On Completing Partial Latin Squares with Prescribed Diagonals

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Abstract

Necessary and sufficient numerical conditions are known for the embedding of an incomplete latin square L of order n into a latin square T of order $t \geqslant 2n+1$ in which each symbol is prescribed to occur in a given number of cells on the diagonal of T outside of L. This includes the classic case where T is required to be idempotent.

If $t < 2n$ then no such numerical sufficient conditions exist since it is known that the arrangement of symbols within the given incomplete latin square can determine the embeddibility. All examples where the arrangement is a factor share the common feature that one symbol is prescribed to appear exactly once in the diagonal of T outside of L, resulting in a conjecture over 30 years ago stating that it is only this feature that prevents numerical conditions sufficing for all $t \geqslant n$.

In this paper we prove this conjecture, providing necessary and sufficient numerical conditions for the embedding of an incomplete latin square L of order n into a latin square T of order t for all $t \geqslant n$ in which the diagonal of T outside of L is prescribed in the case where no symbol is required to appear exactly once in the diagonal of T outside of L.

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1 Introduction

Historically, a *(partial) latin square* L of order n is an $n \times n$ array in which each cell contains (at most) one symbol in $S(n) = \{1, 2, \ldots, n\}$ and each of the symbols in $S(n)$ occurs (at most) once in each row and (at most) once in each column. Let $L(i, j)$ denote the symbol in cell (i, j) of L, and let $N_L(i)$ (or simply $N(i)$ if L is clear) be the number of cells that contain symbol i in L. A *(partial) incomplete latin square* of order n (also referred to as a *(partial) latin array* of order n) on the symbols in $S(t)$ is an $n \times n$ array in which each cell contains (at most) one symbol in $S(t)$ and each of the symbols in $S(t)$ occurs at most once in each row and at most once in each column. A partial or incomplete latin square L of order n is said to be *embedded* in the latin square T of order t if for each cell (i, j) of L that contains a symbol, $L(i, j) = T(i, j)$. The cells (i, i) for $n+1 \leqslant i \leqslant t$ are said to be *the diagonal of* T *outside* L. A latin square of order n is said to be *idempotent* if $L(i, i) = i$ for $1 \leqslant i \leqslant n$, and is said to be *symmetric* if $L(i, j) = L(j, i)$ for $1 \leqslant i \leqslant j \leqslant n$.

There is a rich history of papers that consider the embedding of partial and incomplete latin squares; the following is a sample of such results. The classic result of Ryser [12] shows that an incomplete latin square L of order n on the symbols in $S(t)$ can be embedded in a latin square of order t if and only if $N_L(i) \geqslant 2n - t$ for $1 \leqslant i \leqslant t$. This condition is known as the *Ryser condition*. Evans [7] obtained a related result for partial latin squares, proving that any partial latin square of order n can be embedded in a latin square of order t for any $t \geqslant 2n$. This result is best possible in that there are partial latin squares of order n that cannot be embedded in a latin square of order t if $t < 2n$. Cruse [6] then found necessary and sufficient conditions for a partial latin square of order n to be embedded in a symmetric latin square of order t, and also to be embedded in an idempotent symmetric latin square of order t, where in both cases $t > n$ is arbitrary. It turns out that embedding partial and incomplete latin squares in an idempotent latin square is a very difficult problem. The Ryser conditions can naturally be extended to provide a necessary condition for an incomplete idempotent latin square L of order n with symbol set $S(t)$ to be embedded in an idempotent latin square of order t with symbol set $S(t)$, namely that $N_L(i) \geqslant 2n - t + f(i)$ for $1 \leqslant i \leqslant t$, where $f(i) = 0$ for $1 \leqslant i \leqslant n$ and $f(i) = 1$ for $n + 1 \leqslant i \leqslant t$. It was shown by Andersen et al. [3, 4] that for all $t < 2n$ these Ryser-type conditions are not sufficient: there exists an incomplete idempotent latin square of order n which cannot be embedded in an idempotent latin square of order t. In some cases, just swapping the placement of symbols in two cells results in one which does have an idempotent embedding. So, for the first time in these sorts of embedding problems, the arrangement of the symbols in L can determine its embeddibility, thus making the idempotent setting quite special. The case where $t \geqslant 2n$ was finally settled after various results reduced the bound on t. Treash [13] showed that a finite embedding of a partial idempotent latin square was always possible, Lindner [8] reduced the bound to around $6n$, conjecturing that $2n+1$ was the right lower bound (the Ryser-type conditions come into play when $t \leqslant 2n$), Andersen [5] further reduced it to $t \geqslant 4n$ and $t \neq 4n + 1$, and finally Andersen et al. [2] settled the Lindner conjecture which states that any partial idempotent latin square can be embedded in an idempotent latin square of order t, for any $t \geqslant 2n+1$. The idempotent embedding for incomplete idempotent latin squares was then settled for all $t \geqslant 2n$ by Rodger [10].

A natural generalization to embedding an incomplete latin square L of order n with symbol set $S(t)$ into an idempotent latin square T of order t is to more generally prescribe what is to occur on the diagonal: suppose it is required that for $1 \leqslant i \leqslant t$ symbol i should occur $f(i)$ times in the diagonal cells of T outside L. Then the Ryser-type conditions are again necessary, and if $t \geqslant 2n+1$ then Rodger [11] proved they, along with two other necessary conditions, are also sufficient. It is the case that if $f(i) = 1$ for some symbol, i, then Andersen et al. [3] again showed that when $t < 2n$ the arrangement of symbols in L can determine if L can be embedded in T with the given prescribed diagonal of T outside L. Rodger conjectured that if $f(i) \neq 1$ for $1 \leqslant i \leqslant t$ then, even when $t \leqslant 2n$, the Ryser-type conditions are sufficient. It is this 30 year old conjecture that we prove in this paper.

2 Previous Results

Before proving the main result, Theorem 4, we note the following three results.

Andersen et al. [1] proved Theorem 1, which completely settles the embedding problem providing not all of the diagonal is prescribed.

Theorem 1 ([1]). *Let $t \geqslant n > 0$. Let L be an incomplete latin square of order n on the symbols in $S(t)$. Let $f : \{1, 2, \ldots, t\} \mapsto \mathbb{N}$ satisfy $\sum_{i=1}^{n} f(i) \leqslant t - n - 1$. Then L can be embedded in a latin square T of order t on the same symbols in which each symbol i appears at least $f(i)$ times on diagonal of T outside L if and only if $N_L(i) \geqslant 2n - t + f(i)$ for $1 \leqslant i \leqslant t$.*

The following classic theorem, proven by Ryser [12], will be used in Step 1 of the proof of Theorem 4.

