Linear Bounds for Cycle-Free Saturation Games

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Abstract

Given a family of graphs $\mathcal{F}$, we define the $\mathcal{F}$-saturation game as follows. Two players alternate adding edges to an initially empty graph on n vertices, with the only constraint being that neither player can add an edge that creates a subgraph in $\mathcal{F}$. The game ends when no more edges can be added to the graph. One of the players wishes to end the game as quickly as possible, while the other wishes to prolong the game. We let $\mathrm{sat}_g(n, \mathcal{F})$ denote the number of edges that are in the final graph when both players play optimally.

In general there are very few non-trivial bounds on the order of magnitude of $\mathrm{sat}_g(n, \mathcal{F})$. In this work, we find collections of infinite families of cycles $\mathcal{C}$ such that $\mathrm{sat}_g(n, \mathcal{C})$ has linear growth rate.

Mathematics Subject Classifications: 05C57, 05C35, 05C38

1 Introduction

Given a family of graphs $\mathcal{F}$, we say a graph G is $\mathcal{F}$-saturated if G does not contain a subgraph isomorphic to any $F \in \mathcal{F}$, but adding any edge to G creates a copy of some $F \in \mathcal{F}$. The study of $\mathcal{F}$-saturated graphs is one of the main topics of interest in extremal combinatorics. Particularly well studied quantities include the *extremal number* $\mathrm{ex}(n, \mathcal{F})$, which denotes the maximum number of edges in an n-vertex $\mathcal{F}$-saturated graph, and the *saturation number* $\mathrm{sat}(n, \mathcal{F})$, which denotes the minimum number of edges in an n vertex $\mathcal{F}$-saturated graph. If $\mathcal{F} = \{F\}$, then we will often denote $\mathcal{F}$ simply by F.

For a family of graphs $\mathcal{F}$, we define the $\mathcal{F}$-saturation game as follows. The game is played by two players, Max and Mini. Starting with Max, the two players alternate adding edges to an empty graph on n vertices, with the only constraint being that neither player can add an edge that creates a subgraph in $\mathcal{F}$. The game ends once the graph becomes $\mathcal{F}$-saturated. We define the $\mathcal{F}$-game saturation number $\mathrm{sat}_g(n, \mathcal{F})$ to be the number of edges in the graph at the end of the game when both players play optimally.

It is worth noting that

$$\mathrm{sat}(n, \mathcal{F}) \leqslant \mathrm{sat}_g(n, \mathcal{F}) \leqslant \mathrm{ex}(n, \mathcal{F}). \tag{1}$$

The first $\mathcal{F}$-saturation game to be considered was the C_3-saturation game, which was introduced by Füredi, Reimer, and Seress [7] based on another game proposed by Hajnal. Despite being introduced nearly 30 years ago, very little is known about $\mathrm{sat}_g(n, C_3)$. In their original paper, Füredi, Reimer, and Seress proved that $\mathrm{sat}_g(n, C_3) \geqslant \frac{1}{2} n \log(n) + o(n \log(n))$. The only other non-trivial bound for $\mathrm{sat}_g(n, C_3)$ was obtained recently by Biró, Horn, and Wildstrom [1] who showed that $\mathrm{sat}_g(n, C_3) \leqslant \frac{26}{121} n^2 + o(n^2)$.

The systematic study of saturation games was initiated by Carraher, Kinnersley, Reiniger, and West [2]. In particular, they obtained bounds for the game saturation number of paths and stars, and these results were refined by Lee and Riet [11]. Hefetz, Krivelevich, Naor, and Stojaković [8] further generalized saturation games to avoiding other graph properties such as colorability, and in particular Keusch [9] proved asymptotically tight bounds for the game where both players must keep the graph 4-colorable.

Saturation games have also been generalized to other combinatorial structures such as hypergraphs and directed graphs [10, 12].

In view of (1), the problem of determining the order of magnitude of $\text{sat}_g(n, \mathcal{F})$ is trivial whenever $\text{ex}(n, \mathcal{F})$ and $\text{sat}(n, \mathcal{F})$ have the same order of magnitude. For example, many families which contain a tree will have linear extremal and saturation number; see [5] for further details. Perhaps the simplest non-trivial case of this problem is to try and determine the order of magnitude of $\text{sat}_g(n, \mathcal{C})$ when $\mathcal{C}$ is a family of cycles.

A basic question in this direction that one could ask is: what families of cycles $\mathcal{C}$ have quadratic game saturation number? It is well known that $\text{ex}(n, C_{2k}) = o(n^2)$ for all k, so by (1) a necessary condition to have $\text{sat}_g(n, \mathcal{C}) = \Theta(n^2)$ is that $\mathcal{C}$ consists only of odd cycles. The last author [13] showed that a sufficient condition for a set of odd cycles $\mathcal{C}$ to have quadratic game-saturation number is to have $C_3, C_5 \in \mathcal{C}$, in which case we have $\text{sat}_g(n, \mathcal{C}) \geqslant \frac{6}{25}n^2 + o(n^2)$.

Carraher, Kinnersley, Reiniger, and West [2] showed that $\text{sat}_g(n, \mathcal{C}^o) = \lfloor n^2/4 \rfloor$ where $\mathcal{C}^o$ is the set of all odd cycles [2], though the last author [13] showed that in general a set of odd cycles $\mathcal{C}$ containing C_3 and C_5 need not have $\text{sat}_g(n, \mathcal{C}) = (1 + o(1))\frac{1}{4}n^2$.

It is straightforward to show that $\text{sat}(n, \mathcal{C}) = \Omega(n)$ for any set of cycles $\mathcal{C}$, so it is natural to ask for necessary and sufficient conditions for $\text{sat}_g(n, \mathcal{C})$ to be linear. Much less is known about this problem. Let $\mathcal{C}_{\geqslant k}$ (respectively, $\mathcal{C}_{\geqslant k}^o$) denote the set of all cycles (respectively, all odd cycles) of length at least k. The following result of Erdős and Gallai shows that a trivial condition to have $\text{sat}_g(n, \mathcal{C}) = O(n)$ is for $\mathcal{C}$ to contain every cycle which is at least as large as some cutoff value k.

Theorem 1 ([4, Theorem 2.7]). *For all n, k we have*

$$\text{ex}(n, \mathcal{C}_{\geqslant k}) \leqslant \frac{1}{2}(k-1)(n-1).$$

Prior to this work, the only known non-trivial example of a set of cycles with linear game saturation number was the following.

Theorem 2 ([13, Theorem 1.4]).

$$\text{sat}_g(n, \mathcal{C}_{\geqslant 5}^o) \leqslant 2n - 2.$$

The key idea in the proof of Theorem 2 is that Mini is able to play in the $\mathcal{C}_{\geqslant 5}^o$-saturation game such that the graph stays $\mathcal{C}_{\geqslant 5}$-free throughout the game, so the result follows by Theorem 1. Our main goal is this paper is to generalize the approach used in Theorem 2 to prove that $\text{sat}_g(n, \mathcal{C}) = O(n)$ for many more families of cycles $\mathcal{C}$.

1.1 Our Results

Our first result shows that if $\mathcal{C}$ includes roughly half the cycles of length at least as large as some cutoff value k, then it has game saturation number which grows linearly with n.

Theorem 3. *Let $\mathcal{C}$ be a collection of cycles such that $C_3 \notin \mathcal{C}$, and such that there exists some $k \geqslant 3$ so that for all $\ell \geqslant k$, either $C_\ell \in \mathcal{C}$ or $C_{\ell+1} \in \mathcal{C}$. Then*

$$\mathrm{sat}_g(n, \mathcal{C}) \leqslant \frac{1}{2}(3k-1)(n-1).$$

Applying this theorem with $2k+1$ in place of k immediately gives the following.

