility that $\mathcal{P} \in \mathsf{PRJ}^*$, whose projections are implications, does not satisfy ($\dagger$), yet $\mathcal{F}(\mathcal{P})$ is a convex geometry; however, it follows that in that case $\mathcal{P}(\mathcal{F}(\mathcal{P})) \neq \mathcal{P}$. An example of this situation is $U = \{0, 1, 2\}$ and $\mathcal{P} = \{0 \to 1, 1 \to 2\}$. Then $\mathcal{F}(\mathcal{P}) = \{2, 12, 012\}$ and $\mathcal{P}(\mathcal{F}(\mathcal{P})) = \{0 \to 1, 1 \to 2, 0 \to 2\} \lll_d \mathcal{P}$.

3 Forbidden projections characterization of shattering-extremal systems

For projections (P, h_P) and (Q, h_Q), let us call the set $P \cup Q$ the *support* of P and Q, denoted $\mathbf{sup}(P, Q)$; the set $\{x \in P \cap Q \mid h_P(x) \neq h_Q(x)\}$ the *disagreement set* of P and Q, denoted $\mathbf{dis}(P, Q)$; and the set $\mathbf{sup}(P, Q) - \mathbf{dis}(P, Q)$ the *agreement set* of P and Q, denoted $\mathbf{agr}(P, Q)$. We say that P and Q *agree* if $\mathbf{dis}(P, Q) = \varnothing$ (alternatively, if $\mathbf{agr}(P, Q) = \mathbf{sup}(P, Q)$); and that P and Q *agree on $x \in U$ (on $X \subseteq U$)* if $x \in \mathbf{agr}(P, Q)$ ($X \subseteq \mathbf{agr}(P, Q)$). Thus, P and Q automatically agree on a symmetric difference of their supports and disagree on any point outside of $P \cup Q$, although the latter will not be of importance throughout the paper. Now, we can formulate the forbidden projections characterization of shattering-extremal systems in general.

Theorem 5. *If $\mathcal{P} \in \mathsf{PRJ}^*$ satisfies:*

$$\text{for any } A, B \in \mathcal{P} \text{ and } x \in \mathbf{dis}(A,B) \text{ there is } C \in \mathcal{P} \text{ such that } C \subseteq \mathbf{sup}(A,B) - x \text{ and } C \text{ agrees with } A \text{ and } B \text{ on } \mathbf{agr}(A,B) \cap C, \tag{‡}$$

then $\mathcal{P} \in \mathsf{PRJ}^\diamond$ and the system $\mathcal{F}(\mathcal{P})$ is shattering-extremal.

Conversely, for a shattering-extremal family $\mathcal{F}$, $\mathcal{P}_\mathcal{F} \in \mathsf{PRJ}^\diamond$ satisfies (‡).

It is a straightforward exercise to show that Theorem 3 is a particular case of Theorem 5 in case all forbidden projections are implications. We do not prove this connection as it serves little purpose in the context of this paper. Prior to going to the proof of Theorem 5, we will set up some terminology and intermediate results. The following proposition is an analog of Proposition 4.

Proposition 6. *If $\mathcal{P} \in \mathsf{PRJ}^*$ satisfies (‡) then all supports of the projections from $\mathcal{P}$ are distinct and form an antichain.*

Proof. Suppose $\mathcal{P}$ has different projections with comparable supports, and let us pick $(A, h_A) \neq (B, h_B) \in \mathcal{P}$ such that $A \subseteq B$, and $|\mathbf{dis}(A,B)|$ is minimal among such pairs. Notice that $\mathbf{dis}(A,B) \neq \varnothing$, as otherwise $(A, h_A) \ll_d (B, h_B)$, and let $x \in \mathbf{dis}(A,B)$. Then by (‡) there is $(C, h_C) \in \mathcal{P}$ such that $C \subseteq A \cup B - x = B - x$, and that C agrees with B on $\mathbf{agr}(A,B) \cap C$, in particular, C agrees with B on any $x \in B - A$. But then $\mathbf{dis}(C,B) \subseteq \mathbf{dis}(A,B) - x$, contradicting the minimality of $|\mathbf{dis}(A,B)|$. □

Recall that for a family of projection $\mathcal{P}$, the statement $P \in \mathcal{P}(\mathcal{F}(\mathcal{P}))$ is somewhat similar to semantic inference. Along with it, it is convenient to have a syntactic one, that is, one or several rules of the form $\mathcal{P}' \subseteq \mathcal{P}$ implies $Q = Q(\mathcal{P}') \in \mathcal{P}(\mathcal{F}(\mathcal{P}))$, where both $\mathcal{P}'$ and the construction of $Q(\mathcal{P}')$ is relatively simple. As it turns out, there is a simple syntactic rule of this form that is both sound and complete in the following sense.

Lemma 7. *Let $A, B \in \mathcal{P} \in \mathsf{PRJ}^*$ be two projections that disagree on a single element x. Then there is $C \in \mathcal{P}(\mathcal{F}(\mathcal{P}))$ such that $C \subseteq \mathbf{sup}(A,B) - x$ and C agrees with both A and B.*

Conversely, let $\mathcal{P} \in \mathsf{PRJ}^*$ be such that for any $A, B \in \mathcal{P}$ that disagree on a single element x there is $C \in \mathcal{P}$ such that $C \subseteq \mathbf{sup}(A,B) - x$ and C agrees with both A and B. Then $\mathcal{P} \in \mathsf{PRJ}^\diamond$.

We note that the condition in this lemma is similar to (‡), but, in contrast, applies to pairs of projections that disagree on one element only. Additionally, it is easy to see that Lemma 7 is equivalent to a well-known fact that the DavisPutnamLogemannLoveland (DPLL) algorithm is complete for the satisfiability of CNF formulas [6, 7]. However, making this parallel explicit would require too much off the point effort, so we will not follow this lead.

Proof. ($\Rightarrow$). Without losing generality, assume $h_A(x) = 0$ and $h_B(x) = 1$. Let us define a projection (C', h'_C) where $C' = A \cup B - x$ and $h'_C(y) = h_A(y)$ for $y \in A - x$ and $h_B(y)$ for $y \in B - x$. Note that $h_A(y) = h_B(y)$ for $y \in A \cap B - x$ makes sure that h'_C is well defined. Also, thus defined, C' agrees with both A and B.

Now, for any $W \subseteq U$ such that W invalidates C', either $x \notin W$ or $x \in W$. It is then easy to see that in the first case W invalidates A, and in the second B, and hence $W \notin \mathcal{F}(\mathcal{P})$. Thus, no set from $\mathcal{F}(\mathcal{P})$ invalidates C', and so $C' \in \mathcal{P}^u(\mathcal{F}(\mathcal{P}))$. Then there is $C \in \mathcal{P}(\mathcal{F}(\mathcal{P}))$ such that $C \leqslant C'$, and such C obviously is contained in $\mathbf{sup}(A, B) - x$ and agrees with both A and B.

($\Leftarrow$). Suppose $\mathcal{P}$ satisfies the stated condition. We want to show that $\mathcal{P}(\mathcal{F}(\mathcal{P})) = \mathcal{P}$. By Proposition 2, part 4, $\mathcal{P}(\mathcal{F}(\mathcal{P})) \ll_d \mathcal{P}$, that is, $\mathcal{P}^u(\mathcal{F}(\mathcal{P})) \supseteq \mathcal{P}^u$, and we only need to show the inverse inclusion $\mathcal{P}^u(\mathcal{F}(\mathcal{P})) \subseteq \mathcal{P}^u$.

Contrapositively, we are going to show that $P \notin \mathcal{P}^u$ implies $P \notin \mathcal{P}^u(\mathcal{F}(\mathcal{P}))$, where the latter means that there is $X \subseteq U$ such that X invalidates P, but satisfies any $P' \in \mathcal{P}$. We are going to define X by defining its characteristic function $h_X \colon U \to \{0, 1\}$. Let us enumerate the elements of $U - X$, in an arbitrary way, as $\{y_1, \dots, y_n\}$. Let $X_0 = X$, $X_i = X_{i-1} + y_i$, for $i = 1, \dots, n$, in particular, $X_n = U$. Also, let $\mathcal{P}_i = \{P' \in \mathcal{P} \mid P' \subseteq X_i\}$, $i = 0, \dots, n$, in particular, $\mathcal{P}_n = \mathcal{P}$. Finally, we are going to define a family of partial characteristic functions $h_i \colon X_i \to \{0, 1\}$, where $h_0 = h_P$, and h_{i+1} is obtained from h_i by defining the latter on x_{i+1}, that is, $h_{i+1}(x) = h_i(x)$ for $x \in X_i$, for $i = 0, \dots, n-1$. Then h_n is defined on the whole U, and we are going to put $h_X = h_n$. Let us note that then $h_X|_P = h_P$, and hence X invalidates P.

Now, we are going to inductively define h_{i+1} from h_i by picking the value for h_{i+1} on x_{i+1} with the inductive hypothesis: $h_i|_{P'} \neq h_{P'}$ for any $P' \in \mathcal{P}_i$. Note that this would prove that X satisfies all forbidden projections from $\mathcal{P}_n = \mathcal{P}$ and thus the statement of the lemma holds. For the base of induction, let us take $P' \in \mathcal{P}_0$. Then $P' \subseteq P$, and if $h_0|_{P'} = h_{P'}$ then this means precisely that $P` \leqslant P$. But this is impossible as $P \in \mathcal{P}$ and $P \notin \mathcal{P}^u$.

