Towards the Small Quasi-Kernel Conjecture

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Abstract

Let $D = (V, A)$ be a digraph. A vertex set $K \subseteq V$ is a *quasi-kernel* of D if K is an independent set in D and for every vertex $v \in V \setminus K$, v is at most distance 2 from K. In 1974, Chvátal and Lovász proved that every digraph has a quasi-kernel. P. L. Erdős and L. A. Székely in 1976 conjectured that if every vertex of D has a positive indegree, then D has a quasi-kernel of size at most $|V|/2$. This conjecture is only confirmed for narrow classes of digraphs, such as semicomplete multipartite, quasi-transitive, or locally semicomplete digraphs. In this note, we state a similar conjecture for all digraphs, show that the two conjectures are equivalent, and prove that both conjectures hold for a class of digraphs containing all orientations of 4-colorable graphs (in particular, of all planar graphs).

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1 Introduction and notation

The digraphs in this note may have antiparallel arcs, but do not have loops. Let D be a digraph. We denote by $V(D)$ and $A(D)$ the vertex set and the arc set of D, respectively.

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We say D is *weakly connected* if the underlying graph of D is connected. Let $x \in V(D)$. The *open (closed) outneighborhood* and *inneighborhood* of x in D, denoted $N_D^+(x)$ ($N_D^+[x]$) and $N_D^-(x)$ ($N_D^-[x]$) are defined as follows.

$$N_D^+(x) = \{y \in V(D) \mid xy \in A(D)\}, \quad N_D^+[x] = N_D^+(x) \cup \{x\},$$
$$N_D^-(x) = \{y \in V(D) \mid yx \in A(D)\}, \quad N_D^-[x] = N_D^-(x) \cup \{x\}.$$

The *outdegree* of x in D is $d_D^+(x) = |N_D^+(x)|$, and the *indegree* of x in D is $d_D^-(x) = |N_D^-(x)|$. Vertices of indegree zero in D are called *sources* of D and vertices of outdegree zero in D are called *sinks* of D. By $\delta^+(D)$ (respectively, $\delta^-(D)$) we denote the minimum outdegree (respectively, indegree) in D among all vertices of D. For each $X \subseteq V(D)$, we let

$$N_D^+(X) = \bigcup_{x \in X} N_D^+(x) \setminus X, \quad N_D^+[X] = N_D^+(X) \cup X,$$
$$N_D^-(X) = \bigcup_{x \in X} N_D^-(x) \setminus X, \quad N_D^-[X] = N_D^-(X) \cup X.$$

Let $u, v \in V(D)$ and $K \subseteq V(D)$. The distance from u to v in D, denoted $\mathrm{dist}_D(u, v)$, is the length of a shortest directed path from u to v. The distance from K to v in D, is $\mathrm{dist}_D(K, v) = \min\{\mathrm{dist}_D(x, v) \mid x \in K\}$. We say K is a *kernel* of D if K is independent in D and for every $v \in V(D) \setminus K$, $\mathrm{dist}_D(K, v) = 1$. We say K is a *quasi-kernel* of D if K is independent in D and for every $v \in V(D) \setminus K$, $\mathrm{dist}_D(K, v) \leqslant 2$. [1]

A digraph D is *kernel-perfect* if every induced subdigraph of it has a kernel. Richardson proved the following result.

Theorem 1 (Richardson [10]). *Every digraph without directed odd cycles is kernel-perfect.*

The proof gives rise to an algorithm to find one. On the other hand, Chvátal [4] showed that in general it is NP-complete to decide whether a digraph has a kernel, and by a result of Fraenkel [6] it is NP-complete even in the class of planar digraphs of degree at most 3. While not every digraph has a kernel, Chvátal and Lovász [5] proved that every digraph has a quasi-kernel. In 1976, P.L. Erdős and S. A. Székely made the following conjecture on the size of a quasi-kernel in a digraph.

Conjecture 2 (Erdős–Székely [1]). Every n-vertex digraph D with $\delta^+(D) \geqslant 1$ has a quasi-kernel of size at most $\frac{n}{2}$.

If D is an n-vertex digraph consisting of the disjoint union of directed 2- and 4-cycles, then every kernel or quasi-kernel of D has size exactly $\frac{n}{2}$. Thus, Conjecture 2 is sharp.

