Towards the Small Quasi-Kernel Conjecture

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Abstract

Let $D = (V, A)$ be a digraph. A vertex set $K \subseteq V$ is a quasi-kernel of $D$ if $K$ is an independent set in $D$ and for every vertex $v \in V \setminus K$, $v$ is at most distance 2 from $K$. In 1974, Chvátal and Lovász proved that every digraph has a quasi-kernel. P. L. Erdős and L. A. Székely in 1976 conjectured that if every vertex of $D$ has a positive indegree, then $D$ has a quasi-kernel of size at most $|V|/2$. This conjecture is only confirmed for narrow classes of digraphs, such as semicomplete multipartite, quasi-transitive, or locally semicomplete digraphs. In this note, we state a similar conjecture for all digraphs, show that the two conjectures are equivalent, and prove that both conjectures hold for a class of digraphs containing all orientations of 4-colorable graphs (in particular, of all planar graphs).

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1 Introduction and notation

The digraphs in this note may have antiparallel arcs, but do not have loops. Let $D$ be a digraph. We denote by $V(D)$ and $A(D)$ the vertex set and the arc set of $D$, respectively.

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We say $D$ is weakly connected if the underlying graph of $D$ is connected. Let $x \in V(D)$. The open (closed) outneighborhood and inneighborhood of $x$ in $D$, denoted $N_D^+(x)$ ($N_D^-(x)$) and $N_D^-(x)$ ($N_D^+(x)$) are defined as follows.

$$N_D^+(x) = \{ y \in V(D) \mid xy \in A(D) \}, \quad N_D^+(x) = N_D^+(x) \cup \{ x \},$$
$$N_D^-(x) = \{ y \in V(D) \mid yx \in A(D) \}, \quad N_D^-(x) = N_D^-(x) \cup \{ x \}.$$

The outdegree of $x$ in $D$ is $d_D^+(x) = |N_D^+(x)|$, and the indegree of $x$ in $D$ is $d_D^-(x) = |N_D^-(x)|$.

Vertices of indegree zero in $D$ are called sources of $D$ and vertices of outdegree zero in $D$ are called sinks of $D$. By $\delta^+(D)$ (respectively, $\delta^-(D)$) we denote the minimum outdegree (respectively, indegree) in $D$ among all vertices of $D$. For each $X \subseteq V(D)$, we let

$$N_D^+(X) = \bigcup_{x \in X} N_D^+(x) \setminus X, \quad N_D^+(X) = N_D^+(X) \cup X,$$
$$N_D^-(X) = \bigcup_{x \in X} N_D^-(x) \setminus X, \quad N_D^-(X) = N_D^-(X) \cup X.$$

Let $u, v \in V(D)$ and $K \subseteq V(D)$. The distance from $u$ to $v$ in $D$, denoted $d_D(u, v)$, is the length of a shortest directed path from $u$ to $v$. The distance from $K$ to $v$ in $D$, is $d_D(K, v) = \min \{ d_D(x, v) \mid x \in K \}$. We say $K$ is a kernel of $D$ if $K$ is independent in $D$ and for every $v \in V(D) \setminus K$, $d_D(K, v) = 1$. We say $K$ is a quasi-kernel of $D$ if $K$ is independent in $D$ and for every $v \in V(D) \setminus K$, $d_D(K, v) \leq 2$.

A digraph $D$ is kernel-perfect if every induced subdigraph of it has a kernel. Richardson proved the following result.

**Theorem 1** (Richardson [10]). *Every digraph without directed odd cycles is kernel-perfect.*

The proof gives rise to an algorithm to find one. On the other hand, Chvátal [4] showed that in general it is NP-complete to decide whether a digraph has a kernel, and by a result of Fraenkel [6] it is NP-complete even in the class of planar digraphs of degree at most 3. While not every digraph has a kernel, Chvátal and Lovász [5] proved that every digraph has a quasi-kernel. In 1976, P.L. Erdős and S. A. Székely made the following conjecture on the size of a quasi-kernel in a digraph.

**Conjecture 2** (Erdős–Székely [1]). *Every $n$-vertex digraph $D$ with $\delta^+(D) \geq 1$ has a quasi-kernel of size at most $\frac{n}{2}$.*

If $D$ is an $n$-vertex digraph consisting of the disjoint union of directed 2- and 4-cycles, then every kernel or quasi-kernel of $D$ has size exactly $\frac{n}{2}$. Thus, Conjecture 2 is sharp.

