On certain edge-transitive bicirculants of twice odd order

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Submitted: Oct 28, 2021; Accepted: Jul 5, 2022; Published: Aug 26, 2022

Abstract

A graph admitting an automorphism with two orbits of the same length is called a bicirculant. Recently, Jajcay et al. initiated the investigation of the edge-transitive bicirculants with the property that at least one of the subgraphs induced by the latter orbits is a cycle and the valence is at least 6 (Electron. J. Combin., 2019). We show that the complement of the Petersen graph is the only such graph whose order is twice an odd number.

Mathematics Subject Classifications: 05C25, 20B25

1 Introduction

All groups and graphs in this paper will be finite. A graph admitting an automorphism with two orbits of the same length is called a *bicirculant*. The symmetry properties of bicirculants have attracted considerable attention (see, e.g., [1, 4, 6, 14, 19, 20, 22, 27]). Recently, Jajcay et al. [10] initiated the investigation of the edge-transitive bicirculants with the property that at least one of the subgraphs induced by the two orbits of the semiregular automorphism is a cycle and the valence is at least 6. Motivated by this, we set the following notation.

Notation. For a positive integer $d \geqslant 3$, denote by $\mathcal{F}(d)$ the family of regular graphs having valence d and admitting an automorphism with two orbits of the same length such that at least one of the subgraphs induced by these orbits is a cycle.

*Partially supported by the Slovenian Research Agency (research program P1-0285, research projects N1-0062, J1-9108, J1-1695, J1-2451, N1-0140 and N1-0208).

†Partially supported by the ARRS-NKFIH Slovenian-Hungarian Joint Research Project, grant no. SNN 132625.

The graphs in the family $\mathcal{F}(3)$ are the well studied *generalised Petersen graphs*, which were introduced by Watkins [25] in 1969. The graphs in $\mathcal{F}(4)$ were defined under the name *Rose Window graphs* by Wilson [26] and those in $\mathcal{F}(5)$ under the name *Tabačjn graphs* by Arroyo et al. [2]. The question which of these graphs are edge-transitive has been answered in [9, 13, 2]. Moreover, the automorphism groups of all (not only the edge-transitive) graphs in the families $\mathcal{F}(d)$, $d = 3, 4, 5$, are also known (see [9, 13, 8, 15]).

Jajcay et al. [10] focused primarily on the family $\mathcal{F}(6)$, they called the members of this family *Nest graphs* (see also [24]). Their main result was the classification of the edge-transitive Nest graphs of girth 3, the task to classify all edge-transitive Nest graphs was posed as [10, Problem 1.2]. Regarding the families $\mathcal{F}(d)$ with $d > 6$, the following questions were raised (see [10, Question 1.1]):

1. For which $d > 6$ does the family $\mathcal{F}(d)$ contain at least one edge-transitive graph?
2. For which $d > 6$ does the family $\mathcal{F}(d)$ contain infinitely many edge-transitive graphs?

Jajcay et al. [10] also carried out an exhaustive computer search for edge-transitive graphs of order at most 220 and belonging to $\mathcal{F}(6)$, and also for edge-transitive graphs of order at most 100 and belonging to the families $\mathcal{F}(d)$ with $7 \leqslant d \leqslant 10$. By the *order* of a graph we mean the number of its vertices. They obtained 66 graphs in $\mathcal{F}(6)$ (see [10, Table 1]) and none in the families $\mathcal{F}(d)$, $7 \leqslant d \leqslant 10$. Among the 66 graphs, only one has twice odd order, and this graph is the complement of the *Petersen graph*. Motivated by these observations, in this paper we focus on the edge-transitive graphs in the families $\mathcal{F}(d)$, $d \geqslant 6$, whose order is twice an odd number.

Our main result is the following theorem.

Theorem 1. *The family $\mathcal{F}(d)$ with $d > 6$ contains no edge-transitive graph of twice odd order. Furthermore, the complement of the Petersen graph is the only edge-transitive graph in family $\mathcal{F}(6)$ of twice odd order.*

The paper is organised as follows. Section 2 contains the needed results from graph and group theory. The next two sections are devoted to the preparation for the proof of our main theorem. The main result in Section 3 is Lemma 11, which contains some necessary conditions for a graph in $\mathcal{F}(6)$ to be edge-transitive. In Section 4 we analyse the blocks of imprimitivity for a group of automorphisms acting transitively on the edges of a graph from $\mathcal{F}(d)$. The core of our proof lies in this analysis, in which we rely on the classification of primitive permutation groups containing a semiregular cyclic subgroup with two orbits (see [21]), and the classification of arc-transitive circulants (see [12, 16, 17]). The proof of Theorem 1 is presented in Section 5.

2 Preliminaries

2.1 Graph theory

For a graph Γ, let $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ denote its *vertex set*, *edge set*, *arc set* and *automorphism group*, respectively. The set of vertices adjacent with a given vertex v

is denoted by $\Gamma(v)$.

Let $G \leqslant \mathrm{Aut}(\Gamma)$ and let $v \in V(\Gamma)$. The *stabiliser* of v in G is denoted by G_v and the *orbit* of v under G by v^G. For a subset $B \subseteq V(\Gamma)$, the *set-wise stabiliser* of B in G is denoted by $G_{\{B\}}$. If G is transitive on $V(\Gamma)$, then Γ is said to be *G-vertex-transitive*, and Γ is simply called *vertex-transitive* when it is $\mathrm{Aut}(\Gamma)$-vertex-transitive. The *(G-)edge-* and *(G-)arc-transitive* graphs are defined correspondingly.

Let Γ be a G-vertex-transitive graph. A subset $B \subseteq V(\Gamma)$ is called a *block* for G (the term *block of imprimitivity* is also commonly used) if $B^g \cap B = \emptyset$ or $B^g = B$ holds for every $g \in G$. The block B is *non-trivial* if $1 < |B| < |V(\Gamma)|$, and it is *minimal* if it is non-trivial and no non-trivial block is contained properly in B. The *block system* induced by B is the partition of $V(\Gamma)$ consisting of the images B^g, where g runs over G. A block system is called *normal* if it consists of the orbits of a normal subgroup of G.

Let π be an arbitrary partition of $V(\Gamma)$. For a vertex $v \in V(\Gamma)$, let $\pi(v)$ denote the class containing v. The *quotient graph* of Γ with respect to π, denoted by Γ/π, is defined to have vertex set π, and edges $\{\pi(u), \pi(v)\}$, where $\pi(u) \neq \pi(v)$ and $\{u', v'\} \in E(\Gamma)$ for some $u' \in \pi(u)$ and $v' \in \pi(v)$. If there exists a constant r such that

$$\forall \{u, v\} \in E(\Gamma) : \ \pi(u) \neq \pi(v) \text{ and } |\Gamma(u) \cap \pi(v)| = r,$$

then Γ is called an *r-cover* of Γ/π (our definition of an r-cover generalises the definition given in [6], where π is also assumed to be a block system). The term *cover* will also be used instead of 1-cover. In the special case when π is formed by the orbits of an intransitive normal subgroup $N \lhd \mathrm{Aut}(\Gamma)$, Γ/N will also be written for Γ/π, and the term *normal r-cover* (*normal cover*, respectively) will also be used instead of r-cover (cover, respectively). The following properties are well-known.