So suppose the inductive hypothesis holds for i. We claim that it is possible to pick the value for $h_{i+1}(x_{i+1})$ such that it would be satisfied for $i+1$. Suppose not, that is, for both potential values 0 and 1 there are forbidden projections $P_0, P_1 \in \mathcal{P}_{i+1}$, invalidating the hypothesis for h^0_{i+1} and h^1_{i+1}, where the latter are the two possible extensions of h_i. Note that $x_{i+1} \in P_0$, as otherwise $h_{P_0} = h^0_{i+1}|_{P_0} = h^0_i|_{P_0}$, invalidating the induction hypothesis; similarly, $x_{i+1} \in P_1$. Also, $h_{P_0}(x_{i+1}) = h^0_{i+1}|_{P_0}(x_{i+1}) = 0$, and similarly $h_{P_1}(x_{i+1}) = 1$. Additionally, for any $y \in X_{i+1} - x_{i+1} = X_i$, if $y \in P_0 \cap P_1$ then $h_{P_0}(y) = h_{P_1}(y) = h_i(y)$. So, P_0 and P_1 disagree on exactly one element, and, by the condition on $\mathcal{P}$, there is $Q \in \mathcal{P}$ such that $Q \subseteq \mathbf{sup}(P_0, P_1) - x_{i+1} \subseteq X_i$ such that Q agrees with both P_0 and P_1. But then Q agrees with h_i, which contradicts the induction hypothesis. $\square$

For $x \in U$, the *downshift* operation $\mathcal{D}_x \colon 2^U \to 2^U$ is defined as:

$$\mathcal{D}_x(\mathcal{F}) = \big\{F - x \mid F \in \mathcal{F}\big\} \cup \big\{F \mid F \in \mathcal{F}, x \in F, F - x \in \mathcal{F}\big\}.$$

Notice that $\mathcal{D}_x(\mathcal{F})$ might be regarded as an image of injective map $\Phi \colon \mathcal{F} \to 2^U$, defined as:

$$\Phi(F) = \begin{cases} F, & x \notin F; \\ F, & x \in F, F - x \in \mathcal{F}; \\ F - x, & x \in F, F - x \notin \mathcal{F}; \end{cases} \tag{3}$$

thus, $|\mathcal{D}_x(\mathcal{F})| = |\mathcal{F}|$. The definition of $\mathcal{D}_x$ is from [18, Definition 2.30], however it goes back to at least Frankl's proof of SSP lemma [11]. Downshifts are known to preserve shattering-extremality [18]. With an abuse of notation, we define downshifts also for families of forbidden projections in the following way. For $X \subseteq U$, $x \in U$, $h\colon X \to \{0,1\}$, and $v \in \{0,1\}$ we define $h[x \mapsto v]\colon X \to \{0,1\}$ as $h[x \mapsto v](y) = h(y)$ for $y \in X - x$ and $h[x \mapsto v](x) = v$; in particular, for $x \notin X$, $h = h[x \to v]$. And $\mathcal{D}_x\colon \mathsf{PRJ}^* \to \mathsf{PRJ}^*$ is defined as:

$$\mathcal{D}_x(\mathcal{P}) = \mathrm{Min}\left(\left\{(P, h_P[x \mapsto 1]) \mid P \in \mathcal{P}\right\}\right).$$

Also, on occasion, we write $P[x \mapsto v]$, for $P \in \mathsf{PRJ}$, to denote $P[x \mapsto v] = (P, h_P[x \mapsto v])$.

The family $\mathrm{Str}(\mathcal{F})$ of shattered sets of $\mathcal{F}$ can be trivially defined via $\mathcal{P}_{\mathcal{F}}$. For $\mathcal{Q} \in \mathsf{PRJ}^*$ (or for $\mathcal{Q} \subseteq 2^{\mathsf{PRJ}}$) let us define $\mathcal{A}_{\sup}(\mathcal{Q})$ as an antichain (in 2^U) of minimal supports of the projections of $\mathcal{Q}$; note that it is different from the set of supports of minimal projections, which, in general, is not an antichain. The following proposition is trivial.

Proposition 8. *For a system $\mathcal{F}$,*

$$\begin{aligned}\mathrm{Str}(\mathcal{F}) &= \{S \subseteq U \mid S \not\supseteq P, \text{ for any } (P, h_P) \in \mathcal{P}_{\mathcal{F}}\}\\ &= \{S \subseteq U \mid S \not\supseteq P, \text{ for any } P \in \mathcal{A}_{\sup}(\mathcal{P}_{\mathcal{F}})\}.\end{aligned}$$

Lemma 9.

1. *For any family of forbidden projections $\mathcal{P}$, $\mathcal{F} \subseteq 2^U$, and $x \in U$, it holds*

$$\mathcal{P}(\mathcal{D}_x(\mathcal{F}_{\mathcal{P}})) \ll_d \mathcal{D}_x(\mathcal{P}), \tag{4}$$

$$\mathcal{P}(\mathcal{D}_x(\mathcal{F})) \ll_d \mathcal{D}_x(\mathcal{P}_{\mathcal{F}}), \tag{5}$$

$$\mathcal{D}_x(\mathcal{F}_{\mathcal{P}}) \subseteq \mathcal{F}(\mathcal{D}_x(\mathcal{P})), \tag{6}$$

$$\mathcal{D}_x(\mathcal{F}) = \mathcal{F}(\mathcal{D}_x(\mathcal{P}_{\mathcal{F}})). \tag{7}$$

Moreover, for a shattering-extremal $\mathcal{F}$:

2. *$\mathcal{D}_x(\mathcal{F})$ is shattering-extremal;*
3. *$\mathcal{A}_{\sup}(\mathcal{P}(\mathcal{D}_x(\mathcal{F}))) = \mathcal{A}_{\sup}(\mathcal{D}_x(\mathcal{P}_{\mathcal{F}}))$.*

Prior to the proof, let us discuss the statements of this lemma. Recalling our logic parallel, we note that the operation $\mathcal{D}_x$ is defined both semantically, that is, on set families, and syntactically, that is, as a simple rewriting rule on projections. This parallel gets a little stretched here, as we can note that $\mathcal{D}_x$ is not defined on individual sets, that is, on models, but rather on set families. Let us note that in equations (4)–(6) the left hand side corresponds to semantical, and the right hand side to syntactical application of $\mathcal{D}_x$. Thus, in a way, part 1 says that $\mathcal{D}_x$ preserves soundness, that is, the application of $\mathcal{D}_x$ to both a theory $\mathcal{P}$ and a model family $\mathcal{F}$, satisfying this theory, results in a modified model family satisfying the modified theory.

Let us give a name to the following relaxation of (7):

$$\mathcal{D}_x(\mathcal{F}) \subseteq \mathcal{F}(\mathcal{D}_x(\mathcal{P}_{\mathcal{F}})). \tag{7*}$$

In the proof of part 1, we will note that the equations (4)–(6) and (7*) can be inferred from each other with the following implications: (6)⇔(4)⇒(7*)⇔(5). We will not state it explicitly, but from the proof of part 1, it is straightforward that, for a given $\mathcal{P}$ the equality in (4) implies the equality in (6), but not the other way round. Similarly, for a given $\mathcal{F}$, the equality in (5) implies the equality in (7*). It will take a separate line of argument to prove (7), that is, that (7*) case actually achieves equality. In all other cases, as we are going to show, the inequality can be strict.

Now, for the shattering-extremal systems, part 4 is a proxy for the equality in (5). As it turns out, shattering-extremal systems obtain the equality in (5) in general. Indeed, assuming Theorem 5, for a shattering-extremal $\mathcal{F}$, both $\mathcal{P}(\mathcal{D}_x(\mathcal{F}))$ and $\mathcal{P}_{\mathcal{F}}$ satisfy (‡), and hence, by Proposition 6, the supports of the projections from both of these projection families are distinct and form an antichain; this statement also trivially extends to $\mathcal{D}_x(\mathcal{P}_{\mathcal{F}})$. But, with this condition in place, $\mathcal{P}(\mathcal{D}_x(\mathcal{F})) \ll_d \mathcal{D}_x(\mathcal{P}_{\mathcal{F}})$ together with $\mathcal{A}_{\sup}(\mathcal{P}(\mathcal{D}_x(\mathcal{F}))) = \mathcal{A}_{\sup}(\mathcal{D}_x(\mathcal{P}_{\mathcal{F}}))$ easily imply $\mathcal{P}(\mathcal{D}_x(\mathcal{F})) = \mathcal{D}_x(\mathcal{P}_{\mathcal{F}})$. However, proving $\mathcal{P}(\mathcal{D}_x(\mathcal{F})) = \mathcal{D}_x(\mathcal{P}_{\mathcal{F}})$ for shattering-extremal families directly is hard, that is why we settle for part 4 in its current form.

We will now give several examples that will illustrate how the $\mathcal{D}_x$ operations work and show that the statements of the lemma cannot be strengthened.

First, let us take $U = \{1\}$, $\mathcal{P} = \{(1, \varnothing), (1, 1)\}$, and $x = 1$. Then $\mathcal{F}_{\mathcal{P}} = \mathcal{D}_x(\mathcal{F}_{\mathcal{P}}) = \varnothing$, implying $\mathcal{P}(\mathcal{D}_x(\mathcal{F}_{\mathcal{P}})) = \{(\varnothing, \varnothing)\}$. However, $\mathcal{D}_x(\mathcal{P}) = \{(1, 1), (1, 1)\} = \{(1, 1)\}$, and so $\mathcal{F}(\mathcal{D}_x(\mathcal{P})) = \{\varnothing\}$, which gives an example of $\mathcal{P}$ for which the inequalities in both (6) and (4) are strict; note that $\mathcal{D}_x(\mathcal{F}_{\mathcal{P}}) = \varnothing \subsetneq \mathcal{F}(\mathcal{D}_x(\mathcal{P})) = \{\varnothing\}$. Let us also mention that $\mathcal{F}_{\mathcal{P}} = \varnothing$ is shattering-extremal, and so the inequalities in (6) and (4) can be proper in shattering-extremal case as well.

Second, let $U = \{1, 2\}$, $\mathcal{P} = \{(12, \varnothing), (12, 12)\}$, and $x = 2$. Then $\mathcal{F}_{\mathcal{P}} = \{1, 2\}$, and $\mathcal{D}_x(\mathcal{F}_{\mathcal{P}}) = \{1, \varnothing\}$. Also, $\mathcal{D}_x(\mathcal{P}) = \{(12, 2), (12, 12)\}$, and so $\mathcal{F}(\mathcal{D}_x(\mathcal{P})) = \{\varnothing, 1\} = \mathcal{F}_{\mathcal{P}}$, so there is equality in (4). However, $\mathcal{P}(\mathcal{D}_x(\mathcal{F}_{\mathcal{P}})) = \{(2, 2)\}$, and so the inequality in (4) is strict. We also note that $\mathrm{Str}(\mathcal{F}_{\mathcal{P}}) = \{\varnothing, 1, 2\}$, and so $\mathcal{F}_{\mathcal{P}}$ is not shattering-extremal.