Theorem 2 ([12]). *An incomplete latin square L of order n on the symbols in $S(t)$ can be embedded in a latin square of order t on the same symbols if and only if $N_L(i) \geqslant 2n - t$ for $1 \leqslant i \leqslant t$.*

A family $\mathcal{L}$ of sets is said to be a *laminar* set if $X, Y \in \mathcal{L}$ implies that $X \subseteq Y$, $Y \subseteq X$, or $X \cap Y = \emptyset$. Nash-Williams [9] proved the following result which will play a critical role in Step 3 of the proof of Theorem 4.

Theorem 3 ([9]). *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are laminar sets of subsets of a finite set M, then for each integer $h > 0$ there exists $J \subseteq M$ such that*

$$\left\lfloor \frac{|Z|}{h} \right\rfloor \leqslant |J \cap Z| \leqslant \left\lceil \frac{|Z|}{h} \right\rceil$$

for every $Z \in \mathcal{L}_1 \cup \mathcal{L}_2$.

3 Main Result

In the following proof, $f(i)$ will be modified in various ways. With this in mind, the incomplete latin square L of order n is said to be (f,t)-*satisfied* if $N_L(i) \geqslant 2n - t + f(i)$ for $1 \leqslant i \leqslant t$. We say a symbol i satisfies *Ryser's condition* if $N_L(i) \geqslant 2n - t + f(i)$.

Theorem 4. *Let $t \geqslant n > 0$. Let L be an incomplete latin square of order n on the symbols in $S(t)$. Let $f : S(t) \mapsto \mathbb{N}$ such that $\sum_{i=1}^{n} f(i) = t - n$ and $f(i) \neq 1$ for $1 \leqslant i \leqslant t$. Then L can be embedded in a latin square T of order t on the same symbols in which each symbol i appears $f(i)$ times on the diagonal of T outside L if and only if $N_L(i) \geqslant 2n - t + f(i)$ for $1 \leqslant i \leqslant t$.*

Proof. The necessity is well known, so assume that $N_L(i) \geqslant 2n - t + f(i)$ for $1 \leqslant i \leqslant t$.

Suppose there exists a symbol α for which $f(\alpha) \geqslant 3$. Let $f'(\alpha) = f(\alpha) - 1$ and $f'(i) = f(i)$ for $1 \leqslant i \leqslant t$, $i \neq \alpha$. Thus $\sum_{i=1}^{t} f'(i) = t - n - 1$. Then by Theorem 1, L can be embedded in a latin square T' of order t in which for $1 \leqslant i \leqslant t$, symbol i occurs at least $f'(i)$ times on the diagonal of T' outside L. By a permutation of rows and columns if needed, assume $T'(n+1, n+1) = \alpha$. Define the incomplete latin square L' of order $n+1$ by $L'(a,b) = T'(a,b)$ for $1 \leqslant a, b \leqslant n+1$. We now show that L' is (f', t)-satisfied. Because $T'(n+1, n+1) = \alpha$,

$$\begin{aligned} N_{L'}(\alpha) &= N_L(\alpha) + 1 \\ &\geqslant 2n - t + f(\alpha) + 1 \\ &= 2(n+1) - t + f'(\alpha). \end{aligned}$$

Also, since L' is embedded in T', by the necessary condition in Theorem 1, $N_{L'}(i) \geqslant 2(n+1) - t + f'(i)$ for $1 \leqslant i \leqslant t$, $i \neq \alpha$. Thus L' is an incomplete latin square of order $n+1$ satisfying the conditions of the theorem. Therefore, by repeating this process, we can assume that $f(i) \in \{0, 2\}$ for $1 \leqslant i \leqslant t$; so $t - n$ is even.

The remainder of the proof is completed in three steps. In each step, two rows and columns are added so that the resulting incomplete latin square satisfies the necessary condition after appropriately modifying f to allow for the symbol placed in both the added diagonal cells.

Step 1. Suppose $t - n = 2$. Then $f(\alpha) = 2$ for exactly one symbol α, and $f(i) = 0$ for all symbols $i \neq \alpha$. By assumption, $N_L(\alpha) \geqslant 2n - t + f(\alpha) = 2n - (n+2) + 2 = n$. Because L is of order n, $N_L(\alpha) = n$. Use Theorem 2 to embed L in a latin square T of order t. Because $N_L(\alpha) = n$, symbol α must appear twice in the 2×2 square formed with rows and columns $t-1$ and t of T. If α is on the diagonal, we are done. If not, then permute columns $t-1$ and t to obtain the required embedding. Thus we can assume $t - n \geqslant 4$.

Step 2. Suppose $t - n \geqslant 8$. Let $s = (t-n)/2$. By renaming symbols, we can assume that $f(i) = 2$ for $1 \leqslant i \leqslant s$ and $f(i) = 0$ for $s+1 \leqslant i \leqslant t$. We wish to extend L by 2 rows and 2 columns embedding L in a latin square of order $n+2$ that satisfies the conditions of the theorem. Define $f'(i) = 2$ for $1 \leqslant i \leqslant s-1$, $f'(s) = 1$, and $f'(i) = 0$ for $s+1 \leqslant i \leqslant t$. So, $\sum_{i=1}^{t} f'(i) = t - n - 1$. Thus by Theorem 1 and

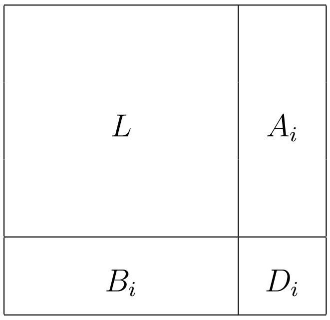

Figure 1: L_i

a permutation of rows and columns if needed, we can embed L in a latin square T' of order t with $T'(n+2i-1, n+2i-1) = i = T'(n+2i, n+2i)$ for $1 \leqslant i \leqslant s-1$ and $T'(n+2s-1, n+2s-1) = s$. (So at this stage we do not know what symbol appears in cell (t,t).) Define the sets of cells $\mathcal{A}_i$, $\mathcal{B}_i$ and $\mathcal{D}_i$ for $1 \leqslant i \leqslant s-1$ as follows:

$$\begin{aligned}
\mathcal{A}_i &:= \{(a,b) : 1 \leqslant a \leqslant n, n+2i-1 \leqslant b \leqslant n+2i\}, \\
\mathcal{B}_i &:= \{(a,b) : n+2i-1 \leqslant a \leqslant n+2i, 1 \leqslant b \leqslant n\}, \text{ and} \\
\mathcal{D}_i &:= \{(a,b) : n+2i-1 \leqslant a, b \leqslant n+2i\}.
\end{aligned}$$