In 1996, Jacob and Meyniel [9] showed that a digraph without a kernel contains at least three distinct quasi-kernels. Gutin et al. [7] characterized digraphs with exactly one and two quasi-kernels, thus provided necessary and sufficient conditions for a digraph to have

[1] Our definition of a kernel is the digraph dual of what was originally defined in [6], and it is "consistent" with the definition of a quasi-kernel.

at least three quasi-kernels. However, these results do not discuss the sizes of the quasi-kernels. Heard and Huang [8] in 2008 showed that each digraph D with $\delta^+(D) \geqslant 1$ has two disjoint quasi-kernels if D is semicomplete multipartite (including tournaments), quasi-transitive (including transitive digraphs), or locally semicomplete. As a consequence, Conjecture 2 is true for these three classes of digraphs.

We propose a conjecture which formally implies Conjecture 2. It suggests a bound for digraphs that may have sources. Note that each quasi-kernel of a digraph contains all of its source vertices and hence contains no outneighbors of the source vertices.

Conjecture 3. Let D be an n-vertex digraph, and let S be the set of sources of D. Then D has a quasi-kernel K such that

$$|K| \leqslant \frac{n + |S| - |N_D^+(S)|}{2}.$$

To show that the upper bound above is best possible, consider the following examples.

- Let S be a nonempty set of isolated vertices, and let D be a digraph obtained from a directed triangle by adding an arc from every vertex in S to the same vertex in the triangle. Then every quasi-kernel of D has size $|S| + 1 = \frac{(|S|+3)+|S|-1}{2}$.
- Let D be an orientation of a connected bipartite graph with parts S and T where each arc goes from S to T. Then S forms a quasi-kernel of D of size $|S| = \frac{(|S|+|T|)+|S|-|T|}{2}$.

In this paper, we support Conjectures 2 and 3 by showing the following results.

Theorem 4. *Let D be an n-vertex digraph and S be the set of sources of D. Suppose that $V(D) \setminus N_D^+[S]$ has a partition $V_1 \cup V_2$ such that $D[V_i]$ is kernel-perfect for each $i = 1, 2$. Then D has a quasi-kernel of size at most $\frac{n+|S|-|N_D^+(S)|}{2}$.*

Since by Theorem 1, every digraph without directed odd cycles is kernel-perfect, Theorem 4 immediately yields:

Corollary 5. *Conjectures 2 and 3 hold for every orientation of each graph with chromatic number at most* 4*.*

By the Four Color Theorem [2, 3], Corollary 5 yields that Conjectures 2 and 3 hold for every digraph whose underlying graph is planar.

Theorem 6. *If Conjecture 3 fails and D is a counterexample to it with the minimum number of vertices, then D has no source.*

Since Conjecture 3 implies Conjecture 2, Theorem 6 implies that the two conjectures are equivalent.

In the next section we prove Theorem 4 and in Section 3 prove Theorem 6.

2 Proof of Theorem 4

Let $D' = D - N_D^+[S]$ be the digraph obtained by removing the source vertices and their outneighbors, and $V_1 \cup V_2 = V(D')$ be a partition of $V(D')$ such that $D[V_i]$ is kernel-perfect for each $i = 1, 2$. In addition, we choose such a partition so that $|V_2|$ is as small as possible. Observe that adding a source vertex v to a kernel-perfect digraph H results in a new kernel-perfect digraph: let H' be the resulting digraph, and let F be a subdigraph of H' that contains v. Then $K \cup \{v\}$ is a kernel of F where K is any kernel of $F - N_{H'}^+[v]$ in H.

If there exists some $v \in V_2$ with no inneighbors in V_1, then we may move v from V_2 to V_1, and obtain a new partition of $V(D')$ into kernel-perfect subgraphs with a smaller V_2 by Theorem 1. Thus, by the choice of V_2,

$$N_{D'}^-(v) \cap V_1 \neq \varnothing \quad \text{for every } v \in V_2. \tag{1}$$

For a digraph F and an independent set $R \subseteq V(F)$, we say $R_0 \subseteq R$ is a *concise set* of R in F if $N_F^+(R_0) = N_F^+(R)$ and $|R_0| \leqslant |N_F^+(R_0)|$. Indeed, every independent set has a concise set—iteratively add vertices v from R to R_0 if and only if $|N_F^+(R_0 \cup \{v\})| > |N_F^+(R_0)|$.