Alternatively, in this case we can start not with $\mathcal{P}$, but with $\mathcal{F}$, that is, take $U = \{1, 2\}$, $\mathcal{F} = \{1, 2\}$, and $x = 2$. Then $\mathcal{P}_{\mathcal{F}} = \{(12, \varnothing), (12, 12)\}$ and $\mathcal{D}_x(\mathcal{P}_{\mathcal{F}}) = \{(12, 1), (12, 12)\}$. At the same time, $\mathcal{D}_x(\mathcal{F}) = \{1, \varnothing\}$ and hence $\mathcal{P}(\mathcal{D}_x(\mathcal{F})) = \{(2, \varnothing)\}$, making the inequality in (5) proper.

Finally, let us give an example of both semantic and syntactic downshifts on a not so trivial system. So let $\mathcal{F}$ be a family of intervals on $U = \{1, 2, 3, 4\}$ with $1 < 2 < 3 < 4$, that is, $234 = [2, 4] \in \mathcal{F}$, but $24 \notin \mathcal{F}$. As a poset, $\mathcal{F}$ is an interval lattice, which is one of the simplest examples of convex geometries, in particular, $\mathcal{F}$ is shattering-extremal, it has 11 elements, and shatters 11 elements, where the latter are all subsets of U of size at most two. Now, let $x = 3$, and let Φ be as in (3) for $\mathcal{D}_3$. Then $\Phi(234) = 24$, $\Phi(1234) = 124$, and $\Phi(F) = F$ otherwise; the latter is by definition for $F \in \mathcal{F}$ such that $3 \notin F$, but otherwise should be checked manually. For example, for 123, $123 - 3 = 12 \in \mathcal{F}$, and so $\Phi(123) = 123$. Thus, $\mathcal{D}_3(\mathcal{F}) = \mathcal{F} - 234 - 1234 + 24 + 124$. Note that downshifts trivially preserve the size of the family, and thus $|\mathcal{D}_3(\mathcal{F})| = |\mathcal{F}| = 11$. Additionally, it can be checked that $\mathcal{D}_3(\mathcal{F})$ shatters the same 11 sets as $\mathcal{F}$, and is thus shattering-extremal, which illustrates part 2 of Lemma 9. Note also that $\mathcal{D}_3(\mathcal{F})$ is no longer a closure system and is thus not a convex geometry: Although $\mathcal{D}_3(\mathcal{F})$ is still intersection-closed, it no longer contains the maximal element U. We note, without proving it, that downshifts preserve the property of being intersection-closed in general.

Now, let us consider $\mathcal{P}_{\mathcal{F}}$ for this $\mathcal{F}$. Figure 1 below shows both $\mathcal{P}_{\mathcal{F}}$ and $\mathcal{D}_3(\mathcal{P}_{\mathcal{F}})$, where, for a forbidden projection (P, h_P), a black point indicates $x \in P$, $h_P(x) = 1$, a white point $x \in P$, $h_P(x) = 0$, and a dot $x \notin P$.

$$\mathcal{P}_{\mathcal{F}} \quad \begin{array}{cccc} \bullet & \circ & \bullet & \cdot \\ \bullet & \circ & \cdot & \bullet \\ \bullet & \cdot & \circ & \bullet \\ \cdot & \bullet & \circ & \bullet \end{array} \qquad\qquad \mathcal{D}_3(\mathcal{P}_{\mathcal{F}}) \quad \begin{array}{cccc} \bullet & \circ & \bullet & \cdot \\ \bullet & \circ & \cdot & \bullet \\ \bullet & \cdot & \bullet & \bullet \\ \cdot & \bullet & \bullet & \bullet \end{array}$$

Figure 1: Forbidden projections of $\mathcal{P}_{\mathcal{F}}$ and $\mathcal{D}_3(\mathcal{P}_{\mathcal{F}})$.

It can be checked explicitly that in this case $\mathcal{D}_3(\mathcal{F}) = \mathcal{F}(\mathcal{D}_3(\mathcal{P}_{\mathcal{F}}))$, confirming (7). Also, $\mathcal{D}_3(\mathcal{P}_{\mathcal{F}}) \in \mathsf{PRJ}^{\diamond}$, and $\mathcal{P}(\mathcal{D}_3(\mathcal{F})) = \mathcal{D}_3(\mathcal{P}_{\mathcal{F}})$; the latter, as we have noted earlier, will be proven to be a general case for the shattering-extremal systems.

Proof (of Lemma 9). 1. We are going to prove (6) and then show that the remaining inequalities, with (7) substituted by (7*), follow from it; the equality in (7) will be proven separately. So, suppose (6) does not hold, that is, there is $F \in \mathcal{D}_x(\mathcal{F}_{\mathcal{P}})$ such that $F \notin \mathcal{F}(\mathcal{D}_x(\mathcal{P}))$. This means that there is $P \in \mathcal{P}$ such that $P' = (P, h_P[x \mapsto 1])$ is invalidated by F. Suppose $P' = P$, which happens if either $x \notin P$ or $h_P(x) = 1$. In this case F invalidates P itself, and hence $F \in \mathcal{D}_x(\mathcal{F}_{\mathcal{P}}) - \mathcal{F}_{\mathcal{P}}$. By the definition of $\mathcal{D}_x$ it means that $x \notin F$, $F + x \in \mathcal{F}_{\mathcal{P}}$ and $F \notin \mathcal{F}_{\mathcal{P}}$. In particular, as F invalidates P, it cannot be the case that $x \in P$ and $h_P(x) = 1$. Thus, $x \notin P$, but then F invalidates P iff $F + x$ invalidates P, and hence $F + x \notin \mathcal{F}_{\mathcal{P}}$, a contradiction.

Now, suppose $P' \neq P$, that is, $x \in P$, $h_{P'}(x) = 1$ and $h_P(x) = 0$. As P' is invalidated by F, it follows that $x \in F$. But then P is invalidated by $F - x$, hence $F - x \notin \mathcal{F}_{\mathcal{P}}$. But $F \in \mathcal{D}_x(\mathcal{F}_{\mathcal{P}})$ implies, by the definition of $\mathcal{D}_x$, that $F, F - x \in \mathcal{F}_P$, a contradiction.

After that, the remaining inequalities follow by a straightforward application of properties from Proposition 2. Indeed, (7*) follows from (6) by taking $\mathcal{P} = \mathcal{P}_{\mathcal{F}}$. By a similar trick, (5) follows from (4). Also, (6) implies (4) and (7*) implies (5) by applying $\mathcal{P}$ to both sides, using its monotonicity, and applying $\mathcal{P}(\mathcal{F}(\mathcal{P})) \ll_d \mathcal{P}$ to the right hand side. A similar application of $\mathcal{F}$, followed by using $\mathcal{F}(\mathcal{P}(\mathcal{F})) = \mathcal{F}$ equation, gives (4) $\Rightarrow$ (6) and (5) $\Rightarrow$ (7*) implications.

Finally, let us show that (7*) can be strengthened to (7), that is, that $\mathcal{D}_x(\mathcal{F}) = \mathcal{F}(\mathcal{D}_x(\mathcal{P}_{\mathcal{F}}))$, for which it is enough to prove $\mathcal{D}_x(\mathcal{F}) \supseteq \mathcal{F}(\mathcal{D}_x(\mathcal{P}_{\mathcal{F}}))$. So, let us fix a set family $\mathcal{F}$, and, contrapositively, let $X \notin \mathcal{D}_x(\mathcal{F})$; our goal is then to prove that $X \notin \mathcal{F}(\mathcal{D}_x(\mathcal{P}_{\mathcal{F}}))$. Now, $X \notin \mathcal{D}_x(\mathcal{F})$ implies that either i) $X \notin \mathcal{F}$, or ii) $X \in \mathcal{F}$ but $X - x \notin \mathcal{F}$: Indeed, if it is not the first option then $X \in \mathcal{F}$. But then $X \notin \mathcal{D}_x(\mathcal{F})$ implies $\Phi(X) \neq X$, where $\Phi = \Phi(\mathcal{F})$ is from (3), which can only happen if $X - x \notin \mathcal{F}$.

Suppose $x \in X$. Then, if i) $X \notin \mathcal{F}$, there is $(P, h_P) \in \mathcal{P}_{\mathcal{F}}$ invalidated by X. Then either $x \notin P$, or $x \in P$ and $h_P(x) = 1$. In both cases, $(P, h_P) = (P, h_P[x \to 1]) \in \mathcal{D}_x(\mathcal{P}_{\mathcal{F}})$ invalidates X, and hence $X \notin \mathcal{F}(\mathcal{D}_x(\mathcal{P}_{\mathcal{F}}))$. And if ii) $X \in \mathcal{F}$ but $X - x \notin \mathcal{F}$ then there is $(P, h_P) \in \mathcal{P}_{\mathcal{F}}$ invalidated by $X - x$, but not by X, which implies $x \in X$, $x \in P$, $h_P(x) = 0$, and for each $y \in P - x$, $h_P(y) = h_X(y)$, where h_X is a characteristic function of X. But then $(P, h_P[x \to 1]) \in D_x(\mathcal{P}_{\mathcal{F}})$ invalidates X, and hence again $X \notin \mathcal{F}(\mathcal{D}_x(\mathcal{P}_{\mathcal{F}}))$.

So let $x \notin X$. Note that in this case ii) is impossible, and so i) $X \notin \mathcal{F}$ holds. Also, $X + x \notin \mathcal{F}$, as otherwise, $\Phi(X + x) = X \in \mathcal{D}_x(\mathcal{F})$. Let $(P_1, h_1), (P_2, h_2) \in \mathcal{P}_{\mathcal{F}}$ be the projection invalidating X and $X + x$ respectively. If $x \notin P_1$ then $(P_1, h_1) = (P_1, h_1[x \to 1]) \in \mathcal{D}_x(\mathcal{P}_{\mathcal{F}})$ invalidates X, and hence $X \notin \mathcal{F}(\mathcal{D}_x(\mathcal{P}_{\mathcal{F}}))$. So we can assume

$x \in P_1$; by a similar argument, $x \in P_2$. This also implies $h_1(x) = 0$ and $h_2(x) = 1$. Finally, for all $y \in P_1 \cap P_2 - x$, it holds $h_1(y) = h_2(y) = h_X(y)$, that is, (P_1, h_1) and (P_2, h_2) disagree just on x.

