

# A Fourfold Refined Enumeration of Alternating Sign Trapezoids

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## Abstract

Alternating sign trapezoids have recently been introduced as a generalisation of alternating sign triangles. Fischer established a threefold refined enumeration of alternating sign trapezoids and provided three statistics on column strict shifted plane partitions with the same joint distribution. In this paper, we are able to add a new pair of statistics to these results. More precisely, we consider the number of  $-1$ s on alternating sign trapezoids and introduce a corresponding statistic on column strict shifted plane partitions that has the same distribution. More generally, we show that the joint distributions of the two quadruples of statistics on alternating sign trapezoids and column strict shifted plane partitions, respectively, coincide. In addition, we provide a closed-form expression for the 2-enumeration of alternating sign trapezoids.

**Mathematics Subject Classifications:** 05A05, 05A15, 15B35

## 1 Introduction

Since their introduction in the early 1980s, alternating sign matrices have generated great interest among combinatorialists. Mills, Robbins and Rumsey [MRR83] conjectured them to be equinumerous with descending plane partitions, which had been enumerated by Andrews [And79] a few years earlier; this was finally proved over a decade later first by Zeilberger [Zei96] and shortly thereafter by Kuperberg [Kup96]. Since then, the spell-binding research of alternating sign matrices has revealed new equinumerous classes of combinatorial objects but finding bijections remains one of the most challenging problems.

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Equally distributed statistics on these objects might finally lead to those eagerly awaited bijections. Embracing this idea, we provide a fourfold refined enumeration of alternating sign trapezoids, a recently defined generalisation of alternating sign triangles. Moreover, we establish four statistics on certain column strict shifted plane partitions with the same joint distribution. Thus, we generalise the recent refined enumerations of alternating sign trapezoids and of column strict shifted plane partitions by Fischer [Fis19].

We start by introducing alternating sign trapezoids and column strict shifted plane partitions together with four statistics on each of these classes of objects.

**Definition 1.** For given integers  $n \geq 1$  and  $l \geq 2$ , an  $(n, l)$ -alternating sign trapezoid is an array of  $-1$ s,  $0$ s and  $+1$ s in a trapezoidal shape with  $n$  rows of the following form

$$\begin{array}{ccccccccccc}
 a_{1,1} & a_{1,2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1,2n+l-2} \\
 & a_{2,2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{2,2n+l-3} & \\
 & & \ddots & & & & & & & \ddots \\
 & & & & a_{n,n} & \cdots & a_{n,n+l-1} & & & & 
 \end{array}$$

such that the following four conditions hold: the nonzero entries alternate in sign in each row and each column; the topmost nonzero entry in each column is  $1$  (if existent); the entries in each row sum to  $1$ ; and the entries in the central  $l - 2$  columns sum to  $0$ .

An  $(n, 1)$ -alternating sign trapezoid is defined as above with the exception that the bottom row, which consists of a single entry in this case, does not have to add up to  $1$ , but where this entry can either be  $1$  or  $0$ .

Note that the notion of  $(n, 1)$ -alternating sign trapezoids coincide with the notion of *quasi alternating sign triangles of order  $n$* . Furthermore,  $(n, 3)$ -alternating sign trapezoids are in bijective correspondence with *alternating sign triangles of order  $n + 1$* . Both alternating sign triangles and quasi alternating sign triangles were introduced by Ayer, Behrend and Fischer [ABF20].

From the definition, it follows that the entries in each column of an alternating sign trapezoid sum to  $0$  or  $1$ . A column whose entries sum to  $1$  is called a *1-column*. If, in addition, the bottom entry of a  $1$ -column is  $0$ , we call the column a *10-column*. Note that the number of  $1$ -columns in any  $(n, l)$ -alternating sign trapezoid is exactly  $n$  if  $l \neq 1$  because the sum of all entries in an  $(n, l)$ -ASTZ is  $n$  as all rows add up to  $1$ ; otherwise, it is  $n$  or  $n - 1$ .

Let  $\text{ASTZ}_{n,l}$  denote the set of  $(n, l)$ -alternating sign trapezoids and consider  $A \in \text{ASTZ}_{n,l}$ . We introduce four different statistics on alternating sign trapezoids. First, we define

$$\begin{aligned}
 \mu(A) &:= \# -1\text{s in } A, \\
 r(A) &:= \# 1\text{-columns among the } n \text{ leftmost columns of } A.
 \end{aligned}$$

Then, we distinguish two cases: For  $l \geq 2$ , we define

$$\begin{aligned}
 p(A) &:= \# 10\text{-columns among the } n \text{ leftmost columns of } A, \\
 q(A) &:= \# 10\text{-columns among the } n \text{ rightmost columns of } A.
 \end{aligned}$$

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
& 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & \\
& & 0 & 0 & 0 & 0 & 1 & 0 & & \\
& & & 1 & 0 & 0 & 0 & & & 
\end{array}$$

Figure 1:  $(4, 4)$ -alternating sign trapezoid  $A$  with  $\mu(A) = 1$ ,  $r(A) = 2$ ,  $p(A) = 0$  and  $q(A) = 2$

An example of a  $(4, 4)$ -alternating sign trapezoid is given in Figure 1. We set the generating function  $\mathcal{Z}_{\text{ASTZ}}(n, l; M, R, P, Q)$  of  $(n, l)$ -alternating sign trapezoids associated to the statistics above to be

$$\sum_{A \in \text{ASTZ}_{n,l}} M^{\mu(A)} R^{r(A)} P^{p(A)} Q^{q(A)}.$$

Table 1 shows all  $(2, 4)$ -alternating sign trapezoids and their respective weights. The sum of these weights yields  $\mathcal{Z}_{\text{ASTZ}}(2, 4; M, R, P, Q) = 1 + 2MR + 2R + R^2 + MP + MQ$ .

Table 1:  $(2, 4)$ -alternating sign trapezoids

1 0 0 0 0 0	0 0 0 0 1 0	0 0 0 0 0 1	1 0 0 0 0 0
1 0 0 0	1 0 0 0	1 0 0 0	0 0 0 1
$R^2$	$RQ$	$R$	$R$
0 1 0 0 0 0	0 0 0 0 0 1	0 0 1 0 0 0	0 0 0 1 0 0
0 0 0 1	0 0 0 1	1 -1 0 1	1 0 -1 1
$RP$	1	$MR$	$MR$

For  $l = 1$ , however, we adapt the statistics in the following way:

$$\begin{aligned}
\tilde{p}(A) &:= \# \text{ 10-columns among the } n - 1 \text{ leftmost columns of } A, \\
\tilde{q}(A) &:= \# \text{ 10-columns among the } n - 1 \text{ rightmost columns of } A.
\end{aligned}$$

We define  $\mathcal{Z}_{\text{ASTZ}}(n, 1; M, R, P, Q)$  as

$$\sum_{A \in \text{ASTZ}_{n,1}} M^{\mu(A)} R^{r(A)} P^{\tilde{p}(A)} Q^{\tilde{q}(A)} (P + Q - M)^{[\text{central column is a 10-column}]},$$

where we use the *Iverson bracket*: For a logical statement  $S$ ,  $[S] = 1$  if  $S$  holds true and  $[S] = 0$  otherwise.

Table 2 provides a complete list of  $(3, 1)$ -alternating sign trapezoids and their respective weights. By adding up all these weights, we see that the generating function  $\mathcal{Z}_{\text{ASTZ}}(3, 1; M, R, P, Q)$  equals

$$1 - MR - MR^2 - MR^2P - MRQ + 3R + 3R^2 + R^3 + 3RP + RPQ + 3R^2P + R^2PQ + R^2P^2 + 3RQ + RQ^2 + 3R^2Q. \quad (1)$$

Table 2:  $(3, 1)$ -alternating sign trapezoids

1 0 0 0 0	0 0 0 1 0	0 0 0 0 1	1 0 0 0 0	0 1 0 0 0
1 0 0	1 0 0	1 0 0	0 0 1	0 0 1
1	1	1	1	1
$R^3$	$R^2Q$	$R^2$	$R^2$	$R^2P$
0 0 0 0 1	0 0 1 0 0	1 0 0 0 0	0 0 1 0 0	0 0 0 1 0
0 0 1	1 -1 1	1 0 0	1 0 0	1 0 0
1	1	0	0	0
$R$	$MR^2$	$R^2$	$R^2(P + Q - M)$	$RQ$
0 0 0 0 1	1 0 0 0 0	0 1 0 0 0	0 0 0 1 0	0 0 0 0 1
1 0 0	0 1 0	0 1 0	0 1 0	0 1 0
0	0	0	0	0
$R$	$R^2(P + Q - M)$	$R^2P(P + Q - M)$	$RQ(P + Q - M)$	$R(P + Q - M)$
1 0 0 0 0	0 1 0 0 0	0 0 1 0 0	0 0 0 0 1	0 0 1 0 0
0 0 1	0 0 1	0 0 1	0 0 1	1 -1 1
0	0	0	0	0
$R$	$RP$	$R(P + Q - M)$	1	$MR$

Clearly,  $p(A) \leq r(A)$  if  $l \geq 2$  and  $\tilde{p}(A) \leq r(A)$  if  $l = 1$  for any  $(n, l)$ -alternating sign trapezoid  $A$ . Moreover, by reflecting along the vertical symmetric axis, we see that

$$\mathcal{Z}_{\text{ASTZ}}(n, l; M, R, P, Q) = R^n \mathcal{Z}_{\text{ASTZ}}(n, l; M, R^{-1}, Q, P).$$

Ayyer, Behrend and Fischer [ABF20] showed that  $n \times n$ -alternating sign matrices are equinumerous with  $(n - 1, 3)$ -alternating sign trapezoids. As a corollary of [ABF20, Theorem 1.2], the statistic  $\mu$  generalises – in the most natural way – a statistic on alternating sign triangles which has the same distribution as the number of  $-1$ s on alternating sign matrices.