In 1996, Jacob and Meyniel [9] showed that a digraph without a kernel contains at least three distinct quasi-kernels. Gutin et al. [7] characterized digraphs with exactly one and two quasi-kernels, thus provided necessary and sufficient conditions for a digraph to have

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1Our definition of a kernel is the digraph dual of what was originally defined in [6], and it is “consistent” with the definition of a quasi-kernel.
at least three quasi-kernels. However, these results do not discuss the sizes of the quasi-kernels. Heard and Huang [8] in 2008 showed that each digraph $D$ with $\delta^+(D) \geq 1$ has two disjoint quasi-kernels if $D$ is semicomplete multipartite (including tournaments), quasi-transitive (including transitive digraphs), or locally semicomplete. As a consequence, Conjecture 2 is true for these three classes of digraphs.

We propose a conjecture which formally implies Conjecture 2. It suggests a bound for digraphs that may have sources. Note that each quasi-kernel of a digraph contains all of its source vertices and hence contains no outneighbors of the source vertices.

**Conjecture 3.** Let $D$ be an $n$-vertex digraph, and let $S$ be the set of sources of $D$. Then $D$ has a quasi-kernel $K$ such that

$$|K| \leq \frac{n + |S| - |N_D^+(S)|}{2}.$$ 

To show that the upper bound above is best possible, consider the following examples.

- Let $S$ be a nonempty set of isolated vertices, and let $D$ be a digraph obtained from a directed triangle by adding an arc from every vertex in $S$ to the same vertex in the triangle. Then every quasi-kernel of $D$ has size $|S| + 1 = \frac{(|S|+3)+|S|-1}{2}$.

- Let $D$ be an orientation of a connected bipartite graph with parts $S$ and $T$ where each arc goes from $S$ to $T$. Then $S$ forms a quasi-kernel of $D$ of size $|S| = \frac{(|S|+|T|)+|S|-|T|}{2}$.

In this paper, we support Conjectures 2 and 3 by showing the following results.

**Theorem 4.** Let $D$ be an $n$-vertex digraph and $S$ be the set of sources of $D$. Suppose that $V(D) \setminus N_D^+[S]$ has a partition $V_1 \cup V_2$ such that $D[V_i]$ is kernel-perfect for each $i = 1, 2$. Then $D$ has a quasi-kernel of size at most $\frac{n + |S| - |N_D^+(S)|}{2}$.

Since by Theorem 1, every digraph without directed odd cycles is kernel-perfect, Theorem 4 immediately yields:

**Corollary 5.** Conjectures 2 and 3 hold for every orientation of each graph with chromatic number at most 4.

By the Four Color Theorem [2, 3], Corollary 5 yields that Conjectures 2 and 3 hold for every digraph whose underlying graph is planar.

**Theorem 6.** If Conjecture 3 fails and $D$ is a counterexample to it with the minimum number of vertices, then $D$ has no source.

Since Conjecture 3 implies Conjecture 2, Theorem 6 implies that the two conjectures are equivalent.

In the next section we prove Theorem 4 and in Section 3 prove Theorem 6.
2 Proof of Theorem 4

Let \( D' = D - N^+_D[S] \) be the digraph obtained by removing the source vertices and their outneighbors, and \( V_1 \cup V_2 = V(D') \) be a partition of \( V(D') \) such that \( D[V_1] \) is kernel-perfect for each \( i = 1, 2 \). In addition, we choose such a partition so that \( |V_2| \) is as small as possible. Observe that adding a source vertex \( v \) to a kernel-perfect digraph \( H \) results in a new kernel-perfect digraph: let \( H' \) be the resulting digraph, and let \( F \) be a subdigraph of \( H' \) that contains \( v \). Then \( K \cup \{v\} \) is a kernel of \( F \) where \( K \) is any kernel of \( F - N^+_H[v] \) in \( H \).

If there exists some \( v \in V_2 \) with no inneighbors in \( V_1 \), then we may move \( v \) from \( V_2 \) to \( V_1 \), and obtain a new partition of \( V(D') \) into kernel-perfect subgraphs with a smaller \( V_2 \) by Theorem 1. Thus, by the choice of \( V_2 \),

\[
N^+_D(v) \cap V_1 \neq \emptyset \quad \text{for every } v \in V_2. \tag{1}
\]

For a digraph \( F \) and an independent set \( R \subseteq V(F) \), we say \( R_0 \subseteq R \) is a concise set of \( R \) in \( F \) if \( N^+_F(R_0) = N^+_F(R) \) and \( |R_0| \leq |N^+_F(R_0)| \). Indeed, every independent set has a concise set—iteratively add vertices \( R \) to \( R_0 \) if and only if \( |N^+_F(R_0 \cup \{v\})| > |N^+_F(R_0)| \).