Then, by Lemma 7, there is $(Q, h_Q) \in \mathcal{P}(\mathcal{F}(\mathcal{P}_{\mathcal{F}})) = \mathcal{P}_{\mathcal{F}}$ such that $Q \subseteq P_1 \cup P_2 - x$ that agrees with both (P_1, h_1) and (P_2, h_2). But then (Q, h_Q) invalidates X and is not changed by $\mathcal{D}_x$, which finishes the proof that $X \notin \mathcal{F}(\mathcal{D}_x(\mathcal{P}_{\mathcal{F}}))$.

2. Note that $\mathcal{F}$ being shattering-extremal implies $|\operatorname{Str}(\mathcal{F})| = |\mathcal{F}| = |\mathcal{D}_x(\mathcal{F})| \leqslant |\operatorname{Str}(\mathcal{D}_x(\mathcal{F}))|$. So to prove the shattering-extremality of $\mathcal{D}_x(\mathcal{F})$ it is enough to prove a well known-fact that $\operatorname{Str}(\mathcal{D}_x(\mathcal{F})) \subseteq \operatorname{Str}(\mathcal{F})$. Indeed, by Proposition 8, $X \notin \operatorname{Str}(\mathcal{F})$ if $P \subseteq X$ for some $P \in \mathcal{A}_{\sup}(\mathcal{P}_F) = \mathcal{A}_{\sup}(\mathcal{D}_x(\mathcal{P}_F))$. But $\mathcal{P}(\mathcal{D}_x(\mathcal{F})) \lll_d \mathcal{D}_x(\mathcal{P}_F)$, which implies $Q \subseteq P \subseteq X$ for some $Q \in \mathcal{A}_{\sup}(\mathcal{P}(\mathcal{D}_x(\mathcal{F})))$. But then $X \notin \operatorname{Str}(\mathcal{D}_x(\mathcal{F}))$.

3. By applying $\mathcal{A}_{\sup}$ to both sides of (5) we get $\mathcal{A}_1 = \mathcal{A}_{\sup}(\mathcal{P}(\mathcal{D}_x(\mathcal{F}))) \lll_d \mathcal{A}_2 = \mathcal{A}_{\sup}(\mathcal{D}_x(\mathcal{P}_{\mathcal{F}}))$. Now

$$
\begin{aligned}
|\mathcal{A}_1^u| &= 2^{|U|} - \left|\{S \subseteq U \mid S \not\supseteq P, \text{ for all } P \in \mathcal{A}_{\sup}(\mathcal{P}(\mathcal{D}_x(\mathcal{F})))\}\right| \\
&= 2^{|U|} - \left|\operatorname{Str}(\mathcal{D}_x(\mathcal{F}))\right| \\
&= \left[\text{by part 2}\right] = 2^{|U|} - \left|\mathcal{D}_x(\mathcal{F})\right| = 2^{|U|} - \left|\mathcal{F}\right| = 2^{|U|} - \left|\operatorname{Str}(\mathcal{F})\right| \\
&= 2^{|U|} - \left|\{S \subseteq U \mid S \not\supseteq P, \text{ for all } P \in \mathcal{A}_{\sup}(\mathcal{P}_{\mathcal{F}})\}\right| \\
&= \left[\text{trivially, } \mathcal{A}_{\sup}(\mathcal{P}_{\mathcal{F}}) = \mathcal{A}_{\sup}(\mathcal{D}_x(\mathcal{P}_{\mathcal{F}}))\right] \\
&= 2^{|U|} - \left|\{S \subseteq U \mid S \not\supseteq P, \text{ for all } P \in \mathcal{A}_{\sup}(\mathcal{D}_x(\mathcal{P}_{\mathcal{F}}))\}\right| = |\mathcal{A}_2^u|.
\end{aligned}
$$

Hence, $\mathcal{A}_1^u = \mathcal{A}_2^u$ and $\mathcal{A}_1 = \mathcal{A}_2$. □

Proof (of Theorem 5). If $\mathcal{F}$ is shattering-extremal then $\mathcal{P}_{\mathcal{F}}$ satisfies (‡). Let us define a partial order $\preceq$ on the set of systems on U as a transitive closure of the downshift operation, namely $\mathcal{G} \preceq \mathcal{F}$ if there is a sequence $x_0, \ldots, x_{k-1} \in U$, $k \geqslant 0$, such that $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_{i+1} = \mathcal{D}_{x_i}(\mathcal{F}_i)$, and $\mathcal{F}_k = \mathcal{G}$. This relation is reflexive and transitive by construction, and it is also antisymmetric, as for $\Sigma(\mathcal{F}) = \sum\{|F| \mid F \in \mathcal{F}\}$, it holds: $\Sigma(\mathcal{D}_x(\mathcal{F})) \leqslant \Sigma(\mathcal{F})$, and the equality happens iff $\mathcal{D}_x(\mathcal{F}) = \mathcal{F}$. The poset of all systems with the partial order $\preceq$ is denoted by DNS.

We prove the claim by a poset induction on DNS, that is, we prove it for minimal elements, and then we prove that for $\mathcal{F} \in \mathsf{DNS}$, if the claim holds for all $\mathcal{G} \precneqq \mathcal{F}$, then it holds for $\mathcal{F}$.

The minimal elements of DNS are exactly hereditary systems, which are shattering-extremal. However, for a hereditary $\mathcal{H} \in \mathsf{DNS}$, $\mathcal{P}_{\mathcal{H}} = \{(P, P) \mid P \subseteq U, P \text{ is minimal such that } P \notin \mathcal{H}\}$. Then, for any $A, B \in \mathcal{P}_{\mathcal{H}}$, $\mathbf{dis}(A, B) = \varnothing$ and (‡) holds. This establishes the base of induction. Also, the claim trivially holds for all non shattering-extremal systems. As downshifts preserve shattering-extremality, any $\mathcal{G} \precneqq \mathcal{F}$ is shattering-extremal whenever $\mathcal{F}$ is. Thus, for the induction step, we should prove the following:

$$
\begin{array}{l}
\text{Let } \mathcal{F} \in \mathsf{DNS} \text{ be shattering-extremal and such that for} \\
\text{any } \mathcal{G} \precneqq \mathcal{F}\ \mathcal{P}_{\mathcal{G}} \text{ satisfies (‡). Then } \mathcal{P}_{\mathcal{F}} \text{ satisfies (‡).}
\end{array}
\tag{8}
$$

Suppose (8) does not hold, and let us pick $\mathcal{F}$ invalidating it, that is, $\mathcal{F}$ is shattering-extremal and such that (‡) does not hold for $\mathcal{P}_{\mathcal{F}}$, but holds for $\mathcal{P}_{\mathcal{G}}$, for any $\mathcal{G} \precneqq \mathcal{F}$. We can also assume that $\mathcal{F}$ is not hereditary.

We claim that $\mathcal{P}(\mathcal{D}_x(\mathcal{F})) = \mathcal{D}_x(\mathcal{P}_\mathcal{F})$, for any $x \in U$. Indeed, suppose first that $\mathcal{D}_x(\mathcal{F}) = \mathcal{F}$. Then $\mathcal{P}_\mathcal{F} = \mathcal{P}(\mathcal{D}_x(\mathcal{F})) \ll_d \mathcal{D}_x(\mathcal{P}_\mathcal{F})$, where the second inequality is by Lemma 9, part 1. But this can only happen if $\mathcal{D}_x(\mathcal{P}_\mathcal{F}) = \mathcal{P}_\mathcal{F}$. Indeed, if there is some $(P, h_P) \in \mathcal{P}_\mathcal{F}$ which is affected by $\mathcal{D}_x$, that is, such that $x \in P$ and $h_P(x) = 0$, then $\mathcal{P}_\mathcal{F} \ll_d \mathcal{D}_x(\mathcal{P}_\mathcal{F})$ implies there is $(Q, h_Q) \in \mathcal{P}_\mathcal{F}$ such that $(Q, h_Q) \leqslant (P, h_P[x \to 1])$. If $x \notin Q$ then $(Q, h_Q) \lneq (P, h_P)$, which is impossible, so $x \in Q$ and $h_Q(x) = 1$. Moreover, $Q \subseteq P$, and (P, h_P) and (Q, h_Q) disagree only on x. Hence, by Lemma 7 there is $(R, h_R) \in \mathcal{P}_\mathcal{F}$ such that $R \subseteq P \cap Q - x = Q - x$ which agrees with (P, h_p). But then $(R, h_R) \lneq (P, h_P)$, a contradiction.

So suppose $\mathcal{D}_x(\mathcal{F}) \precneqq \mathcal{F}$. Then, by Lemma 9, part 2 and 3, $\mathcal{P}(\mathcal{D}_x(\mathcal{F})) \ll_d \mathcal{D}_x(\mathcal{P}_\mathcal{F})$ and $\mathcal{A} = \mathcal{A}_{\sup}(\mathcal{P}(\mathcal{D}_x(\mathcal{F}))) = \mathcal{A}_{\sup}(\mathcal{D}_x(\mathcal{P}_\mathcal{F})))$. For any $A \in \mathcal{A}$, let us take $(A, h_A) \in \mathcal{D}_x(\mathcal{P}_F)$. By the definition of $\ll_d$, there is $(A', h_{A'}) \in \mathcal{P}(\mathcal{D}_x(\mathcal{F}))$ such that $(A', h_{A'}) \leqslant (A, h_A)$. But $A \in \mathcal{A}_{\sup}(\mathcal{P}(\mathcal{D}_x(\mathcal{F})))$, hence $A' = A$ and $h_{A'} = h_A$, that is, $(A, h_A) \in \mathcal{P}(\mathcal{D}_x(\mathcal{F}))$. By the induction hypothesis, $\mathcal{P}(\mathcal{D}_x(\mathcal{F}))$ satisfies (‡), and, by Proposition 6, supports of the projections of $\mathcal{P}(\mathcal{D}_x(\mathcal{F}))$ are all distinct and form an antichain $\mathcal{A}$. Thus,

$$\mathcal{P}(\mathcal{D}_x(\mathcal{F})) = \{(A, h_A) \in \mathcal{D}_x(\mathcal{P}_\mathcal{F}) \mid A \in \mathcal{A}\} \subseteq \mathcal{D}_x(\mathcal{P}_\mathcal{F}).$$

This, together with $\mathcal{P}(\mathcal{D}_x(\mathcal{F})) \ll_d \mathcal{D}_x(\mathcal{P}_\mathcal{F})$, implies $\mathcal{P}(\mathcal{D}_x(\mathcal{F})) = \mathcal{D}_x(\mathcal{P}_\mathcal{F})$. This finishes the proof of $\mathcal{P}(\mathcal{D}_x(\mathcal{F})) = \mathcal{D}_x(\mathcal{P}_\mathcal{F})$ claim.