Corollary 4. *For $k \geqslant 2$,*

$$\mathrm{sat}_g(n, \mathcal{C}^o_{\geqslant 2k+1}) \leqslant (3k+1)(n-1).$$

Our next result applies to infinite families of cycles which are much sparser than those considered in Theorem 3.

Definition 5. For $k \geqslant 5$, a family of cycles $\mathcal{C}$ is said to be *k-dense* if the following properties hold:

1. The cycles $C_k, C_{k+1} \in \mathcal{C}$, unless $k = 5$ in which case we only require $C_5 \in \mathcal{C}$,

2. For all $\ell < k$, we have $C_\ell \notin \mathcal{C}$,

3. For all $s \geqslant 3$ there exists ℓ with $s + 2 \leqslant \ell \leqslant 3 + (k-2)(s-2)$ and $C_\ell \in \mathcal{C}$.

Roughly speaking, a family $\mathcal{C}$ is k-dense if it contains no cycle up to length k, we have $C_k, C_{k+1} \in \mathcal{C}$, and the gaps between consecutive cycle lengths in $\mathcal{C}$ grow no faster than an exponential function in $k - 2$.

Theorem 6. *If $\mathcal{C}$ is a k-dense family of cycles, then*

$$\mathrm{sat}_g(n, \mathcal{C}) \leqslant \frac{1}{2}(k-1)(n-1) + 1.$$

Corollary 7. *Let $\mathcal{C} = \bigcup_{r \geqslant 0}\{C_{3^r+4}\} = \{C_5, C_7, C_{13}, C_{31}, C_{85}, \ldots\}$. Then*

$$\mathrm{sat}_g(n, \mathcal{C}) \leqslant 2n - 1.$$

Note that the gaps between the cycle lengths of this family $\mathcal{C}$ grow exponentially large. Moreover, $\mathrm{ex}(n, \mathcal{C}) = \Theta(n^2)$ because $\mathcal{C}$ consists only of odd cycles, so in theory its game saturation number could have been much larger than linear.

We will prove Theorem 6 by showing that when $\mathcal{C}$ is a k-dense family of cycles, either player can play in the $\mathcal{C}$-saturation game so that the graph never contains a cycle of length at most k, and given this the result will essentially follow from Theorem 1. We will show (see Proposition 20) that the bounds in the definition of $\mathcal{C}$ being k-dense are close to best possible for such a strategy to exist.

The rest of the paper is organized as follows. In Section 1.2 we establish some basic graph theory notation and facts. In Section 2 we give a short proof of Theorem 3. The majority of the paper is dedicated to the proof of Theorem 6 in Section 3. Finally, we close with some concluding remarks and open problems in Section 4.

1.2 Preliminaries

Given a graph $G = (V, E)$, we say that a graph $G' = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. We say that G' is an *induced subgraph* of G if $E' = \{uv \in E : u, v \in V'\}$. For a set $S \subseteq V$, we define the *induced subgraph on* S to be $G[S] := (S, E_S)$ where $E_S = \{uv \in E : u, v \in S\}$. We will often write $G + e$ and $G - e$ to denote the graph G obtained by adding or deleting the edge e respectively. Given two vertices $u, v \in V(G)$, a path from u to v is a *u–v geodesic* if the path is of length $d(u, v)$. A path that is a u–v geodesic for some choice of u and v will just be called a *geodesic*. The *circumference* of a graph G is the length of the longest cycle in G. We say that a path is *v-avoiding* if it does not contain the vertex v.

Many of our proofs will rely heavily on the block structure of our underlying graph. We recall that a *block* B of a graph G is a maximal subgraph of G which is either a K_2 or 2-connected, and that the edges of the blocks partition the edges of G. By a slight abuse of notation, we will often refer to blocks, vertex sets of blocks, and edge sets of blocks simply as "blocks".

We recall some basic facts about blocks, and we refer the reader to Diestel [3] for more information on blocks. Whitney's Theorem states that a graph on at least 3 vertices is 2-connected if and only if every pair of vertices is connected by two internally vertex-disjoint paths. Thus given a block B on at least three vertices and $u, v \in B$, there exists a pair of internally disjoint paths from u to v. A *dominating vertex* in a block B is a vertex adjacent to all other vertices in B. We say two blocks are *adjacent* if their vertex sets have a nonempty intersection. Note that two blocks can overlap in at most one vertex, so two blocks being adjacent is equivalent to them sharing exactly one vertex.

We now move onto some less standard terminology. Given a path $P = (u_1, u_2, \ldots, u_\ell)$ in G with $\ell \geqslant 2$, note that there is a unique list of blocks $B_1, B_2, \ldots, B_r$ of G and integers $1 \leqslant s_1 < s_2 < \cdots < s_r \leqslant \ell - 1$ such that for all $1 \leqslant j \leqslant r$ and $s_j \leqslant i < s_{j+1}$, the edge $u_i u_{i+1} \in B_j$. We will say $(B_1, B_2, \ldots, B_r)$ is the *block geodesic associated with P*. Note further that if P_1 and P_2 are two u–v paths, then P_1 and P_2 are associated with the same block geodesic, which we will call the *u–v block geodesic*. Furthermore, if the u–v block geodesic consists of r blocks, we will say that the *block distance from u to v*, denoted $bd(u, v)$, is r. In the $\mathcal{C}$-saturation game, we let $G^{(t)}$ be the graph after the tth edge has been added in the game. We let $G^{(0)}$ be the initial (empty) graph of the game, and we let $G^{(T)}$ denote the graph once it has become $\mathcal{C}$-saturated.

2 Proof of Theorem 3

We use the Erdős-Gallai theorem, Theorem 1, to give a relatively short proof of Theorem 3. We will say that a path is a *path of K_2 blocks* if every edge in the path is a K_2 block. We will call such a path trivial whenever it contains only a single block.

Proof of Theorem 3. First, we will show that we can play such that at the end of each of our turns the following hold for $G^{(t)}$:

(i) Every block that is not a K_2 block contains a triangle, and

(ii) There is at most one non-trivial maximal path of K_2 blocks, with this path containing at most three edges. When it exists, we denote this path by P.
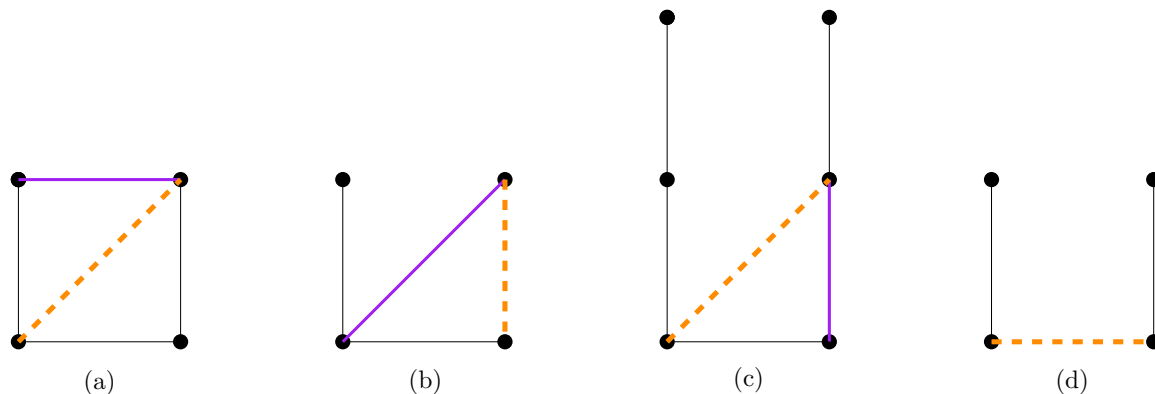

Figure 1: An illustration of our moves in different situations depicted by dashed orange lines. The previous move of our opponent (if it is relevant) is highlighted by a thick purple line. Situation (a) shows a case when property (i) is violated. Situations (b) and (c) show cases when property (ii) is violated (in (b) due to multiple non-trivial paths of K_2 blocks, in (c) due to a long path of K_2 blocks). Situation (d) handles one possible case when properties (i) and (ii) are satisfied before our move. Note, that (d) is the only place when a path of K_2 blocks of length 3 is present after our move.