**Definition 2.** For a *strict partition*  $\lambda = (\lambda_1, \dots, \lambda_m)$ , that is, a sequence  $\lambda_1 > \dots > \lambda_m > 0$  of strictly decreasing positive integers, a *shifted Young diagram* of shape  $\lambda$  is a finite collection of cells arranged in  $m$  rows such that row  $i$  has length  $\lambda_i$  and each row is indented by one cell compared to the row above.



Figure 2: A shifted Young diagram of shape  $(5, 3, 2)$  and a column strict shifted plane partition  $\pi$  of the same shape and of class 4 with  $\mu_3(\pi) = 2$ ,  $r(\pi) = 3$ ,  $p_3(\pi) = 1$  and  $q(\pi) = 1$

A filling of a shifted Young diagram with positive integers such that the entries, termed *parts*, weakly decrease along each row and strictly decrease down each column is called a *column strict shifted plane partition*. It is of *class*  $k$  if the first part of each row  $i$  is exactly  $k + \lambda_i$ , that is, exactly  $k$  plus its corresponding row length.

For any  $k$ , we consider the collection of zero cells to be a column strict shifted plane partition with zero rows and of class  $k$ . Note that we usually omit the drawings of the cells. Furthermore, we cannot always associate a class to a given column strict shifted plane partition. Column strict shifted plane partitions of class 2 correspond to *descending plane partitions* as defined by Andrews [And79].

Let  $\text{CSSPP}_{n,k}$  denote the set of column strict shifted plane partitions of class  $k$  with at most  $n$  parts in the first row. Following the standard labelling of entries of matrices, we refer to the part in row  $i$  and column  $j$  of a column strict shifted plane partition  $\pi$  as  $\pi_{i,j}$ . We introduce four different statistics on  $\pi \in \text{CSSPP}_{n,k}$  of which two depend on a fixed parameter  $d \in \{1, \dots, k\}$ :

$$\begin{aligned} \mu_d(\pi) &:= \# \text{ parts } \pi_{i,j} \in \{2, 3, \dots, j - i + k\} \setminus \{j - i + d\}, \\ r(\pi) &:= \# \text{ rows of } \pi, \\ p_d(\pi) &:= \# \text{ parts } \pi_{i,j} = j - i + d, \\ q(\pi) &:= \# \text{ parts } \pi_{i,j} = 1. \end{aligned}$$

We define the generating function  $\mathcal{Z}_{\text{CSSPP}}(n, k, d; M, R, P, Q)$  associated to these statistics as

$$\sum_{\pi \in \text{CSSPP}_{n,k}} M^{\mu_d(\pi)} R^{r(\pi)} P^{p_d(\pi)} Q^{q(\pi)}.$$

For  $d = 0$ , we add another family of statistics by defining

$$\begin{aligned} \mu_0(\pi) &:= \# \text{ parts } \pi_{i,j} \in \{2, 3, \dots, j - i + k\} \setminus \{j - i\}, \\ p_0(\pi) &:= \# \text{ parts } \pi_{i,j} = j - i > 1, \\ q_0(\pi) &:= \# \text{ parts } \pi_{i,j} = 1 \text{ such that } j - i > 1. \end{aligned}$$

In this case, the generating function  $\mathcal{Z}_{\text{CSSPP}}(n, k, 0; M, R, P, Q)$  is defined as

$$\sum_{\pi \in \text{CSSPP}_{n,k}} M^{\mu_0(\pi)} R^{r(\pi)} P^{p_0(\pi)} Q^{q_0(\pi)} (P + Q - M)^{[\pi_{i,j} = 1 \text{ such that } j - i = 1]}.$$

Note that  $j - i = 1$  means that the part  $\pi_{i,j}$  is in the second position of a row. A 1 at the second position of a row can only occur in the bottom row. Also note that there is never a 1 at the first position of a row if the class of the column strict shifted plane partition is greater than 0.

An example of a column strict shifted plane partition is presented in Figure 2. Note that the parts counted by the statistic  $\mu$  generalise the *special parts* in descending plane partitions that were first defined by Mills, Robbins and Rumsey [MRR83]. They presented without proof a determinantal formula for an associated generating function that incorporates a total of three statistics. Subsequently, Behrend, Di Francesco and Zinn-Justin [BDZ12] obtained a closely related determinantal formula and identified the connection to the one found by Mills, Robbins and Rumsey.

Table 3: Column strict shifted plane partitions of class 3 with at most two parts in the first row

	$\emptyset$	4	5 5	5 4	5 3	5 2	5 1	$\begin{matrix} 5 & 5 \\ & 4 \end{matrix}$
$d = 0 :$	1	$R$	$R$	$MR$	$MR$	$MR$	$R(P + Q - M)$	$R^2$
$d = 1 :$	1	$R$	$R$	$MR$	$MR$	$RP$	$RQ$	$R^2$
$d = 2 :$	1	$R$	$R$	$MR$	$RP$	$MR$	$RQ$	$R^2$
$d = 3 :$	1	$R$	$R$	$RP$	$MR$	$MR$	$RQ$	$R^2$

Table 3 presents all column strict shifted plane partitions of class 3 with at most two parts in the first row. The weights with respect to all possible choices of the parameter  $d \in \{0, 1, 2, 3\}$  are listed below. By summing the weights for each  $d$ , we obtain  $\mathcal{Z}_{\text{CSSPP}}(2, 3, d; M, R, P, Q) = 1 + 2MR + 2R + R^2 + RP + RQ$ , which is independent of  $d$ . Similarly, the column strict shifted plane partitions of class 0 with at most three parts in the first row are shown in Table 4.

It turns out that the generating functions  $\mathcal{Z}_{\text{ASTZ}}(2, 4; M, R, P, Q)$  and  $\mathcal{Z}_{\text{CSSPP}}(2, 3, d; M, R, P, Q)$  for any  $d \in \{0, 1, 2, 3\}$  as well as  $\mathcal{Z}_{\text{ASTZ}}(3, 1; M, R, P, Q)$  and  $\mathcal{Z}_{\text{CSSPP}}(3, 0, 0; M, R, P, Q)$  coincide. Fischer [Fis19] established refined enumerations of alternating sign trapezoids and column strict shifted plane partitions by showing that

$$\mathcal{Z}_{\text{ASTZ}}(n, l; 1, R, P, Q) = \mathcal{Z}_{\text{CSSPP}}(n, l - 1, d; 1, R, P, Q)$$

for any  $d \in \{0, 1, \dots, l - 1\}$ . We extend her proof by adding the fourth statistic and show that the joint distribution of the corresponding statistics on alternating sign trapezoids and on column strict shifted plane partitions coincide:

**Theorem 3.** *Let  $n, l \geq 1$  and  $0 \leq d \leq l - 1$ . Then*

$$\mathcal{Z}_{\text{ASTZ}}(n, l; M, R, P, Q) = \mathcal{Z}_{\text{CSSPP}}(n, l - 1, d; M, R, P, Q).$$

The proof of Theorem 3 is organised as follows: We provide explicit determinantal expressions for  $\mathcal{Z}_{\text{ASTZ}}(n, l; M, R, P, Q)$  and  $\mathcal{Z}_{\text{CSSPP}}(n, l - 1, d; M, R, P, Q)$  in Section 2 and

Table 4: Column strict shifted plane partitions of class 0 with at most three parts in the first row

	$\emptyset$	1	2 2	2 1	3 3 3
$d = 0 :$	1	$R$	$R$	$R(P + Q - M)$	$R$
	3 3 2	3 3 1	3 2 2	3 2 1	3 1 1
$d = 0 :$	$RP$	$RQ$	$RP$	$RQ$	$RQ(P + Q - M)$
	2 2 1	3 3 3 1	3 3 3 2 2	3 3 3 2 1	3 3 2 1
$d = 0 :$	$R^2$	$R^2$	$R^2$	$R^2(P + Q - M)$	$R^2P$
	3 3 2 2 1	3 3 1 1	3 2 2 1	3 2 1 1	3 3 3 2 2 1
$d = 0 :$	$R^2P(P + Q - M)$	$R^2Q$	$R^2P$	$R^2Q$	$R^3$

Section 3, respectively. These expressions are then each found to be equal to (14). The expression for  $\mathcal{Z}_{\text{ASTZ}}(n, l; M, R, P, Q)$  is obtained using a bijection between alternating sign trapezoids and truncated monotone triangles, together with operator formulae for the weighted enumeration of monotone triangles due to Fischer and Riegler [FR15]. The expression for  $\mathcal{Z}_{\text{CSSPP}}(n, l - 1, d; M, R, P, Q)$  is obtained using a bijection between column strict shifted plane partitions and families of nonintersecting lattice paths, together with the Lindström–Gessel–Viennot lemma for the weighted enumeration of such families.

To conclude the present paper, we also present a result for the 2-enumeration of  $(n, l)$ -alternating sign trapezoids with  $l \geq 2$  in Section 4.

