Shellability of Polyhedral Joins of Simplicial Complexes and its Application to Graph Theory

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Abstract
We investigate the shellability of the polyhedral join $Z^*_M(K, L)$ of simplicial complexes $K, M$ and a subcomplex $L \subset K$. We give sufficient conditions and necessary conditions on $(K, L)$ for $Z^*_M(K, L)$ being shellable. In particular, we show that for some pairs $(K, L)$, $Z^*_M(K, L)$ becomes shellable regardless of whether $M$ is shellable or not. Polyhedral joins can be applied to graph theory as the independence complex of a certain generalized version of lexicographic products of graphs which we define in this paper. The graph obtained from two graphs $G, H$ by attaching one copy of $H$ to each vertex of $G$ is a special case of this generalized lexicographic product and we give a result on the shellability of the independence complex of this graph by applying the above results.

Mathematics Subject Classifications: 05E45, 05C76

1 Introduction
A finite simple graph $G$ is a pair $G = (V(G), E(G))$ of a finite set $V(G)$ and a set $E(G) \subset \{ e \subset V(G) \mid |e| = 2 \}$. We drop adjectives “finite simple” and call $G$ a graph. $v \in V(G)$ is called a vertex of $G$ and $e \in E(G)$ is called an edge of $G$. The independence complex of $G$ is an abstract simplicial complex $I(G)$ defined by

$$I(G) = \{ \sigma \subset V(G) \mid \{ u, v \} \notin E(G) \text{ for any } u, v \in \sigma \}.$$ A simplex of $I(G)$ is called an independent set of $G$.

A simplicial complex $K$ is shellable if its facets can be arranged in a linear order $F_1, F_2, \ldots, F_t$ (which we call a shelling) in such a way that the subcomplex $\left( \bigcup_{i=1}^{k-1} F_i \right) \cap$
A graph is called shellable if its independence complex is shellable. The shellability (including vertex decomposability, which is one of the sufficient conditions for a graph being shellable) of graphs has been studied by many researchers, such as [6, 8, 9, 11]. In this paper, we focus on the following result by Hibi, Higashitani, Kimura, and O’Keefe [6]. Here, a graph $G$ is called well-covered if every maximal independent set of $G$ has the same cardinality.

**Theorem 1** (Based on Hibi, Higashitani, Kimura, and O’Keefe [6, Theorem 1.1]). Let $G$ be a graph on a vertex set $V(G) = \{u_1, \ldots, u_n\}$. Let $k_1, \ldots, k_n \geqslant 2$ be integers. Then the graph $G'$ obtained from $G$ by attaching the complete graph $K_{k_i}$ to $u_i$ for $i = 1, \ldots, n$ is well-covered and shellable.

Motivated by Theorem 1, we consider the graph $G[H; \{v_0\}]$ defined as follows. Let $G$, $H$ be graphs and $v_0$ be a vertex of $H$. Define $G[H; \{v_0\}]$ as the graph obtained from $G \sqcup \left( \bigcup_{u \in V(G)} H_u \right)$, where $H_u$ is a copy of $H$, by identifying $u \in G$ with $v_0 \in H_u$. Theorem 1 implies that if $H$ is a complete graph, then $G[H; \{v_0\}]$ is well-covered and shellable for any graph $G$.

In this paper, we obtain a necessary and sufficient condition on $H$ and $v_0$ for $G[H; \{v_0\}]$ being well-covered and shellable for any graph $G$.

**Theorem 2.** Let $H$ be a graph and $v_0$ be a vertex of $H$. Then the following two conditions are equivalent.

1. For any graph $G$, $G[H; \{v_0\}]$ is well-covered and shellable.
2. $H$ is well-covered, both $H$ and $H \setminus \{v_0\}$ are shellable, and for any maximal independent set $\tau$ of $H \setminus \{v_0\}$, there exists $v \in \tau$ such that $\{v_0, v\} \in E(H)$.

In order to prove Theorem 2, we need to investigate the independence complex of $G[H; \{v_0\}]$. With $I(G)$ and $I(H)$, the independence complex $I(G[H; \{v_0\}])$ is described as a polyhedral join. Polyhedral join is a construction of simplicial complexes introduced by Ayzenberg [1, Definition 4.2, Observation 4.3]. It is similar to polyhedral product $Z_K(\mathbf{X}, \mathbf{A})$, a well-known construction of spaces, where $K$ is a simplicial complex and $(\mathbf{X}, \mathbf{A}) = \{(X_u, A_u)\}_{u \in V(K)}$ is a family of pairs of spaces. The definition of polyhedral joins is obtained from the definition of polyhedral products by replacing “pairs of spaces” and “product of spaces” with “pairs of simplicial complex and its subcomplex” and “join of simplicial complexes”, respectively. Polyhedral joins appear in previous studies, including when they are called by other names. Bahri, Bendersky, Cohen, and Gitler [2, Definition 2.1] defined a simplicial complex $K(J)$ for a simplicial complex $K$ on $V(K) = \{1, \ldots, n\}$ and a tuple $J = (j_1, \ldots, j_n)$ of positive integers. Using our notation of polyhedral joins, $K(J)$ is denoted by