We prove this claim inductively. For the base cases, $G^{(1)}$ and $G^{(2)}$ have properties (i) and (ii), since there are not enough edges to negate the claim. In the induction we describe moves that maintain properties (i) and (ii) in all possible situations; see also Figure 1. Inductively assume that at the end of our turn, $G^{(t)}$ has properties (i) and (ii) for some time $t \geqslant 2$.. Then the only way a block without a triangle could form in $G^{(t+1)}$ would be for this block to consist of the new edge in $G^{(t+1)}$ and some number of K_2 blocks from $G^{(t)}$. Since by property (ii) there is only one non-trivial path of K_2 blocks, P, and as this has length at most 3, the only blocks that could have formed in $G^{(t+1)}$ in this way are triangle blocks or (a unique) C_4 block. If a unique C_4 block forms, we can add a diagonal to this block, which then contains a triangle and satisfies property (i), and it is not difficult to see that (ii) is satisfied. Thus we can maintain the desired conditions if $G^{(t+1)}$ contains a non K_2 block without a triangle.

With this we can assume that every non K_2 block of $G^{(t+1)}$ contains a triangle, so property (i) is satisfied. If $G^{(t+1)}$ contains more than one non-trivial path of K_2 blocks, or a path of K_2 blocks of length 4 or more, this must have been created by our opponent's last turn. In either case, there exists a choice of edge we can add which connects our opponent's most recent edge to another edge within this K_2 block path such that we create a triangle containing the edge our opponent just played (See Figure 1 for a few example situations). In this case, the K_2 blocks in $G^{(t+2)}$ form a subset of the K_2 blocks

in $G^{(t)}$, so property (ii) follows from the fact that $G^{(t)}$ satisfied property (ii), and the only new block created was a triangle, so property (i) remains satisfied.

Next, we must handle the case that $G^{(t+1)}$ already satisfies properties (i) and (ii). If there exists a non-trivial path of K_2 blocks, then we can add an edge between two vertices on that path such that we create a triangle. The resulting graph satisfies both properties. Thus we may assume $G^{(t+1)}$ has no two adjacent K_2 blocks. Any move that does not create a forbidden cycle cannot create a path of K_2 blocks on more than three edges, so any move we make satisfies property (ii). Also, such a move can create at most one block B which was not present in $G^{(t+1)}$. If B is a K_2 block, then property (i) is satisfied. If the vertices of B were contained in a single block in $G^{(t+1)}$, then that block must have contained a triangle, so property (i) is satisfied. Thus B must contain the vertices from at least two blocks in $G^{(t+1)}$. If any of those blocks contained a triangle in $G^{(t+1)}$, then B contains a triangle and property (i) is satisfied. Thus all the blocks from $G^{(t+1)}$ contained in B are K_2 blocks. However, no two K_2 blocks are adjacent in $G^{(t+1)}$, and it is impossible to form a single block using only non-adjacent K_2 blocks and one additional edge. This completes the proof of the claim.

Assume we play according to this strategy throughout the game, which implies that every block of $G^{(T)}$ is a K_2 block or contains a triangle. We claim that this implies that $G^{(T)}$ is $\mathcal{C}_{\geqslant 3k}$-free. If some block B of $G^{(T)}$ contained a cycle C of length $\ell \geqslant 3k$ and a triangle xyz, then we will show that this block also contains a cycle of length ℓ' for some $\ell' \geqslant \ell/3 + 2$ which contains the three vertices in the triangle as consecutive vertices. This implies that $G^{(T)}$ also contains a cycle of length $\ell' - 1$, which contradicts $G^{(T)}$ being $\mathcal{C}$-free.

Suppose the triangle xyz intersects C in two or three vertices, not necessarily sequentially along C. This yields two or three paths along C between each of the vertices of x, y, z that it intersects. One of these paths must have length at least $\frac{\ell}{3}$, call it P_1 with endpoints say x, y, without loss of generality. Then the cycle formed by P_1 and the edges xz, zy give the desired cycle.

Next, suppose that C and xyz have exactly one vertex in common, say x without loss of generality. Since B is 2-connected, there exists some shortest path Q in $B - x$ from a vertex in $\{y, z\}$ to a vertex of $C - x$. Since Q is a shortest path, it contains at most one vertex from $\{y, z\}$ and $C - x$, say these vertices are y, v, which must be the endpoints of Q. Then x and v split C into two paths, one of which will have length at least $\frac{\ell}{2} \geqslant \frac{\ell}{3}$. This long path, together with Q and the edges yz and zx, gives the desired cycle.

Finally, suppose that C and xyz are disjoint. Then since B is 2-connected, we can find two paths, disjoint from each other and internally-disjoint with both xyz and C, connecting distinct vertices of the triangle to distinct vertices of the cycle. This again partitions C into two paths, one of which will have length at least $\frac{\ell}{3}$, and using this together with the two paths from xyz and the third vertex of xyz gives the desired cycle.

Thus $G^{(T)}$ is $\mathcal{C}_{\geqslant 3k}$-free. Using Theorem 1 gives

$$|E(G^{(T)})| \leqslant \frac{(3k-1)(n-1)}{2}. \qquad \square$$

3 k-Dense Families of Cycles

In this section, we consider the cycle saturation game for families of cycles with sufficiently small gaps between consecutive forbidden cycle lengths.

Let us briefly outline the strategy we use in the $\mathcal{C}$-saturation game when $\mathcal{C}$ is a k-dense family of cycles. Ideally we would like to play such that at the end of each of our turns, $G^{(t)}$ is a connected graph where every block B is dominated, i.e. B contains a dominating vertex r_B, and moreover r_B is the unique vertex of B which is closest to some special vertex h. Assuming this holds, if our opponent connects two vertices u and v with block distance s, then the path through their dominating vertices creates a cycle of length $s + 1$. We further require that each dominating vertex r_B be the endpoint of a path of length $k - 3$ in another block, which will allow us to extend this path through dominating vertices in such a way that we can actually get any cycle length between $s+1$ and $3 + (k - 2)(s - 2)$, with at least one of these lengths forbidden by assumption of $\mathcal{C}$ being k-dense. Thus distant blocks can not be connected to one another. This will imply that $G^{(t)}$ has small circumference, and hence relatively few edges.

It remains to describe how we can maintain this ideal structure (or at least something close to it). If our opponent ever connects an isolated vertex x to one of these blocks B, then we can try to make x adjacent to the dominating vertex of B. If this is impossible then it will turn out that x's neighbor y dominates the xy block and y is the end of a long path in B, so this new block maintains the desired properties. The issue will be when our opponent connects two isolated vertices. To maintain connectedness we are forced to make one of these vertices adjacent to h. By doing this repeatedly the opponent can create some non-desirable structures, but we can at least maintain that any that do appear are incident to h.