## 2 Weighted Enumeration of Alternating Sign Trapezoids

First, we provide a formula for the generating function of alternating sign trapezoids. For this purpose, we heavily exploit the correspondence between alternating sign trapezoids and certain truncated *monotone triangles*, where the latter are defined below.

### 2.1 Correspondence between trees and alternating sign trapezoids

**Definition 4.** For a given integer  $n \geq 1$ , a *monotone triangle of order  $n$*  is an array of integers in a triangular shape with  $n$  rows of the following form

$$\begin{array}{ccccccc}
& & & & a_{1,1} & & \\
& & & & a_{2,1} & & a_{2,2} \\
& & & \dots & \dots & & \dots \\
& & a_{n-1,1} & & a_{n-1,2} & & \dots & & a_{n-1,n-1} \\
a_{n,1} & & a_{n,2} & & a_{n,3} & & \dots & & a_{n,n}
\end{array}$$

such that the entries strictly increase along all rows other than the bottom row and weakly increase both along  $\nearrow$ -diagonals and  $\searrow$ -diagonals.

**Definition 5.** For given integers  $p, q \geq 0$  and  $n \geq 1$  such that  $p + q \leq n$  as well as a weakly decreasing sequence  $\mathbf{s} = (s_1, s_2, \dots, s_p)$  and a weakly increasing sequence  $\mathbf{t} = (t_{n-q+1}, t_{n-q+2}, \dots, t_n)$  of nonnegative integers, we define an  $(\mathbf{s}, \mathbf{t})$ -tree as an array of integers which arises from a monotone triangle of order  $n$  by truncating the diagonals as follows: for each  $1 \leq i \leq p$ , we delete the  $s_i$  bottom entries of the  $i^{\text{th}}$   $\nearrow$ -diagonal; for each  $n - q + 1 \leq i \leq n$ , we delete the  $t_i$  bottom entries of the  $i^{\text{th}}$   $\searrow$ -diagonal. All diagonals are counted from left to right.

We say that an  $(\mathbf{s}, \mathbf{t})$ -tree has *bottom row*  $\mathbf{k} = (k_1, \dots, k_n)$  if the following holds true: for all  $i$  such that  $1 \leq i \leq n - q$  or  $n - q + 1 \leq i \leq n$ , the integer  $k_i$  is the bottom entry of the  $i^{\text{th}}$   $\nearrow$ -diagonal or the  $i^{\text{th}}$   $\searrow$ -diagonal, respectively.

Figure 3 gives an examples of a  $((6, 3, 1, 1), (1, 4, 4))$ -tree. By way of illustration, we indicate the deleted parts of the truncated diagonals by  $\bullet$ . In addition, we add an edging to the shape of the tree in order to visualise the naming.

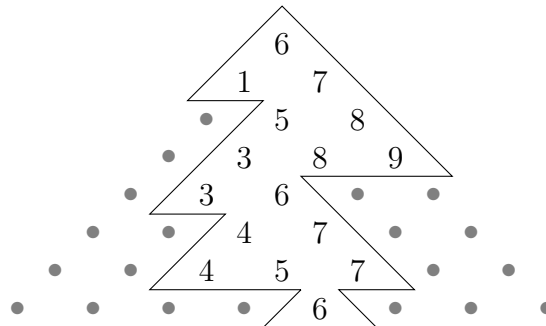


Figure 3: Example of a  $((6, 3, 1, 1), (1, 4, 4))$ -tree with eight rows ( $n = 8$ ) and bottom row  $(1, 3, 4, 5, 6, 7, 8, 9)$

The definition of monotone triangles in the present paper differs from the original definition given by Mills, Robbins and Rumsey in [MRR83], where they also impose strict increase on the bottom row. Although we predominantly consider strictly increasing bottom rows, monotone triangles and trees with weakly increasing bottom rows do appear in Section 2.3, where we investigate quasi alternating sign triangles. Note that the monotonicity conditions on the diagonals of monotone triangles guarantee that the bottom row is at least weakly increasing.

Next, we show how to transform an alternating sign trapezoid into a tree: Given an  $(n, l)$ -alternating sign trapezoid such that  $l \geq 2$ , we pad the array with additional zeroes





define the *symmetriser*  $\mathbf{Sym}$  and the *antisymmetriser*  $\mathbf{ASym}$  of a function  $f(X_1, \dots, X_n)$ . Let  $\mathfrak{S}_n$  be the symmetric group of degree  $n$ . Then

$$\begin{aligned}\mathbf{Sym}_{X_1, \dots, X_n} f(X_1, \dots, X_n) &:= \sum_{\sigma \in \mathfrak{S}_n} f(X_{\sigma(1)}, \dots, X_{\sigma(n)}) \text{ and} \\ \mathbf{ASym}_{X_1, \dots, X_n} f(X_1, \dots, X_n) &:= \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) f(X_{\sigma(1)}, \dots, X_{\sigma(n)}).\end{aligned}$$

We use  $\mathbf{Sym}_{\mathbf{X}}$  and  $\mathbf{ASym}_{\mathbf{X}}$  as an abbreviation if  $\mathbf{X} = (X_1, \dots, X_n)$  is clear from the context. Furthermore,  $\text{CT}_{\mathbf{X}} f(\mathbf{X}) = \text{CT}_{X_1, \dots, X_n} f(X_1, \dots, X_n)$  denotes the constant term of the function  $f$  with respect to the variables  $X_1, \dots, X_n$ . Finally, we define the *shift operator*  $E_X$ , the *forward difference operator*  $\Delta_X$  and the *backward difference operator*  $\delta_X$ :

$$\begin{aligned}E_X [f(X)] &:= f(X + 1), \\ \Delta_X &:= E_X - \text{id}, \\ \delta_X &:= \text{id} - E_X^{-1},\end{aligned}$$

where  $\text{id}$  denotes the standard identity operator. The  $M$ -*forward difference operator*  ${}^M\Delta_X$  and the  $M$ -*backward difference operator*  ${}^M\delta_X$  are defined as follows:

$$\begin{aligned}{}^M\Delta_X &:= (M \text{id} - (1 - M) \Delta_X)^{-1} \Delta_X, \\ {}^M\delta_X &:= (M \text{id} - (M - 1) \delta_X)^{-1} \delta_X.\end{aligned}$$

They generalise the previous difference operators as they reduce to  $\Delta_X$  and  $\delta_X$ , respectively, if we set  $M = 1$ . We use the notation  $E_x [f(x)] := E_X [f(X)]|_{X=x}$  for a given a variable  $X$  and an integer  $x$ . This abbreviatory notation is correspondingly used for other operator expressions. Note that for a function of several variables, any of these operators which is associated with a variable  $X$  commutes with any of these operators which is associated with a different variable  $Y$ .

Fischer and Riegler [FR15] provided a weighted enumeration of monotone triangles:

**Theorem 6.** *The generating function  $\mathcal{Z}_{\text{MT}}(n, \mathbf{k}; M)$  of monotone triangles  $T$  of order  $n$  with strictly increasing bottom row  $\mathbf{k} = (k_1, \dots, k_n)$  with respect to the weight  $M^{\mu(T)}$  is given by*

$$\begin{aligned}\text{CT}_{\mathbf{Y}} \left( \mathbf{ASym}_{\mathbf{Y}} \left( \prod_{i=1}^n (1 + Y_i)^{k_i} \right. \right. \\ \left. \left. \times \prod_{1 \leq i < j \leq n} (M - (1 - M)Y_i + Y_j + Y_i Y_j) \right) \prod_{1 \leq i < j \leq n} (Y_j - Y_i)^{-1} \right). \quad (2)\end{aligned}$$

To obtain a enumeration formula for trees, we need to apply generalised difference operators to  $\mathcal{Z}_{\text{MT}}(n, \mathbf{k}; M)$ . The crucial observation is that if we repeatedly apply  $-\Delta_{k_i}$  and  $\delta_{k_i}$  to  $\mathcal{Z}_{\text{MT}}(n, \mathbf{k}; 1)$ , we enumerate monotone triangles with truncated diagonals. By using the aforementioned generalisations of the difference operators, [Fis18, Theorem 5] implies the following result:

**Theorem 7.** *The generating function of  $(\mathbf{s}, \mathbf{t})$ -trees  $T$  with  $\mathbf{s} = (s_1, s_2, \dots, s_p)$ ,  $\mathbf{t} = (t_{n-q+1}, t_{n-q+2}, \dots, t_n)$  and strictly increasing bottom row  $\mathbf{k} = (k_1, \dots, k_n)$  with respect to the weight  $M^{\mu(T)}$  is given by*

$$\prod_{i=1}^p (-{}^M\Delta_{k_i})^{s_i} \prod_{i=n-q+1}^n M\delta_{k_i}^{t_i} [\mathcal{Z}_{\text{MT}}(n, \mathbf{k}; M)]. \quad (3)$$

Note that Theorems 6 and 7 are also true for weakly increasing bottom rows  $\mathbf{k}$  when we consider the straight enumeration of monotone triangles and trees by setting  $M = 1$ .

### 2.3 Enumeration of alternating sign trapezoids with prescribed 1-column vector

We use the correspondence between alternating sign trapezoids and trees to obtain enumeration formulae. First, we consider alternating sign trapezoids with prescribed 1-column vectors. For this purpose, we make use of Theorems 6 and 7.