$$Z_K(\Delta^{J-1}, \partial \Delta^{J-1}), \text{ where } (\Delta^{J-1}, \partial \Delta^{J-1}) = \{(\Delta_i^{J-1}, \partial \Delta_i^{J-1})\}_{i \in \{1, \ldots, n\}}.$$
Here, $\Delta^d$ is the $d$-simplex and $\partial \Delta^d$ is its boundary. They obtained the decomposition of polyhedral products, more precisely, moment-angle complexes, denoted by

$$
Z_K(D^2, S^1) = Z^*_K(\Delta^1, \partial \Delta^1)(D^1, S^0).
$$

We note that the above observation is mentioned by Vidaurre [10], who investigated the polyhedral products over polyhedral joins. Another example is $(j_1, \ldots, j_n)$-expansion of $K$, which is introduced by Moradi and Khosh-Ahang [7, Definition 2.1]. It is denoted by

$$
Z^*_K(pt_{j_i}, \emptyset) = \{ (\bigcup_{i \in \{1, \ldots, n\}} pt_{j_i}, \emptyset) \}.
$$

They studied the shellability and vertex decomposability of expansions. We generalize one of their results [7, Theorem 2.12].

This paper is organized as follows. In Section 2, we define terminologies and notations on simplicial complexes and state some basic properties of shellable simplicial complexes. Section 3 provides the definition of polyhedral joins and the explicit description of simplices and facets of polyhedral joins. Section 4 is the main part of this paper. Here we obtain two necessary conditions and two sufficient conditions for polyhedral joins being shellable, giving counterexamples of converse propositions. Note that some of the results are not relevant to Theorem 2. Finally, in Section 5, we apply the results obtained in Section 4 to the independence complexes of graphs and prove Theorem 2.

2 Preliminaries

In the following, for a positive integer $m$, we set $[m] = \{1, 2, \ldots, m\}$.

An abstract simplicial complex $K$ is a collection of finite subsets of a given set $V(K)$ such that if $\sigma \in K$ and $\tau \subset \sigma$, then $\tau \in K$. In this paper, we drop the adjective “abstract”.

An element of $K$ is called a simplex of $K$. An element of $V(K)$ is called a vertex of $K$.

We suppose that $\{v\} \in K$ for any $v \in V(K)$. We set $\dim K = \max_{\sigma \in K} |\sigma| - 1$, where $|\sigma|$ is the cardinality of $\sigma \subset V(K)$. If $\dim K = d$, then $K$ is called $d$-dimensional. A maximal simplex with respect to the inclusion is called a facet of $K$. $K$ is called pure if every facet of $K$ has the same cardinality.

$L \subset K$ is called a subcomplex of $K$ if $L$ is a simplicial complex. In this paper, we call $(K, L)$ a pair of simplicial complexes if $K$ is a simplicial complex and $L \subset K$ is a subcomplex of $K$. For a vertex $v \in V(K)$ of $K$, we define a subcomplex $dl_K(v)$ of $K$ by

$$
dl_K(v) = \{ \sigma \in K \mid v \notin \sigma \}.
$$

Let $\{K_i\}_{i \in [m]}$ be a family of simplicial complexes. We define a simplicial complex $K_1 \ast \cdots \ast K_m$, which we call the join of $K_1, K_2, \ldots, K_m$, by

$$
K_1 \ast \cdots \ast K_m = \left\{ \sigma \subset \bigcup_{i \in [m]} V(K_i) \middle| \sigma \cap V(K_i) \in K_i \text{ for any } i \in [m] \right\}.
$$
Let \( V \) be a finite set and \( F_1, \ldots, F_t \subset V \) be a collection of subsets of \( V \) such that \( F_i \not\subset F_j \) for any \( i \neq j \). Define a simplicial complex \( \langle F_1, \ldots, F_t \rangle \) on \( V \) by
\[
\langle F_1, \ldots, F_t \rangle = \{ \sigma \subset V \mid \sigma \subset F_i \text{ for some } i \in [t] \}.
\]

As defined in Section 1, a simplicial complex \( K \) is shellable if its facets can be arranged in a linear order \( F_1, F_2, \ldots, F_t \) (which we call a shelling) in such a way that the subcomplex \((\bigcup_{i=1}^{k-1} \langle F_i \rangle) \cap \langle F_k \rangle\) is pure and \((\dim F_k - 1)\)-dimensional for all \( k = 2, \ldots, t \). Here we state some of the properties of shellable simplicial complexes without proofs.