As our second step, we use the assumption that (‡) does not hold for $\mathcal{P}_\mathcal{F}$. So let us fix $A, B \in \mathcal{P}_\mathcal{F}$ and $x \in \mathbf{dis}(A, B)$ invalidating (‡), such that $|\mathbf{dis}(A, B)|$ is minimal. Without losing generality we assume that $h_A(x) = 0$ and $h_B(x) = 1$. We claim that in that case $\mathcal{F}$ is downshifted outside of $\mathbf{sup}(A, B)$, that is, for any $u \in U - \mathbf{sup}(A, B)$, $\mathcal{D}_u(\mathcal{F}) = \mathcal{F}$, in particular, for any $(P, h_P) \in \mathcal{P}_\mathcal{F}$ and any $u \in P - \mathbf{sup}(A, B)$, $h_P(u) = 1$. Indeed, if not, take $u \notin \mathbf{sup}(A, B)$ such that $\mathcal{D}_u(\mathcal{F}) \precneqq \mathcal{F}$. As $u \notin \mathbf{sup}(A, B)$, $A, B \in \mathcal{P}(\mathcal{D}_u(\mathcal{F})) = \mathcal{D}_u(\mathcal{P}_\mathcal{F})$. But then, by the induction hypothesis, there is $C' \in \mathcal{P}(\mathcal{D}_u(\mathcal{F}))$, validating (‡) for A and B. But then $C \subseteq \mathbf{sup}(A, B) - x$, hence the preimage of C under $\mathcal{D}_u$ was not affected by this downshift, so $C \in \mathcal{P}_\mathcal{F}$ and (‡) holds for A and B in $\mathcal{P}_\mathcal{F}$.

Let us note that Lemma 7 immediately implies that A and B cannot disagree on a single point $x \in U$. Indeed, otherwise there is $C \in \mathcal{P}_\mathcal{F}$ such that $C \subseteq A \cup B - x$ which agrees with both A and B, that is, C witnesses (‡) for A and B. So suppose that $|\mathbf{dis}(A, B)| > 1$, the argument for which is illustrated in Figure 2 below.

Let us take $y \in \mathbf{dis}(A, B) - x$, and use it in the induction hypothesis for $\mathcal{D}_x(\mathcal{P}_\mathcal{F}) = \mathcal{P}(\mathcal{D}_x(\mathcal{F}))$. Thus, there is $(C', h_{C'}) \in \mathcal{D}_x(\mathcal{P}_\mathcal{F})$ such that $C' \subseteq \mathbf{sup}(A, B) - y$, which agrees with $A' = \mathcal{D}_x(A)$ and $B' = \mathcal{D}_x(B)$ on $\mathbf{agr}(A', B') = \mathbf{agr}(A, B) + x$. Notice that $x \in C$, as otherwise $C \in \mathcal{P}_\mathcal{F}$ witnesses (‡) for A, B, and x. Let $C_0 = C'[x \mapsto 0]$ and $C_1 = C'[x \mapsto 1]$ be the two possible preimages of C' under $\mathcal{D}_x$.

If the preimage is C_0, then it disagrees with B on x, and $\mathbf{dis}(B, C_0) \subseteq \mathbf{dis}(A, B) - y$, and thus $|\mathbf{dis}(B, C_0)| < |\mathbf{dis}(A, B)|$. Similarly, C_1 disagrees with A on x and $|\mathbf{dis}(A, C_1)| < |\mathbf{dis}(A, B)|$. From our assumption that A and B have the smallest disagreement sets among the pairs breaking (‡) it follows that there is either $D_0 \subseteq \mathbf{sup}(B, C_0) - x$, which agrees with B and C_0 on $\mathbf{agr}(B, C_0) \cap D_0$, or D_1 with similar properties with respect to A and C_1.

We argue that both D_0 and D_1 witness (‡) for A, B, and x. Indeed, $D_0 \subseteq \mathbf{sup}(B, C_0) - x = B \cup C_0 - x \subseteq B \cup A - x = \mathbf{sup}(A, B) - x$. Now, suppose D_0 disagrees with A or with B on some $w \in \mathbf{agr}(A, B) \cap D_0$. If $w \in A - B$ then on w D_0 agrees with B and

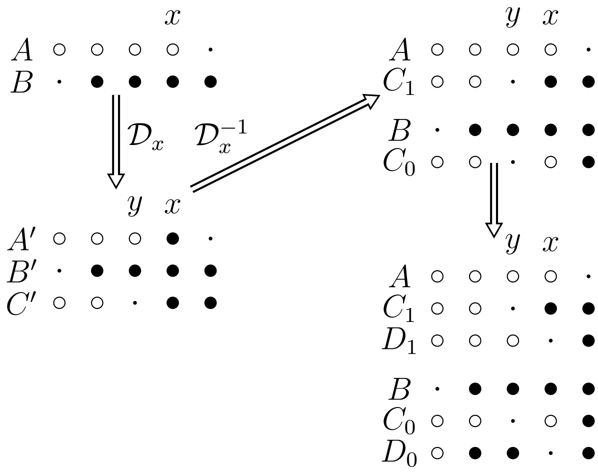

Figure 2: Illustration for the $|\mathbf{dis}(A, B)| > 1$ case of the first part of the proof of Theorem 5.

hence disagrees with A. Also, $w \in \mathbf{sup}(B, C_0) - x$, and, as $w \notin B$, $w \in C_0$, and so $w \in \mathbf{agr}(B, C_0)$. As D_0 agrees with C_0 on $\mathbf{agr}(B, C_0) \cap D_0$, $h_{D_0}(w) = h_{C_0}(w)$. Now, C_0 agrees with A on $\mathbf{agr}(A, B) \cap C_0 - x$, hence $h_A(w) = h_{C_0}(w)$. But then D_0 agrees with A on w, a contradiction.

On the other hand, if $w \in B - A$, then on w D_0 agrees with A and hence disagrees with B. And if $w \in A \cap B$ then on w D_0 disagrees with both A and B. We use the fact that it disagrees with B to reach the contradiction for both cases. As D_0 agrees with B on $\mathbf{agr}(B, C_0) \cap D_0$, it follows that $w \in \mathbf{dis}(B, C_0)$, that is, $w \in B \cap C_0$, $h_B(w) \neq h_{C_0}(w)$. But $w \in \mathbf{agr}(A, B) \cap C_0 - x$, hence C_0 agrees on it with B, a contradiction.

This argument finishes the proof of the induction step and of the first part of the theorem.

If $\mathcal{P} \in \mathsf{PRJ}^*$ satisfies (‡) then $\mathcal{P} \in \mathsf{PRJ}^\diamond$ and $\mathcal{F}_\mathcal{P}$ is shattering-extremal. Let us first note that (‡) trivially implies, by Lemma 7, that $\mathcal{P} \in \mathsf{PRJ}^\diamond$. Suppose now that $\mathcal{P}$ is not shattering-extremal, that is, $|\operatorname{Str}(\mathcal{F}_\mathcal{P})| > |\mathcal{F}_\mathcal{P}|$. Let us enumerate $U = \{x_1, \ldots, x_n\}$ somehow, and define $\mathcal{P}_0 = \mathcal{P}$, $\mathcal{P}_{i+1} = \mathcal{D}_{x_i}(\mathcal{P}_i)$, $\mathcal{P}^* = \mathcal{P}_{n+1}$; and $\mathcal{F}_0 = \mathcal{F}_\mathcal{P}$, $\mathcal{F}_{i+1} = \mathcal{D}_x(\mathcal{F}_i)$, and $\mathcal{F}^* = \mathcal{F}_{n+1}$. Notice that, by Lemma 9, $\mathcal{F}(\mathcal{P}_i) \supseteq \mathcal{F}_i$ and $|\mathcal{F}_i| = |\mathcal{F}_\mathcal{P}|$, for all i. Finally, $\mathcal{P}^* = \{(P, P) \mid (P, H_P) \in \mathcal{P}\}$, hence $\mathcal{F}(\mathcal{P}^*)$ is hereditary, $\mathcal{P}^* = \mathcal{P}(\mathcal{F}(\mathcal{P}^*))$ and $\operatorname{Str}(\mathcal{F}(\mathcal{P}^*)) = \mathcal{F}(\mathcal{P}^*)$. Also, by Proposition 8

$$\begin{aligned}
\operatorname{Str}(\mathcal{F}(\mathcal{P}^*)) &= \{S \subseteq U \mid S \not\supseteq P, \text{ for all } P \in \mathcal{A}_{\sup}(\mathcal{P}^*)\} \\
&= \{S \subseteq U \mid S \not\supseteq P, \text{ for all } P \in \mathcal{A}_{\sup}(\mathcal{P})\} \\
&\supseteq \{S \subseteq U \mid S \not\supseteq P, \text{ for all } P \in \mathcal{A}_{\sup}(\mathcal{P}(\mathcal{F}_\mathcal{P}))\} = \operatorname{Str}(\mathcal{F}_\mathcal{P}).
\end{aligned}$$

Then $|\operatorname{Str}(\mathcal{F}_\mathcal{P})| > |\mathcal{F}_\mathcal{P}|$ implies $|\mathcal{F}(\mathcal{P}^*)| \geqslant |\operatorname{Str}(\mathcal{F}_\mathcal{P})| > |\mathcal{F}_\mathcal{P}| = |\mathcal{F}^*|$, and so $\mathcal{F}(\mathcal{P}^*) \supsetneq \mathcal{F}^*$. Let i be minimal such that $\mathcal{F}(\mathcal{P}_i) \supsetneq \mathcal{F}_i$; $i > 0$ as $\mathcal{F}(\mathcal{P}_0) = \mathcal{F}_0$ by definition. Notice that (‡) is preserved under downshifts, so it holds for $\mathcal{P}_{i-1}$. Then, without losing generality, we may assume $i = 1$, that is, there is $x \in U$ such that $\mathcal{F}(\mathcal{D}_x(\mathcal{P})) \supsetneq \mathcal{D}_x(\mathcal{F}_\mathcal{P})$, and let us take $X \in \mathcal{F}(\mathcal{D}_x(\mathcal{P})) - \mathcal{D}_x(\mathcal{F}_\mathcal{P})$, as illustrated in Figure 3 below.