**Lemma 8.** *The generating function of  $(n, l)$ -alternating sign trapezoids  $A$  with 1-column vector  $\mathbf{c}$  with respect to the weight  $M^{\mu(A)} P^{p(A)} Q^{q(A)}$  is given by*

$$\prod_{i=1}^m \left( \text{id} - \frac{P}{M} \delta_{c_i} \right) (\text{id} + {}^M\Delta_{c_i}) (-{}^M\Delta_{c_i})^{-c_i-1} \\ \times \prod_{i=m+1}^n \left( \text{id} + \frac{Q}{M} \Delta_{c_i} \right) (\text{id} - {}^M\delta_{c_i}) M\delta_{c_i}^{c_i-1} [\mathcal{Z}_{\text{MT}}(n, \tilde{\mathbf{c}}; M)], \quad (4)$$

where  $\tilde{\mathbf{c}} = (c_1, \dots, c_m, c_{m+1} + l - 3, \dots, c_n + l - 3)$ .

*Proof.* First, let us assume  $l \geq 2$ . In order to enumerate  $(n, l)$ -alternating sign trapezoids with 1-column vector  $\mathbf{c}$  such that  $-n \leq c_1 < \dots < c_m < 0 < c_{m+1} < \dots < c_n \leq n$  for some  $0 \leq m \leq n$ , it suffices to enumerate  $((-c_1 - 1, \dots, -c_m - 1), (c_{m+1} - 1, \dots, c_n - 1))$ -trees with bottom row  $\tilde{\mathbf{c}} = (c_1, \dots, c_m, c_{m+1} + l - 3, \dots, c_n + l - 3)$ .

Let  $L_{10} \subseteq \{1, \dots, m\}$  and  $R_{10} \subseteq \{m+1, \dots, n\}$ . Consider an  $(n, l)$ -alternating sign trapezoid with 1-column vector  $\mathbf{c}$  such that  $c_i$  is a 10-column if and only if  $i \in L_{10} \cup R_{10}$ . By the correspondence between alternating sign triangles and trees, we have to apply a generalised difference operator for every 10-column, namely  $-{}^M\Delta_{c_i}$  if  $i \in L_{10}$  and  ${}^M\delta_{c_i}$  if  $i \in R_{10}$ . For every other column, we apply  $\text{id} + {}^M\Delta_{c_i}$  or  $\text{id} - {}^M\delta_{c_i}$ , respectively. In total, we obtain the following generating function by Theorem 7:

$$\prod_{i \in L_{10}} (-{}^M\Delta_{c_i}) \prod_{\substack{1 \leq i \leq m, \\ i \notin L_{10}}} (\text{id} + {}^M\Delta_{c_i}) \prod_{i=1}^m (-{}^M\Delta)^{-c_i-1} \\ \times \prod_{i \in R_{10}} {}^M\delta_{c_i} \prod_{\substack{m+1 \leq i \leq n, \\ i \notin R_{10}}} (\text{id} - {}^M\delta_{c_i}) \prod_{i=m+1}^n M\delta_{c_i}^{c_i-1} [\mathcal{Z}_{\text{MT}}(n, \tilde{\mathbf{c}}; M)]. \quad (5)$$

Since  $-{}^M\Delta_X = -M^{-1}\delta_X(\text{id} + {}^M\Delta_X)$  and  ${}^M\delta_X = M^{-1}\Delta_X(\text{id} - {}^M\delta_X)$ , (5) is equal to

$$\prod_{i \in L_{10}} (-M^{-1}\delta_{c_i}) \prod_{i=1}^m (\text{id} + {}^M\Delta_{c_i}) (-{}^M\Delta)^{-c_i-1} \\ \times \prod_{i \in R_{10}} (M^{-1}\Delta_{c_i}) \prod_{i=m+1}^n (\text{id} - {}^M\delta_{c_i}) {}^M\delta^{c_i-1} [\mathcal{Z}_{\text{MT}}(n, \tilde{\mathbf{c}}; M)].$$

As an intermediate step, we enumerate all alternating sign trapezoids with  $p$  10-columns on the left side and  $q$  on the right side, that is,  $|L_{10}| = p$  and  $|R_{10}| = q$ . The corresponding generating function is given by

$$e_p(-M^{-1}\delta_{c_1}, \dots, -M^{-1}\delta_{c_m}) \prod_{i=1}^m (\text{id} + {}^M\Delta_{c_i}) (-{}^M\Delta)^{-c_i-1} \\ \times e_q(M^{-1}\Delta_{c_{m+1}}, \dots, M^{-1}\Delta_{c_n}) \prod_{i=m+1}^n (\text{id} - {}^M\delta_{c_i}) {}^M\delta^{c_i-1} [\mathcal{Z}_{\text{MT}}(n, \tilde{\mathbf{c}}; M)],$$

where  $e_p$  denotes the  $p^{\text{th}}$  elementary symmetric function. To sum over all possible positions of 10-columns, we notice that  $e_p(X_1, \dots, X_n)$  is the coefficient of  $P^p$  in  $\prod_{i=1}^n (1 + PX_i)$ . Hence, we finally obtain (4).

Finally, let us have a look at the case  $l = 1$ . We distinguish three sets of quasi alternating sign triangles which are all illustrated by an example in Figure 5: (i) those with bottom entry 1, (ii) those with bottom entry 0 and central column sum 0, and (iii) those with bottom entry 0 and central column sum 1.

We generalise the construction of transforming  $(n, l)$ -alternating sign trapezoids into monotone triangles for the case  $l \geq 2$  presented in Section 2.1 to  $l = 1$  as follows; this generalisation first appeared in the proof of [Fis19, Lemma 2.2]. As before, we add zeroes to the quasi alternating sign triangle of order  $n$  such that we obtain a rectangular  $n \times (2n - 1)$  array. We number the columns from  $-n$  to  $n - 2$  from left to right and compute the array of partial column sums. There are exactly  $i$  1s in the  $i^{\text{th}}$  row except in the last row in the cases (ii) and (iii), where the number of 1s is  $n - 1$ . In order to record the positions of the 1s in a triangular shape, we compensate for the missing 1 by adding a  $-1$  in the bottom row of the monotone triangle. In case (iii), this results in a bottom row which is not strictly but only weakly increasing instead. By removing the entries that correspond to the initially added zeroes, we then obtain trees as demonstrated in Figure 5.

In order to associate  $\mathbf{c} = (c_1, \dots, c_n)$  to any quasi alternating sign triangle of order  $n$ , we first number the  $n$  leftmost columns from  $-n$  to  $-1$  and the  $n$  rightmost columns from  $1$  to  $n$ , and we then list the positions of the 1-columns. By our numbering, the central column is equipped with two labels, namely  $-1$  and  $1$ . In case (i), we record the position of the central 1-column by  $-1$ , whilst we record it both by  $-1$  and by  $1$  in case (iii). In the remaining case (ii), we add the entry  $1$  to  $\mathbf{c}$  to ensure that  $\mathbf{c}$  consists of  $n$  entries in total. Thus, we obtain  $\mathbf{c} = (c_1, \dots, c_n)$  with  $-n \leq c_1 < \dots < c_m < 0 < c_{m+1} < \dots < c_n \leq n$

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
& 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
& & 1 & -1 & 0 & 0 & 1 & \\
& & & & 0 & 0 & 1 & \\
& & & & & & & 1
\end{array}
\longleftrightarrow
\begin{array}{cc}
& & 2 \\
& -2 & 2 \\
-3 & 1 & \\
& 0 & \\
& & -1
\end{array}$$

(i)  $\mathbf{c} = (-3, -1, 2, 3, 4)$   
bottom row  $(-3, -1, 0, 1, 2)$

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
& 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
& & 1 & -1 & 0 & 0 & 1 & \\
& & & & 0 & 0 & 1 & \\
& & & & & & & 0
\end{array}
\longleftrightarrow
\begin{array}{cc}
& & 2 \\
& -2 & 2 \\
-3 & 1 & \\
& 0 & \\
& & -1
\end{array}$$

(ii)  $\mathbf{c} = (-3, 1, 2, 3, 4)$   
bottom row  $(-3, -1, 0, 1, 2)$

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
& 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
& & 1 & -1 & 1 & 0 & 0 & \\
& & & & 0 & 0 & 1 & \\
& & & & & & & 0
\end{array}
\longleftrightarrow
\begin{array}{cc}
& & 2 \\
& -2 & 2 \\
-3 & -1 & \\
& -1 & 0 \\
& & -1 & -1
\end{array}$$

(iii)  $\mathbf{c} = (-3, -1, 1, 2, 4)$   
bottom row  $(-3, -1, -1, 0, 2)$

Figure 5: Three  $(5, 1)$ -alternating sign trapezoids with corresponding trees

for some  $0 \leq m \leq n$  such that  $(c_1, \dots, c_m, c_{m+1} - 2, \dots, c_n - 2)$  is the bottom row of the corresponding tree.

