A Counterexample to the Shuffle Compatiblity Conjecture

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Submitted: Jan 7, 2022; Accepted: May 30, 2022; Published: Sep 9, 2022

Abstract

The shuffle product has a connection with several useful permutation statistics such as descent and peak, and corresponds to the multiplication operation in the corresponding descent and peak algebras. Gessel and Zhuang formalized the notion of shuffle-compatibility and studied various permutation statistics from this viewpoint. They further conjectured that any shuffle compatible permutation statistic is a descent statistic. In this note we construct a counter-example to this conjecture.

Mathematics Subject Classifications: 05A05

1 Introduction

For the purposes of this note, we define a *permutation* $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ to be a sequence of any n distinct integers, not necessarily the numbers between 1 and n. We call $|\sigma| = n$ the *size* of the permutation σ and denote the set of permutations of size n with P_n. We further set $P = \bigcup_n P_n$. Two permutations σ and ϕ of the same size are said to be *order-isomorphic*, denoted $\sigma \sim \phi$, if they have the same relative order, i. e. $\sigma_i > \sigma_j$ if and only $if \phi_i > \phi_j$ for all $i, j \leqslant n$. For example we have $1342 \sim 2894$, but they are not order-isomorphic to 2891, as the order of the first and last integers is different. Note that every permutation of size n is order-isomorphic to exactly one permutation of the numbers $1, 2, \ldots, n$.

Two permutations said to be *disjoint* if they do not share a number. Given two disjoint permutations σ and ϕ, a *shuffle* of σ and ϕ is a permutation of size $|\sigma| + |\phi|$ that contains both σ and ϕ as subsequences. We denote the set of shuffles of σ and ϕ with $\sigma \sqcup \phi$.

A function st on permutations is said to be a *permutation statistic* if $st(\sigma) = st(\phi)$ whenever $\sigma \sim \phi$. Some examples of permutation statistics are defined below:

$$\begin{aligned} \imath(\sigma) &= 1, \\ \mathrm{Des}(\sigma) &= \{i \mid \sigma_i > \sigma_{i+1}\}, \\ \mathrm{Inv}(\sigma) &= \{(i,j) \mid i < j \text{ and } \sigma_i > \sigma_j\}. \end{aligned}$$

We will also make use of *multisets*, where unlike the regular sets we are allowed to repeat elements. We will use to notation $\cdots$ when describing a multiset instead of a regular set.

Definition 1 ([1])**.** A permutation statistic st is said to be shuffle-compatible if for all disjoint permutations σ and ϕ, the multiset $\{\{st(\gamma) | \gamma \in \sigma \sqcup \phi\}\}$ depends only on $st(\sigma), st(\phi), |\sigma|$ and $|\phi|$.

From the examples above, $\imath$ and Des are shuffle-compatible, whereas Inv is not.The paper [1] by Gessel and Zhuang provides an in-depth exploration of shuffle-compatible permutation statistics. In that, they conjecture [1, Conjecture 6.11] that any shuffle-compatible permutation statistic st is a *descent statistic*, meaning if $\sigma, \phi \in P_n$ satisfy $\mathrm{Des}(\sigma) = \mathrm{Des}(\phi)$, then $st(\sigma) = st(\phi)$. In this note we disprove the conjecture by constructing a permutation statistic that is shuffle-compatible, but not a descent statistic.

Proposition 2. *Let st be a shuffle-compatible statistic. For $|\sigma| = |\phi| < 4$, $\mathrm{Des}(\sigma) = \mathrm{Des}(\phi)$ implies $st(\sigma) = st(\phi)$.*

Proof. For sizes 0, 1 and 2, any two permutations with the same descent set are order-isomorphic, so there is nothing to show. Let us focus on size 3. As these are permutation statistics, we can limit our attention to permutations of $1, 2, 3$. There are two pairs of non-order-isomorphic permutations with the same descent set: $213 - 312$ and $231 - 132$. The calculations below show that $st(213) = st(312)$ and $st(231) = st(132)$.

$$\begin{aligned} &\{\{st(\sigma) \mid \sigma \in 12 \sqcup 3\}\} = \{\{st(\sigma) \mid \sigma \in 13 \sqcup 2\}\}, \\ \Rightarrow &\{\{st(123), st(132), st(312)\}\} = \{\{st(132), st(123), st(213)\}\}, \\ &\{\{st(\sigma) \mid \sigma \in 23 \sqcup 1\}\} = \{\{st(\sigma) \mid |\sigma \in 13 \sqcup 2\}\}, \\ \Rightarrow &\{\{st(231), st(213), st(123)\}\} = \{\{st(132), st(123), st(213)\}\}. \end{aligned}$$ $\square$

This proposition shows that the minimum size we can have permutations that have the same descent set, but different values for some shuffle-compatible statistic is 4.