Let us denote $X_0 = X - x$ and $X_1 = X + x$. If $X = X_1$, that is, if $x \in X$, then neither X_0 nor X_1 invalidate $\mathcal{P}$. Indeed, if some $P \in \mathcal{P}$ is invalidated by X_0 or X_1, then

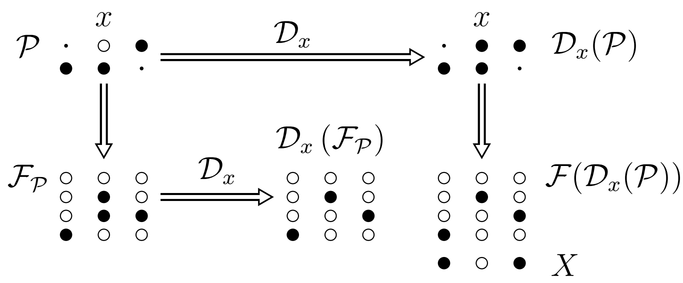

Figure 3: Illustration for the second part of the proof of Theorem 5. Transformations of $\mathcal{S}$ and $\mathcal{F}_{\mathcal{S}}$ under downshift.

$P[x \mapsto 1]$ is invalidated by $X_0 + x = X$ or $X_1 + x = X$. But then both X_0 and X_1 are in $\mathcal{F}_{\mathcal{P}}$, and, consequently, in $\mathcal{D}_x(\mathcal{F}_{\mathcal{P}})$, which contradicts to the fact that $X \notin \mathcal{D}_x(\mathcal{F}_{\mathcal{P}})$.

Thus, $X = X_0$. This implies that neither X_0 nor X_1 lie in $\mathcal{F}_{\mathcal{P}}$. If X_0 and X_1 are invalidated by the same $P \in \mathcal{P}$, then $x \notin P$, and consequently $P = P[x \mapsto 1] \in \mathcal{D}_x(\mathcal{P})$, which invalidates X in $\mathcal{D}_x(\mathcal{P})$. Hence, there are two different forbidden projections (A, h_A) and (B, h_B) in $\mathcal{P}$, invalidating X_0 and X_1 correspondingly, such that $x \in A \cap B$. Then $\mathbf{dis}(A, B) = \{x\}$: indeed, h_A and h_B disagree on x, and, as X_0 and X_1 agree on all points except for x, h_A and h_B must also agree everywhere, except for x.

Now, by (‡), there is $(C, h_C) \in \mathcal{P}$ such that $C \in A \cup B - x$, which agrees with A and B on all points. But then h_C agrees with both X_0 and X_1, and, consequently, C is invalidated by both of them. Finally, $C = C[x \mapsto 1]$, which means that $C \in \mathcal{D}_x(\mathcal{P})$, and thus X invalidates $\mathcal{D}_x(\mathcal{P})$, a contradiction. □

Corollary 10. *If a system $\mathcal{F}$ is shattering-extremal, then*

1. *$\mathcal{D}_x$ and $\mathcal{D}_y$ commute on $\mathcal{F}$, that is, $\mathcal{D}_x(\mathcal{D}_y(\mathcal{F})) = \mathcal{D}_y(\mathcal{D}_x(\mathcal{F}))$, for all $x, y \in U$;*
2. *$\mathcal{P}(\mathcal{D}_x(\mathcal{F})) = \mathcal{D}_x(\mathcal{P}_{\mathcal{F}})$;*
3. *The supports of forbidden projections of $\mathcal{P}_{\mathcal{F}}$ are all distinct and form an antichain.*

4 Topological properties of shattering-extremal systems

In this section, we are trying to work along the lines of characterization of convex geometries in terms of maximal chains, given by the following abridged version of [15, Theorem III.1.1.], which, in turn, refers to [10]:

Theorem 11. *A closure system $\mathcal{F}$ is a convex geometry iff for every $X \in \mathcal{F}$, $X \neq U$, there is $x \notin X$ such that $X + x \in \mathcal{F}$.*

Alternatively, a union-closed system $\mathcal{F}$ containing $\varnothing$ is an antimatroid iff for every $X \in \mathcal{F}$, $X \neq \varnothing$, there is $x \in X$ such that $X - x \in \mathcal{F}$.

Somewhat parallel to the definition of the downshift $\mathcal{D}_x$, we define a *bit flip* operation $\mathcal{B}_X$. For $X \subseteq U$, and $\mathcal{F} \subseteq 2^U$, let

$$\mathcal{B}_X(\mathcal{F}) = \{F \triangle X \mid F \in \mathcal{F}\}.$$

And for a set of forbidden projections $\mathcal{P}$, let

$$\mathcal{B}_X(\mathcal{P}) = \big\{(P, H_P \triangle (P \cap X)) \mid (P, H_P) \in \mathcal{F}\big\}.$$

When $X = \{x\}$, the latter can be reformulated as:

$$\mathcal{B}_x(\mathcal{P}) = \big\{(P, H_P \triangle x) \mid (P, H_P) \in \mathcal{F}, x \in P\big\} \cup \big\{(P, H_P) \mid (P, H_P) \in \mathcal{F}, x \notin P\big\}.$$

The following proposition gives some obvious properties of bit flips.

Proposition 12. *For any system $\mathcal{F}$, set of forbidden projections $\mathcal{P}$, and $X \subseteq U$, it holds:*

1. *$\mathcal{F}$ is shattering-extremal iff $\mathcal{B}_X(\mathcal{F})$ is shattering-extremal;*
2. *$\mathcal{P}$ satisfies (‡) iff $\mathcal{B}_X(\mathcal{P})$ satisfies (‡). Recall that (‡) is from the characterization of shattering-extremal projection families, that is, from Theorem 5;*
3. *$\mathcal{P} \in \mathsf{PRJ}^\diamond$ iff $\mathcal{B}_X(\mathcal{P}) \in \mathsf{PRJ}^\diamond$;*
4. *$\mathcal{B}_X(\mathcal{P}_\mathcal{F}) = \mathcal{P}(\mathcal{B}_X(\mathcal{F}))$;*
5. *$\mathcal{B}_X(\mathcal{F}(\mathcal{P})) = \mathcal{F}(\mathcal{B}_X(\mathcal{P}))$;*
6. *$\mathcal{B}_X$ is involutive, that is, $\mathcal{B}_X(\mathcal{B}_X(\mathcal{F})) = \mathcal{F}$ and $\mathcal{B}_X(\mathcal{B}_X(\mathcal{P})) = \mathcal{P}$.*

Although we find bit flips to be rather intuitive, let us still illustrate them using the same system that we used for the illustration of downshifts in the discussion after the statement of Lemma 9. Recall that $\mathcal{F}$ is a family of intervals on $U = \{1, 2, 3, 4\}$ linearly ordered as $1 < 2 < 3 < 4$; $\mathcal{F}$ is a convex geometry, is thus shattering-extremal, has 11 elements, and shatters all subsets of U of size at most two. Now, let $X = 24$. Figures 4 and 5 below show how this operator applies to $\mathcal{F}$ and $\mathcal{P}_\mathcal{F}$ respectively.

$\mathcal{P}_\mathcal{F}$

○	○	○	○		●	●	○	○
●	○	○	○		○	●	●	○
○	●	○	○		○	○	●	●
○	○	●	○		●	●	●	○
○	○	○	●		○	●	●	●
					●	●	●	●

$\mathcal{D}_3(\mathcal{P}_\mathcal{F})$

○	●	○	●		●	○	○	●
●	●	○	●		○	○	●	●
○	○	○	●		○	●	●	○
○	●	●	●		●	○	●	●
○	●	○	○		○	○	●	○
					●	○	●	○

Figure 4: Bit flip $\mathcal{B}_{24}$ applied to $\mathcal{F}$.

$\mathcal{P}_\mathcal{F}$

●	○	●	·
●	○	·	●
●	·	○	●
·	●	○	●

$\mathcal{D}_3(\mathcal{P}_\mathcal{F})$

●	●	●	·
●	●	·	○
●	·	○	○
·	○	○	○

Figure 5: Bit flip $\mathcal{B}_{24}$ applied to $\mathcal{P}_\mathcal{F}$.

For $A, B \subseteq U$, the *Hamming distance* $d(A, B)$ between A and B is $|d(A, B)| = |A \triangle B|$. For a system $\mathcal{F}$, let us define $\Gamma_\mathcal{F}$ as a simple graph with vertex set $\mathcal{F}$ and edge set $E_\mathcal{F} = \{(F, G) \mid F, G \in \mathcal{F}, d(F, G) = 1\}$. For $A, B \in \mathcal{F}$ we define $d_\mathcal{F}(A, B)$ as a distance between A and B in $\Gamma_\mathcal{F}$; if A and B are in different connected components of $\Gamma_\mathcal{F}$, then $d_\mathcal{F}(A, B) = \infty$. Trivially, $d_\mathcal{F}(A, B) \leqslant d_\mathcal{F}(A, B)$, for all $A, B \in \mathcal{F}$.

Lemma 13. *Let $\mathcal{F}$ be shattering-extremal, then $d(A,B) = d_{\mathcal{F}}(A,B)$, for all $A, B \in \mathcal{F}$.*

Let us give an example. For $U = \{1,2,3,4,5\}$, let $\mathcal{F}$ be a system containing U together with all sets of size at most three. Then $d(123,345) = d_{\mathcal{F}}(123,345) = 2$ with $123, 234, 345$ being a path in $\Gamma_{\mathcal{F}}$ of length two. However, $d(123,12345) = 2$ and $d_{\mathcal{F}}(123,12345) = \infty$, as 12345 is an isolated edge in $\Gamma_{\mathcal{F}}$. In the light of Lemma 13 this means that $\mathcal{F}$ is not shattering-extremal. Indeed, it can be noticed that $\mathcal{F}$ shatters all sets except for U, and thus $|\operatorname{Str}(\mathcal{F})| = 2^5 - 1 > |F| = 2^5 - \binom{5}{4}$.