To sum up, the cases (i), (ii) and (iii) correspond to the following sets of trees: the first one corresponds to the case  $c_m = -1$  and  $c_{m+1} > 1$ , the second one corresponds to the case  $c_m < -1$  and  $c_{m+1} = 1$ , and the third one corresponds to the case  $c_m = -1$  and  $c_{m+1} = 1$ . If  $c_m = -1$  and  $c_{m+1} = 1$ , then all but two entries in the bottom row of the monotone triangle are truncated. The two remaining entries are identically equal to  $-1$  and force an entry in the upper row to be  $-1$ , too. In this case, (3) produces a factor  $2 - M$ , whereas (4) gives a factor  $P + Q - M$ . This justifies the modification of the weights in the generating function of  $(n, l)$ -alternating sign trapezoids for  $l = 1$ .  $\square$

Instead of evaluating the polynomial (4) at  $\tilde{\mathbf{c}}$ , we can shift the argument by suitable

operators and evaluate it at  $\mathbf{x} = \mathbf{0}$ . Therefore, (4) is equal to

$$\prod_{i=1}^m E_{x_i}^{c_i} \left( \text{id} - \frac{P}{M} \delta_{c_i} \right) (\text{id} + {}^M \Delta_{c_i}) (-{}^M \Delta_{c_i})^{-c_i-1} \\ \times \prod_{i=m+1}^n E_{x_i}^{c_i+l-3} \left( \text{id} + \frac{Q}{M} \Delta_{c_i} \right) (\text{id} - {}^M \delta_{c_i}) {}^M \delta_{c_i}^{c_i-1} [\mathcal{Z}_{\text{MT}}(n, \mathbf{x}; M)] \Big|_{\mathbf{x}=\mathbf{0}},$$

which we rewrite as

$$\prod_{i=1}^m E_{x_i}^{-1} (M \text{id} - (P - M) \Delta_{x_i}) (M \text{id} - (1 - M) \Delta_{x_i})^{c_i} (-\delta_{x_i})^{-c_i-1} \\ \times \prod_{i=m+1}^n E_{x_i}^{l-2} (M \text{id} + (Q - M) \delta_{x_i}) (M \text{id} + (1 - M) \delta_{x_i})^{-c_i} (\Delta_{x_i})^{c_i-1} [\mathcal{Z}_{\text{MT}}(n, \mathbf{x}; M)] \Big|_{\mathbf{x}=\mathbf{0}} \quad (6)$$

by using the fact that

$$\text{id} + {}^M \Delta_x = M (\text{id} + \Delta_x) (M \text{id} - (1 - M) \Delta_x)^{-1} = M E_x (M \text{id} - (1 - M) \Delta_x)^{-1}$$

and

$$\text{id} - {}^M \delta_x = M (\text{id} - \delta_x) (M \text{id} + (1 - M) \delta_x)^{-1} = M E_x^{-1} (M \text{id} + (1 - M) \delta_x)^{-1}.$$

We analyse how the operators in (6) interact with the argument of the antisymmetriser in (2): The effect of the shift operator  $E_{x_i}$  is the multiplication by  $1 + Y_i$ . Therefore, the application of the forward difference operator  $\Delta_{x_i}$  or of the backward difference operator  $\delta_{x_i}$  is equivalent to the multiplication by  $Y_i$  or by  $Y_i(1 + Y_i)^{-1}$ , respectively.

This observation implies that (6) is equal to

$$\text{CT}_{\mathbf{Y}} \left( \mathbf{ASym}_{\mathbf{Y}} \left( \prod_{i=1}^m (-Y_i)^{-c_i-1} (1 + Y_i)^{c_i} (M - (1 - M)Y_i)^{c_i} (M - (P - M)Y_i) \right. \right. \\ \times \prod_{i=m+1}^n Y_i^{c_i-1} (1 + Y_i)^{c_i+l-3} (M + Y_i)^{-c_i} (M + QY_i) \\ \left. \left. \times \prod_{1 \leq i < j \leq n} (M - (1 - M)Y_i + Y_j + Y_i Y_j) \right) \prod_{1 \leq i < j \leq n} (Y_j - Y_i)^{-1} \right). \quad (7)$$

## 2.4 Sum over all 1-column vectors

Thus far, we have considered  $(n, l)$ -alternating sign trapezoids with prescribed 1-column vector  $\mathbf{c}$ . To sum over all  $c_i$  such that  $-n \leq c_1 < \dots < c_m < 0 < c_{m+1} < \dots < c_n \leq n$ , we ignore the upper and lower bound in the summation since the polynomial in (7) has

no constant term if  $c_1 < n$  or  $c_n > n$ . We use the following identity which we obtain by repeated geometric series evaluations:

$$\sum_{0 \leq x_1 < \dots < x_k} X_1^{x_1} \dots X_k^{x_k} = \prod_{i=1}^k X_i^{i-1} \left( 1 - \prod_{j=i}^k X_j \right)^{-1}. \quad (8)$$

Hence, by applying (8), we obtain that

$$\begin{aligned} & \sum_{c_1 < \dots < c_m < 0} \left( \prod_{i=1}^m (-Y_i)^{-c_i-1} (1+Y_i)^{c_i} (M - (1-M)Y_i)^{c_i} (M - (P-M)Y_i) \right) \\ & \times \sum_{0 < c_{m+1} < \dots < c_n} \left( \prod_{i=m+1}^n Y_i^{c_i-1} (1+Y_i)^{c_i+l-3} (M+Y_i)^{-c_i} (M+QY_i) \right) \\ & \times \prod_{1 \leq i < j \leq n} (M - (1-M)Y_i + Y_j + Y_i Y_j) (Y_j - Y_i)^{-1}. \end{aligned}$$

is equal to

$$\begin{aligned} & \prod_{i=1}^m \frac{1}{1+Y_i} \left( \frac{-Y_i}{(1+Y_i)(M - (1-M)Y_i)} \right)^{m-i} \\ & \times \left( 1 - \prod_{j=1}^i \left( \frac{-Y_j}{(1+Y_j)(M - (1-M)Y_j)} \right) \right)^{-1} \frac{M - (P-M)Y_i}{M - (1-M)Y_i} \\ & \times \prod_{i=m+1}^n (1+Y_i)^{l-2} \left( \frac{Y_i(1+Y_i)}{M+Y_i} \right)^{i-m-1} \left( 1 - \prod_{j=i}^n \left( \frac{Y_j(1+Y_j)}{M+Y_j} \right) \right)^{-1} \frac{M+TY_i}{M+Y_i} \\ & \times \prod_{1 \leq i < j \leq n} (M - (1-M)Y_i + Y_j + Y_i Y_j) (Y_j - Y_i)^{-1}. \quad (9) \end{aligned}$$

Before summing over all  $m$  such that  $0 \leq m \leq n$ , we apply the symmetriser to the expression (9). To this end, we use the following trick by Fischer [Fis19]: We set

$$\mathfrak{S}_n^m := \{ \sigma \in \mathfrak{S}_n \mid \sigma(i) < \sigma(j) \text{ for all } 1 \leq i < j \leq m \text{ and all } m+1 \leq i < j \leq n \}$$

and define

$$\mathbf{Subsets}_{X_1, \dots, X_m}^{X_{m+1}, \dots, X_n} f(X_1, \dots, X_n) := \sum_{\sigma \in \mathfrak{S}_n^m} f(X_{\sigma(1)}, \dots, X_{\sigma(n)}).$$

It follows that  $\mathbf{Sym}_{X_1, \dots, X_n} f(X_1, \dots, X_n)$  can be evaluated as

$$\mathbf{Subsets}_{X_1, \dots, X_m}^{X_{m+1}, \dots, X_n} \mathbf{Sym}_{X_1, \dots, X_m} \mathbf{Sym}_{X_{m+1}, \dots, X_n} f(X_1, \dots, X_n).$$

That is, we first apply  $\mathbf{Sym}_{Y_1, \dots, Y_m}$  and  $\mathbf{Sym}_{Y_{m+1}, \dots, Y_n}$  to (9) by means of the following antisymmetriser lemma [Hön21, Lemma 5.9] and its variation [Hön21, (5.19)] by setting  $X_i \mapsto -(X_{m+1-i})(1+X_{m+1-i})^{-1}$ :

**Lemma 9.** *Let  $n \geq 1$ . Then*

$$\begin{aligned} \mathbf{ASym}_{\mathbf{x}} \left( \prod_{i=1}^n \frac{\left(\frac{X_i(1+X_i)}{M+X_i}\right)^{i-1}}{1 - \prod_{j=i}^n \frac{X_j(1+X_j)}{M+X_j}} \prod_{1 \leq i < j \leq n} (M - (1-M)X_i + X_j + X_iX_j) \right) \\ = \prod_{i=1}^n \frac{M + X_i}{M - X_i^2} \prod_{1 \leq i < j \leq n} \frac{(M(1+X_i)(1+X_j) - X_iX_j)(X_j - X_i)}{M - X_iX_j}. \end{aligned}$$

Eventually, we obtain

$$\begin{aligned} \prod_{i=1}^m \frac{M - (P-M)Y_i}{M(1+Y_i)^2 - Y_i^2} \prod_{1 \leq i < j \leq m} \frac{M - Y_iY_j}{M(1+Y_i)(1+Y_j) - Y_iY_j} \prod_{i=m+1}^n (1+Y_i)^{l-2} \frac{M + QY_i}{M - Y_i^2} \\ \times \prod_{m+1 \leq i < j \leq n} \frac{M(1+Y_i)(1+Y_j) - Y_iY_j}{M - Y_iY_j} \prod_{i=1}^m \prod_{j=m+1}^n \frac{M - (1-M)Y_i + Y_j + Y_iY_j}{Y_j - Y_i}. \quad (10) \end{aligned}$$