Let A_i, B_i, and D_i be the $n \times 2$, $2 \times n$, and 2×2 latin subrectangles of T' formed by the cells in $\mathcal{A}_i$, $\mathcal{B}_i$, and $\mathcal{D}_i$ respectively. Similarly, let $A_i \cup B_i \cup D_i$ be the array formed by the cells in $\mathcal{A}_i$, $\mathcal{B}_i$, and $\mathcal{D}_i$. For $1 \leqslant i \leqslant s-1$, let L_i be the incomplete latin square of order $n+2$ depicted in Figure 1. We now have $s-1$ candidates for extending L by two rows and two columns, namely $L_1, \ldots, L_{s-1}$. We now show that at least one of them must satisfy the necessary conditions of the theorem. (It is only symbol s that is potentially problematic because $f'(s) \neq f(s)$. However, we show for at least one value of i, $1 \leqslant i \leqslant s-1$, s appears the necessary number of times in $A_i \cup B_i \cup D_i$, so L_i meets the necessary conditions of the theorem.)

Suppose $1 \leqslant i \leqslant s-1$. Permute the rows and columns of T' to produce a latin square T_i such that L_i is embedded in T_i and for $1 \leqslant j \leqslant t$, $j \neq i$, symbol j appears in at least $f'(j)$ diagonal cells of T_i outside L_i. Define $f_i(j) = f(j)$ for $1 \leqslant j \leqslant t$, $j \neq i$, and define $f_i(i) = f(i) - 2 = 0$. Because i appears 2 more times on the diagonal of L_i than it did in L,

$$\begin{aligned}
N_{L_i}(i) &= N_L(i) + 2 \\
&\geqslant 2n - t + f(i) + 2 \\
&= 2n - t + (f_i(i) + 2) + 2 \\
&= 2(n+2) - t + f_i(i).
\end{aligned}$$

Since L_i is embedded in T_i, by the necessity of Theorem 1, for $1 \leqslant j \leqslant t, j \notin \{i, s\}$,

$$\begin{aligned}
N_{L_i}(j) &\geqslant 2(n+2) - t + f'(j) \\
&= 2(n+2) - t + f(j) \\
&= 2(n+2) - t + f_i(j).
\end{aligned}$$

Also, by the necessity of Theorem 1,

$$\begin{aligned}N_{L_i}(s) &\geqslant 2(n+2)-t+f'(s)\\ &= 2(n+2)-t+(f(s)-1)\\ &= 2(n+2)-t+f_i(s)-1.\end{aligned}$$

We claim that for some i, $1 \leqslant i \leqslant s-1$, s satisfies Ryser's condition in L_i, so in actuality $N_{L_i}(s) \geqslant 2(n+2)-t+f_i(s)$. Assume otherwise; so for all i, $1 \leqslant i \leqslant s-1$, assume that $N_{L_i}(s) = 2(n+2)-t+f_i(s)-1 = 2(n+2)-t+1$. But then,

$$\begin{aligned}\sum_{i=1}^{s-1} N_{L_i}(s) &= (s-1)(2(n+2)-t+1)\\ &= (s-1)(2n-t+5).\end{aligned}$$

Symbol s appears n times in the first n rows of T_i (by the definition of a latin square), but does not appear in the $(t-1)^{th}$ column of the first n rows because symbol s appears on the diagonal in that column. Symbol s could possibly appear in the t^{th} column of the first n rows. Thus $N_L(s)+\sum_{i=1}^{s-1} N_{A_i}(s) \geqslant n-1$. Similarly, $N_L(s)+\sum_{i=1}^{s-1} N_{B_i}(s) \geqslant n-1$. Therefore,

$$\begin{aligned}\sum_{i=1}^{s-1} N_{L_i}(s) &= \sum_{i=1}^{s-1}(N_L(s)+N_{A_i}(s)+N_{B_i}(s)+N_{D_i}(s))\\ &\geqslant (s-3)N_L(s)+(n-1)+(n-1)+\sum_{i=1}^{s-1} N_{D_i}(s)\\ &\geqslant (s-3)N_L(s)+(n-1)+(n-1),\end{aligned}$$

implying

$$\begin{aligned}(s-3)N_L(s) &\leqslant (s-1)(2n-t+5)-2n+2\\ &= (s-3)(2n-t+5)+4n-2t+10-2n+2\\ &= (s-3)(2n-t+5)-4s+12\\ &= (s-3)(2n-t+1).\end{aligned}$$

So, because $s \geqslant 4$, $N_L(s) \leqslant (2n-t+1)$, contradicting our original assumption. Therefore, for some value of i, $1 \leqslant i \leqslant s-1$, say $i=\alpha$, $N_{L_\alpha}(s) \geqslant 2(n+2)-t+f_\alpha(s)$. Also, as already stated, $N_{L_\alpha}(j) \geqslant 2(n+2)-t+f_\alpha(j)$ for $1 \leqslant j \leqslant t, j \neq s$. Thus L_α is an incomplete latin square of order $n+2$ that is (f_α, t)-satisfied and thus satisfies the conditions of the theorem. By repeating this process, we may now assume $t-n \leqslant 6$.