3.1 k-Fantastic Graphs

We now move onto our formal definitions. Throughout the remainder of this section, we will assume we are playing the $\mathcal{C}$-saturation game for a k-dense family of cycles $\mathcal{C}$. To define the structure that we wish to maintain in this game, we work with a graph G with a specified vertex h. We say a block B in G is *rooted* if there exists a vertex r_B that is a dominating vertex in B, and for which $d(r_B, h) < d(u, h)$ for all $u \in B - r_B$. When B is rooted, we say r_B is the *root* of B and that r_B *roots* the block B. Note that if $h \in B$ and B is rooted, then h is the root. Furthermore, note that if every block in G has a root, then every vertex $v \in V(G)\backslash\{h\}$ is in exactly one block for which it is not the root, namely the first block in the v–h block geodesic. If $v \neq h$ is a vertex that roots every block containing it except for one block B, then we will call this block the *stem of v* and denote it by B_v.

We say a vertex v in a rooted block B is *finished* if v is the endpoint of a path of length at least $k - 3$ in the induced subgraph $B - r_B$. We say the block B is *finished* if all vertices in $B - r_B$ are finished. If a vertex or block is not finished, we call it *unfinished*. We say a block B is *nearly h-dominated* if B contains h, and all but one vertex in B is

adjacent to h. In this case we define $r_B := h$ (even though h is technically not a root of the block).

Definition 8. Two blocks B_1 and B_2 are together called an *h-umbrella* if the following hold:

1. The vertex h dominates B_1,

2. The block B_2 is a K_2 block,

3. The blocks B_1 and B_2 intersect at a vertex $u \neq h$,

4. For any other block B, if B intersects B_1, then they intersect at h,

5. The vertex in $B_2 \setminus B_1$ has degree 1, and we will refer to this vertex as the handle of the h-umbrella.

We say that an h-umbrella is finished if the unique neighbor of the handle is finished in B_1, and unfinished otherwise.

See Figure 2 for an illustration of an h-umbrella. Note that if B_1 and B_2 constitute an h-umbrella, then $r_{B_1} = h$ and $r_{B_2} = u$.

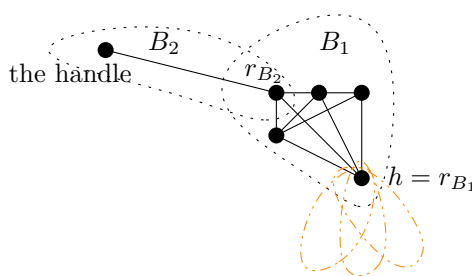

Figure 2: An illustration of Definition 8. The h-umbrella is in black. The vertex r_{B_2} is the unique neighbor of the handle. Other blocks that intersects B_1 are depicted in orange.

We are now ready to state the definition of a k-fantastic graph, which is the main structural tool we will need in this section.

Definition 9. Given a graph G with specified vertex h, let H be the subgraph induced by the set of vertices which are contained in a block which is nearly h-dominated or part of an unfinished h-umbrella, and if no such blocks exist set $H = \{h\}$. Let F be the subgraph induced by the set consisting of h and all of the vertices contained in the blocks of G not contained in H. We say G is *k-fantastic* whenever F has the following properties:

Property 1. F is connected,

Property 2. Every block in F is rooted,

Property 3. If a vertex $v \neq h$ is the root of some block in F, then v is finished in its stem B_v.

Property 4. Each vertex $v \neq h$ roots at most one unfinished block in F,

Property 5. The vertex h is adjacent to at most one vertex of degree 1 in F.

We note that the graph G consists of $H \cup F$ along with a set of isolated vertices. Note that in a slight abuse of notation, we will sometimes use H and F to refer to $V(H)$ and $V(F)$, respectively.

3.2 Preliminary work towards k-Dense results

We will ultimately show that for any k-dense family of cycles $\mathcal{C}$, either player can play the $\mathcal{C}$-saturation game such that the graph $G^{(t)}$ is k-fantastic at the end of their turns. To this end, throughout this subsection we let G refer to a k-fantastic graph, and we consider h, H, F, and I as defined above.

Lemma 10. *Let G' be a $\mathcal{C}$-free graph with specified vertex h. If B is a block of G' that is either rooted or nearly h-dominated, then the longest r_B-avoiding path in B is of length at most $k - 3$.*

Proof. Assume to the contrary that B contains an r_B-avoiding path $(v_0, v_1, \ldots, v_{k-2})$. If the edges $v_0 r_B$ and $v_{k-2} r_B$ are in $E(G')$, then $(r_B, v_0, v_1, \ldots, v_{k-2}, r_B)$ is a cycle of length k, which contradicts the fact that G' is $\mathcal{C}$-free. This implies that B must be nearly h-dominated, $r_B = h$, and exactly one of the edges $v_0 h$ or $v_{k-2} h$ are not in $E(G')$. We assume without loss of generality $v_{k-2} h \notin E(G')$. Since B is non-trivial and 2-connected, v_{k-2} must have at least two neighbors in B, and specifically at least one neighbor v with $v \neq v_{k-3}$. If $v \neq v_i$ for any $1 \leqslant i \leqslant k - 4$, then $(h, v_1, v_2 \ldots, v_{k-2}, v, h)$ is a cycle of length k, a contradiction. If $v = v_i$ for some $1 \leqslant i \leqslant k - 4$, then $(h, v_{i+1}, v_{i+2}, \ldots, v_{k-2}, v_i, v_{i-1}, \ldots, v_0, h)$ is a cycle of length k in G', a contradiction. We conclude that for all rooted and nearly h-dominated blocks B, we have that B does not contain an r_B-avoiding path of length $k - 2$. $\square$

Lemma 11. *The circumference of any k-fantastic graph is at most $k - 1$.*

Proof. Let G be a k-fantastic graph, and note that every block of a k-fantastic graph is either rooted or nearly h-dominated. If G had a cycle of length $\ell \geqslant k$, then this cycle would have to be in some block B because cycles are 2-connected and blocks are maximal 2-connected subgraphs. Thus, B would necessarily contain an r_B-avoiding path of length $\ell - 2 > k - 3$, a contradiction with Lemma 10. So, the circumference of G is at most $k - 1$. $\square$

We continue our discussion on the structural properties of k-fantastic graphs with the following two lemmas regarding which path lengths are attainable between specific vertices.

Lemma 12. *Let G be k-fantastic. If a vertex v is finished in a rooted block B and $u \in B$, then for each ℓ with $d(u,v) \leqslant \ell \leqslant k-2$ there is a u–v path of length ℓ in B.*

Proof. If $\ell = 1$ the result is immediate, so assume $\ell \geqslant 2$. Since v is finished there exists an r_B-avoiding path $(v, v_1, \ldots, v_{k-3})$ in B. If $u = r_B$ then the path $(v, v_1, \ldots, v_{\ell-1}, r_B)$ works. If $u \neq r_B$ and $u \neq v_i$ for any $i < \ell$, then $(v, v_1, \ldots, v_{\ell-2}, r_B, u)$ is a path of length ℓ. If $u = v_i$ with $2 \leqslant i < \ell$, then we use $(v, v_1, \ldots, v_{i-1}, r_B, v_{\ell-1}, v_{\ell-2}, \ldots, v_{i+1}, u)$. In each case we find a path of length ℓ, thus completing the proof. $\qquad\square$

Recall that we denote the block distance from vertex u to vertex v as $bd(u,v)$, a vertex v can only have one stem, and that v roots all other blocks containing it.

Lemma 13. *Let G be k-fantastic, and let $u, v \in F$ with $u \neq h$ be such that $s := bd(u,v) \geqslant 2$. Then there is a u–v path of length ℓ in G for every ℓ with*

$$s + 1 \leqslant \ell \leqslant 2 + (k-2)(s-2).$$

Moreover, if $v = h$ we can do this for all ℓ with

$$s \leqslant \ell \leqslant 1 + (k-2)(s-1).$$

Proof. Let $(B_1, \ldots, B_s)$ be the u–v block geodesic, and observe that these blocks are all rooted since $u, v \in F$. For $1 \leqslant i \leqslant s-1$, let v_i be the vertex in both B_i and B_{i+1}, and note that e.g. $u \neq v_1$ as otherwise $(B_2, \ldots, B_s)$ would be a shorter path of blocks from u to v. We will first prove, regardless of if $v = h$ or $v \neq h$, that for all $d(v_i, v_{i+1}) \leqslant k_i \leqslant k-2$ there is a u–v path of length

$$d(u, v_1) + \sum_{i=2}^{s-1} k_i + d(v_{s-1}, v),$$

and moreover the path ends in a v_{s-1}–v geodesic.