Since \( D[V_1] \) is kernel-perfect, it has a kernel \( R \). Let \( R_0 \) be a concise set of \( R \) in \( D' \). Let \( D'' = D' - (R_0 \cup N^+_D(R_0)) = D' - N^+_D[R_0] \). We partition \( R \setminus R_0 \) into sets \( S'' \) and \( T \) of sources and non-sources in \( D'' \) respectively. Note that since each \( v \in S'' \) was not a source in the original digraph \( D, v \) must have an innneighbor in \( V(D) - V(D'') \).

Set \( K = S \cup R_0 \cup T \). We will show that \( K \) is a quasi-kernel of \( D \). We first show that it is independent. Indeed, \( K \cap R \) is independent, since \( R \) was a kernel of \( D[V_1] \). There are no arcs from \( K \cap R \) to \( K \setminus R = S \) because each vertex in \( S \) is a source in \( D \). Similarly, there are no arcs from \( S \) to \( K \setminus S \). Finally, there are no arcs from \( S \) to \( K \setminus S \) because \( K \setminus S \subseteq V(D') = V(D) - N^+_D[S] \).

Now we check that each vertex is at distance at most 2 from \( K \). For any \( v \in N^+_D[K] \), we have \( \text{dist}_D(K,v) \leq 1 \). Consider \( v \in V_1 \setminus N^+_D[K] \). Recall that \( R \) is a kernel of \( D[V_1] \), so \( V_1 \subseteq N^+_D[R] \). It follows that since \( R_0 \) is a concise set of \( R \), the vertex \( v \) must be contained in \( R \setminus K = S'' \). Therefore \( v \) has an innneighbor in \( N^+_D[S] \cup N^+_D[R_0] \subseteq N^+_D[K] \), hence \( \text{dist}_D(K,v) \leq 2 \).

Now suppose \( v \in V_2 \setminus N^+_D[K] \). By (1), \( v \) has an innneighbor \( u \in V_1 \). If \( u \in N^+_D[K] \), then \( \text{dist}_D(K,v) \leq 2 \). So we may assume \( u \in V_1 \setminus N^+_D[K] = S'' \). Since \( S'' \subseteq R \), \( v \in N^+_D[R] \). But \( R_0 \) is a concise set of \( R \), so \( v \in N^+_D[R_0] \subseteq N^+_D[K] \). We get \( \text{dist}_D(K,v) \leq 1 \).

Therefore, \( K \) is a quasi-kernel of \( D \). If \( |T| \leq |V(D'') \setminus T| \) (so \( 2|T| \leq |V(D'') \cup T| = |V(D'')|) \), then using the fact that \( R_0 \) is a concise set,

\[
|K| = |S| + |R_0| + |T| \leq |S| + \frac{1}{2}|R_0 \cup N^+_D(R)| + \frac{1}{2}|V(D'')| \\
\leq |S| + \frac{1}{2}|V(D) \setminus N^+_D[S]| \leq \frac{1}{2}(n + |S| - |N^+_D(S)|),
\]

and the theorem holds. Thus, assume that \( |T| > |V(D'') \setminus T| \) (so \( |V(D'') \setminus T| < |V(D'')|/2 \)). Note that \( V(D'') \setminus T = (V_2 \setminus N^+_D[R]) \cup S'' \). Since \( D[V_2] \) is kernel-perfect and adding source
vertices preserves kernel-perfectness, the digraph $D'' - T$ is also kernel-perfect. Let $W$ be a kernel of $D'' - T$ and set $K' = (S \cup R_0 \cup W) \setminus N_D^+(W)$.

Similarly to $K$, the set $K'$ is independent in $D$. Since $|T| > |V(D'') \setminus T|$, 

$$|K'| \leq |S| + |R_0| + |W| \leq |S| + \frac{1}{2}|R_0 \cup N_D^+(R)| + \frac{1}{2}|V(D'')| \leq \frac{n + |S| - |N_D^+(S)|}{2}.$$ We now show that $dist_D(K', v) \leq 2$ for every $v \in V(D) \setminus K'$.