Step 3. Suppose $t-n \in \{4,6\}$. Form a bipartite multigraph G_c^* with bipartition $C=\{c_1,c_2,\ldots,c_n,c^*\}$ and $S=\{\sigma_1,\sigma_2,\ldots,\sigma_t\}$ of the vertex set as follows. For $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant t$, join c_i to σ_j if and only if symbol j is missing from column i of L and join c^* to σ_j with $f(j)$ edges. Similarly, form a bipartite multigraph G_ρ^* with bipartition

$R = \{\rho_1, \rho_2, \ldots, \rho_n, \rho^*\}$ and $S' = \{\sigma'_1, \sigma'_2, \ldots, \sigma'_t\}$ of the vertex set as follows. For $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant t$, join ρ_i to σ'_j if and only if symbol j is missing from row i of L, and join ρ^* to σ'_j with $f(j)$ edges. Because each column and row of L contains n symbols, for $1 \leqslant i \leqslant n$,

$$\deg_{G_c^*}(c_i) = \deg_{G_\rho^*}(\rho_i) = t - n. \tag{1}$$

Because $\sum_{i=1}^{n} f(i) = t - n$,

$$\deg_{G_c^*}(c^*) = \deg_{G_\rho^*}(\rho^*) = t - n. \tag{2}$$

For $1 \leqslant j \leqslant t$, symbol j is missing from $n - N_L(j)$ rows of L and $n - N_L(j)$ columns of L, so

$$\deg_{G_c^*}(\sigma_j) = \deg_{G_\rho^*}(\sigma'_j) = n - N_L(j) + f(j). \tag{3}$$

For $1 \leqslant j \leqslant t$, let $z(j) = N_L(j) - (2n - t + f(j))$. So $0 \leqslant z(j) \leqslant n - (2n - t + f(j)) = t - n - f(j)$. Thus, by (3),

$$\begin{aligned} \deg_{G_c^*}(\sigma_j) = \deg_{G_{\rho^*}}(\sigma'_j) &= n - N_L(j) + f(j) \\ &= n - (2n - t + f(j) + z(j)) + f(j) \\ &= t - n - z(j) \\ &\leqslant t - n. \end{aligned} \tag{4}$$

So, $\Delta(G_{c^*}) = \Delta(G_{\rho^*}) = t - n$ and $z(j)$ measures how far σ_j and σ'_j are from this maximum degree.

Define a laminar set $\mathcal{L}_1$ of subsets of $E(G_c^*) \cup E(G_\rho^*)$ as follows. For $1 \leqslant i \leqslant n$, let $C_i \in \mathcal{L}_1$ be the set of edges incident to c_i. Let $C^* \in \mathcal{L}_1$ be the set of edges incident to c^*. For $1 \leqslant j \leqslant t$ such that $f(j) > 0$, let $C_j^* \in \mathcal{L}_1$ be the subset of C^* given by the two element set of the pair of edges joining c^* and σ_j. Similarly, for $1 \leqslant i \leqslant n$, let $R_i \in \mathcal{L}_1$ be the set of edges incident to ρ_i. Let $R^* \in \mathcal{L}_1$ be the set of edges incident to ρ^*. For $1 \leqslant j \leqslant t$ such that $f(j) > 0$, let $R_j^* \in \mathcal{L}_1$ be the subset of R^* given by the two element set of the pair of edges joining ρ^* and σ_j. Define a second laminar set $\mathcal{L}_2$ of subsets of $E(G_c^*) \cup E(G_\rho^*)$ as follows. For $1 \leqslant j \leqslant t$, let $S_j \in \mathcal{L}_2$ be the set of edges incident to σ_j, $S'_j \in \mathcal{L}_2$ be the set of edges incident to σ'_j, and $\Sigma_j \in \mathcal{L}_2$ be the set of all edges incident to either σ_j or σ'_j. By Theorem 3, there exists a set $J \subseteq (E(G_c^*) \cup E(G_\rho^*))$ for which

$$\left\lfloor \frac{|Z|}{(t-n)/2} \right\rfloor \leqslant |J \cap Z| \leqslant \left\lceil \frac{|Z|}{(t-n)/2} \right\rceil$$

for every $Z \in (\mathcal{L}_1 \cup \mathcal{L}_2)$.

Let G_J be the graph induced by the edges of G_c^* and G_ρ^* in J. Later, a modified version of G_J will be colored with 2 colors and be used to fill rows and columns $n+1$ and $n+2$ to embed L in an incomplete latin square of order $n+2$. But first we explore G_J to see what modifications are needed.

By (1), for $1 \leqslant i \leqslant n$, $\deg_{G_c^*}(c_i) = \deg_{G_\rho^*}(\rho_i) = t - n$; so, because $C_i, R_i \in \mathcal{L}_1$, $\deg_{G_J}(c_i) = \deg_{G_J}(\rho_i) = 2$. By (2), $\deg_{G_c^*}(c^*) = \deg_{G_\rho^*}(\rho^*) = t - n$; so, because $C^*, R^* \in$

$\mathcal{L}_1$, $\deg_{G_J}(c^*) = \deg_{G_J}(\rho^*) = 2$. By (4), $\deg_{G_c^*}(\sigma_j) = \deg_{G_\rho^*}(\sigma'_j) = t-n-z(j)$; so, because $S_j, S'_j \in \mathcal{L}_2$, $\deg_{G_J}(\sigma_j) \leqslant \lceil 2 - \frac{2z(j)}{t-n} \rceil \leqslant 2$ and $\deg_{G_J}(\sigma'_j) \leqslant \lceil 2 - \frac{2z(j)}{t-n} \rceil \leqslant 2$. Also, because $\Sigma_j \in \mathcal{L}_2$, $\deg_{G_J}(\sigma_j) + \deg_{G_J}(\sigma'_j) \geqslant \lfloor 4 - \frac{4z(j)}{t-n} \rfloor \geqslant 4 - z(j)$. So, for $1 \leqslant j \leqslant t$,

$$\begin{aligned} N_L(j) + \deg_{G_J}(\sigma_j) + \deg_{G_J}(\sigma'_j) &\geqslant (2n - t + f(j) + z(j)) + (4 - z(j)) \\ &= 2(n+2) - t + f(j). \end{aligned} \tag{5}$$

Recall, $\deg_{G_J}(c^*) = 2$. Because $C_j^* \in \mathcal{L}_1$ at most one edge $\{c^*, \sigma_j\} \in C_j^*$ is in J. So, the two edges in J incident to c^* are incident to two different vertices in S. Similarly, there are exactly two edges in J incident to ρ^*, each of which is incident to two different vertices in S'. Because c^* and ρ^* are incident to σ_j and σ'_j respectively for the same two (if $t-n=4$) or three (if $t-n=6$) values of j, there must exist an α such that $1 \leqslant \alpha \leqslant t$, $\{c^*, \sigma_\alpha\} \in J$ and $\{\rho^*, \sigma'_\alpha\} \in J$.