Indeed, starting from u we transverse a u–v_1 geodesic. If $v_1 \neq r_{B_2}$, then by definition $B_{v_1} = B_2$, i.e. B_2 is the unique block containing v_1 which v_1 is not the root of. The uniqueness of this block implies $v_1 = r_{B_1}$, and by Property 9 this means that v_1 is finished in B_2. Thus we can apply Lemma 12 to find a path of length k_2 from v_1 to v_2 in B_2. If $v_1 = r_{B_2}$, then $v_2 \neq r_{B_2}$ and a symmetric argument gives the same conclusion. Continuing in this manner, Lemma 12 gives for each $3 \leqslant i \leqslant s-1$ a path of length k_i from v_{i-1} to v_i in B_i, and once we reach v_{s-1} we can traverse a v_{s-1}–v geodesic to complete the path. This proves the claim.

By choosing the appropriate values for the k_i's, we can find a u–v path of length ℓ for every ℓ with

$$d(u,v) \leqslant \ell \leqslant d(u, v_1) + d(v_{s-1}, v) + (k-2)(s-2).$$

To finish the proof for the $v \neq h$ case, it suffices to show $d(u, v_1) + d(v_{s-1}, v) \geqslant 2$ and $d(u,v) \leqslant s+1$. The first part is immediate, and the second part follows by considering

the path $(u, r_{B_1}, \ldots, r_{B_s}, v)$ after deleting duplicated vertices if, say, $r_{B_i} = r_{B_{i+1}}$ for some i.

In the case where $v = h$, we can again apply the claim to find u–h paths of length ℓ for all ℓ with $d(u, h) \leqslant \ell \leqslant d(u, v_1) + d(v_{s-1}, h) + (k-2)(s-2)$ and which end in the edge $v_{s-1}h$. Note that v_{s-1} does not root B_s (since $h \in B_s$), and thus it must root B_{s-1} so it will be be finished in B_s. By Lemma 12 we can replace the edge $v_{s-1}h$ with a path of length k_s for any $1 \leqslant k_s \leqslant k - 2$, thus allowing us to build a u–h path of every length ℓ with $d(u, h) \leqslant \ell \leqslant d(u, v_1) + (k-2)(s-1)$. Since $d(u, h) = s$ and $d(u, v_1) \geqslant 1$, the result follows. $\qquad\square$

We show that we can always find long paths between pairs of vertices in adjacent blocks, except for one exceptional case.

Lemma 14. *Let G be k-fantastic. Let $u, v \in F$ with $bd(u, v) = 2$, and let (B_1, B_2) be the u–v block geodesic. Then there is a u–v path of length $k - 1$ in G, unless h roots B_1 and B_2 and both u and v are unfinished.*

Proof. Let $x \in B_1 \cap B_2$. Note that G being k-fantastic implies that x roots at least one of these blocks, and that $x \neq u, v$ due to u and v having block distance 2. First consider the case that x roots both B_1 and B_2. We claim that either u or v are finished. If $x = h$, then this is true by assumption, and if $x \neq h$, then one of the blocks B_1 or B_2 must be finished by Property 9, so one of u or v must be finished, and we assume without loss of generality that u is finished in B_1. Then by Lemma 12, there is a u–x path of length $k-2$ which can be extended using the edge xv to a path of length $k-1$ that ends at v, so we are done in this case.

Now assume x roots only one of the blocks, say without loss of generality B_2. Since B_1 is the stem of x, x is finished in B_1 by Property 9, so by Lemma 12 there is a path of length $k-2$ from u to x, which can be extended to a u–v path of length $k-1$ using the edge xv. Thus in all cases we are done. $\qquad\square$

The next two lemmas will help us in situations in which our opponent plays an edge that is incident with a nearly h-dominated block.

Lemma 15. *Let G' be a $\mathcal{C}$-free graph with specified vertex h. If B is a nearly h-dominated block and $u \in B$ is the vertex not adjacent to h, then $G' + uh$ is $\mathcal{C}$-free.*

Proof. If adding the edge uh creates a cycle in $\mathcal{C}$, this cycle would be contained in the vertices of B, which implies that there must be a u–h path of length at least $k-1$ in B, and hence an h-avoiding path of length $k-2$ in B, but this contradicts Lemma 10. $\qquad\square$

We remind the reader that only unfinished H-umbrellas are in H, which in particular means the unique neighbor of the handle in an h-umbrella of H must be unfinished. We also recall for $k \geqslant 6$, we require that both C_k and C_{k+1} are in $\mathcal{C}$ for $\mathcal{C}$ to be k-dense (but for $k = 5$ we do not require $C_6 \in \mathcal{C}$).

Lemma 16. *Let G be k-fantastic for $k \geqslant 6$. Let $u, v \in H - h$ be distinct vertices such that one of the following holds:*

1. u and v are in distinct nearly h-dominated blocks,

2. u and v are both handles in distinct h-umbrellas, or

3. One of u, v is in a nearly h-dominated block while the other is the handle in an h-umbrella.

Let B be the block containing the edge uv in $G' := G + uv$, and let $a, b \in B - h$ be the vertices such that $ah, bh \notin E(G')$. If G' is $\mathcal{C}$-free, then both $G' + ah$ and $G' + bh$ are also $\mathcal{C}$-free.

Proof. Towards a contradiction, we may assume $G' + ah$ contains a cycle from $\mathcal{C}$, so there is an a–h path in B of length ℓ with $\ell \geqslant k - 1$ such that $C_{\ell+1} \in \mathcal{C}$, say the path $P = (v_0 := a, v_1, v_2, \ldots, v_{\ell-1}, h)$. We can assume without loss of generality that the vertices a and u are both in the same nearly h-dominated block or h-umbrella in G (and consequently the same is true for b and v).

We claim that this path contains the vertices u and v in that order. If not, then P is either completely contained inside a nearly h-dominated block, which contradicts Lemma 15, or P is completely contained inside an h-umbrella in H. By the definition of H, the neighbor of a in this h-umbrella v_1 must be unfinished, but $(v_1, v_2, \ldots, v_{\ell-1})$ is an h-avoiding path of length $\ell - 2 \geqslant k - 3$, giving us a contradiction and proving the claim.

Let A_1 denote the sets of vertices in the nearly h-dominated block or h-umbrella in G containing a and u, and let A_2 denote the set of vertices in the nearly h-dominated block or h-umbrella in G containing b and v. Then $u = v_i$ and $v = v_{i+1}$ for some $0 \leqslant i \leqslant \ell - 1$, and note that $\{v_j \mid 0 \leqslant j \leqslant i\} \subseteq A_1$ while $\{v_j \mid i + 1 \leqslant j \leqslant \ell - 1\} \subseteq A_2$.

We now claim that we can find a cycle of length $\ell + 2$ in G' of the form $(h, x, a = v_0, v_1, \ldots, v_{\ell-1}, h)$ for some vertex $x \in A_1$. Indeed, if A_1 is an h-umbrella, we can choose x to be the unique neighbor of a in A_1, then by the definition of u, we have that $u = a$, so $(h, x, a = u = v_0, v = v_1, v_2, \ldots, v_{\ell-1}, h)$ is such a cycle. Thus we may assume that A_1 is a nearly h-dominated block. Since A_1 is 2-connected and non-trivial, a has a neighbor x that is not v_1 (nor h since $ah \notin E(G')$). We consider two cases based on x.