Proposition 2. *Let Γ be a connected G-vertex- and G-edge-transitive graph, let $\mathcal{B}$ be a normal block system of G, and let K be the kernel of the action of G on $\mathcal{B}$.*

(1) Γ is a normal r-cover of $\Gamma/\mathcal{B}$, where $r = |\Gamma(v) \cap B|$, v is any vertex and B is any block in $\mathcal{B}$ containing a neighbour of v.

(2) If Γ is a normal cover of $\Gamma/\mathcal{B}$, then Γ and $\Gamma/\mathcal{B}$ have the same valence, the kernel K is regular on every block in $\mathcal{B}$, and $\Gamma/\mathcal{B}$ is G/K-edge-transitive.

Let $S \subset H$ be a subset of a group H such that $1_H \notin S$, where 1_H denotes the identity element of H. The *Cayley digraph* $\mathrm{Cay}(H, S)$ is defined to have vertex set H and arcs (h, sh), where $h \in H$ and $s \in S$. In the case when S is inverse-closed, we regard $\mathrm{Cay}(H, S)$ as an undirected graph and use the term *Cayley graph*. It is a well-known observation (see Sabidussi [23]) that, if Γ is any graph, v is any vertex, and $H \leqslant \mathrm{Aut}(\Gamma)$ is a regular subgroup, then

$$\Gamma \cong \mathrm{Cay}(H, S), \text{ where } S = \{x \in H : v^x \in \Gamma(v)\}. \tag{1}$$

The Cayley digraphs of cyclic groups are shortly called *circulants*. A recursive classification of finite arc-transitive circulants was obtained independently by Kovács [12] and

Li [16]. The paper [12] also provides an explicit characterisation (see [12, Theorem 4]), which was rediscovered recently by Li et al. [17]. The characterisation presented below follows from the proof of [12, Theorem 4] or from [17, Theorem 1.1]. In order to state this characterisation, some further definitions are introduced next.

For a finite group H, denote by $H^{\#}$ the set of all non-identity elements of H, and by H_R the group of all *right multiplications* by the elements of H. If $K \leqslant H$, then let $[H : K]$ denote the set of *right K-cosets* in H. In the case when K is a block for $\mathrm{Aut}(\mathrm{Cay}(H, S))$, the block system induced by K is equal to $[H : K]$. Now, if $K \lhd H$ also holds, then the image of the action of H_R on $[H : K]$ is regular, in particular, $\Gamma/[H : K]$ becomes a Cayley graph of the group H/K.

A Cayley graph $\mathrm{Cay}(H, S)$ is called *normal* if $H_R \lhd \mathrm{Aut}(\mathrm{Cay}(H, S))$. Note that, if $\mathrm{Cay}(H, S)$ is a normal arc-transitive Cayley graph, then S is equal to an A-orbit for some subgroup $A \leqslant \mathrm{Aut}(H)$.

Theorem 3. ([12]) *Let $\Gamma = \mathrm{Cay}(C, S)$ be a connected arc-transitive graph, where C is a cyclic group of order n. Then one of the following holds.*

(a) Γ is the complete graph.

(b) Γ is normal.

(c) There exists a subgroup $1 < D < C$ such that D is a block for $\mathrm{Aut}(\Gamma)$ and $\Gamma/[C : D]$ is a connected arc-transitive circulant. Furthermore, S is a union of D-cosets.

(d) There exist subgroups $1 < D, E < C$ such that both D and E are blocks for $\mathrm{Aut}(\Gamma)$, $C = D \times E$, $|D| > 3$ and $\gcd(|D|, |E|) = 1$. Furthermore, $S = D^{\#}R$, where $R \subseteq E^{\#}$, R is inverse-closed, $\mathrm{Cay}(E, R)$ is connected and arc-transitive.

Besides the Petersen graph, two further small arc-transitive graphs will appear later. The *Clebsch graph* is obtained from the 4 dimensional cube graph Q_4 by adding the edges connecting antipodal points; the *lattice graph* $L_2(4)$ is defined to have vertices the ordered pairs (i, j), $0 \leqslant i, j \leqslant 3$, and two vertices are adjacent if and only if either their first or second coordinates are the same. The graph $L_2(4)$ is depicted in Figure 1. It can be easily checked that the mapping $\sigma : (i, j) \mapsto (j + 1, i)$, where the addition is computed modulo 4, is an automorphism of $L_2(4)$, σ has two orbits of the same length, and one of the subgraphs induced by these orbits is a cycle. This shows that $L_2(4)$ is an example of an edge-transitive graph from the family $\mathcal{F}(6)$.

2.2 Group theory

Our terminology and notation are standard and we follow the books [7, 11]. The *socle* of a group G, denoted by $\mathrm{soc}(G)$, is the subgroup generated by the set of all minimal normal subgroups (see [7, page 111]). The group G is called *almost simple* if $\mathrm{soc}(G) = T$, where T is a non-abelian simple group. In this case G is embedded in $\mathrm{Aut}(T)$ so that its socle is embedded via the inner automorphisms of T, and we also write $T \leqslant G \leqslant \mathrm{Aut}(T)$ (see [7, page 126]).

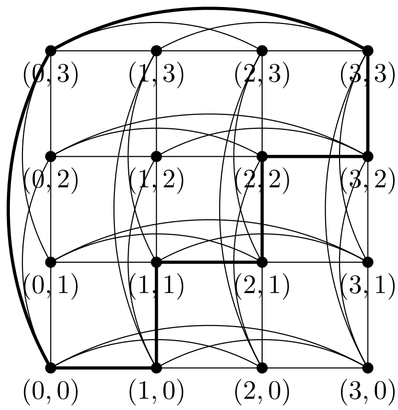

Figure 1: The lattice graph $L_2(4)$ and its subgraph induced by the orbit of the vertex $(1, 1)$ under the automorphism $\sigma : (i, j) \mapsto (j + 1, i)$, which is shown with thick lines.

Our proof of Theorem 1 relies on the classification of primitive groups containing a cyclic subgroup with two orbits due to Müller [21]. Here we need only the special case when the cyclic subgroup is semiregular.

Theorem 4. ([21, Theorem 3.3]) *Let G be a primitive permutation group of degree $2n$ containing an element with two orbits of the same length. Then one of the following holds.*

(1) (Affine action) G contains a regular normal subgroup isomorphic to $\mathbb{Z}_2^m$, where $m \in \{2, 3, 4\}$.[1]

(2) (Almost simple action) G is an almost simple group and one of the following holds.