In what follows we construct another set of edges J' through a modest modification of J so both edges in C_α^* and both edges in R_α^* will be in J'. The graph G_J is a bipartite graph with maximum degree 2. Thus, the edges of G_J can be properly colored with 2 colors, say 1 and 2. Consider the graphs $G_c^* - J$ and $G_\rho^* - J$. They are bipartite graphs with maximum degree $t-n-2$. Thus, the edges of $G_c^* - J$ and $G_\rho^* - J$ can be properly colored with $t-n-2$ colors, say $3, \ldots, t-n$. These two edge-colorings naturally induce a proper $(t-n)$-edge-coloring of $G_c^* \cup G_\rho^*$, $X : E(G_c^* \cup G_\rho^*) \to \{1, 2, \ldots, t-n\}$, in which all edges in J are colored 1 or 2.

In what follows we construct an edge-coloring $X' : E(G_c^* \cup G_\rho^*) \to \{1, 2, \ldots, t-n\}$ by interchanging colors on two 2-colored trails, T_1 and T_2, in X. In X' the edges in C_α will be colored 1 and 2 and the edges in R_α will be colored 1 and 2. Suppose the edges in C_α are colored 1 and 3 by X. Consider the maximal trail, T_1, containing the edge $\{c^*, \sigma_\alpha\}$ colored 3, in which the edges are alternately colored 2 and 3 by X. Because the edge-coloring is proper, T_1 is either a cycle or a path. Interchange the colors on T_1 and let this new edge-coloring be X' on the edges in G_c^*. The edges in C_α are now colored 1 and 2 by X'. If T_1 is a cycle, interchanging colors did not impact the number of edges of each color incident to each vertex. Suppose T_1 is a path. Interchanging colors did not impact the number of edges of each color incident to each vertex in the interior of T_1, but did impact the endpoints. For each $c \in C$, by (1) and (2), $\deg(c) = t-n$. So there is exactly one edge colored 2 and one edge colored 3 by X incident to vertex c. Thus c is not an endpoint of T_1, so the endpoints of T_1 must be in S. Because G_c^* is bipartite and both ends of T_1 are in S, exactly one of the ends was incident to an edge colored 2 by X. This end cannot be σ_α because σ_α was incident to an edge colored 3 by X. So one end of T_1 is a vertex $\sigma_u \in S \setminus \{\sigma_\alpha\}$ that now does not have an edge colored 2 by X' incident to it. The other end of T_1 was incident to an edge colored 3 by X. So this vertex now is incident to an edge colored 2 by X'. Similarly, we can use a trail T_2 to modify the proper edge-coloring, X, of G_ρ^* and define X' on the edges of G_ρ^* so the edges in R_α are now colored 1 and 2 in X'. After recoloring, at most one vertex in $S' \setminus \{\sigma'_\alpha\}$ has lost an edge colored 2 incident to it in X'. If such a vertex exists, name it σ'_v. All other vertices in $G_c^* \cup G_\rho^*$ have an equal or greater number of edges colored 2 incident to them. Thus, we have the revised edge-coloring $X' : E(G_c^* \cup G_\rho^*) \to \{1, 2, \ldots, t-n\}$. Define J' to be the set of edges colored 1 and 2 by X', and let $G_{J'}$ be the graph induced by the edges in J'.

It is important to note a property that will be used later in the proof if σ_u and/or σ'_v have been defined. In X', σ_u and σ'_v do not have an edge colored 2 incident to them, so $\deg_{G_c^*}(\sigma_u) = \deg_{G_\rho^*}(\sigma'_u) < t-n$ and $\deg_{G_\rho^*}(\sigma'_v) = \deg_{G_c^*}(\sigma_v) < t-n$. So, for each $j \in \{u, v\}$, by (3),

$$N_L(j) \geqslant 2n - t + f(j) + 1. \tag{6}$$

To finally arrive at the desired edge-coloring, $X'' : E(G_c^* \cup G_\rho^*) \to \{1, 2, \ldots, t-n\}$, X' is to be modified in the situation where $\deg_{G_{J'}}(\sigma_u) = \deg_{G_{J'}}(\sigma_v) = 1$ and/or $\deg_{G_{J'}}(\sigma'_u) = \deg_{G_{J'}}(\sigma'_v) = 1$ (as in Case 3 below). To do this we construct the edge-coloring X'' by interchanging colors on up to two trails, T_3 and T_4, whose edges are colored 1 and 2 in X' as follows.

We first define X'' on $E(G_c^*)$. Suppose $\deg_{G_{J'}}(\sigma_u) = \deg_{G_{J'}}(\sigma_v) = 1$. Let e_u and e_v be the edges incident to σ_u and σ_v in $G_{J'}$ respectively. In X'', we make sure that one of e_u and e_v is colored 1 and the other is colored 2. If $X'(e_u) \neq X'(e_v)$, then we already have the desired property, so define $X''(e) = X'(e)$ for all $e \in E(G_c^*)$. Otherwise, $X'(e_u) = X'(e_v)$. Take a maximal trail, T_3, in $G_{J'}$ that begins with e_u. Because $\Delta(G_{J'}) = 2$, the trail T_3 is necessarily a path. For each $c \in C$, $\deg_{G_{J'}}(c) = 2$. Thus c is not an endpoint of T_3, so the endpoint of T_3 must be in S. The path T_3 cannot end with e_v because $X'(e_u) = X'(e_v)$ and $G_{J'}$ is a bipartite graph. So, because $\deg_{G_{J'}}(\sigma_v) = 1$, T_3 does not include e_v. Interchange colors along T_3. All interior vertices of T_3 still have exactly one incident edge colored 1 and exactly one incident edge colored 2. The endpoints of T_3 now have the opposite color incident to them. Define X'' on the edges in G_c^* to be this new edge-coloring. Thus, in any case, if e_u and e_v exists, we can assume

$$X''(e_u) \neq X''(e_v). \tag{7}$$

For convenience, if $\deg_{G_{J'}}(\sigma_u) \neq 1$ or $\deg_{G_{J'}}(\sigma_v) \neq 1$, so we are not in the above case, then define $X''(e) = X'(e)$ for all $e \in E(G_c^*)$.

Similarly, we can define X'' on the edges in G_ρ^* by interchanging colors on a trail T_4 in X' if needed. So, if $\deg_{G_{J'}}(\sigma'_u) = 1$ and $\deg_{G_{J'}}(\sigma'_v) = 1$ and we let e'_u and e'_v be the edges incident to σ'_u and σ'_v in $G_{J'}$ respectively, then

$$X''(e'_u) \neq X''(e'_v). \tag{8}$$

For convenience, if $\deg_{G_{J'}}(\sigma'_u) \neq 1$ or $\deg_{G_{J'}}(\sigma'_v) \neq 1$, so we are not in the above case, then define $X''(e) = X'(e)$ for all $e \in E(G_\rho^*)$.