Let σ be a permutation of $a_1 < a_2 < a_3 < a_4$. Set

$$\begin{aligned} \mathrm{Inv}_{12}(\sigma) &= \begin{cases} 1 & \text{if } a_1 \text{ is to the left of } a_2 \text{ in } \sigma, \\ -1 & \text{otherwise,} \end{cases} \\ \mathrm{Adj}_{34}(\sigma) &= \begin{cases} 1 & \text{if } a_3 \text{ is adjacent to } a_4 \text{ in } \sigma, \\ -1 & \text{otherwise,} \end{cases} \\ \Lambda(\sigma) &= \mathrm{Inv}_{21}(\sigma) \cdot \mathrm{Adj}_{34}(\sigma). \end{aligned}$$

For example, $\sigma = 2413$ has $\mathrm{Inv}_{12}(\sigma) = -1$ and $\mathrm{Adj}_{34}(\sigma) = -1$, so $\Lambda(2413) = (-1) \cdot (-1) = 1$. For $\phi = 1423$, $\mathrm{Inv}_{12}(\phi) = 1$ and $\mathrm{Adj}_{34}(\phi) = -1$, so $\Lambda(1423) = 1 \cdot (-1) = 1$.

Definition 3. We define a permutation statistic $\Psi : P \longrightarrow \mathbb{Z}$ as follows:

$$\Psi(\sigma) = \begin{cases} \Lambda(\sigma) & \text{if } |\sigma| = 4, \\ 1 & \text{otherwise.} \end{cases}$$

Proposition 4. *The function Ψ is not a descent statistic.*

Proof. For $\sigma = 2413$ and $\phi = 1423$, $\mathrm{Des}(\sigma) = \mathrm{Des}(\phi) = \{2\}$, but $\Psi(\sigma) = 1 \neq \Psi(\phi) = -1$. $\square$

Theorem 5. *The function Ψ is shuffle-compatible.*

Proof. Let σ and ϕ be two permutations. Note that if $|\sigma| + |\phi| \neq 4$, the multiset $\{\{\Psi(\gamma) | \gamma \in \sigma \sqcup\!\sqcup \phi\}\}$ contains only 1s, and the number of those depends only on $|\sigma|$ and $|\phi|$. So we can focus on when $|\sigma| + |\phi| = 4$. As we are working with a permutation statistic, it is enough to verify the result when the integers used are $1, 2, 3$ and 4. Further note that the shuffle operation is symmetric, and Ψ is symmetric under exchanging 3 and 4.

Case 1: $|\sigma| = 3$, $|\phi| = 1$. We claim that in this case $\{\{\Psi(\gamma) | \gamma \in \sigma \sqcup\!\sqcup \phi\}\} = \{\{-1, -1, 1, 1\}\}$. If $\phi = 3$, then independent of the placement of 4, of the four elements of $\sigma \sqcup\!\sqcup \phi$, exactly two have 3 and 4 adjacent, and the order of 1 and 2 is the same for all of them, so the claim holds. Same argument applies for the case $\phi = 4$ by symmetry. The 6 other possibilities are illustrated at Table 1, left.

Case 2: $|\sigma| = |\phi| = 2$. We claim that $\{\{\Psi(\gamma) | \gamma \in \sigma \sqcup\!\sqcup \phi\}\} = \{\{-1, -1, -1, 1, 1, 1\}\}$. As exchanging 3 and 4 does not alter the Ψ value, there are 6 possible pairings we need to consider, all illustrated in Table 1, right. $\square$

	$\Psi = +1$	$\Psi = -1$
$134 \sqcup\!\sqcup 2$	$1234, 1342$	$1324, 2134$
$314 \sqcup\!\sqcup 2$	$2314, 3214$	$3124, 3142$
$341 \sqcup\!\sqcup 2$	$3412, 3241$	$3421, 2341$
$234 \sqcup\!\sqcup 1$	$1234, 2314$	$2341, 2134$
$324 \sqcup\!\sqcup 1$	$3241, 3214$	$3124, 1324$
$342 \sqcup\!\sqcup 1$	$3412, 1342$	$3421, 3142$

	$\Psi = +1$	$\Psi = -1$
$12 \sqcup\!\sqcup 34$	$1234, 1342, 3412$	$1324, 3124, 3142$
$13 \sqcup\!\sqcup 24$	$1234, 1243, 2413$	$2134, 2143, 1324$
$13 \sqcup\!\sqcup 42$	$4213, 1432, 1342$	$4132, 4123, 1423$
$21 \sqcup\!\sqcup 34$	$2314, 3241, 3214$	$2134, 2341, 3421$
$23 \sqcup\!\sqcup 14$	$2314, 1243, 1234$	$1423, 2134, 2143$
$23 \sqcup\!\sqcup 41$	$4213, 2413, 4231$	$4123, 2341, 2431$

Table 1: Ψ values of shuffles of pairs σ and ϕ.

Corollary 6. *Conjecture 6.11 from [1] is incorrect.*

Note that this counterexample mainly depends on how Ψ acts on permutations of size 4. Even though taking only two order-isomorphism classes at size 4 limits our options for larger sizes, it does not force the existence of only one order-isomorphism class at each level, that is just selected for simplicity.