- *(a) $n \geqslant 3$, $\mathrm{soc}(G) = A_{2n}$, and $A_{2n} \leqslant G \leqslant S_{2n}$, where G acts on $2n$ elements.*
- *(b) $n = 5$, $\mathrm{soc}(G) = A_5$, and $A_5 \leqslant G \leqslant S_5$, where G acts on the set of 2-subsets of $\{1, 2, 3, 4, 5\}$.*
- *(c) $n = (q^d - 1)/2(q - 1)$, $\mathrm{soc}(G) = PSL_d(q)$, and $PSL_d(q) \leqslant G \leqslant P\Gamma L_d(q)$ for some odd prime power q and even number $d \geqslant 2$ such that $(d, q) \neq (2, 3)$.*
- *(d) $n = 6$ and $\mathrm{soc}(G) = G = M_{12}$.*
- *(e) $n = 11$, $\mathrm{soc}(G) = M_{22}$, and $M_{22} \leqslant G \leqslant \mathrm{Aut}(M_{22})$.*
- *(f) $n = 12$ and $\mathrm{soc}(G) = G = M_{24}$.*

If G is a group in one of the families (a)-(f) above, then it follows from [7, Theorem 4.3B] that $\mathrm{soc}(G)$ is the unique minimal normal subgroup of G. Therefore, we have the following corollary.

Corollary 5. *Let G be a primitive permutation group in one of the families (a)-(f) in part (2) of Theorem 4, and let $N \lhd G$, $N \neq 1$. Then N is also primitive.*

[1]The group G is from a short list, but as this possibility will not occur later, we omit the details.

For a transitive permutation group $G \leqslant \mathrm{Sym}(\Omega)$, the *subdegrees* of G are the lengths of the orbits of a point stabiliser G_ω, $\omega \in \Omega$. Since G is transitive, it follows that the subdegrees do not depend on the choice of ω (see [7, page 72]). The number of orbits of G_ω is called the *rank* of G. The actions of a group G on sets Ω and Ω' are said to be *equivalent* if there is a bijection $\varphi : \Omega \to \Omega'$ such that

$$\forall \omega \in \Omega,\ \forall g \in G :\ \varphi(\omega^g) = (\varphi(\omega))^g.$$

Now, suppose that G is a group in one of the families (a)-(f) in part (2) of Theorem 4. If G is in family (a), then the action is unique up to equivalence and G is clearly 2-transitive. If G is in family (b), then the action is unique up to equivalence and the subdegrees are $1, 3$ and 6. Let G be in family (c). The semiregular cyclic subgroup of G with two orbits is contained in a regular cyclic group, called the *Singer subgroup* of $PGL_d(q)$ (see [11, Chapter 2, Theorem 7.3]). In this case the action is unique up to equivalence if and only if $d = 2$. If $d \geqslant 4$, then the action of G is equivalent to either its natural action on the set of points of the projective geometry $PG_d(q)$, or to its natural action on the set of hyperplanes of $PG_d(q)$. In both actions G is 2-transitive. Finally, if G is in the families (d)-(f), then the action is unique up to equivalence and G is 2-transitive (this can also be read off from [5]). All this information is summarised in the lemma below.

Lemma 6. *Let G be a primitive permutation group in one of the families (a)-(f) in part (2) of Theorem 4.*

(1) G is 2-transitive, unless G belongs to family (b). In the latter case the subdegrees are $1, 3$ and 6.

(2) The action of G is unique up to equivalence, unless G is in family (c) and $d \geqslant 4$. In the latter case G admits two inequivalent faithful actions, namely, the natural actions on the set of points and the set of hyperplanes, respectively, of the projective geometry $PG_d(q)$.

The following result about G-arc-transitive bicirculants was proved by Devillers et al. [6], but the proof works also for the edge-transitive bicirculants as well. In fact, it is an easy consequence of Theorem 4.

Proposition 7. ([6, part (1) of Proposition 4.2]) *Let Γ be a G-edge-transitive bicirculant such that G is a primitive group. Then Γ is one of the following graphs:*

(1) The complete graph, and G is one of the 2-transitive groups described in part (2) of Theorem 4.

(2) The Petersen graph or its complement, and $A_5 \leqslant G \leqslant S_5$.

(3) The lattice graph $L_2(4)$ or its complement, and G is a rank 3 subgroup of $AGL(4, 2)$.

(4) The Clebsch graph or its complement, and G is a rank 3 subgroup of $AGL(4, 2)$.

One can easily check which of the graphs in the families (1)–(4) above belongs also to the family $\mathcal{F}(d)$ for some $d \geqslant 3$.

Corollary 8. *Let $\Gamma \in \mathcal{F}(d)$ be a G-edge-transitive graph for some $d \geqslant 3$. If G is primitive on $V(\Gamma)$, then Γ is isomorphic to K_6, or the Petersen graph, or its complement, or the lattice graph $L_2(4)$.*

3 A lemma on the graphs in the family $\mathcal{F}(6)$

In this section we derive some necessary conditions for a graph in $\mathcal{F}(6)$ to be edge-transitive. Our main tool is the coset graph construction defined next.

Let G be a group, let H be a core-free subgroup of G, and let S be a subset of G such that $S \cap H = \emptyset$ and $HSH = HS^{-1}H$. By *core-free* we mean that H contains no non-trivial normal subgroup of G. The *coset graph* $\mathrm{Cos}(G, H, HSH)$ is defined to have vertex set $[G : H]$ (the set of all right H-cosets in G), and edges $\{Hx, Hy\}$, where $x, y \in G$ and $yx^{-1} \in HSH$. The action of G on $[G : H]$ defined by right multiplication is transitive and as H is core-free in G, it is also faithful. The corresponding image of G is a group of automorphisms of $\mathrm{Cos}(G, H, HSH)$. The lemma below is a folklore result.

Lemma 9. *Let Γ be a both G-vertex- and G-edge-transitive graph. Write H for the vertex-stabiliser G_v, and let $g \in G$ such that $\{v, v^g\}$ is an edge. Then the mapping*

$$\varphi : [G : H] \to V(\Gamma),\ Hx \mapsto v^x \text{ for } x \in G$$

is an isomorphism between $\mathrm{Cos}(G, H, H\{g, g^{-1}\}H)$ and Γ.

The valence of the graph $\mathrm{Cos}(G, H, H\{g, g^{-1}\}H)$ is given below.

Lemma 10. ([18, Lemma 2.4]) *The valence of the coset graph $\mathrm{Cos}(G, H, H\{g, g^{-1}\}H)$ is equal to $|H|/|H \cap H^g|$ if $HgH = Hg^{-1}H$, or $2|H|/|H \cap H^g|$ otherwise.*

Suppose that Γ is a G-edge-transitive graph in $\mathcal{F}(6)$ such that G contains a semiregular cyclic subgroup C with two orbits and the subgraph of Γ induced by at least one of the C-orbits is a cycle. Note that Γ is then G-vertex-transitive as well. This can be seen by observing that Γ admits an edge with end-vertices lying in distinct C-orbits and also an edge with end-vertices from the same C-orbit. Now as G acts transitively on $E(\Gamma)$, the two C-orbits are merged into one G-orbit.

The main result of this section is the following lemma.