Next, we need to apply the operator  $\mathbf{Subsets}_{Y_1, \dots, Y_m}^{Y_{m+1}, \dots, Y_n}$  to (10) and evaluate at  $\mathbf{Y} = 0$ . To simplify the computation, we divide (10) by the polynomial  $\prod_{1 \leq i < j \leq n} (M(1+Y_i)(1+Y_j) - Y_iY_j)(M - Y_iY_j)$ , which is symmetric and, thus, invariant under the application of  $\mathbf{Subsets}_{Y_1, \dots, Y_m}^{Y_{m+1}, \dots, Y_n}$ . However, we need to incorporate its constant term  $M^{n(n-1)}$ . We obtain

$$\begin{aligned} M^{n(n-1)} \prod_{i=1}^m (M - (P-M)Y_i) \prod_{i,j=1}^m \frac{1}{M(1+Y_i)(1+Y_j) - Y_iY_j} \\ \times \prod_{i=m+1}^n (1+Y_i)^{l-2} (M + QY_i) \prod_{i,j=m+1}^n \frac{1}{M - Y_iY_j} \\ \times \prod_{i=1}^m \prod_{j=m+1}^n \frac{M - (1-M)Y_i + Y_j + Y_iY_j}{(Y_j - Y_i)(M(1+Y_i)(1+Y_j) - Y_iY_j)(M - Y_iY_j)}. \quad (11) \end{aligned}$$

This expression can be written in determinantal form. For this purpose, we consider the *Cauchy determinant*

$$\det_{1 \leq i, j \leq n} \left( \frac{1}{X_i + Y_j} \right) = \frac{\prod_{1 \leq i < j \leq n} (X_j - X_i)(Y_j - Y_i)}{\prod_{i,j=1}^n (X_i + Y_j)}$$

and set  $X_i = \frac{M(1+Y_i)}{M-(1-M)Y_i}$  for all  $1 \leq i \leq m$  and  $X_i = -\frac{M}{Y_i}$  for all  $m+1 \leq i \leq n$ . This yields that

$$\det_{1 \leq i, j \leq n} \left( \begin{cases} \frac{M-(1-M)Y_i}{M(1+Y_i)(1+Y_j)-Y_iY_j}, & 1 \leq i \leq m \\ \frac{-Y_i}{M-Y_iY_j}, & m+1 \leq i \leq n \end{cases} \right) \quad (12)$$



is equal to

$$\begin{aligned}
& (-1)^{n-m} M^{\binom{n}{2}} \prod_{i=1}^m (M - (1 - M)Y_i) \prod_{i,j=1}^m \frac{1}{M(1 + Y_i)(1 + Y_j) - Y_i Y_j} \prod_{1 \leq i < j \leq m} (Y_j - Y_i)^2 \\
& \quad \times \prod_{i=m+1}^n Y_i \prod_{i,j=m+1}^n \frac{1}{M - Y_i Y_j} \prod_{m+1 \leq i < j \leq n} (Y_j - Y_i)^2 \\
& \quad \times \prod_{i=1}^m \prod_{j=m+1}^n \frac{(Y_j - Y_i)(M - (1 - M)Y_i + Y_j + Y_i Y_j)}{(M(1 + Y_i)(1 + Y_j) - Y_i Y_j)(M - Y_i Y_j)}.
\end{aligned}$$

Next, we perform the following row operations on the determinant's underlying matrix in (12): For  $1 \leq i \leq m$ , we multiply the  $i^{\text{th}}$  row by  $(M - (P - M)Y_i)/(M - (1 - M)Y_i)$ , whereas, for  $m + 1 \leq i \leq n$ , we multiply the  $i^{\text{th}}$  row by  $-(1 + Y_i)^{l-2}(M + QY_i)/Y_i$ . Finally, by multiplying the entire determinant by  $M^{\binom{n}{2}}/\prod_{1 \leq i < j \leq n} (Y_j - Y_i)^2$ , we see that (11) is equal to

$$\frac{M^{\binom{n}{2}}}{\prod_{1 \leq i < j \leq n} (Y_j - Y_i)^2} \det_{1 \leq i, j \leq n} \left( \begin{cases} \frac{M - (P - M)Y_i}{M(1 + Y_i)(1 + Y_j) - Y_i Y_j}, & 1 \leq i \leq m \\ (1 + Y_i)^{l-2} \frac{M + QY_i}{M - Y_i Y_j}, & m + 1 \leq i \leq n \end{cases} \right).$$

To apply the subset operator, we observe that the antisymmetry of the determinant implies that it follows for any  $\sigma \in \mathfrak{S}_n^m$  that

$$\frac{M^{\binom{n}{2}}}{\prod_{1 \leq i < j \leq n} (Y_j - Y_i)^2} \det_{1 \leq i, j \leq n} \left( \begin{cases} \frac{M - (P - M)Y_i}{M(1 + Y_i)(1 + Y_j) - Y_i Y_j}, & i \in \sigma(\{1, \dots, m\}) \\ (1 + Y_i)^{l-2} \frac{M + QY_i}{M - Y_i Y_j}, & i \in \sigma(\{m + 1, \dots, n\}) \end{cases} \right)$$

is equivalent to applying  $\sigma$  to the variables  $Y_1, \dots, Y_n$  of (11). Hence, the application of **Subsets** $_{Y_1, \dots, Y_m}^{Y_{m+1}, \dots, Y_n}$  and the summation over all  $1 \leq m \leq n$  yield

$$\frac{M^{\binom{n}{2}}}{\prod_{1 \leq i < j \leq n} (Y_j - Y_i)^2} \det_{1 \leq i, j \leq n} \left( R \frac{M - (P - M)Y_i}{M(1 + Y_i)(1 + Y_j) - Y_i Y_j} + (1 + Y_i)^{l-2} \frac{M + QY_i}{M - Y_i Y_j} \right), \quad (13)$$

where the insertion of the factor  $R$  into the first term of the matrix entries gives the required factors of  $R^m$  in the sum over  $m$ .

## 2.5 Transformation of the determinantal formula

The constant term  $\text{CT}_{\mathbf{Y}}$  of the determinant expression in (13) provides our first expression for the generating function  $\mathcal{Z}_{\text{ASTZ}}(n, l; M, R, P, Q)$  for the fourfold refined enumeration of  $(n, l)$ -alternating sign trapezoids. We transform it into a determinant involving binomial coefficients. Our key tool is the following formula by Behrend, Di Francesco and Zinn-Justin [BDZ12, (43)-(47)]:

**Lemma 10.** For a given power series  $f$  in variables  $X$  and  $Y$ , it holds that

$$\frac{\det_{1 \leq i, j \leq n} (f(X_i, Y_j))}{\prod_{1 \leq i < j \leq n} (X_j - X_i)(Y_j - Y_i)} \Big|_{\mathbf{x}=\mathbf{y}=\mathbf{0}} = \det_{0 \leq i, j \leq n-1} ([X^i Y^j] f(X, Y));$$

here,  $[X^i Y^j] f(X, Y)$  denotes the coefficient of  $X^i Y^j$  in the series expansion of  $f$ .

We set

$$f(X, Y) = R \frac{M - (P - M)X}{M(1 + X)(1 + Y) - XY} + (1 + X)^{l-2} \frac{M + QX}{M - XY}$$

and extract the coefficient  $[X^i Y^j] f(X, Y)$ . It equals  $R(-1)^j + [j = 0]$  if  $i = 0$ ; otherwise,  $[X^i Y^j] f(X, Y)$  is given by

$$R(-1)^{i+j} \sum_{k \geq 0} \binom{j}{k} M^{-k} \left( \binom{i-1}{k-1} + \binom{i-1}{k} P M^{-1} \right) + \binom{l-2}{i-j} M^{-j} + \binom{l-2}{i-j-1} Q M^{-j-1}.$$

Note that we set the binomial coefficient  $\binom{n}{k} := 0$  for  $k < 0$ .

The determinant of the matrix  $([X^i Y^j] f(X_i, Y_j))_{0 \leq i, j \leq n-1}$  remains invariant under the left multiplication by the triangular matrix  $\left( \binom{2-l}{i-j} \right)_{0 \leq i, j \leq n-1}$  with determinant 1. Thus, we obtain

$$R \sum_{k \geq 0} (-1)^{j-k} \binom{j}{k} M^{-k} \left( \binom{2-l-k}{i-k} - \binom{1-l-k}{i-k-1} P M^{-1} \right) + M^{-j} \delta_{i,j} + Q M^{-j-1} \delta_{i,j+1}.$$

In addition, we multiply the  $i^{\text{th}}$  row by  $(-1)^i$  and the  $j^{\text{th}}$  row by  $(-1)^j$  for all  $0 \leq i, j \leq n-1$  and use  $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}$  to obtain

$$R \sum_{k \geq 0} \binom{j}{k} M^{-k} \left( \binom{i+l-3}{i-k} + \binom{i+l-3}{i-k-1} P M^{-1} \right) + M^{-j} \delta_{i,j} - Q M^{-j-1} \delta_{i,j+1}.$$

Next, define the matrix

$$K(n) := (M^{-j} \delta_{i,j} - M^{-j-1} Q \delta_{i,j+1})_{0 \leq i, j \leq n-1},$$

whose determinant evaluation yields  $M^{-\binom{n}{2}}$  and whose inverse is

$$K(n)^{-1} = \begin{cases} Q^{i-j} M^j, & i \geq j, \\ 0, & i < j. \end{cases}$$

We left multiply by  $K(n)^{-1}$ , and, thus, we finally obtain by Lemma 10 that  $\mathcal{Z}_{\text{ASTZ}}(n, l; Q, R, S, T)$  is equal to

$$\det_{0 \leq i, j \leq n-1} \left( R \sum_{k=0}^i Q^{i-k} \sum_{m=0}^j \binom{j}{m} M^{k-m} \left( \binom{k+l-3}{k-m} + \binom{k+l-3}{k-m-1} P M^{-1} \right) + \delta_{i,j} \right). \quad (14)$$

### 3 Weighted Enumeration of Column Strict Shifted Plane Partitions

In order to enumerate column strict shifted plane partitions, we interpret them as a family of nonintersecting lattice paths which can be enumerated by the Lindström–Gessel–Viennot lemma.