Thus, we have the revised edge-coloring $X'' : E(G_c^* \cup G_\rho^*) \to \{1, 2, \ldots, t-n\}$. Define J'' to be the set of edges colored 1 and 2 by X'', and let $G_{J''}$ be the graph induced by the edges in J''.

We will use $G_{J''}$ to fill in rows and columns $n+1$ and $n+2$ and extend L to an incomplete latin square of order $n+2$. For $1 \leqslant i \leqslant t$, $\deg_{G_{J''}}(c_i) = \deg_{G_{J''}}(\rho_i) = 2$. For

$1 \leqslant j \leqslant t$, $\deg_{G_{J''}}(\sigma_j) \leqslant 2$ and $\deg_{G_{J''}}(\sigma'_j) \leqslant 2$. Also, for $1 \leqslant j \leqslant t$, by (5),

$$\begin{aligned} N_L(j) &+ \deg_{G_{J''}}(\sigma_j) + \deg_{G_{J''}}(\sigma'_j) \\ &\geqslant N_L(j) + \deg_{G_J}(\sigma_j) + \deg_{G_J}(\sigma'_j) - \epsilon_j \\ &\geqslant 2(n+2) - t + f(j) - \epsilon_j, \end{aligned} \tag{9}$$

where $\epsilon_j = 0$ if $j \notin \{u, v\}$, $\epsilon_j = 1$ if $j \in \{u, v\}$ and $u \neq v$, and $\epsilon_j = 2$ if $j = u = v$.

Form a partial incomplete latin square L_α of order $n+2$ by adding two new rows and columns to L using J'' as follows. Let $L_\alpha(a,b) = L(a,b)$ for $1 \leqslant a \leqslant n$ and $1 \leqslant b \leqslant n$. Define the sets of cells $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ as follows:

$$\begin{aligned} &\mathcal{A} := \{(a,b) : 1 \leqslant a \leqslant n, n+1 \leqslant b \leqslant n+2\}, \\ &\mathcal{B} := \{(a,b) : n+1 \leqslant a \leqslant n+2, 1 \leqslant b \leqslant n\}, \text{ and} \\ &\mathcal{D} := \{(a,b) : n+1 \leqslant a, b \leqslant n+2\}. \end{aligned}$$

Let A, B, and D be the $n \times 2$, $2 \times n$, and 2×2 latin subrectangles of L_α formed by the cells in $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ respectively. Similarly, let $A \cup B$ be the array formed by the cells in $\mathcal{A}$ and $\mathcal{B}$. We now fill A and B using $G_{J''}$. For $1 \leqslant k \leqslant 2$ and $1 \leqslant i \leqslant n$, let $L_\alpha(n+k, i) = j$ if and only if $\{c_i, \sigma_j\}$ is colored k in $G_{J''}$. Similarly, for $1 \leqslant k \leqslant 2$ and $1 \leqslant i \leqslant n$, let $L_\alpha(i, n+k) = j$ if and only if $\{\rho_i, \sigma'_j\}$ is colored k in $G_{J''}$. Every cell in $A \cup B$ is filled because $\deg_{G_{J''}}(c_i) = \deg_{G_{J''}}(\rho_i) = 2$ for $1 \leqslant i \leqslant n$. So, for $1 \leqslant j \leqslant t$, by (9),

$$N_{L_\alpha}(j) \geqslant 2(n+2) - t + f(j) - \epsilon_j. \tag{10}$$

In a later modification of L_α we will place α in the two new diagonal cells, so define $f'(\alpha) = f(\alpha) - 2$ and $f'(j) = f(j)$ for $1 \leqslant j \leqslant t$ and $j \neq \alpha$. For $1 \leqslant j \leqslant t$ and $j \notin \{u, v, \alpha\}$, by (10),

$$N_{L_\alpha}(j) \geqslant 2(n+2) - t + f'(j). \tag{11}$$

Thus, L_α is a partial incomplete latin square of order $n+2$ with all cells except those in D filled and all symbols satisfying Ryser's condition except possibly α (which will satisfy Ryser's condition once placed twice on the diagonal in D) and possibly u and v if they exist. The aim is to construct L'_α through a modest modification of L_α to form a partial incomplete latin square of order $n+2$ on $S(t)$ which is (f', t)-satisfied.

By (10), $N_{L_\alpha}(u) \geqslant 2(n+2) - t + f'(u) - \epsilon_u$ and $N_{L_\alpha}(v) \geqslant 2(n+2) - t + f'(v) - \epsilon_v$. We will now modify L_α to form L'_α so that if $N_{L_\alpha}(j) < 2(n+2) - t + f'(j)$ for any $j \in \{u, v\}$, u and/or v will be placed in cells $(n+1, n+2)$ and/or $(n+2, n+1)$ of L'_α as needed to ensure that $N_{L'_\alpha}(u) \geqslant 2(n+2) - t + f'(u)$ and $N_{L'_\alpha}(v) \geqslant 2(n+2) - t + f'(v)$.

Let $j \in \{u, v\}$. If $N_{L_\alpha}(j) < 2(n+2) - t + f'(j)$, by (10), $N_{L_\alpha}(j) = 2n - t + f'(j) + 2$ or $N_{L_\alpha}(j) = 2n - t + f'(j) + 3$. These two cases of $N_{L_\alpha}(j)$ correspond to L'_α needing 2 or 1 more occurrence of j respectively. To reveal more about these potentially problematic cases, consider the following properties. Recall, by (6), $N_L(j) \geqslant 2n - t + f(j) + 1$.

(i) Suppose $N_{L_\alpha}(u) < 2(n+2) - t + f'(u)$ and $N_L(u) = 2n - t + f(u) + 1$. By (3), $\deg_{G^*_c}(\sigma_u) = n - N_L(u) + f(u) = t - n - 1$. In the edge-coloring X'', σ_u is missing exactly one of the colors 1 or 2 (because T_1 ended on this vertex). So, since σ_u is incident to an edge of every color in G_c^* except one, σ_u must be incident to the other color (1 or 2). Therefore, $\deg_{G_{J''}}(\sigma_u) = 1$, and so because u appears at most 2 times (because $N_{L_\alpha}(u) - N_L(u) \leqslant 2$) in $A \cup B$, $\deg_{G_{J''}}(\sigma'_u) \leqslant 1$.