Lemma 11. *Let Γ be a G-edge-transitive graph in $\mathcal{F}(6)$ of order $2n$ such that G contains a semiregular cyclic subgroup C with two orbits, and the subgraph of Γ induced by at least one of the C-orbits is a cycle. Furthermore, let H be a vertex-stabiliser in G. Then G contains an element g of order n satisfying one of the following sets of conditions:*

(1) $HgH = Hg^{-1}H$ and $|H| = 6|H \cap H^g| = \frac{1}{2}|H\langle g\rangle \cap HgH|$.

(2) $HgH \neq Hg^{-1}H$ *and* $|H| = 3|H \cap H^g| = |H\langle g\rangle \cap HgH|$*.*

Proof. There is a vertex $v \in V(\Gamma)$ such that the subgraph of Γ induced by the orbit v^C is a cycle. Choose $c \in C$ such that $\{v, v^c\}$ is an edge. Note that c has order n. As it has been observed above, Γ is not only G-edge-, but also G-vertex-transitive. Thus Lemma 9 can be applied to $G, K := G_v$ and c, and this yields

$$\Gamma \cong \Gamma' := \mathrm{Cos}(G, K, K\{c, c^{-1}\}K).$$

Note that, as both H and K are vertex-stabilisers, $H = K^{g'}$ for some $g' \in G$.

Assume first that $KcK = Kc^{-1}K$. Using the formula for the valence of Γ' given in Lemma 10, we find that

$$|K| = 6|K \cap K^c|. \tag{2}$$

Let φ be the isomorphism between Γ' and Γ defined in Lemma 9. Since $KcK = Kc^{-1}K$, it follows that Γ' is G-arc-transitive. This means that Γ is also G-arc-transitive, hence the stabiliser K is transitive on $\Gamma(v)$. It follows that $v^C \cap (v^c)^K = \{v^c, v^{c^{-1}}\}$. Applying φ^{-1}, we get

$$\{Kc^i : 0 \leqslant i \leqslant n-1\} \cap \{Kck : k \in K\} = \{Kc, Kc^{-1}\}.$$

This shows that $K\langle c\rangle \cap KcK = Kc \cup Kc^{-1}$ holds in G, and so

$$|K\langle c\rangle \cap KcK| = 2|K|. \tag{3}$$

Using also the condition that $H = K^{g'}$, (2) and (3) show that choosing g to be $c^{g'}$, part (1) of the lemma holds.

Now assume that $KcK \neq Kc^{-1}K$. Using again Lemma 10, we find that

$$|K| = 3|K \cap K^c|. \tag{4}$$

In this case Γ' is not arc-transitive. Thus neither is Γ, and $\Gamma(v)$ splits into two K-orbits of the same size. We claim that v^c and $v^{c^{-1}}$ belong to different K-orbits. For otherwise, $v^{c^{-1}} = v^{cg''}$ for some $g'' \in K$, and this would imply that the automorphism cg'' inverts the arc $(v^{c^{-1}}, v)$, contradicting the assumption that Γ is not arc-transitive. Then $v^C \cap (v^c)^K = \{v^c\}$, and this yields $K\langle c\rangle \cap KcK = Kc$, and so

$$|K\langle c\rangle \cap KcK| = |K|. \tag{5}$$

Then (4) and (5) show that part (2) of the lemma holds for $g = c^{g'}$. □

4 Blocks

Throughout this section we keep the following notation:

$\Gamma \in \mathcal{F}(d)$ is a G-edge-transitive graph of oder $2n$ for some $d \geqslant 6$.

$C \leqslant G$ is a cyclic semiregular subgroup with two orbits and at least one of the subgraphs induced by these orbits is a cycle.

B is a non-trivial block for G and $\mathcal{B}$ is the block system induced by B.

We say that B is *cyclic* when it is contained in one of the C-orbits, and *non-cyclic* otherwise.

Recall that, $C_{\{B\}}$ is the set-wise stabiliser of B in C, and $\mathcal{B}$ is said to be normal when there is a normal subgroup N of G such that $\mathcal{B}$ consists of the N-orbits.

Lemma 12. *Suppose that B is cyclic such that $|B| < n/2$. Then the kernel of the action of G on $\mathcal{B}$ is equal to $C_{\{B\}}$. Furthermore, Γ is a normal cover of $\Gamma/\mathcal{B}$ and $\Gamma/\mathcal{B} \in \mathcal{F}(d)$.*

Proof. Denote by V_i the C-orbits, $i = 1, 2$, and by K the kernel of the action of G on $\mathcal{B}$. It is clear that any block in $\mathcal{B}$ is contained in either V_1 or V_2. Consider the blocks contained in V_1. These form a block system for C, and as C is regular on V_1, it follows that these blocks are the $C_{\{B_1\}}$-orbits, where B_1 is any block contained in V_1. The group $C_{\{B_1\}}$ is regular on B_1, hence $|C_{\{B_1\}}| = |B_1| = |B|$, from which it follows that $C_{\{B\}} = C_{\{B_1\}}$. The same applies to V_2, and we conclude that $\mathcal{B}$ consists of the $C_{\{B\}}$-orbits. Thus $K \geqslant C_{\{B\}}$, in particular, $\mathcal{B}$ is normal.

It can be assumed w.l.o.g. that the subgraph of Γ induced by V_1 is a cycle. Now, fix an edge $\{u, v\}$ such that $u, v \in V_1$. Since $|B| < n/2$, it follows that $\Gamma(u) \cap B' = \{v\}$, where B' is the block containing v. By Proposition 2(1), Γ is a normal cover of $\Gamma/\mathcal{B}$. Then part (2) of the same proposition shows that K is regular on every block, and so we have $K = C_{\{B\}}$.

In order to see that $\Gamma/\mathcal{B}$ belongs to $\mathcal{F}(d)$, one only needs to observe that $\Gamma/\mathcal{B}$ has valence d, $C/C_{\{B\}}$ is semiregular with two orbits, the induced cycle of Γ on V_1 projects to an induced cycle of $\Gamma/\mathcal{B}$, and V_1 projects to a $C/C_{\{B\}}$-orbit. □

Recall that, B is minimal if no non-trivial block for G is contained properly in B (see the third paragraph in Section 2.1).

Lemma 13. *Suppose that B is non-cyclic.*

(1) B is a union of two $C_{\{B\}}$-orbits. The group C acts transitively on $\mathcal{B}$ with kernel equal to $C_{\{B\}}$.

(2) If $|B| > 2$ and B is minimal, then $\mathcal{B}$ is normal.

Proof. (1): Since there are two C-orbits on $V(\Gamma)$ of the same size and B has a point in common with both, it follows that B splits into two $C_{\{B\}}$-orbits, hence $|B| = 2|C_{\{B\}}|$.

Let $\bar{C}$ and K be the image and the kernel, respectively, of the action of C on $\mathcal{B}$. It is clear that C acts transitively on $\mathcal{B}$. This shows that $\bar{C}$ is regular, hence $C_{\{B\}} \leqslant K$, and we can write

$$|C|/|K| = |\bar{C}| = |\mathcal{B}| = 2n/|B| = |C|/|C_{\{B\}}|.$$

This shows that $|K| = |C_{\{B\}}|$ also holds, and so $K = C_{\{B\}}$.