#### 3.1 Interpretation of column strict shifted plane partitions as families of non-intersecting lattice paths

We transform column strict shifted plane partitions into a family of nonintersecting lattice paths as follows: Each row corresponds to a path that consists of vertical and horizontal unit steps. If  $(\pi_{i,i}, \pi_{i,i+1}, \dots, \pi_{i,i+\lambda_i-1})$  denotes the  $i^{\text{th}}$  row of a column strict shifted plane partition  $\pi$  of shape  $\lambda$ , then the associate path starts at  $(\lambda_i - 1, 0)$  on the  $x$ -axis and ends at  $(-1, \pi_{i,i} - 1)$ ; the heights of the vertical steps are the parts of the corresponding row in reverse order and reduced by 1, that is,  $\pi_{i,i+\lambda_i-1} - 1, \dots, \pi_{i,i+1} - 1$  and  $\pi_{i,i} - 1$ . Figure 6 displays the family of nonintersecting lattice paths corresponding to the column strict shifted plane partition in Figure 2.

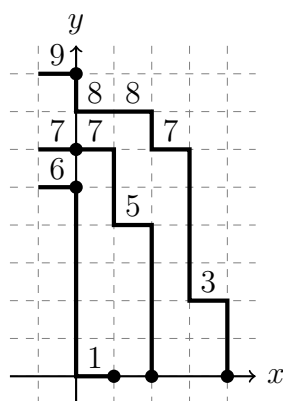


Figure 6: Family of lattice paths corresponding to the corresponding to the column strict shifted plane partition in Figure 2

This construction yields a bijective correspondence between column strict shifted plane partitions of class  $l - 1$  with at most  $n$  parts in the first row and the family of nonintersecting lattice paths consisting of horizontal and vertical unit steps with starting points  $S \subseteq \{S_i := (i, 0) \mid 0 \leq i \leq n - 1\}$  and endpoints  $E \subseteq \{E_i := (0, i + l - 1) \mid 0 \leq i \leq n - 1\}$  such that  $S_i \in S$  if and only if  $E_i \in E$ .

By this interpretation of column strict shifted plane partitions as a family of nonintersecting lattice paths and by the Lindström–Gessel–Viennot lemma, it can be shown that column strict shifted plane partitions of class  $l - 1$  with at most  $n$  parts in the first row are enumerated by

$$\det_{0 \leq i, j \leq n-1} \left( \binom{i+j+l-1}{i} + \delta_{i,j} \right). \quad (15)$$

This was first proved by Andrews [And79]. In fact, the determinant (15) can be obtained from (14) by setting  $M = R = P = Q = 1$ .

### 3.2 Weighted enumeration for $d \geq 1$

We generalise Andrews' result. To this end, consider an element of  $\text{CSSPP}_{n,l-1}$  and  $L$  be the corresponding family of nonintersecting lattice paths. For the time being, fix  $d \in \{1, \dots, l-1\}$ . We investigate how to translate the statistics on column strict shifted plane partitions into statistics on lattice paths: The weight of  $L$  is

$$M^{\mu_d(L)} R^{r(L)} P^{p_d(L)} Q^{q(L)},$$

where

$\mu_d(L) := \#$  horizontal steps of  $L$  below the line  $y = x + l - 1$  neither lying on the  $x$ -axis nor touching the line  $y = x + d$  from the right,

$r(L) := \#$  paths of  $L$ ,

$p_d(L) := \#$  horizontal steps of  $L$  touching the line  $y = x + d$  from the right,

$q(L) := \#$  horizontal steps of  $L$  on the  $x$ -axis.

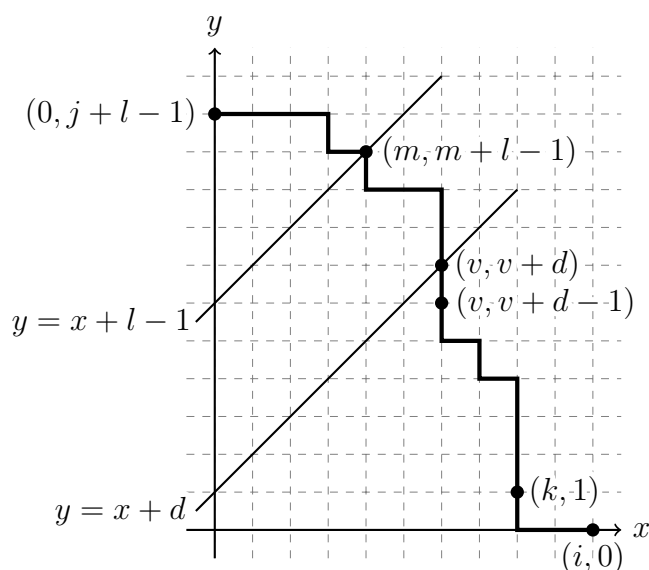


Figure 7: Lattice path from  $(i, 0)$  to  $(0, j + l - 1)$  with  $i - k$  horizontal steps at height 0 that intersects the line  $y = x + d$  from the right side with a vertical step

Figure 7 shows a single lattice path from  $(i, 0)$  to  $(0, j + l - 1)$  that has  $i - k$  horizontal steps at height 0. In addition, the line  $y = x + d$  intersects the path with a vertical step from the right side. Let  $\mathcal{P}((x_1, y_1) \rightarrow (x_2, y_2))$  denote the number of paths from  $(x_1, y_1)$  to  $(x_2, y_2)$  with horizontal steps  $(-1, 0)$  and vertical steps  $(0, 1)$ ; that is,  $\mathcal{P}((x_1, y_1) \rightarrow$

$(x_2, y_2) = \binom{(x_1-x_2)+(y_2-y_1)}{x_1-x_2}$ ). We extend the previously defined statistics  $\mu_d$ ,  $r$ ,  $p_d$  and  $q$  to single paths from  $(i, 0)$  to  $(0, j + l - 1)$  and derive the generating function of all paths from  $(i, 0)$  to  $(0, j + l - 1)$  which intersect the line  $y = x + d$  with a vertical step from the right as

$$\begin{aligned} R \sum_{k=0}^i Q^{i-k} \sum_{v=0}^k \mathcal{P}((k, 1) \rightarrow (v, v + d - 1)) \\ \times \sum_{m=0}^j M^{k-m} \mathcal{P}((v, v + d) \rightarrow (m, m + l - 1)) \mathcal{P}((m, m + l - 1) \rightarrow (0, j + l - 1)) \\ = R \sum_{k=0}^i Q^{i-k} \sum_{v=0}^k \binom{k + d - 2}{k - v} \sum_{m=0}^j M^{k-m} \binom{l - 1 - d}{v - m} \binom{j}{m}, \end{aligned}$$

which we simplify to

$$R \sum_{k=0}^i Q^{i-k} \sum_{m=0}^j M^{k-m} \binom{j}{m} \binom{k + l - 3}{k - m} \quad (16)$$

by using the Chu-Vandermonde identity.

If the line  $y = x + d$  intersects the path with a horizontal step from the right side, the generating function of such paths is given by

$$\begin{aligned} RP \sum_{k=0}^i Q^{i-k} \sum_{v=0}^k \mathcal{P}((k, 1) \rightarrow (v + 1, v + d)) \\ \times \sum_{m=0}^j M^{k-m-1} \mathcal{P}((v, v + d) \rightarrow (m, m + l - 1)) \mathcal{P}((m, m + l - 1) \rightarrow (0, j + l - 1)), \end{aligned}$$

which is equal to

$$RP \sum_{k=0}^i Q^{i-k} \sum_{m=0}^j M^{k-m-1} \binom{j}{m} \binom{k + l - 3}{k - m - 1}. \quad (17)$$

The sum of (16) and (17) yields the generating function of all single lattice paths from  $(i, 0)$  to  $(0, j + l - 1)$ :

$$R \sum_{k=0}^i Q^{i-k} \sum_{m=0}^j \binom{j}{m} M^{k-m} \left( \binom{k + l - 3}{k - m} + \binom{k + l - 3}{k - m - 1} P M^{-1} \right). \quad (18)$$

By the well-known Lindström–Gessel–Viennot lemma [GV85, GV89, Lin73], it follows

that  $\mathcal{Z}_{\text{CSSPP}}(n, l - 1, d; M, R, P, Q)$  is equal to

$$\sum_{\substack{0 \leq u_1 < \dots < u_r \leq n-1, \\ 0 \leq r \leq n-1}} \det_{1 \leq i, j \leq r} \left( R \sum_{k=0}^{u_i} Q^{u_i-k} \sum_{m=0}^{u_j} \binom{u_j}{m} M^{k-m} \right. \\ \left. \times \left( \binom{k+l-3}{k-m} + \binom{k+l-3}{k-m-1} P M^{-1} \right) \right),$$

which simplifies to (14).

### 3.3 The case $d = 0$

In the case  $d = 0$ , a family  $L$  of nonintersecting lattice paths is assigned the weight

$$M^{\mu_0(L)} R^{r(L)} P^{p_0(L)} Q^{q_0(L)} (P + Q - M)^{[L \text{ contains } (1,0) \rightarrow (0,0)]},$$

where

$\mu_0(L) := \#$  horizontal steps of  $L$  below the line  $y = x + l - 1$  neither lying on the  $x$ -axis nor touching the line  $y = x$  from the right,

$p_0(L) := \#$  horizontal steps of  $L$  not lying on the  $x$ -axis but touching the line  $y = x$  from the right,

$q_0(L) := \#$  horizontal steps of  $L$  on the  $x$ -axis except  $(1,0) \rightarrow (0,0)$ .