As before, prior to going into the proof of Lemma 13, we prove an intermediate result:

Lemma 14. *Let system $\mathcal{F}$ be shattering-extremal, then*

$$\mathcal{P}' = \{(P, H_P) \in \mathcal{P}_{\mathcal{F}} \mid |H_P| \leqslant 1\}$$

satisfies (‡), hence $\mathcal{F}(\mathcal{P}') \supseteq \mathcal{F}$ is shattering-extremal. Moreover, if $\mathcal{F}$ contains $\varnothing$ then $\mathcal{F}(\mathcal{P}') \supseteq \mathcal{F}$ is an antimatroid.

Proof. Indeed, for $A, B \in \mathcal{P}'$, let $x \in \mathbf{dis}(A,B)$ and let $C \in \mathcal{P}_{\mathcal{F}}$ validate (‡) for A, B and x. As $h_A(x) \neq h_B(x)$, let us assume without loss of generality that $h_A(x) = 1$. Then x is a unique element in A on which h_A is non-zero. Now, $C \subseteq \mathbf{sup}(A,B) - x$ and C agrees with A and B on $\mathbf{agr}(A,B)$. Let $w \in C$ is such that $h_C(w) = 1$. If $w \in A$ then $h_A(w) = 0$, hence $w \in \mathbf{dis}(A,B)$, which means that $w = y$ where $y \in B$ is a unique element such that $h_B(y) = 1$. Alternatively, if $y \notin A$ then $y \in B$ and $y \in \mathbf{agr}(A,B)$. Hence $1 = h_C(w) = h_B(w)$, and again $w = y$. Thus, $C \in \mathcal{P}'$ and hence P' satisfies (‡).

Now, if $\mathcal{F}$ contains $\varnothing$ then there is no projection $P \in \mathcal{F}_{\mathcal{P}}$ for which $H_P = \varnothing$. Thus, for all $P' \in \mathcal{P}'$ it holds $|H_{P'}| = 1$. Then, after bit-flipping on the entire U, $\mathcal{B}_U(\mathcal{P}')$ satisfies the conditions of Theorem 3, hence $\mathcal{B}_U(\mathcal{P}') \in \mathsf{PRJ}^{\diamond}$ and $\mathcal{F}(\mathcal{B}_U(\mathcal{P}'))$ is a convex geometry. Then, by definition, $\mathcal{B}_U(\mathcal{F}(\mathcal{B}_U(\mathcal{P}'))) = \mathcal{F}(\mathcal{P}')$ is an antimatroid. $\square$

Proof (of Lemma 13). As noted, $d(A,B) \leqslant d_{\mathcal{F}}(A,B)$, for all $A, B \in \mathcal{F}$. The claim of the lemma then follows from the following statement:

$$\begin{aligned}&\text{for any } A, B \in \mathcal{F} \text{ such that } d(A,B) \geqslant 2 \text{ there is } C \in \mathcal{F},\\ &C \neq A, B, \text{ such that } d(A,B) = d(A,C) + d(C,B).\end{aligned} \tag{9}$$

Suppose (9) does not hold and let us pick A and B invalidating it. By bit flipping, we might assume that $A = \varnothing$ and $|B| \geqslant 2$. Let $\mathcal{P}'$ be defined as in Lemma 14, i.e.

$$\mathcal{P}' = \{(P, H_P) \in \mathcal{P}_{\mathcal{F}} \mid |H_P| \leqslant 1\}.$$

Then $\mathcal{F}' = \mathcal{F}(\mathcal{P}') \supseteq \mathcal{F}$ is an antimatroid.

Notice that, by assumption, for any $x \in B$ it holds $\{x\} \notin \mathcal{F}$, hence there is $P_x \in \mathcal{P}_{\mathcal{F}}$ invalidated by it. However, as $\varnothing \in \mathcal{F}$ and $\{x\} \notin \mathcal{F}$, P_x should tell these sets apart, and hence $x \in P_x$ and $h_{P_x}(w) = 0$, for all $w \in P_x - x$ and $h_{P_x}(x) = 1$. Thus, $P_x \in \mathcal{P}'$ and $\{x\} \notin \mathcal{F}'$, for all $x \in B$. But as $\mathcal{F}'$ is an antimatroid, Theorem 11 trivially implies that there is $x \in B$ such that $\{x\} \in \mathcal{F}'$, a contradiction. $\square$

The following reformulation of Lemma 13 makes the parallel with Theorem 11 more visible:

Corollary 15. *If $\mathcal{F}$ is shattering-extremal then for all $A, B \in \mathcal{F}$, $A \neq B$ there is $x \in U$ such that $A + x \in \mathcal{F}$ and $d(A + x, B) = d(A, B) - 1$.*

We define a *minimal path* between $A, B \in \mathcal{F}$ as a path in $\Gamma_{\mathcal{F}}$ of length $d(A, B)$, in particular, if $A = B$, then the minimal path consists of a single vertex A. We denote the set of all paths between A and B by $\mathsf{PAT}(A, B)$, and the set of minimal paths between A and B by $\mathsf{MPAT}(A, B)$; moreover, for $C \in \mathcal{F}$ such that $d(A, B) = d(A, C) + d(C, B)$, we denote by $\mathsf{MPAT}(A, C, B)$ the set of minimal paths from A to B through C. Also, for such C, for $\chi \in \mathsf{MPAT}(A, C)$ and $\gamma \in \mathsf{MPAT}(C, B)$ we denote by $\chi + \gamma \in \mathsf{MPAT}(A, C, B)$ the minimal path obtained by concatenating χ and γ.

Let $d(A, B) = k$, and $\chi = (X_0, \dots, X_k)$, $\gamma = (Y_0, \dots, Y_k) \in \mathsf{MPAT}(A, B)$. We say that χ can be 1-*deformed* into γ if either $\chi = \gamma$, or there is j, $0 < j < k$, such that $X_i = Y_i$ for all $i \neq j$; and that χ can be *deformed* into γ if there is a sequence $\chi_0 = \chi, \dots, \chi_n = \gamma$ from $\mathsf{MPAT}(A, B)$ such that χ_i can be 1-deformed into χ_{i+1}, for all $i = 0, \dots, n-1$. For $\chi \in \mathsf{PAT}(A, B)$, $\chi^{-1} \in \mathsf{PAT}(B, A)$ denotes a path obtained by traversing χ backwards.

We denote a loop ω in $\Gamma_{\mathcal{F}}$ of length k by $\omega = (W_0, \dots, W_{k-1})$, where $W_0 \neq W_{k-1}$, but instead $d(W_i, W_{i+1}) = 1$ for $i = 0, \dots, k-1$, where lower index is taken is modulo k; in particular, $d(W_{k-1}, W_0) = 1$. A *cyclic shift* of ω is defined in an obvious way. We consider an empty sequence to be a loop of length 0. For $\chi \in \mathsf{PAT}(A, B)$ and $\gamma \in \mathsf{PAT}(B, A)$, we denote by $\chi + \gamma$ the loop obtained by concatenating χ and γ.

Let $\omega = (W_0, \dots, W_{l-1})$ and $\zeta = (Z_0, \dots, Z_{k-1})$ be loops in $\Gamma_{\mathcal{F}}$. We say that ω can be 1-*deformed* into ζ if:

1. either $l = k$ and for some cyclic shift ω' of ω there is $0 \leqslant i < l$ such that $W'_j = Z_j$ for all $i \neq j$;

2. or $k = l-2$, and for some cyclic shifts ω' of ω and ζ' of ζ, $W'_{l-3} = W'_{l-1}$ and $W'_j = Z'_j$ for all $0 \leqslant j \leqslant l-3$;

3. or 2. holds for ζ and ω.

We say that ω can be deformed into ζ if there is a sequence $\omega_0 = \omega, \dots, \omega_n = \zeta$ of loops such that ω_i can be 1-deformed into ω_{i+1}, for all $i = 0, \dots, n-1$. We say that a loop ω is *contractible* if it can be deformed into an empty loop.

Theorem 16. *If $\mathcal{F}$ is a shattering-extremal system, then*

1. *For any $A, B \in \mathcal{F}$, any two minimal paths between A and B can be deformed into each other;*

2. *Any loop in $\mathcal{F}$ is contractible.*

Proof. (1). Suppose not, that is, there are $A, B \in \mathcal{F}$, $d(A, B) = k$, and $\chi = (X_0, \dots, X_k)$, $\gamma = (Y_0, \dots, Y_k) \in \mathsf{MPAT}(A, B)$ that cannot be deformed into each other. Take k to be minimal for which there exist such $\mathcal{F}, A, B, \chi$, and γ. Using bit flipping we can assume that $A = \varnothing$ and $|B| = k$. Trivially, in this case $X_i, Y_i \subseteq B$, and $|X_i| = |Y_i| = i$ for all $i = 1, \dots, k$.