(2): For a subgroup $X \leqslant G_{\{B\}}$, denote by X^* the image of the action of X on B. Let M be the kernel of the action of G on $\mathcal{B}$. By the minimality of B, $(G_{\{B\}})^*$ is primitive. On the other hand, as $1 < (C_{\{B\}})^* \leqslant M^* \lhd (G_{\{B\}})^*$, M^* is transitive on B. This shows that $\mathcal{B}$ is normal. □

From now on we focus on the case when n is odd.

Lemma 14. *If n is odd, then $|B| > 2$.*

Proof. Assume on the contrary that $|B| = 2$. Write $B = \{u, v\}$. Then $u^C \neq v^C$, and we may assume that the subgraph of Γ induced by u^C is a cycle. Let $c \in C$ such that $\{u, u^c\}$ is an edge. Clearly, c has order n.

There is a unique number $1 \leqslant k \leqslant (n-1)/2$ such that v^{c^k} and $v^{c^{-k}}$ are the neighbours of v. Define the subset $S \subseteq C$ as

$$S = \{x \in C : \{u, v^x\} \in E(\Gamma)\}.$$

It is clear that $|S| = d - 2$. Also, $1_C \notin S$, where 1_C is the identity element of C. For otherwise, $\{u, v\}$ is an edge, but as it is also a block, $G_u = G_v$, and this contradicts the fact that Γ is G-edge-transitive.

We say that two blocks in $\mathcal{B}$ are adjacent when these are adjacent as vertices of $\Gamma/\mathcal{B}$. It can be easily seen that any two subgraphs of Γ induced by the union of two adjacent blocks are isomorphic to the same graph, say Δ. We claim that

$$\Delta \cong K_2 \cup 2K_1 \text{ or } 2K_2.$$

Assume for the moment that there exists some $s \in S$ such that $s \notin \{c, c^{-1}, c^k, c^{-k}\}$. Then the subgraph induced by $\{u, v, u^s, v^s\} \cong 2K_2$ or $K_2 \cup 2K_1$ depending on whether $s^{-1} \in S$ or not, and the claim follows. Now, as $|S| = d - 2 \geqslant 4$, we are left with the case when $k \neq 1$ and $S = \{c, c^{-1}, c^k, c^{-k}\}$. In this case the subgraph induced by $\{u, u^c, v, v^c\}$ is the 3-path (v, u^c, u, v^c). Since Γ is G-edge-transitive, there is some $g \in G$ mapping $\{u, u^c\}$ to $\{u, v^c\}$. This implies that g maps the 3-path to itself, hence it induces an automorphism of it. This is clearly impossible, and so the claim is proved.

Moreover, the argument above also shows that we have the following options:

$$(k = 1 \text{ and } S = S^{-1}) \text{ or } (k \neq 1 \text{ and } S \cap S^{-1} = \emptyset). \tag{6}$$

Now, define the permutation t of $V(\Gamma)$ as

$$t = (u\,v)(u^c\,v^c)\cdots(u^{c^{n-1}}\,v^{c^{n-1}}).$$

Observe that t commutes with any element of G. In particular, $\hat{C} := \langle c, t\rangle$ is a regular cyclic group.

Define next the graph Γ' by

$$V(\Gamma') = V(\Gamma) \text{ and } E(\Gamma') = \{\{u, u^c\}^g : g \in \langle G, t\rangle\}.$$

Using the fact that $tg = gt$ for any $g \in G$, we find that

$$E(\Gamma') = \{\{u, u^c\}^g : g \in G\} \ \cup \ \{\{v, v^c\}^g : g \in G\}.$$

This can be used to find the neighbourhood $\Gamma'(u)$. If $k = 1$, then $E(\Gamma') = E(\Gamma)$, and so $\Gamma'(u) = \Gamma(u)$. If $k \neq 1$, then $E(\Gamma')$ splits into two edge-orbits under G, and t swaps these edge-orbits. This yields that $\Gamma'(u) = \Gamma(u) \cup \Gamma(u^t)^t = \Gamma(u) \cup \Gamma(v)^t$. Now, according to (1), $\Gamma' \cong \mathrm{Cay}(\hat{C}, S' \cup S'')$, where

$$S' = \{c, c^{-1}, c^k, c^{-k}\} \text{ and } S'' = tS \cup tS^{-1}.$$

It follows from (6) that the valence of Γ' is $2 + |S|$ if $k = 1$, and $4 + 2|S|$ if $k \neq 1$.

By definition, Γ' is edge-transitive. It is well-known that it must be then arc-transitive as well, and therefore, Γ' belongs to one of the families (a)-(d) in Theorem 3. We consider below all possibilities case by case.

Family (a): Γ' is the complete graph. This contradicts the fact that Γ' has even valence and order.

Family (b): Γ' is normal. Then $S' \cup S'' = c^A$ for some subgroup $A \leqslant \mathrm{Aut}(\hat{C})$. This contradicts the fact that c has order n, while ts has even order for each $s \in S$.

Family (c): There exists a subgroup $1 < D < \hat{C}$ such that $S' \cup S''$ is a union of D-cosets. If $|D|$ is odd, then $D \leqslant C$, and so S' would be a union of D-cosets. This is clearly impossible. Hence, $|D|$ is even. Then $t \in D$, implying $tS' = S''$ and $tS'' = S'$, and so $|S'| = |S''|$. This contradicts the conditions that $|S'| = 2$ and $|S''| = |S| \geqslant 4$ if $k = 1$, and $|S'| = 4$ and $|S''| = 2|S| \geqslant 8$ otherwise.

Family (d): There exist subgroups $1 < D, E < \hat{C}$ such that $\hat{C} = D \times E$, $|D| > 3$, $\gcd(|D|, |E|) = 1$, and $S' \cup S'' = D^{\#}R$ for some subset $R \subseteq E^{\#}$.

Suppose first that $|D|$ is odd. Then $D \leqslant C$. For every $i \in \{1, -1, k, -k\}$, $|Dc^i \cap S'| = |D| - 1 \geqslant 4$. It follows that $|D| = 5$, $k \neq 1$, and $S' \subset Dc$. This shows that $c^2 \in D$, whence $D = C$. On the other hand, D is a block for $\mathrm{Aut}(\Gamma')$, and so C is a block for G. This is impossible.

Now suppose that $|D|$ is even. Then $t \in D$ and D can be written as $D = \langle t \rangle \times D'$. Also, $R \subset E \leqslant C$. As $S' \cup S'' = D^{\#}R$ is inverse-closed, so is R, in particular, $|R|$ is even. Also, $S' = D^{\#}R \cap C = (D^{\#} \cap C)R = (D')^{\#}R$. Thus $|S'| = (|D'| - 1)|R|$, and these imply in turn that $|R| = 2$, $|D'| = 3$, $|S'| = 4$, and $|S''| = |D^{\#}R| - |S'| = 6$. We have seen above that this is impossible. ☐

Our last lemma is one of the crucial steps towards Theorem 1.