(ii) Similarly, if $N_{L_\alpha}(v) < 2(n+2) - t + f'(v)$ and $N_L(v) = 2n - t + f(v) + 1$, then $\deg_{G_{J''}}(\sigma'_v) = 1$ and $\deg_{G_{J''}}(\sigma_v) \leqslant 1$.

(iii) If $N_L(u) = 2n - t + f(u) + 1$, $N_{L_\alpha}(u) < 2(n+2) - t + f'(u)$, and $u = v$, then by (i-ii), $\deg_{G_{J''}}(\sigma_u) = 1 = \deg_{G_{J''}}(\sigma'_u)$. So, $N_{L_\alpha}(u) = 2n - t + f'(u) + 3$.

(iv) If $j \in \{u, v\}$, $N_{L_\alpha}(j) < 2(n+2) - t + f'(j)$, and $2n - t + f(j) + 2 \leqslant N_L(j) \leqslant 2n - t + f(j) + 3$, then j appears at most 1 time (because $N_{L_\alpha}(u) - N_L(u) \leqslant 1$) in $A \cup B$.

If $N_{L_\alpha}(j) \geqslant 2(n+2) - t + f'(j)$ for $j \in \{u, v\}$, then let $L'_\alpha(a,b) = L_\alpha(a,b)$ for $1 \leqslant a, b \leqslant n+2$. Otherwise we will make use of (i-iv) to modify L_α to define L'_α and place u and/or v in cells $(n+1, n+2)$ and/or $(n+2, n+1)$ of L'_α as needed to ensure that, for $j \in \{u, v\}$, $N_{L'_\alpha}(j) \geqslant 2(n+2) - t + f'(j)$. The following three cases are considered. The first two cases consider if exactly one of u or v, say u, does not meet Ryser's condition. So, by (10), $N_{L_\alpha}(u) = 2(n+2) - t + f'(u) - 2$ or $N_{L_\alpha}(u) = 2(n+2) - t + f'(u) - 1$. The third case considers when both u and v do not meet Ryser's Condition, so, by (10), $N_{L_\alpha}(u) = 2(n+2) - t + f'(u) - 1$ and $N_{L_\alpha}(v) = 2(n+2) - t + f'(v) - 1$.

Case 1: Suppose $N_{L_\alpha}(u) = 2(n+2) - t + f'(u) - 2$. Thus, by (10), $u = v$. By (iii), $N_L(u) = 2n - t + f(u) + 2$. Then u does not appear in A nor in B. Define $L'_\alpha(a,b) = L_\alpha(a,b)$ for $(a,b) \in A \cup B$ and for $1 \leqslant a, b \leqslant n$. Also, define $L'_\alpha(n+1, n+2) = L'_\alpha(n+2, n+1) = u$. Thus $N_{L'_\alpha}(u) = N_{L_\alpha}(u) + 2 = 2(n+2) - t + f'(u)$.

Case 2: Suppose that for exactly one of u or v, say u, $N_{L_\alpha}(u) = 2(n+2) - t + f'(u) - 1$ and for the other, say v, $N_{L_\alpha}(v) \geqslant 2(n+2) - t + f'(v)$, $u = v$, or v does not exist. By (i,iii,iv), u is in at most one row of B, say $n+2$, and at most one column of A, say $n+1$ (permuting the columns and/or rows of A and/or B respectively if need be). Define $L'_\alpha(a,b) = L_\alpha(a,b)$ for $(a,b) \in A \cup B$ or $1 \leqslant a, b \leqslant n$. Also, define $L'_\alpha(n+1, n+2) = u$. Thus $N_{L'_\alpha}(u) = N_{L_\alpha}(u) + 1 = 2(n+2) - t + f'(u)$.

Case 3: Suppose $u \neq v$, $N_{L_\alpha}(u) = 2(n+2) - t + f'(u) - 1$, and $N_{L_\alpha}(v) = 2(n+2) - t + f'(v) - 1$. By (i-ii, iv), u and v each appear at most once in A and at most once in B. By (7) and (8), we can assume u and v appear in different rows of B and different columns of A. Thus, permuting rows and/or columns if necessary we can assume u does not appear in row $n+1$ nor in column $n+2$ of L_α and v does not appear in row $n+2$ nor in column $n+1$ of L_α. Define $L'_\alpha(a,b) = L_\alpha(a,b)$ for $(a,b) \in A \cup B$ or $1 \leqslant a, b \leqslant n$. Also, define $L'_\alpha(n+1, n+2) = u$ and $L'_\alpha(n+2, n+1) = v$. So $N_{L'_\alpha}(u) = N_{L_\alpha}(u) + 1 = 2(n+2) - t + f'(u)$ and $N_{L'_\alpha}(v) = N_{L_\alpha}(v) + 1 = 2(n+2) - t + f'(v)$.

Thus, in any case, we can place u and/or v in cells $(n+1, n+2)$ and/or $(n+2, n+1)$ if needed so that for $j \in \{u, v\}$,

$$N_{L'_\alpha}(j) \geqslant 2(n+2) - t + f'(j).$$

Define $L'_\alpha(n+1, n+1) = L'_\alpha(n+2, n+2) = \alpha$. So

$$N_{L_\alpha}(\alpha) = N_L(\alpha) + 2 \geqslant 2(n+2) - t + f'(j).$$

Also for $1 \leqslant j \leqslant t$ and $j \notin \{u, v, \alpha\}$, $N_{L'_\alpha}(j) = N_{L_\alpha}(j)$, so by (11),

$$\begin{aligned} N_{L'_\alpha}(j) &= N_{L_\alpha}(j) \\ &\geqslant 2(n+2) - t + f'(j). \end{aligned}$$

Thus, L'_α is a partial incomplete latin square of order $n+2$ with all symbols satisfying Ryser's condition and all cells filled except possibly cells $(n+1, n+2)$ and $(n+2, n+1)$.