We claim that $X_i = B - Y_{k-i}$ for all $i = 0, \dots k$. Suppose not, that is, there is $i > 0$ such that $X_i \neq B - Y_{k-i}$. Then $d(X_i, Y_{k-i}) = l < k$, and let $\rho = (P_0, \dots, P_l)$ be a minimal path between X_i and Y_{k-i}. Notice that $P_j \neq A, B$ for all $0 \leqslant j \leqslant l$. Indeed, $d(X_i, P_j) + d(P_j, Y_{k-i}) = d(X_i, Y_{k-i}) = l$, for all j; however, $d(X_i, A) + d(A, Y_{k-i}) = i + (k - i) = k > l$; similarly $P_j \neq B$. Now we argue that any minimal path from $\mathsf{MPAT}(A, P_j, B)$ can be deformed into a path from $\mathsf{MPAT}(A, P_{j+1}, B)$, for all $0 \leqslant j < l$. Indeed, let $\nu \in \mathsf{MPAT}(A, P_j, B) = \nu_b + \nu_t$, for $\nu_b \in \mathsf{MPAT}(A, P_j)$ and $\nu_t \in \mathsf{MPAT}(P_j, B)$. As $d(P_j, P_{j+1}) = 1$, it is either $P_j \subsetneq P_{j+1}$ or $P_j \supsetneq P_{j+1}$. Without losing generality, let us assume that it is the former. Then there is some $\eta_t \in \mathsf{MPAT}(P_j, P_{j+1}, B)$. As $P_j \neq A$, $d(P_j, B) < k$, and, by the assumption of minimality of k, ν_t can be deformed into η_t; hence, $\nu = \nu_b + \nu_t$ can be deformed into $\nu_b + \eta_t \in \mathsf{MPAT}(A, P_{j+1}, B)$. Finally, again by the minimality of k, any paths from $\mathsf{MPAT}(A, X_i, B)$ can be deformed into each other, and same for $\mathsf{MPAT}(A, Y_{k-i}, B)$. Combining, we get that A can be deformed into B, a contradiction. This argument is illustrated in Figure 6

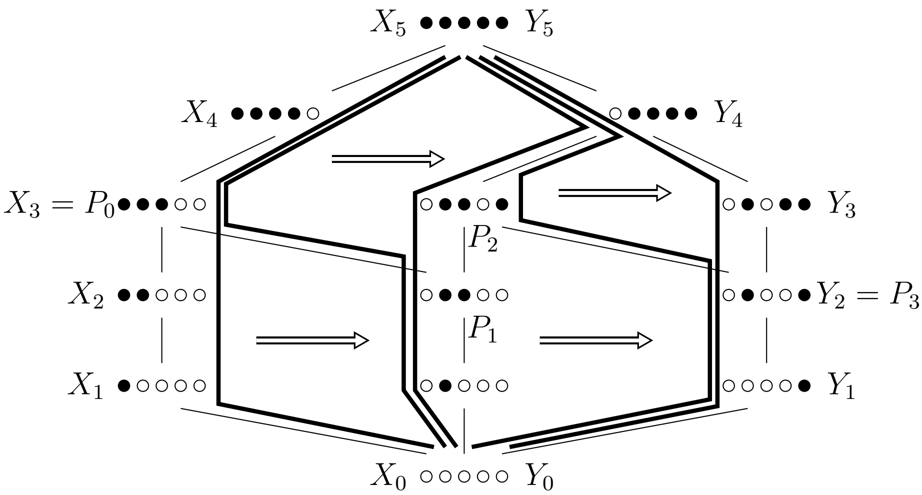

Figure 6: Illustration for the path deformation argument in Theorem 16.

Thus, $X_i = B - Y_{k-i}$ for all $i = 0, \dots k$. Notice that it also follows that there is no $C \in \mathcal{F}$, $C \subseteq B$, $C \notin \chi, \gamma$, as otherwise there is $\nu \in \mathsf{MPAT}(A, C, B)$ and, by the previous argument, χ can be deformed into ν and ν into γ. Let us enumerate elements of $B = \{b_0, \dots, b_{k-1}\}$ such that $X_i = \{b_0, \dots, b_{i-1}\}$ and $Y_i = \{b_{k-i}, \dots, b_{k-1}\}$, for $i = 0, \dots, k$.

Let $\mathcal{P} = \mathcal{P}_{\mathcal{F}}$. We claim that for any p, q, r, $0 \leqslant p < q < r \leqslant k - 1$ there is $P_{p,q,r} = (P, h_P) \in \mathcal{F}_{\mathcal{P}}$ such that $P \cap B = \{b_p, b_q, b_r\}$, $h_P(p) = h_P(r) = 1$, $h_P(q) = 0$, and $h_P(w) = 0$ for all $w \in U - B$. Indeed, let $(P', h_{P'}) \in \mathsf{PRJ}$ be defined as $P' = \{b_p, b_q, b_r\} \cup (U - B)$, and $h_{P'}(p) = h_{P'}(r) = 1$, $h_{P'}(q) = 0$, and $h_{P'}(w) = 0$ for all $w \in U - B$. Notice that a $D \subseteq U$ invalidates P' iff $D \subseteq B$, $p, r \in D$, and $q \notin D$; however, by the above argument, there is no such $D \in \mathcal{F}$, and hence P' is a forbidden projection for $\mathcal{F}$. Hence, there is $P_{p,q,r} \in \mathcal{P}_{\mathcal{F}}$, $P \ll_d P'$. But if $P \cap B$ does not contain b_p, b_q, or b_r, then $P_{p,q,r}$ is invalidated by Y_{k-r}, B, or X_{p+1} correspondingly.

By a similar argument, for any p, q, r, $0 \leqslant p < q < r \leqslant k-1$ there is $Q_{p,q,r} = (Q, h_Q) \in \mathcal{F}_{\mathcal{Q}}$ such that $Q \cap B = \{b_p, b_q, b_r\}$, $h_Q(p) = h_Q(r) = 0$, $h_Q(q) = 1$, and $h_Q(w) = 0$ for all $w \in U - B$. Now, let us take $P_{0,1,2}$ and $Q_{0,1,2}$. Applying (‡) to them with $x = b_0$ we get $R \in \mathcal{P}_{\mathcal{F}}$ such that $|R \cap B| \leqslant 2$ and $h_R(w) = 0$ for all $w \in R - B$. The latter is impossible, a contradiction.

(2). For a loop $\omega = (W_0, \ldots, W_{n-1})$ let $\chi = (W_k, \ldots, W_{k+l})$ be a subpath of ω, which is maximal such that $d(W_k, W_{k+l}) = l$. Then $\chi \in \mathsf{MPAT}(W_k, W_{k+l})$ and $d(W_{k-1}, W_{k+l}) = l - 1$, and hence there is a path $\chi' \in \mathsf{MPAT}(W_k, W_{k-1}, W_{k+l})$. Then χ can be deformed into χ', and if we represent $\omega = \gamma + \chi$ (up to cyclic shift) then ω can be deformed to $\omega' = \gamma + \chi'$. But then, up to a cyclic shift, $\omega' = \omega'' + (W_{k-1}, W_k, W_{k-1})$, and hence ω' can be 1-deformed to ω''. As ω'' has a smaller length, the proof can be finished by induction. $\square$

Note that, unlike Theorem 11, which is a characterization, neither Lemma 13 (Corollary 15), nor Theorem 16 give a sufficient condition for a system to be shattering-extremal. A non shattering-extremal system $\mathcal{F} = 2^{\{1,2,3\}} - \{\varnothing, 123\}$ satisfies the conclusion of Lemma 13, however the graph $\Gamma_{\mathcal{F}}$ is a six-element loop that is not contractible; and a non shattering-extremal system $\mathcal{G} = 2^{\{1,2,3,4\}} - \{\varnothing, 1234\}$ satisfies the conclusion of both Lemma 13 and Theorem 16.

5 Open problems

Although Lemma 7 enables us to tell if $\mathcal{P} \in \mathsf{PRJ}^*$ is actually in $\mathsf{PRJ}^\diamond$, we note that constructing $\mathcal{P}(\mathcal{F}(\mathcal{P}))$ from $\mathcal{P} \in \mathsf{PRJ}^*$ cannot be easy. Indeed, $\mathcal{P} \in \mathsf{PRJ}$ can be reinterpreted as a CNF-boolean formula

$$\phi(\mathcal{P}) = \bigwedge_{P \in \mathcal{P}} \big(\bigvee_{x \in P} x \neq h_P(x) \big).$$

Then checking that $\mathcal{P}(\mathcal{F}(\mathcal{P})) = \{(\varnothing, \varnothing)\}$ is equivalent to checking that $\phi(\mathcal{P})$ is unsatisfiable, and is thus a co-NP complete problem. It does not, however, automatically imply that checking that $\mathcal{F}(\mathcal{P})$ is shattering-extremal for $\mathcal{P} \in \mathsf{PRJ}^*$ is intractable, although the previous remark suggests that simply constructing $\mathcal{P}(\mathcal{F}(\mathcal{P}))$ and applying Theorem 5 to it is computationally hard. Hence

Problem 17. Given $\mathcal{P} \in \mathsf{PRJ}^*$, is there an easy criterion (or a polynomial-time algorithm) to tell whether $\mathcal{F}(\mathcal{P})$ is extremal?

We say that $\mathcal{P} \in \mathsf{PRJ}^*$ is *minimal* if there is no $\mathcal{P}' \in \mathsf{PRJ}^*$ such that $P' \subsetneq P$ and $\mathcal{F}(\mathcal{P}') = \mathcal{F}(\mathcal{P})$. Seemingly related to Problem 17 is the following:

Problem 18. Is there a characterization of minimal families of forbidden projections for shattering-extremal systems?

It seems that some insight into these two problems can be found in [15], in consideration of *critical circuits* of antimatroids, which are related to minimal families of forbidden projections. Moreover, we can define the size $s(\mathcal{P})$ of a family of forbidden projections as $S(\mathcal{P}) = \sum |P| \mid P \in \mathcal{P}$, and say that $\mathcal{P}$ is *optimal* for $\mathcal{F} = \mathcal{F}_{\mathcal{P}}$ if it has the minimal size among all families of forbidden projections for $\mathcal{F}$. It is easy to see that optimality implies minimality, and thus the problem of finding an optimal family of forbidden projections refines Problem 18. The problem of finding the optimal families of implications for lattices in general and convex geometries in particular has a long history. For example, [1] gives several characterizations of this sort. However, it is not clear whether these results can be translated to finding optimal families of projections for shattering-extremal systems.

Another obvious loose end is the following:

Problem 19. Is there a topological characterization of shattering-extremal systems in the spirit of Lemma 13 and Theorem 16?

In particular, Theorem 16 can be understood as that the first homotopy group of a shattering-extremal system is trivial.

Problem 20. Can higher homotopies be reasonably defined for set families? Can it be the case that all homotopies of shattering-extremal systems are trivial, and could it also be a sufficient condition?

An interesting example of a shattering-extremal system exhibiting some nontrivial topological properties was given in [4, Theorem 4.5], as a counterexample to a corner peeling conjecture. This paper, in turn, attributes the topological approach to [22]. Thus, taking a closer look at this counterexample might provide some guidance toward Problems 19 and 20.

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