Lemma 15. *If $n > 5$ is odd, then G admits a non-trivial cyclic block.*

Proof. Since $n > 5$, it follows from Corollary 8 that G is imprimitive. Choose a minimal non-trivial block B for G, denote by $\mathcal{B}$ the block system induced by B, and let K denote the kernel of the action of G on $\mathcal{B}$.

We are done if B is cyclic, hence we assume that B is non-cyclic. By Lemma 14, $|B| > 2$. As before, for a subgroup $X \leqslant G_{\{B\}}$, X^* denotes the image of the action of X on B.

Apply Lemma 13 to B. This shows that $\mathcal{B}$ is normal and $(C_{\{B\}})^*$ is a cyclic semiregular subgroup of $(G_{\{B\}})^*$ with two orbits. As B is minimal, $(G_{\{B\}})^*$ is also primitive, and therefore described by Theorem 4. The fact that n is odd shows that $(G_{\{B\}})^*$ is one of the groups in the families (a)-(f) in part (2) of Theorem 4. Then $K^* \lhd (G_{\{B\}})^*$. Note that, K^* is non-trivial by Lemma 13(2) since $|B| > 2$. By Corollary 5, K^* is also primitive. We derive the lemma in three steps.

Step 1. K acts faithfully on every block in $\mathcal{B}$.

Assume on the contrary that K acts unfaithfully on some block in $\mathcal{B}$. Using the connectedness of Γ, it is easy to show that there are adjacent blocks $B', B'' \in \mathcal{B}$ so that the kernel of the action of K on B' is non-trivial on B''. Denote by N the latter kernel. Now, as $N \lhd K$ and K is primitive on B'', N is transitive on B''. This implies that any vertex in B' is adjacent with any vertex in B''. This contradicts the facts that the subgraph of Γ induced by at least one of the C-orbits is a cycle and that $|B| = 2\ell$ for some odd $\ell \geqslant 3$.

Fix a vertex $u \in B$.

Step 2. For each block $B' \in \mathcal{B}$ there exists a unique vertex $u' \in B'$ such that $K_u = K_{u'}$.

Define the binary relation $\sim$ on $\mathcal{B}$ by letting $B' \sim B''$ if and only if the action of K on B' and B'', respectively, are equivalent. It is easy to show that $\sim$ is an equivalence relation.

Let $B', B'' \in \mathcal{B}$ be such that $B' \sim B''$ and let $g \in G$. We claim that $(B')^g \sim (B'')^g$. There is a bijective mapping φ from B' to B'' such that

$$\forall v \in B',\ \forall k \in K:\ \varphi(v^k) = (\varphi(v))^k.$$

Now, pick arbitrary $w \in (B')^g$ and $k \in K$. Let γ_1 be the bijection from B' to $(B')^g$ defined by $\gamma_1(x) = x^g$ for each $x \in B'$, and let γ_2 be the bijection from B'' to $(B'')^g$ defined by $\gamma_2(x) = x^g$ for each $x \in B''$. We finish the proof of the claim by showing $\psi(w^k) = (\psi(w))^k$, where ψ is the bijection defined as the composition $\psi = \gamma_2 \circ \varphi \circ \gamma_1^{-1}$. Then $w = v^g$ for some $v \in B'$ and $gk = k'g$ for some $k' \in K$ because $K \lhd G$. Thus

$$\psi(w^k) = \psi(v^{k'g}) = (\varphi(v^{k'}))^g = \varphi(v)^{k'g} = (\varphi(v)^g)^k = (\psi(w))^k.$$

Thus $\sim$ is G-invariant, and therefore, it is a G-congruence. Due to [7, Exercise 1.5.4], the $\sim$-classes form a block system for G with respect to its action on $\mathcal{B}$. Denote by m the number of $\sim$-classes. As $|\mathcal{B}|$ is odd, so is m. On the other hand, by Lemma 6(2), K has at most two inequivalent faithful actions, and we conclude that $m = 1$.

Let $B' \in \mathcal{B}'$ be an arbitrary block. Since K acts equivalently on B and B', it follows by [7, Lemma 1.6B] that there is an element $u' \in B'$ such that $K_u = K_{u'}$. By Lemma 6(1), K is 2-transitive on B', unless $|B'| = 10$, $K = A_5$ or S_5, and it has subdegrees $1, 3$ and 6. This shows that $K_x \neq K_{u'}$ for any vertex $x \in B'$ such that $x \neq u'$. On the other hand, $K_u = K_{u'}$ and this finishes off the proof of Step 2.

Step 3. The set of all vertices u' defined in Step 2 is a cyclic block.

Denote by $\hat{B}$ the set of all vertices u' defined in Step 2. The cardinality $|\hat{B}| = |\mathcal{B}|$, and as $|\mathcal{B}|$ is odd, we are done if we show that $\hat{B}$ is a block. Equivalently, $\hat{B}^g = \hat{B}$ or $\hat{B}^g \cap \hat{B} = \emptyset$ holds for each $g \in G$.

Suppose that $v^g \in \hat{B}$ for some $v \in \hat{B}$ and $g \in G$. We have to show that $\hat{B}^g = \hat{B}$. In fact, it is enough to show that $\hat{B}^g \subseteq \hat{B}$. Choose an arbitrary element $w \in \hat{B}$. Then $K_u = K_v = K_{v^g} = K_w$. Using also the normality of K in G, we can write that

$$K_{w^g} = K \cap G_{w^g} = K \cap (G_w)^g = (K \cap G_w)^g = (K_w)^g.$$

The same argument shows that $K_{v^g} = (K_v)^g$, and thus $K_u = K_{v^g} = (K_v)^g = (K_w)^g = K_{w^g}$. By the definition of the set $\hat{B}$, $w^g \in \hat{B}$, and $\hat{B}^g \subseteq \hat{B}$ follows. □

5 Proof of Theorem 1

Assume on the contrary that there is an edge-transitive graph $\Gamma \in \mathcal{F}(d)$ of order $2n$ such that $d \geqslant 6$, n is odd and $n > 5$. Here we use the fact that there is no edge-transitive graph in the class $\mathcal{F}(7)$ with $n = 5$ (see [10]). Choose n to be the smallest possible, i.e., whenever $\Gamma' \in \mathcal{F}(d')$ is edge-transitive of order $2n'$ such that $d' \geqslant 6$, $n' < n$ and n' is odd, then Γ' is isomorphic to the complement of the Petersen graph. In what follows, we denote the latter graph by $\overline{\mathsf{Pet}}$.

For the sake of simplicity, write G for $\mathrm{Aut}(\Gamma)$. Let $C \leqslant G$ be a cyclic semiregular subgroup with two orbits such that at least one of the subgraphs induced by these orbits is a cycle.