Since $D[V_1]$ is kernel-perfect, it has a kernel R. Let R_0 be a concise set of R in D'. Let $D'' = D' - (R_0 \cup N_{D'}^+(R)) = D' - N_{D'}^+[R_0]$. We partition $R \setminus R_0$ into sets S'' and T of sources and non-sources in D'' respectively. Note that since each $v \in S''$ was not a source in the original digraph D, v must have an inneighbor in $V(D) - V(D'')$.

Set $K = S \cup R_0 \cup T$. We will show that K is a quasi-kernel of D. We first show that it is independent. Indeed, $K \cap R$ is independent, since R was a kernel of $D[V_1]$. There are no arcs from $K \cap R$ to $K \setminus R = S$ because each vertex in S is a source in D. Similarly, there are no arcs from S to S. Finally, there are no arcs from S to $K \setminus S$ because $K \setminus S \subseteq V(D') = V(D) - N_D^+[S]$.

Now we check that each vertex is at distance at most 2 from K. For any $v \in N_D^+[K]$, we have $\operatorname{dist}_D(K, v) \leqslant 1$. Consider $v \in V_1 \setminus N_D^+[K]$. Recall that R is a kernel of $D[V_1]$, so $V_1 \subseteq N_D^+[R]$. It follows that since R_0 is a concise set of R, the vertex v must be contained in $R \setminus K = S''$. Therefore v has an inneighbor in $N_D^+[S] \cup N_{D'}^+[R_0] \subseteq N_D^+[K]$, hence $\operatorname{dist}_D(K, v) \leqslant 2$.

Now suppose $v \in V_2 \setminus N_D^+[K]$. By (1), v has an inneighbor $u \in V_1$. If $u \in N_D^+[K]$, then $\operatorname{dist}_D(K, v) \leqslant 2$. So we may assume $u \in V_1 \setminus N_D^+[K] = S''$. Since $S'' \subseteq R$, $v \in N_{D'}^+[R]$. But R_0 is a concise set of R, so $v \in N_{D'}^+(R_0) \subseteq N_D^+[K]$. We get $\operatorname{dist}_D(K, v) \leqslant 1$.

Therefore, K is a quasi-kernel of D. If $|T| \leqslant |V(D'') \setminus T|$ (so $2|T| \leqslant |V(D'') \cup T| = |V(D'')|$), then using the fact that R_0 is a concise set,

$$\begin{aligned} |K| = |S| + |R_0| + |T| &\leqslant |S| + \frac{1}{2}|R_0 \cup N_{D'}^+(R)| + \frac{1}{2}|V(D'')| \\ &\leqslant |S| + \frac{1}{2}|V(D) \setminus N_D^+[S]| \leqslant \frac{1}{2}(n + |S| - |N_D^+(S)|), \end{aligned}$$

and the theorem holds. Thus, assume that $|T| > |V(D'') \setminus T|$ (so $|V(D'') \setminus T| < |V(D'')|/2$). Note that $V(D'') \setminus T = (V_2 \setminus N_{D'}^+(R)) \cup S''$. Since $D[V_2]$ is kernel-perfect and adding source vertices preserves kernel-perfectness, the digraph $D'' - T$ is also kernel-perfect. Let W be a kernel of $D'' - T$ and set $K' = (S \cup R_0 \cup W) \setminus N_D^+(W)$.

Similarly to K, the set K' is independent in D. Since $|T| > |V(D'') \setminus T|$,

$$|K'| \leqslant |S| + |R_0| + |W| \leqslant |S| + \frac{1}{2}|R_0 \cup N_{D'}^+(R)| + \frac{1}{2}|V(D'')| \leqslant \frac{n + |S| - |N_D^+(S)|}{2}.$$

We now show that $\mathrm{dist}_D(K', v) \leqslant 2$ for every $v \in V(D) \setminus K'$.