Case 1: The vertex $x = v_j$ for some $0 \leqslant j \leqslant \ell - 1$. Note that $2 \leqslant j \leqslant i$ since $x \in A_1$. This implies that $v_{j-1} \in A_1 - a - h$, and so hv_{j-1} is an edge. Then $(h, v_{j-1}, v_{j-2}, \ldots, v_0 = a, x = v_j, v_{j+1}, \ldots, v_{\ell-1}, h)$ is a $C_{\ell+1}$ in G', a contradiction to $C_{\ell+1} \in \mathcal{C}$.

Case 2: The vertex $x \neq v_j$ for any $0 \leqslant j \leqslant \ell - 1$. Since $x \in A_1 - a - h$, $hx \in E(G)$. Then $(h, x, a = v_0, v_1, \ldots, v_{\ell-1}, h)$ is a $C_{\ell+2}$ in G' with h and a at distance 2 along the cycle.

Thus, we have exhibited a cycle $(h, x, a = u = v_0, v = v_1, v_2, \ldots, v_{\ell-1}, h)$ as claimed. This implies that $C_{\ell+2} \notin \mathcal{C}$. Since $k \geqslant 6$, we have $C_k, C_{k+1} \in \mathcal{C}$. We conclude $\ell \geqslant k$ and that v_{k-2} and v_{k-1} are defined. We now show that G' contains either a C_k or a C_{k+1}, which will give us a contradiction. Indeed, since $v_{k-2}, v_{k-1} \neq a = v_0$, we have that h is adjacent to at least one vertex v_j with $j \in \{k-1, k-2\}$, so $(h, x, v_0, v_1, \ldots, v_j, h)$ is a C_{j+2} in G, where $j + 2 \in \{k, k+1\}$, proving the claim and completing the proof. $\qquad\square$

Our next lemma characterizes what moves within the cycle saturation game will leave the graph k-fantastic. For the rest of the section, we refer to a *legal move* as an allowable move in the $\mathcal{C}$-saturation game for the implied family $\mathcal{C}$ which is k-dense.

Lemma 17. *If G is k-fantastic, and $u, v \in F$ are vertices such that uv is a legal move, then $G + uv$ is also k-fantastic. Further, $G + uv$ does not contain any nearly h-dominated blocks which were not in G.*

Proof. Note that adding an edge within a block does not interfere with any of the properties of being k-fantastic nor create a nearly h-dominated block. Thus we can assume $bd(u, v) = s \geqslant 2$. If $s \geqslant 3$, by Lemma 13 there is a path of length ℓ for every ℓ with $s + 1 \leqslant \ell \leqslant (k-2)(s-2) + 2$, which implies that uv would complete a cycle in $\mathcal{C}$, a contradiction. Thus, we may assume $s = 2$.

Since uv is a legal move there is no u–v path of length $k - 1$. By Lemma 14, both u and v are adjacent to h. Then h still dominates the block resulting from adding u and v (in particular meaning it is not nearly h-dominated), and all the properties of being k-fantastic are retained as desired. $\square$

The following lemma will allow us to focus our attention only on those cases in which our opponent makes a move that results in a graph that is not k-fantastic.

Lemma 18. *If G is k-fantastic but not $\mathcal{C}$-saturated, then there exists a legal move uv such that $G + uv$ is k-fantastic. Moreover, if G has at most one nearly h-dominated block, then uv can be chosen so that $G + uv$ has no nearly h-dominated blocks.*

Proof. First assume G contains a nearly h-dominated block with x the vertex in this block not adjacent to h. By Lemma 15 we can add hx, which creates a rooted block with root h. This makes G k-fantastic with no nearly h-dominated blocks. Thus we may assume that G contains no nearly h-dominated blocks.

Assume G has an isolated vertex x. If h is adjacent to a vertex of degree 1, say y, then xy is a legal move making an h-umbrella. Otherwise hx is a legal move. Thus we may assume G contains no isolated vertices.

Suppose G contains an h-umbrella in H, say with handle y and x its unique neighbor, and recall that x cannot be finished. Thus yh can be added without creating a forbidden cycle or nearly h-dominated block. We can then assume that G contains no h-umbrellas in H, and consequently that $H = \{h\}$ is trivial.

Since H is trivial and there are no isolated vertices, we must have $G = F$. By hypothesis there exists a legal move involving two vertices of F, and by Lemma 17 and any such move leaves the graph k-fantastic, so we are done. $\square$

3.3 Main Results for k-Dense Families

We are now ready to prove our main structural result for this section.

Proposition 19. *Let $\mathcal{C}$ be a k-dense set of cycles for some $k \geqslant 5$. Then either player can play the $\mathcal{C}$-saturation game such that at the end of each of their turns, the graph is*

k-fantastic. Moreover, if $k = 5$, then that player can guarantee that the graph contains no nearly h-dominated blocks at the end of each of their turns.

Proof. Note that $G^{(0)}$ and $G^{(1)}$ are both trivially k-fantastic and do not contain nearly h-dominated blocks. Now let us assume that $G^{(t)}$ is k-fantastic for some $t \geqslant 0$, and if $k = 5$ that further $G^{(t)}$ contains no nearly h-dominated blocks. We will show that we can play such that $G^{(t+2)}$ is k-fantastic, unless $G^{(t+1)}$ is already $\mathcal{C}$-saturated. In the analysis that follows, when $k = 5$ we will not verify that our own move does not create a new nearly h-dominated block, but it is easy to verify that the only time our strategy has our move creating such a block is when $k \geqslant 6$ in Case 5c.

We consider cases based on the edge added at time $t + 1$ which we denote by $e = uv$. We also let I denote the set of isolated vertices of $G^{(t)}$.

Case 0: $G^{(t+1)}$ is k-fantastic. In this case we may apply Lemma 18, and in particular this leaves it so that $G^{(t+2)}$ has no nearly h-dominated blocks when $k = 5$. Note that by Lemma 17 this handles the case that $u, v \in F$.

Case 1: $u, v \in I$. We play the edge uh, which creates an h-umbrella in H and maintains $G^{(t+2)}$ being k-fantastic.

Case 2: $u = h$. First note that we do not need to consider the case when $v \in F$ since $h \in F$ as well.

Case 2a: $v \in H - h$. If hv is contained inside a nearly h-dominated block, then this block becomes rooted with root h, so $G^{(t+1)}$ is k-fantastic and we are in Case 0. The only other possibility is that v is the handle of an h-umbrella since all other vertices in H are adjacent to h. As such, adding the edge hv causes this h-umbrella to become a single block which is rooted with root h, so the graph is still k-fantastic and we are in Case 0.

Case 2b: $v \in I$. If h was not adjacent to a degree 1 vertex in G, then we are still k-fantastic and in Case 0. Otherwise, by Property 9 there is exactly one other degree 1 vertex x adjacent to h, and adding the edge vx creates a block rooted at h, leaving the graph k-fantastic.

Case 3: $u \in F - h$ and $v \in I$. This adds an unfinished K_2 block rooted at u. We only consider the cases where u is the root of another unfinished block (which violates Property 9), and the case where u is an unfinished vertex (which violates Property 9). In any other case, $G^{(t+1)}$ remains k-fantastic since the Proprieties 9 and 9 cannot be affected by a new unfinished K_2 block, and thus we are in Case 0.

Case 3a: u is the root of an unfinished block B. We add an edge from v to some unfinished vertex $x \in B$. This does not create any cycle of length at least k since x was unfinished, so this is a legal move. Now u is only adjacent to at most one unfinished block again, and Property 9 holds. Since v is adjacent to the root of this block, Property 9 holds as well.