It follows from Corollary 8 that G is imprimitive. Choose a minimal non-trivial block B for G, denote by $\mathcal{B}$ be the block system induced by B, and let K be the kernel of the action of G on $\mathcal{B}$.

Due to Lemma 15 we may assume that B is cyclic, i.e., any block in $\mathcal{B}$ is contained in one of the two C-orbits.

As n is odd, $|B| < n/2$. By Lemma 12, $C_{\{B\}} \lhd G$. Let p be a prime divisor of $|B|$, and let $P \leqslant C_{\{B\}}$ be the subgroup of order p. The group P is characteristic in $C_{\{B\}}$ and as $C_{\{B\}} \lhd G$, it follows that $P \lhd G$. The minimality of B implies $P = C_{\{B\}}$ and thus $|B| = p$. By Lemma 12, Γ is a normal cover of Γ/P and $\Gamma/P \in \mathcal{F}(d)$. Due to Proposition 2(2), Γ/P is also G/P-edge-transitive. The order of Γ/P is $2n/p$, hence the minimality of n yields

$$n = 5p, \ \Gamma/P \cong \overline{\mathsf{Pet}}, \ \text{and } G/P \cong A_5 \text{ or } S_5.$$

This also implies that $d = 6$ and Γ is arc-transitive. The last condition follows from the fact that G/P acts transitively on the edges of $\overline{\mathsf{Pet}}$.

If $p = 3$ or 5, then Γ has order 30 or 50 and its valence is 6. It follows from [10, Table 1] that no graph in $\mathcal{F}(6)$ of order 30 or 50 is edge-transitive. Thus $p > 5$, and the Zassenhaus theorem (see [11, Chapter 1, Theorem 18.1]) shows that there exists a subgroup $L < G$ such that $G = P \rtimes L$, $P \cong \mathbb{Z}_p$ and $L \cong G/P \cong A_5$ or S_5.

Fix z to be a generator of P and let L be identified with A_5 or S_5. Note that every element of G can be expressed as a product

$$gz^i\lambda, \text{ where } 0 \leqslant i \leqslant p-1 \text{ and } \lambda \in L.$$

Let $N = C_L(P)$. Since $N_L(P) = L$, it follows that $N \lhd L$ and L/N is isomorphic to a subgroup of $\mathrm{Aut}(P)$, in particular, it is a cyclic group. This implies that either $N = L$, or $L = S_5$ and $N = A_5$. Consequently,

$$G = \begin{cases} P \times L & \text{if } N = L, \\ P \rtimes L & \text{if } L = S_5 \text{ and } N = A_5. \end{cases} \tag{7}$$

Furthermore, in the second case the action of L on P by conjugation is defined by

$$(z^i)^\lambda = \begin{cases} z^i & \text{if } \lambda \text{ is even}, \\ z^{-i} & \text{if } \lambda \text{ is odd}, \end{cases} \tag{8}$$

where $0 \leqslant i \leqslant p-1$ and $\lambda \in L = S_5$.

Let $H = N_L(\langle(1,2,3)\rangle)$. Then

$$H = \begin{cases} \big\langle (1,2,3), (1,2)(4,5) \big\rangle \cong S_3 & \text{if } L \cong A_5, \\ \big\langle (1,2,3), (1,2), (4,5) \big\rangle \cong S_3 \times \mathbb{Z}_2 & \text{if } L \cong S_5. \end{cases}$$

We show next that H is a vertex-stabiliser in G. First, as Γ is a normal cover of $\overline{\mathsf{Pet}}$, the vertex-stabilisers in G are isomorphic to the vertex-stabilisers in the image of G under its action on the quotient graph $\Gamma/P \cong \overline{\mathsf{Pet}}$. Therefore, these vertex-stabilisers in G are isomorphic to H. It follows immediately from (7) and (8) that all the elements of order 3 in G are contained in L, and form a single conjugacy class within G. In particular, there exists a vertex-stabiliser in G containing $(1,2,3)$, let this vertex-stabiliser be denoted by M. Clearly, $M \leqslant N_G(\langle(1,2,3)\rangle)$. Using the identities in (7) and (8), we obtain that

$$N_G(\langle(1,2,3)\rangle) = \begin{cases} P \times H & \text{if } N = L, \\ P \rtimes H & \text{if } L = S_5 \text{ and } N = A_5. \end{cases}$$

If $N_G(\langle(1,2,3)\rangle) = P \times H$, then it is clear that H is the only subgroup of $N_G(\langle(1,2,3)\rangle)$ isomorphic to H, so $M = H$, i.e., H is indeed a vertex-stabiliser.

Let $N_G(\langle(1,2,3)\rangle) = P \rtimes H$ and suppose that $M \neq H$. Then $z^i\lambda \in M$ for some $1 \leqslant i \leqslant p-1$ and $\lambda \in H$. If λ is even, then p divides the order of $z^i\lambda$, which is impossible because $M \cong H$. Thus λ must be odd.

Suppose that $z_1\lambda_1$ and $z_2\lambda_2$ are distinct elements in M for some $z_1, z_2 \in P$ and $\lambda_1, \lambda_2 \in H$. Now if $\lambda_1 = \lambda_2$, then $z_1 \neq z_2$, hence $z_1 z_2^{-1}$ is an element of order p in M, which is impossible. Combining this with the equality $|M| = |H|$, it is not hard to show that for any $\mu \in H$, there is an element $z' \in P$ such that $z'\mu \in M$. It follows from this that $(H \cap A_5) \leqslant M$. If $i = 2j$, then $(z^i\lambda)^{z^j} = \lambda$, and we have

$$M^{z^j} = \big\langle H \cap A_5, z^i\lambda \big\rangle^{z^j} = \big\langle (H \cap A_5)^{z^j}, (z^i\lambda)^{z^j} \big\rangle = \big\langle H \cap A_5, \lambda \big\rangle = H.$$

If $i = 2j+1$, then one finds in the same way that $M^{\lambda z^{(p-1)/2-j}} = H$. In either case we obtain that H is a vertex-stabiliser.

The desired contradiction will arise after applying Lemma 11 to H. Due to this lemma, there is an element $g \in G$ of order $5p$ satisfying all conditions in either part (1) or (2) of Lemma 11. W.l.o.g. we may write $g = z\sigma$, where σ is a 5-cycle in S_5.

Case 1. $N = L$.

By (7), $G = P \times L$. Using the condition that $z\lambda = \lambda z$ for every $\lambda \in H$, it is easy to see that $g^{-1} \notin HgH$. Thus $HgH \neq Hg^{-1}H$, and so we have $|H| = 3|H \cap H^g| = |H\langle g\rangle \cap HgH|$. Since $H^z = H$, it follows that $H^g = H^\sigma$. Also,

$$H\langle g\rangle = \bigcup_{i=0}^{p-1} z^i H\langle\sigma\rangle \text{ and } HgH = zH\sigma H.$$

Thus the equalities $|H| = 3|H \cap H^g| = |H\langle g\rangle \cap HgH|$ reduce to

$$|H| = 3|H \cap H^\sigma| = |H\langle\sigma\rangle \cap H\sigma H|.$$

A computation with the computer package Magma [3] shows that no 5-cycle σ satisfies these conditions.