Observe that $S'' \subseteq W$ since the vertices in S'' are sources in $D'' - T$. Clearly, we have that each vertex $v \in V(D'' - T)$ has $\mathrm{dist}_D(K', v) \leqslant 1$. Now suppose $v \in T$. Since v is not a source in D'', it has an inneighbor in $V(D'')$, and this neighbor cannot be in T because $T \subset R$ is independent. Hence $\mathrm{dist}_D(K', v) \leqslant 2$.

We have $\mathrm{dist}_D(K', v) \leqslant 1$ for all $v \in N_D^+[S]$. It remains to consider $v \in V(D') \setminus V(D'') = N_{D'}^+[R_0]$. If $v \in R_0$, then either $v \in K'$ or $v \in N_D^+(W)$. Hence $\mathrm{dist}_D(K', v) \leqslant 1$. It follows that $\mathrm{dist}_D(K', v) \leqslant 2$ for all $v \in N_{D'}^+(R_0)$. Therefore K' is a quasi-kernel of D. □

3 Proof of Theorem 6

Assume Conjecture 3 fails and D is a counterexample to it with the fewest vertices. Let $n = |V(D)|$. We assume $n \geqslant 4$ as the cases $n \leqslant 3$ are verifiable by hand. By the minimality of n, D is weakly connected. Let S be the set of sources of D. We show that $S = \varnothing$. Assume instead that $S \neq \varnothing$.

Case 1: $|N_D^+[S]| \geqslant 3$. Let D_1 be obtained from D by deleting all vertices in $N_D^+[S]$, adding two new vertices x and y, adding an arc from y to every vertex of $D - N_D^+[S]$ that is an outneighbor of some vertex of $N_D^+(S)$ in D, and adding an arc from x to y. Then x is the only source vertex of D_1, and $N_{D_1}^+(x) = \{y\}$. Since $|V(D_1)| = |V(D)| - |N_D^+[S]| + 2 \leqslant |V(D)| - 1$, the minimality of n implies that D_1 has a quasi-kernel K_1 of size at most $\frac{n - |N_D^+[S]| + 2 + 1 - 1}{2}$. Then $K = (K_1 \setminus \{x\}) \cup S$ is a quasi-kernel of G that has size at most

$$\frac{n - |N_D^+[S]| + 2 + 1 - 1}{2} - 1 + |S| = \frac{n + |S| - |N_D^+(S)|}{2},$$

as desired.

Case 2: $|N_D^+[S]| \leqslant 2$. Since D is weakly connected, and $|S| \geqslant 1$, we get $|S| = 1$ and $|N_D^+(S)| = 1$. Let $D_1 = D - N_D^+[S]$. If D_1 has no sources, then by the minimality of D, digraph D_1 has a quasi-kernel K_1 with $|K_1| \leqslant \frac{n-2}{2}$. Then $K = K_1 \cup S$ is a desired quasi-kernel of D. Therefore, we assume that D_1 has a source. Let

$$S_1 = \{v \in V(D_1) \,|\, d_{D_1}^-(v) = 0\}.$$

If $|N_{D_1}^+(S_1)| \leqslant |S_1|$, we let $D_2 = D_1 - S_1$. By the minimality of D, D_2 has a quasi-kernel K_1 of size at most $\frac{n - 2 - |S_1| + |N_{D_1}(S_1)|}{2} \leqslant \frac{n-2}{2}$. Then $K = K_1 \cup S$ is a desired quasi-kernel of D. Thus, we assume that $|N_{D_1}(S_1)| > |S_1|$. Let D_2 be obtained from D_1 by deleting all vertices in $N^+_{D_1}[S_1]$, adding two new vertices x and y, adding an arc from y to every vertex of $D_1 - N^+_{D_1}[S_1]$ that is an outneighbor of some vertex of $N^+_{D_1}(S_1)$ in D_1, and adding an arc from x to y. Note that x is the only source of D_2, and $N^+_{D_2}(x) = \{y\}$. Again, by the minimality of D, D_2 has a quasi-kernel K_1 of size at most $\frac{n-2-|N^+_{D_1}[S_1]|+2+1-1}{2}$. Then $K = (K_1 \setminus \{x\}) \cup S \cup S_1$ is a quasi-kernel of D that has size at most

$$\frac{n-2-|N^+_{D_1}[S_1]|+2+1-1}{2} - 1 + |S| + |S_1| \leqslant \frac{n-1}{2},$$

as desired. □

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