Observe that $S'' \subseteq W$ since the vertices in $S''$ are sources in $D'' - T$. Clearly, we have that each vertex $v \in V(D'') - T$ has $dist_D(K', v) \leq 1$. Now suppose $v \in T$. Since $v$ is not a source in $D''$, it has an inneighbor in $V(D'')$, and this neighbor cannot be in $T$ because $T \subset R$ is independent. Hence $dist_D(K', v) \leq 2$.

We have $dist_D(K', v) \leq 1$ for all $v \in N_D^+[S]$. It remains to consider $v \in V(D') \setminus V(D'') = N_D^+[R_0]$. If $v \in R_0$, then either $v \in K'$ or $v \in N_D^+(W)$. Hence $dist_D(K', v) \leq 1$. It follows that $dist_D(K', v) \leq 2$ for all $v \in N_D^+[R_0]$. Therefore $K'$ is a quasi-kernel of $D$.

\[\square\]

3 Proof of Theorem 6

Assume Conjecture 3 fails and $D$ is a counterexample to it with the fewest vertices. Let $n = |V(D)|$. We assume $n \geq 4$ as the cases $n \leq 3$ are verifiable by hand. By the minimality of $n$, $D$ is weakly connected. Let $S$ be the set of sources of $D$. We show that $S = \emptyset$. Assume instead that $S \neq \emptyset$.

**Case 1:** $|N_D^+[S]| \geq 3$. Let $D_1$ be obtained from $D$ by deleting all vertices in $N_D^+[S]$, adding two new vertices $x$ and $y$, adding an arc from $y$ to every vertex of $D - N_D^+[S]$ that is an outneighbor of some vertex of $N_D^+(S)$ in $D$, and adding an arc from $x$ to $y$. Then $x$ is the only source vertex of $D_1$, and $N_D^+(x) = \{y\}$. Since $|V(D_1)| = |V(D)| - |N_D^+[S]| + 2 \leq |V(D)| - 1$, the minimality of $n$ implies that $D_1$ has a quasi-kernel $K_1$ of size at most $\frac{n - |N_D^+[S]| + 2 + 1 - 1}{2} - 1 + |S| = \frac{n + |S| - |N_D^+(S)|}{2}$.

Then $K = (K_1 \setminus \{x\}) \cup S$ is a quasi-kernel of $G$ that has size at most 

\[\frac{n - |N_D^+[S]| + 2 + 1 - 1}{2} - 1 + |S| = \frac{n + |S| - |N_D^+(S)|}{2},\]

as desired.

**Case 2:** $|N_D^+[S]| \leq 2$. Since $D$ is weakly connected, and $|S| \geq 1$, we get $|S| = 1$ and $|N_D^+[S]| = 1$. Let $D_1 = D - N_D^+[S]$. If $D_1$ has no sources, then by the minimality of $D$, digraph $D_1$ has a quasi-kernel $K_1$ with $|K_1| \leq \frac{n - 2}{2}$. Then $K = K_1 \cup S$ is a desired quasi-kernel of $D$. Therefore, we assume that $D_1$ has a source. Let

$$S_1 = \{v \in V(D_1) \mid d_{D_1}(v) = 0\}.$$

If $|N_D^+(S_1)| \leq |S_1|$, we let $D_2 = D_1 - S_1$. By the minimality of $D$, $D_2$ has a quasi-kernel $K_1$ of size at most $\frac{n - 2 - |S_1| + |N_D^+(S_1)|}{2} \leq \frac{n - 2}{2}$. Then $K = K_1 \cup S$ is a desired quasi-kernel of $D$. Thus, we assume that $|N_D^+(S_1)| > |S_1|$. Let $D_2$ be obtained from $D_1$ by deleting all
vertices in $N^+_D(S_1)$, adding two new vertices $x$ and $y$, adding an arc from $y$ to every vertex of $D_1 - N^+_D(S_1)$ that is an outneighbor of some vertex of $N^+_D(S_1)$ in $D_1$, and adding an arc from $x$ to $y$. Note that $x$ is the only source of $D_2$, and $N^+_D(x) = \{y\}$. Again, by the minimality of $D$, $D_2$ has a quasi-kernel $K_1$ of size at most $\frac{n-2-|N^+_D(S_1)|+2+1-1}{2}$. Then $K = (K_1 \setminus \{x\}) \cup S \cup S_1$ is a quasi-kernel of $D$ that has size at most

$$n - 2 - |N^+_D(S_1)| + 2 + 1 - 1 - 1 + |S| + |S_1| \leq \frac{n-1}{2},$$

as desired.

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References