A question that arises from this counterexample is whether it is possible to refine the conjecture by adding extra conditions to ensure the resulting statistics only depend on descent. One such result was recently proved by Grinberg in [2]:

Definition 7 ([2])**.** A permutation statistic st is *left-shuffle-compatible* if for disjoint permutations γ and ϕ satisfying $\sigma_1 > \gamma_1$, the multiset $\{\{st(\gamma) \mid \gamma \in \sigma \sqcup\!\sqcup\, \phi \text{ and } \gamma_1 = \sigma_1\}\}$ depends only on $st(\sigma), st(\phi), |\sigma|$ and $|\phi|$.

Proposition 8 ([2])**.** *Any shuffle-compatible and left-shuffle-compatible statistic is a descent statistic.*

Note that our counter-example does not violate this result, as it is not left-shuffle-compatible:

$$\begin{aligned}\{\{\Psi(\gamma) \mid \gamma \in 24 \sqcup\!\sqcup\, 31,\, \gamma_1 = 1\}\} &= \{\{\Psi(2431), \Psi(2341), \Psi(2314)\}\} = \{\{-1,-1,1\}\},\\ \{\{\Psi(\gamma) \mid \gamma \in 34 \sqcup\!\sqcup\, 21,\, \gamma_1 = 3\}\} &= \{\{\Psi(3421), \Psi(3241), \Psi(3214)\}\} = \{\{-1,1,1\}\}.\end{aligned}$$

There are examples of shuffle-compatible descent statistics that are not left-shuffle-compatible, so the above proposition does not offer a complete characterization.

2 Addendum

The following wonderful proof for Theorem 5 was contributed by Darij Grinberg as an alternative that underlines the connection with Lie theory:

Let $\mathcal{F}$ be the free $\mathbb{Q}$-algebra on generators a, b, c, d. We identify each permutation on the letters a, b, c, d with a corresponding word in $\mathcal{F}$, by writing it in one-line notation and replacing the numbers 1, 2, 3, 4 by a, b, c, d respectively. For example, the permutation 3142 becomes identified with $cadb$, whereas the permutation 341 becomes cda. When σ and ϕ are two disjoint permutations, the shuffle product of the corresponding words in the free algebra $\mathcal{F}$ is given by sum of the words corresponding to the permutations in $\sigma \sqcup\!\sqcup\, \phi$. Let $\mathcal{S}$ be the span of all nontrivial shuffles in $\mathcal{F}$ - that is, of all shuffles of the form $u \sqcup\!\sqcup\, v$ where u and v are two nonempty words (We disallow empty words, since otherwise $\mathcal{S}$ would be the whole $\mathcal{F}$).

We also consider the free Lie subalgebra $\mathcal{L}$ of $\mathcal{F}$ generated by a, b, c, d. Note that $\mathcal{L}$ is given by the span of all iterated commutators of a, b, c, d including the elements a, b, c, d themselves. For example, $[[b,[c,[a,d]],[b,d]] \in \mathcal{L}$. The elements of $\mathcal{L}$ are called *Lie polynomials.*

There is an inner product $< \cdot,\cdot >: \mathcal{F} \times \mathcal{F} \to \mathbb{Q}$ with respect to which the words form an orthonormal basis (i.e., for any two words u and v we have $< u, v >= \delta_{u,v}$. Let

$$x := [a,[c[b,d]]] + [d,[c,[b,a]]] + [b,[c,[d,a]]] \in \mathcal{L}.$$

Expanding x, we easily see that x is the signed sum of the words corresponding to all 24 permutations with entries 1, 2, 3, 4. Furthermore, each such permutation σ appears with sign $\Psi\sigma$ in this signed sum. Thus for every permutation σ of 1, 2, 3 and 4, we have

$$\Psi(\sigma) =< x, \sigma > . \tag{1}$$

However, it is a known fact (easily follows from Theorem 1.4 in [3]) that $< \mathcal{S}, \mathcal{L} >= 0$, that is we have $< u \sqcup\!\sqcup\, v, w >= 0$ for every nontrivial shuffle $u \sqcup\!\sqcup\, v \in \mathcal{S}$ and every Lie polynomial $w \in \mathcal{L}$. This is a consequence of the fact that $< \cdot, \cdot >$ is a graded duality pairing between the shuffle Hopf algebra and the tensor Hopf algebra, and that $\mathcal{S}$ is the square of the augmentation ideal of the shuffle Hopf algebra, while $\mathcal{L}$ is the space of primitives of the tensor Hopf algebra.

Thus, for every nontrivial shuffle $u \sqcup\!\sqcup v \in \mathcal{S}$ we have $< u \sqcup\!\sqcup v, x >= 0$. In particular, if σ and ϕ are two disjoint nonempty permutations we have:

$$\sum_{\gamma \in \sigma \sqcup\!\sqcup \phi} \Psi(\gamma) = 0.$$

As Ψ can only take the values ± 1, this completely characterizes the multiset $\sigma \sqcup\!\sqcup \phi$ and proves that Ψ is indeed shuffle compatible.

Acknowledgements

The author would like to thank Sami Assaf and Darij Grinberg for helpful suggestions.

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