Case 2. $L = S_5$ and $N = A_5$.

Let $H_1 = H \cap A_5$. Note that $H = H_1 \cup (4,5)H_1$. By (7), $G = K \rtimes L$, and the action of L on P by conjugation is described in (8).

Then $HgH = Hg^{-1}H$ if and only if $g^{-1} \in HgH$, and so $g^{-1} = z^{-1}\sigma^{-1} = \lambda_1 z\sigma\lambda_2$ for some $\lambda_1, \lambda_2 \in H$. It can be seen that both λ_1 and λ_2 must be odd. Then $\lambda_i = (4,5)\lambda_i'$ for some $\lambda_i' \in H_1$, where $i = 1, 2$, and it holds $\sigma^{-1} = (4,5)\lambda_1'\sigma(4,5)\lambda_2'$. A computation with Magma [3] verifies that such λ_1' and λ_2' exist for any 5-cycle σ. Thus $HgH = Hg^{-1}H$, and so we have $|H| = 6|H \cap H^g| = \frac{1}{2}|H\langle g\rangle \cap HgH|$. Then $H^g = H_1^g \cup ((4,5)H_1)^g = H_1^\sigma \cup z^{-2}((4,5)H_1)^\sigma$, hence the first equality reduces to

$$|H| = 6|H \cap H_1^\sigma|. \tag{9}$$

In order to rewrite the second equality, observe first that

$$H\langle g\rangle = \bigcup_{i=0}^{p-1} Hz^i\langle\sigma\rangle = \bigcup_{i=0}^{p-1} \left(z^i H_1\langle\sigma\rangle \cup z^{-i}(4,5)H_1\langle\sigma\rangle\right) = \bigcup_{i=0}^{p-1} z^i H\langle\sigma\rangle.$$

On the other hand, $HgH = (H_1 \cup (4,5)H_1)z\sigma H = zH_1\sigma H \cup z^{-1}(4,5)H_1\sigma H$. Thus $|H| = \frac{1}{2}|H\langle g\rangle \cap HgH|$ reduces to

$$|H\langle\sigma\rangle \cap H_1\sigma H| + |H\langle\sigma\rangle \cap (4,5)H_1\sigma H| = 2|H|. \tag{10}$$

A computation with Magma [3] shows that no 5-cycle σ satisfies both (9) and (10). This completes the proof of Theorem 1.

Acknowledgements

The authors thank the reviewers for the numerous helpful suggestions, which improved the presentation considerably.

References

[1] I. Antončič, A. Hujdurović, and K. Kutnar. A classification of pentavalent arc-transitive bicirculants. *J. Algebaic. Combin.*, 41:643–668, 2015.

[2] A. Arroyo, I. Hubard, K. Kutnar, E. O'Reilly, and P. Šparl. Classification of symmetric Tabačjn graphs. *Graphs Combin.*, 31:1137–1153, 2015.

[3] W. Bosma, C. Cannon, and C. Playoust. The MAGMA algebra system I: The user language. *J. Symbolic Comput.*, 24:235–265, 1997.

[4] M. Conder, J.-X. Zhou, Y.-Q. Feng, and M.-M. Zhang. Edge-transitive bi-Cayley graphs. *J. Combin. Theory, Ser. B*, 145:264–306, 2020.

[5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. Atlas of finite groups. Clarendon Press, Oxford, 1985.

[6] A. Devillers, M. Giudici, and W. Jein. Arc-transitive bicirculants. *J. London Math. Soc.*, 105:1–23, 2022.

[7] J. D. Dixon and B. Mortimer. Permutation groups. Graduate Text in Mathematics, no 163, Springer-Verlag, New York, 1996.

[8] E. Dobson, I. Kovács, and Š. Miklavič. The automorphism groups of non-edge-transitive rose window graphs. *Ars. Math. Contemp.*, 9:63–75, 2015.

[9] R. Frucht, J. E. Graver, and M. E. Watkins. The group of the generalized Petersen graphs. *Proc. Camb. Philos. Soc.*, 70:211–218, 1971.

[10] R. Jajcay, Š. Miklavič, P. Šparl, and G. Vasiljević. On certain edge-transitive bicirculants. *Electron. J. Combin.*, 26(2):#P2.6, 2019.

[11] B. Huppert. Endliche Gruppen I. Springer-Verlag, Berlin-Heidelberg-New York, 1967.

[12] I. Kovács. Classifying arc-transitive circulants. *J. Algebraic Combin.*, 20:353–358, 2005.

[13] I. Kovács, K. Kutnar, and D. Marušič. Classification of edge-transitive rose window graphs. *J. Graph Theory*, 65:216–231, 2010.

[14] I. Kovács, B. Kuzman, A. Malnič, and S. Wilson. Characterization of edge-transitive 4-valent bicirculants. *J. Graph Theory*, 69:441–463, 2012.

[15] K. Kutnar, D. Marušič, Š. Miklavič, and R. Strašek. Automorphisms of Tabačjn graphs. *Filomat*, 27:1157–1164, 2013.

[16] C. H. Li. Permutation groups with a cyclic regular subgroup and arc transitive circulants. *J. Algebraic Combin.*, 21:131–136, 2005.

[17] C. H. Li, B. Xia, and S. Zhou. An explicit characterisation of arc-transitive circulants. *J. Combin. Theory, Ser. B*, 150:1–16, 2021.

[18] C. H. Li, Z. P. Lu, and D. Marušič. On primitive permutation groups with small suborbits and their orbital graphs. *J. Algebra*, 279:749–770, 2004.

[19] A. Malnič, D. Marušič, P. Šparl, and B. Frelih. Symmetry structure of bicirculants. *Discrete Math.*, 307:409–414, 2007.

[20] D. Marušič and T. Pisanski. Symmetries of hexagonal molecular graphs on the torus. *Croat. Chem. Acta*, 73:969–981, 2000.

[21] P. Müller. Permutation groups with a cyclic two-orbits subgroup and monodromy groups of Laurent polynomials. *Ann. Sc. Norm. Super. Pisa CI Sci.*, 12:369–438, 2013.

[22] T. Pisanski. A classification of cubic bicirculants. *Discrete Math.*, 307:567–578, 2007.

[23] G. Sabidussi. On a class of fixed-point-free graphs. *Proc. Amer. Math. Soc.*, 9:800–804, 1958.

[24] G. Vasiljević. O simetričnih Nest grafih. MSc thesis, Faculty of Education of the University of Ljubljana, Ljubljana, 2017.

[25] M. E. Watkins. A Theorem on Tait colorings with an application to the generalized Petersen graphs. *J. Combin. Theory*, 6:152–164, 1969.

[26] S. Wilson. Rose window graphs. *Ars. Math. Contemp.*, 1:7–18, 2008.

[27] J.-X. Zhou and M.-M. Zhang. The classification of half-arc-regular bi-circulants of valency 6. *European J. Combin.*, 64:45–56, 2017.