To derive the generating function of all single lattice paths from  $(i, 0)$  to  $(0, j + l - 1)$ , we distinguish several cases: If the path contains the horizontal step  $(1, 0) \rightarrow (0, 0)$ , it has the weight  $RQ^{i-1}(P + Q - M)$ . If the path does not contain the horizontal step  $(1, 0) \rightarrow (0, 0)$ , it can intersect the line  $y = x$  with a vertical step from below. In this case, the generating function is given by

$$R \sum_{k=1}^i Q^{i-k} \sum_{v=1}^k \mathcal{P}((k, 1) \rightarrow (v, v - 1)) \\ \times \sum_{m=0}^j M^{k-m} \mathcal{P}((v, v) \rightarrow (m, m + l - 1)) \mathcal{P}((m, m + l - 1) \rightarrow (0, j + l - 1)),$$

which is equal to

$$RQ^{i-1}M + R \sum_{k=1}^i Q^{i-k} \sum_{m=0}^j M^{k-m} \binom{j}{m} \binom{k+l-3}{k-m}. \quad (19)$$

If the path intersects the line  $y = x$  with a horizontal step from the right side, the generating function is given by

$$RP \sum_{k=2}^i Q^{i-k} \sum_{v=1}^k \mathcal{P}((k, 1) \rightarrow (v+1, v)) \\ \times \sum_{m=0}^j M^{k-m-1} \mathcal{P}((v, v) \rightarrow (m, m+l-1)) \mathcal{P}((m, m+l-1) \rightarrow (0, j+l-1)),$$

which is equal to

$$RP \sum_{k=2}^i Q^{i-k} \sum_{m=0}^j M^{k-m-1} \binom{j}{m} \binom{k+l-3}{k-m-1}. \quad (20)$$

As before, the sum of (19), (20) and  $RQ^{i-1}(P+Q-M)$  is equal to (18).

As a result, (14) is the generating function  $\mathcal{Z}_{\text{CSSPP}}(n, l-1, d; M, R, P, Q)$  of column strict shifted plane partitions of class  $l-1$  with at most  $n$  entries in the first row for any  $d \in \{0, \dots, l-1\}$ . This completes the proof of Theorem 3.

## 4 The $x$ -Enumeration

For an integer  $x \in \mathbb{Z}$ , we define the  $x$ -enumeration of  $(n, l)$ -alternating sign trapezoids as

$$\sum_{A \in \text{ASTZ}_{n,l}} x^{\mu(A)}.$$

Ayyer, Behrend and Fischer [ABF20] proved that the analogously defined  $x$ -enumeration of  $n \times n$ -alternating sign matrices is equal to the  $x$ -enumeration of  $(n-1, 3)$ -alternating sign trapezoids. For specific values of  $x$ , the  $x$ -enumeration of alternating sign matrices surprisingly yields round numbers: Mills, Robbins and Rumsey [MRR83] presented a closed expression of the 2-enumeration and Kuperberg [Kup96] for the 3-enumeration.

We prove that the 2-enumeration of  $(n, l)$ -alternating sign trapezoids yields round numbers in general, too.

**Theorem 11.** *For  $l \geq 2$ , the 2-enumeration of  $(2n+1, l)$ -alternating sign trapezoids is given by*

$$2^{n+1} \left( \prod_{1 \leq i \leq j \leq n} \frac{2(i+j)+l-3}{2i-1} \right)^2$$

and the 2-enumeration of  $(2n+2, l)$ -alternating sign trapezoids is given by

$$2^{n+1} \prod_{1 \leq i \leq j \leq n} \frac{2(i+j)+l-3}{2i-1} \prod_{1 \leq i \leq j \leq n+1} \frac{2(i+j)+l-3}{2i-1}.$$

*Proof.* Let  $l \geq 1$ . To prove the conjecture, we provide a product formula for the generating function  $\mathcal{Z}_{\text{ASTZ}}(n, l; 2, 1, 1, 1)$ . To this end, we set

$$f(l; i, j) := \sum_{k=0}^i \sum_{m=0}^j 2^{k-m} \binom{j}{m} \left( \binom{k+l-3}{k-m} + \frac{1}{2} \binom{k+l-3}{k-m-1} \right).$$

Then

$$\mathcal{Z}_{\text{ASTZ}}(n, l; 2, 1, 1, 1) = \det_{0 \leq i, j \leq n-1} (f(l; i, j) + \delta_{i,j}).$$

First, we simplify the expression for  $f(l; i, j)$ . We split the sum up, shift the index in the second summand and make use of  $\binom{j}{m} + \binom{j}{m-1} = \binom{j+1}{m}$  in order to see that

$$\sum_{m=0}^j 2^{k-m} \binom{j}{m} \left( \binom{k+l-3}{k-m} + \frac{1}{2} \binom{k+l-3}{k-m-1} \right) = \sum_{m=0}^{j+1} 2^{k-m} \binom{j+1}{m} \binom{k+l-3}{k-m},$$

which can be restated as an ordinary hypergeometric function<sup>1</sup> as follows by means of an instance of the classical Pfaff transformation [AAR99, (2.2.6)]:

$$2^k \binom{k+l-3}{k} {}_2F_1 \left[ \begin{matrix} -k, -j-1 \\ l-2 \end{matrix} \middle| \frac{1}{2} \right] = \binom{k+l-3}{k} {}_2F_1 \left[ \begin{matrix} -k, j+l-1 \\ l-2 \end{matrix} \middle| -1 \right].$$

Finally, by resorting the order of summation, we further deduce that  $f(l; i, j)$  can be expressed as

$$\begin{aligned} & \sum_{k=0}^i \binom{k+l-3}{k} {}_2F_1 \left[ \begin{matrix} -k, j+l-1 \\ l-2 \end{matrix} \middle| -1 \right] \\ &= \sum_{k=0}^i \binom{k+l-3}{k} \sum_{m=0}^k \binom{k}{m} \frac{(j+l-1)_m}{(l-2)_m} = \sum_{m=0}^i \frac{(j+l-1)_m}{(l-2)_m} \sum_{k=m}^i \binom{k+l-3}{k} \binom{k}{m} \\ &= \sum_{m=0}^i \frac{(j+l-1)_m}{(l-2)_m} \binom{m+l-3}{m} \sum_{k=0}^{i-m} \binom{k+m+l-3}{k} \\ &= \sum_{m=0}^i \frac{(j+l-1)_m}{m!} \binom{i+l-2}{m+l-2}, \end{aligned}$$

which is equal to

$$\binom{i+l-2}{l-2} {}_2F_1 \left[ \begin{matrix} -i, j+l-1 \\ l-1 \end{matrix} \middle| -1 \right]. \tag{21}$$

However, if we set  $a = l - 2$ , then

$$g(a; i, j) := \sum_{m=0}^i \binom{m+j+a}{m} \binom{i+a}{m+a}$$

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<sup>1</sup>The *ordinary hypergeometric function* is defined as  ${}_2F_1 \left[ \begin{matrix} a, b \\ c \end{matrix} \middle| z \right] := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$ , where  $(a)_n := a(a+1) \cdots (a+n-1)$  if  $n \geq 1$  and  $(a)_0 := 1$  denote the *Pochhammer symbol*.



equals (21), too. Hence,  $g(l-2; i, j) = f(l; i, j)$ . Andrews [And87, Theorem 1 & Lemma 2] evaluated the determinant  $D_n(a) := \det_{0 \leq i, j \leq n-1} (g(a; i, j) + \delta_{i,j})$  for  $a \geq 0$  and proved that

$$\frac{D_{2n}(a)}{D_{2n-1}(a)} = 2^n \prod_{k=1}^n \frac{2(n+k) + a - 1}{n+k}, \quad (22)$$

$$\frac{D_{2n+1}(a)}{D_{2n}(a)} = 2^{n+1} \prod_{k=1}^n \frac{2(n+k) + a - 1}{n+k}. \quad (23)$$

Since  $D_1(a) = 2$ , (22) and (23) imply Theorem 11. □

The determinant  $D_n(a)$  originally emerged in [MRR87] as a generating function of a differently weighted enumeration of column strict shifted plane partitions of class  $2a$  with at most  $n$  parts in the first row.

Note that  $\mathcal{Z}_{\text{ASTZ}}(n, 1; x, 1, 1, 1)$  is not equivalent to the  $x$ -enumeration of quasi alternating sign triangles since we altered the weight in the case  $l = 1$ . However, computer experiments suggest that neither the 2-enumeration 2, 5, 22, 188, 3152, 104704, 6905856, ... nor the 3-enumeration 2, 5, 24, 252, 5832, 301077, 34720812, ... of quasi alternating sign triangles yields round numbers. Computer experiments suggest that the 3-enumeration of  $(n, l)$ -alternating sign trapezoids does not provide round numbers either unless  $l = 3$ . Furthermore, note that for  $l = 2$ , the 2-enumeration coincides with the total dimension of the homology of free 2-step nilpotent Lie algebras [GKT02, Theorem 1.1] as well as with the number of rhombus tilings of half-hexagons with glued sides [DZZ05, (3.5)].

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