We now define L''_α through a modest modification of L'_α to fill cells $(n+1, n+2)$ and/or $(n+2, n+1)$ if needed to form an (f', t)-satisfied incomplete latin square. Suppose cell $(n+1, n+2)$ of L'_α is empty. Form the bipartite graph B with bipartition $C' = \{c_i \mid 1 \leqslant i \leqslant n+2\}$ and $S = \{\sigma_j \mid 1 \leqslant j \leqslant t\}$ of the vertex set as follows. For $1 \leqslant i \leqslant n+2$ and $1 \leqslant j \leqslant t$, join c_i to σ_j if and only if symbol j is missing from column i of L'_α or $L'_\alpha(n+1, i) = j$. For $c_i \in C' \setminus \{c_{n+1}\}$, $\deg_B(c_i) = t - n - 1$. Because j appears at most once in row $n+1$, for $\sigma_j \in S$,

$$\begin{aligned} \deg_B(\sigma_j) &\leqslant n + 2 - (N_{L'_\alpha}(j) - 1) \\ &\leqslant n + 2 - (2(n+2) - t + f'(j) - 1) \\ &= t - n - f'(j) - 1 \\ &\leqslant t - n - 1. \end{aligned} \tag{12}$$

Define the matching M by letting $\{c_i, \sigma_j\} \in E(B)$ be in M if and only if $L'_\alpha(n+1, i) = j$. Because symbol α appears in cell $(n+1, n+1)$, $\{c_{n+1}, \sigma_\alpha\} \in M$. Let B' be the induced subgraph of B formed by removing vertices c_{n+1} and σ_α. We wish to find an M-augmenting path in B' starting at c_{n+2}. For a contradiction, suppose there does not exist an M-augmenting path in B' starting at c_{n+2}. Let W be the subgraph of B' induced by the set of vertices that can be reached by an M-alternating path starting at c_{n+2}. All maximal M-alternating paths starting at c_{n+2} end at an M-saturated vertex in $C' \setminus \{c_{n+1}\}$. So $V(W)$ contains say x vertices from $S \setminus \{\sigma_\alpha\}$ and $V(W)$ contains $x+1$ vertices from $C' \setminus \{c_{n+1}\}$, namely c_{n+2} and the M-neighbors of the x vertices from $S \setminus \{\sigma_\alpha\}$. Let $C'_W = C' \cap V(W)$ denote the set of these $x+1$ vertices. By the definition of W, every edge in B' incident to a vertex in C'_W must be an edge in W (which implies the equality in the relations below). Because $\deg_B(\sigma_\alpha) \leqslant t - n - 1$ (by (12)) and σ_α is adjacent to c_{n+1} in B, at most $t - n - 2$ vertices in C'_W have degree $t - n - 2$ in B' and all other vertices in C'_W have degree $t-n-1$ in B'. So,

$$\begin{aligned}(t-n-1)x &\geqslant \sum_{\sigma_j \in V(W)} \deg_{B'}(\sigma_j)\\ &\geqslant e(W)\\ &= \sum_{c_i \in V(W)} \deg_{B'}(c_i)\\ &\geqslant (t-n-1)(x+1)-(t-n-2).\end{aligned}$$

This is a contradiction. Thus, there exists an M-augmenting path in B' starting at c_{n+2}. Form the matching M' from M by interchanging edges in M with edges not in M along this path. Now replace row $n+1$ of L'_α to form L''_α using M' by letting $L''_\alpha(n+1,i)=j$ if and only if $\{c_i,\sigma_j\}$ is in M' for $1 \leqslant i \leqslant n+2$ and $1 \leqslant j \leqslant t$ and letting $L''_\alpha(a,b)=L'_\alpha(a,b)$ for $1 \leqslant a \leqslant n$ or $a=n+2$ and $1 \leqslant b \leqslant n+2$. Thus, L''_α contains the same symbols as L'_α with the addition of one more symbol in row $n+1$. Now, all cells in row $n+1$ are filled. Similarly, if cell $(n+2,n+1)$ is empty, we can modify L''_α using the same approach. So we can assume all cells of L''_α are filled and all symbols satisfy Ryser's condition. Thus, L''_α is an incompete latin square of order $n+2$ that is (f',t)-satisfied and thus satisfies the conditions of the theorem. □

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References

[1] L. D. Andersen, R. Häggkvist, A. J. W. Hilton, and W. B. Poucher. Embedding incomplete Latin squares in Latin squares whose diagonal is almost completely prescribed. *European J. Combin.*, 1(1):5–7, 1980.

[2] L. D. Andersen, A. J. W. Hilton, and C. A. Rodger. A solution to the embedding problem for partial idempotent Latin squares. *J. London Math. Soc. (2)*, 26(1):21–27, 1982.

[3] L. D. Andersen, A. J. W. Hilton, and C. A. Rodger. Small embeddings of incomplete idempotent Latin squares. In *Combinatorial mathematics (Marseille-Luminy, 1981)*, volume 75 of *North-Holland Math. Stud.*, pages 19–31. North-Holland, Amsterdam, 1983.

[4] L.D. Andersen. *Latin squares and their generalizations.* PhD thesis, University of Reading, 1979.

[5] L.D. Andersen. Embedding Latin squares with prescribed diagonal. In *Algebraic and geometric combinatorics*, volume 65 of *North-Holland Math. Stud.*, pages 9–26. North-Holland, Amsterdam, 1982.

[6] A. B. Cruse. On embedding incomplete symmetric Latin squares. *J. Combinatorial Theory Ser. A*, 16:18–22, 1974.

[7] T. Evans. Embedding incomplete latin squares. *Amer. Math. Monthly*, 67:958–961, 1960.

[8] C. C. Lindner. Embedding partial idempotent Latin squares. *J. Combinatorial Theory Ser. A*, 10:240–245, 1971.

[9] C. St. J. A. Nash-Williams. Amalgamations of almost regular edge-colourings of simple graphs. *J. Combin. Theory Ser. B*, 43(3):322–342, 1987.

[10] C. A. Rodger. Embedding incomplete idempotent Latin squares. In *Combinatorial mathematics, X (Adelaide, 1982)*, volume 1036 of *Lecture Notes in Math.*, pages 355–366. Springer, Berlin, 1983.

[11] C. A. Rodger. Embedding an incomplete Latin square in a Latin square with a prescribed diagonal. *Discrete Math.*, 51(1):73–89, 1984.

[12] H. J. Ryser. A combinatorial theorem with an application to latin rectangles. *Proc. Amer. Math. Soc.*, 2:550–552, 1951.

[13] C. Treash. The completion of finite incomplete Steiner triple systems with applications to loop theory. *J. Combinatorial Theory Ser. A*, 10:259–265, 1971.

Corrigendum – Added May 19, 2023

The authors have noted the following typographic errors:

- Page 7, Line -12: Replace σ_j with σ'_j.
- Page 8, Line -21: Replace C_α with C^*_α.
- Page 8, Line -20: Replace R_α with R^*_α.
- Page 8, Line -19: Replace C_α with C^*_α.
- Page 8, Line -16: Replace C_α with C^*_α.
- Page 8, Line -4: Replace R_α with R^*_α.