Case 3b: u is an unfinished vertex in some block B. We add the edge vr_B. This does not create a cycle of length at least k since u was unfinished, so this is a legal move. The resulting block is rooted with root r_B, so properties 9 and 9 hold in $G^{(t+2)}$. Thus, $G^{(t+2)}$ is k-fantastic.

Case 4: $u \in H - h$ and $v \in I$.

<u>Case 4a:</u> u is in a nearly h-dominated block B. We add the edge containing h missing from B, which is legal by Lemma 15. This creates an h-umbrella, and thus $G^{(t+2)}$ is k-fantastic.

<u>Case 4b:</u> u is in an h-umbrella consisting of blocks B_1 and B_2 with $h \in B_1$. Let y be the handle and x its unique neighbor (possibly with $x = u$ or $y = u$). Then by the definition of H, x is unfinished. Then we can add the edge yh as this does not create a cycle in $\mathcal{C}$ due to the fact that x is unfinished. Then $\tilde{B}_1 := B_1 \cup B_2$ becomes a block rooted at h, and so if $\tilde{B}_2$ is the block containing uv, $\tilde{B}_1$ and $\tilde{B}_2$ constitute an h-umbrella, and thus $G^{(t+2)}$ is k-fantastic.

Case 5: $u, v \in H - h$.

<u>Case 5a:</u> u and v are in the same nearly h-dominated block or the same h-umbrella in H. If u and v are both in a nearly h-dominated block, then this block remains nearly h-dominated. If u and v are both in the same h-umbrella, this either remains an h-umbrella or if either u or v was the handle in G, this becomes a nearly h-dominated block. In either case $G^{(t+1)}$ is k-fantastic, so we are in Case 0.

<u>Case 5b:</u> u is in an h-umbrella and is not the handle. By Case 5a, we may assume that v is not in the same h-umbrella as u. If v is in a nearly h-dominated block or v is the handle of an h-umbrella, then the block containing uv in $G^{(t+1)}$ is nearly h-dominated, so by Lemma 15, we can add the edge to turn this block into a rooted block with root h, which results in an h-umbrella so $G^{(t+2)}$ is k-fantastic.

It remains to consider when v is also in an h-umbrella but not the handle. Note that the addition of the edge uv forms a block rooted at h, adjacent to two rooted K_2 blocks in $G^{(t+1)}$. Let x and y be the handles of the original h-umbrellas containing u and v respectively, and let x' and y' be the neighbors of x and y. If either x' or y' are unfinished in $G^{(t+1)}$, then we can add the edge xh or yh creating an h-umbrella and leaving $G^{(t+2)}$ k-fantastic. If both x' and y' are finished in $G^{(t+1)}$, then all these blocks are in F, so $G^{(t+1)}$ is k-fantastic and we are in Case 0.

<u>Case 5c:</u> The conditions of Case 5a and Case 5b are not met. Then one of the following holds:

1. u and v are in distinct nearly h-dominated blocks,

2. u and v are both handles of distinct h-umbrellas, or

3. One of u, v is in a nearly h-dominated block while the other is a handle of an h-umbrella.

Note that if $k = 5$, then we do not have nearly h-dominated blocks, and u and v cannot both be handles of h-umbrellas as uv would create a C_5, which is forbidden. Hence, we may assume $k \geqslant 6$. Let B be the block containing uv in $G^{(t+1)}$, and note that B is only adjacent to other blocks at h. By Lemma 16, in all cases we can add an edge containing h and some other vertex of B, which will turn B into a nearly h-dominated block, leaving $G^{(t+2)}$ k-fantastic. Since this case only happens with $k \geqslant 6$, we do not create h-dominated blocks when $k = 5$, as desired.

Case 6: $u \in F - h$ and $v \in H - h$.

<u>Case 6a:</u> The block distance $s := bd(u, h) \geqslant 2$. By Lemma 13 we have a u–h path of length ℓ for every $s \leqslant \ell \leqslant (k-2)(s-1) + 1$, and thus $G^{(t+1)}$ contains a $C_{\ell'}$ for every ℓ' satisfying

$$s + d(h, v) + 1 \leqslant \ell' \leqslant (k-2)(s-1) + 1 + d(h, v) + 1.$$

If $s = 2$, then since $1 \leqslant d(h, v) \leqslant 2$, we have that $G^{(t+1)}$ contains cycles of every length ℓ' with $5 \leqslant \ell' \leqslant k + 1$. In particular it contains C_k, a contradiction, so we may assume $s \geqslant 3$. If $d(h, v) = 1$, then we have cycles of length ℓ' for all $s + 2 \leqslant \ell' \leqslant 3 + (k-2)(s-2) < 3 + (k-2)(s-1)$, contradicting the fact that $\mathcal{C}$ must contain one such cycle. If $d(h, v) = 2$, then let $s' := s + 1$, and note that we have cycles of all lengths ℓ' with $s' + 2 \leqslant \ell' \leqslant (k-2)(s'-2) + 3 < (k-2)(s'-2) + 4$, again a contradiction to the definition of $\mathcal{C}$ being k-dense.

<u>Case 6b:</u> The block distance $s := bd(u, h) = 1$. Since h roots every block contained in F that h is in, $uh \in E(G)$. Let B be the block containing the edge uh. If v is in a nearly h-dominated block or is the handle of an h-umbrella, then the addition of uv just creates a nearly h-dominated block, leaving $G^{(t+1)}$ k-fantastic. If v is in an h-umbrella but not the handle, then the addition of uv creates a larger h-umbrella, which again leaves $G^{(t+1)}$ k-fantastic. Thus we are in Case 0. $\qquad\square$

We can now prove our main result of this section.

Proof of Theorem 6. By Proposition 19, Mini can play such that at the end of each of her turns, $G^{(t)}$ is k-fantastic. In particular, with this strategy, either $G^{(T)}$ or $G^{(T-1)}$ is k-fantastic. Lemma 11 implies that either $G^{(T)}$ or $G^{(T-1)}$ contains no cycles of length k or more, and so by Theorem 1, we must have

$$|E(G^{(T)})| \leqslant \frac{(k-1)(n-1)}{2} + 1. \qquad\qquad\square$$

3.4 Near Optimality of the Conditions for k-dense Families

Our work in the previous subsection shows that if $\mathcal{C}$ is a family of k-dense cycles, then either player can play so that $G^{(t)}$ always has circumference less than k, which gives our desired bound on $\mathrm{sat}_g(n, \mathcal{C})$ by Theorem 1. In this subsection we show that a slight loosening of the definition of k-fantastic families makes such a strategy impossible. More precise, we prove the following.

Proposition 20. *Let $k \geqslant 5$. If $\mathcal{C}$ is a set of cycles such that there exists an $s \geqslant 3$ with $C_\ell \notin \mathcal{C}$ for all $s \leqslant \ell \leqslant 4 + (k-2)(s-2) + 2(k-2)^2$, then for n sufficiently large, either player can play the $\mathcal{C}$-saturation game such that the game ends with circumference at least k.*

As a point of comparison, we remind the reader that for k-dense families and any $s \geqslant 3$, we have $C_\ell \in \mathcal{C}$ for some

$$s + 2 \leqslant \ell \leqslant 3 + (k-2)(s-2),$$

which is very close to the conditions of Proposition 20 when s is large in terms of k.

To emphasize, Proposition 20 does not say that $\mathrm{sat}_g(n, \mathcal{C})$ will not be linear for $\mathcal{C}$ as in the proposition, only that any strategy in the $\mathcal{C}$-saturation game which tries to guarantee that $G^{(t)}$ has circumference smaller than k is doomed to fail. Thus new ideas would be needed for computing $\mathrm{sat}_g(n, \mathcal{C})$ for families of this form.

To prove Proposition 20, we first establish the following technical lemma.

Lemma 21. *Let G be a graph of circumference less than $k \geqslant 3$ and $P = (x_1, \ldots, x_m)$ a path of maximum length in G. For all $m' \leqslant m$, we have that the longest path between x_1 and $x_{m'}$ has at most $m' - 1 + 2(k-2)^2$ edges.*

Proof. Let Y be a longest path between x_1 and $x_{m'}$ and let $S = V(P) \cap V(Y)$.

If all the vertices in S have indices less than or equal to m', then Y together with $(x_{m'+1}, \ldots, x_m)$ defines a path in G from x_1 to x_m, which cannot be longer than P since P is a path of maximum length. Thus in this case we have $|E(Y)| \leqslant m' - 1$, which satisfies the condition of the lemma.

Thus we can assume there is some vertex in S with index greater than m'. Order the set S based on the order that the vertices appear in Y, so that x_1 is the first vertex of S and $x_{m'}$ the last. Let x_j be the smallest vertex of S which has $j > m'$, and let x_i be the vertex which immediately precedes x_j in S based on the ordering of S. We claim that $x_i \in \{x_{m'-k+3}, \ldots, x_{m'-1}\}$. Indeed, $i < m'$ by definition of j, and if $i < m' - k + 3$, then the portion of the path Y between x_i and x_j together with the portion of the path P between those two vertices forms a cycle of length at least k, a contradiction.

Let $I = \{x_{m'-k+3}, \ldots, x_{m'+k-3}\}$ and let $V(Y) \cap I = \{x_{j_1}, \ldots, x_{j_p}\}$ with the vertices x_{j_i} in the order in which they appear along the path Y. Notice in particular that $x_{j_p} = x_{m'}$, which by the previous claim implies $x_{j_1} \in \{x_{m'-k+3}, \ldots, x_{m'-1}\}$, and also that $V(Y) \cap I \neq \emptyset$. Decompose Y into the subpaths Y', Y'' separated at x_{j_1}. By the argument in the previous paragraph, Y' contains no vertices to the right of x_{j_1} along P, so Y' together with the path along P from x_{j_1} to x_m defines a path in G from x_1 to x_m, which cannot be longer than P. Hence $|Y'| \leqslant m' - 1$. We will now show that $|Y''| \leqslant 2(k-2)^2$.

For each $1 \leqslant i \leqslant p - 1$, let Y_i be the subpath of Y'' from x_{j_i} to $x_{j_{i+1}}$. We claim that $|Y_i| \leqslant k - 2$ for all i. Indeed, Y_i is disjoint from the subpath P_i in P that goes from x_{j_i} to $x_{j_{i+1}}$ because Y_i only intersects I at x_{j_i} and $x_{j_{i+1}}$. Thus P_i and Y_i give rise to a cycle of length at least $|Y_i| + 1$ in G. Since the circumference of G is strictly less than k, Y_i can have at most $k - 2$ edges.

Since $x_{j_1}, \ldots, x_{j_p} \in I$, we have $p \leqslant 2(k-3) + 1$. Thus we have at most $2(k-3)$ paths Y_i, $1 \leqslant p - 1$. In total, Y'' can be decomposed into at most $2(k-3)$ many paths of length at most $k - 2$, so $|Y''| \leqslant 2(k-2)^2$. Combining this with the bound $|Y'| \leqslant m' - 1$ gives the desired result. $\qquad\square$

With this we can now prove our main result for this subsection.

Proof of Proposition 20. Let $k' = 3 + (k-2)(s-2)$ with s as in the hypothesis of the proposition. Initially we play as follows: if there is no path with at least k' vertices in $G^{(t)}$ at the start of our turn, then we add an edge between an isolated vertex and the

endpoint of some longest path in the graph. If n is sufficiently large, then eventually this will lead us to start our turn with some longest path in $G^{(t)}$ of length at least k', call it $(x_1, \ldots, x_m)$ with $m \geqslant k'$. Once this is achieved, we subsequently attempt to add the edge $x_1 x_{k'}$ to $G^{(t)}$. If this is a legal move, then we are done since $k' \geqslant k$, and we have the desired result.

We can thus assume that adding the edge $x_1 x_{k'}$ is not a legal move. That is, $G^{(t)}$ must contain a path $Y = (y_1, y_2, \ldots, y_q)$ with $y_1 = x_1$, $y_q = x_{k'}$ and $C_q \in \mathcal{C}$. Note by the previous lemma and the assumption $(x_1, \ldots, x_m)$ a longest path implies $q \leqslant k' - 1 + 2(k-2)^2$, so by hypothesis of $\mathcal{C}$ we must have $q < s$.

Let i_1 be such that x_{i_1} is the first x_i vertex to appear in Y (so $i_1 = 1$), and define $i_2, \ldots, i_p$ in an analogous same way. This implies for all $1 \leqslant j < p$ that $G^{(t)}$ contains a cycle of length at least $i_j - i_{j-1} + 1$. Thus we can assume $i_j \leqslant i_{j-1} + k - 2$ for all j, as otherwise $G^{(t)}$ contains a cycle of length at least k. By applying this bound repeatedly we find

$$1 + (k-2)(p-1) \geqslant i_p = k' = 3 + (k-2)(s-2).$$

Thus we must have $s \leqslant p \leqslant q$, contradicting that $q < s$ from above. Therefore it must be the case that $x_1 x_{k'}$ is a legal move, allowing us to create a graph with circumference at least k. $\qquad\square$

4 Concluding Remarks

In this paper we considered the saturation game where Max made the first move, and one could instead consider the analogous game where Mini makes the first move. It was shown by Hefetz, Krivelevich, Naor, and Stojaković [8] that in general these two saturation games can have dramatically different scores. However, all of our proofs allowed for either player to implement the proposed strategies, and from this one can easily show that all of the bounds of our theorems continue to hold even if Mini makes the first move.

Many of the results in this paper and in [13] focused on families of odd cycles. This is because in theory the game saturation number of a family of odd cycles could be anywhere between linear and quadratic. Motivated by this, we ask the following.

Problem 22. Determine whether Theorem 6 continues to hold if k-dense families do not require $C_{k+1} \in \mathcal{C}$ for k odd.

We have already shown that this is true for $k = 5$, and we believe that with more work one can use our methods to show that this also holds for $k = 7$ and possibly $k = 9$, but beyond this new ideas are needed.

In this paper we focused primarily on upper bounds for the game saturation number of a family of (odd) cycles. It would be of interest to consider lower bounds as well.

Problem 23. Determine non-trivial asymptotic lower bounds on $\mathrm{sat}_g(n, \mathcal{C}_k^o)$ for $k \geqslant 5$ odd.

We can prove one such bound when $k = 5$. By essentially the same argument used in the proof of Theorem 3, one can show that Mini can maintain that $G^{(t)}$ has circumference at most 4 in the $\mathcal{C}_5^o$-saturation game. It was proven by Ferrara, Jacobson, Milans, Tennenhouse, and Wenger [6, Theorem 2.17] that $\text{sat}(n, \{C_5, C_6, C_7, \ldots\}) = \frac{10}{7}(n - 1)$, which implies that $\text{sat}_g(n, \mathcal{C}_5^o) \geqslant \frac{10}{7}(n - 1)$.

Lastly, we note that all of our examples of families of cycles $\mathcal{C}$ with linear game saturation number contained infinitely many cycles. It is unclear if every family of cycles with linear game saturation must have infinite size.

Problem 24. Determine whether $\text{sat}_g(n, \mathcal{C}) = \omega(n)$ whenever $\mathcal{C}$ is a finite collection of cycles.

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