**Lemma 3** (Based on [3, Lemma 2.3]). Let \( K \) be a simplicial complex. An order \( F_1, F_2, \ldots, F_t \) of the facets of \( K \) is a shelling if and only if for every \( i, k \) with \( 1 \leq i < k \leq t \), there exists an index \( j \) with \( 1 \leq j < k \) and a vertex \( x \in F_k \setminus F_i \) such that \( F_j \cap F_k = F_k \setminus \{x\} \).

**Lemma 4** (Based on [3, Lemma 2.6]). Let \( K \) be a shellable simplicial complex. Then there exists a shelling \( F_1, F_2, \ldots, F_t \) such that \( |F_i| \geq |F_j| \) for any \( 1 \leq i < j \leq t \).

**Lemma 5** ([4, Remark 10.22]). The join of two simplicial complexes is shellable if and only if each of the simplicial complex is shellable.

For a simplicial complex \( X \), we denote the set of all facets of \( X \) by \( F_X \). Note that for a pair \((K, L)\) of simplicial complexes, we have
\[
F_K \setminus F_L = \{ \sigma \in F_K \mid \sigma \not\subset L \},
\]
\[
F_L \setminus F_K = \{ \tau \in F_L \mid \text{there exists } \sigma \in F_K \setminus F_L \text{ such that } \tau \subset \sigma \},
\]
\[
F_K \cap F_L = \{ \sigma \in F_K \mid \sigma \subset L \}.
\]

3 Facets of Polyhedral Joins

The main subject of this paper is the polyhedral join. It is a construction of simplicial complexes, which is based on the definition by Ayzenberg [1, Definition 4.2, Observation 4.3].

**Definition 6.** Let \( M \) be a simplicial complex on \([m]\) and \((K, L) = \{(K_i, L_i)\}_{i \in [m]}\) be a family of pairs of simplicial complexes. For a simplex \( S \in M \), we define a simplicial complex \((K, L)^S\) by
\[
(K, L)^S = X_1 \ast X_2 \ast \cdots \ast X_m, \quad X_i = \begin{cases} K_i & (i \in S), \\ L_i & (i \notin S). \end{cases}
\]

\((K, L)^S\) is a subcomplex of \( K_1 \ast \cdots \ast K_m \).

\( \mathcal{Z}_M^*(K, L) \) is a subcomplex of \( K_1 \ast \cdots \ast K_m \) defined by
\[
\mathcal{Z}_M^*(K, L) = \bigcup_{S \in M} (K, L)^S,
\]
(union is taken in \( K_1 \ast \cdots \ast K_m \)).

Let \((K, L)\) be a pair of simplicial complexes. If \( K_i = K, L_i = L \) for any \( i \in [m] \), we write \( \mathcal{Z}_M^*(K, L) \) instead of \( \mathcal{Z}_M^*(K, L) \).
We give an explicit description of simplices and facets of $Z^*_M(K, L)$.

**Proposition 7.** Let $M$ be a simplicial complex on $[m]$ and $(K, L) = \{(K_i, L_i)\}_{i \in [m]}$ be a family of pairs of simplicial complexes. For $\phi \subset \bigsqcup_{i \in [m]} V(K_i)$, we set

$$\phi_i = \phi \cap V(K_i) \ (i \in [m]),$$

$$\overline{\phi} = \{i \in [m] \mid \phi_i \notin L_i\}.$$  

Then $\phi \in Z^*_M(K, L)$ if and only if $\phi_i \in K_i$ for any $i \in [m]$ and $\overline{\phi} \in M$.

**Proof.** Let $\phi \in Z^*_M(K, L)$. Then $\phi$ is a simplex of $\phi \in (K, L)^sS$ for some $S \in M$. So, by the definition of the join, we have

- $\phi \cap V(K_i) = \phi_i \in K_i$ for any $i \in S$,
- $\phi \cap V(K_i) = \phi_i \in L_i \subset K_i$ for any $i \notin S$.

It follows that $\phi_i \in K_i$ for any $i \in [m]$ and that $\overline{\phi} \subset S$. Since $M$ is a simplicial complex and $S \in M$, we have $\overline{\phi} \in M$.

Conversely, let $\phi \subset \bigsqcup_{i \in [m]} V(K_i)$ be a set such that $\phi_i \in K_i$ for any $i \in [m]$ and $\overline{\phi} \in M$. By the definition of $\overline{\phi}$, $i \notin \overline{\phi}$ implies $\phi_i \in L_i$. So, we have $\phi \in (K, L)^sS \subset Z^*_M(K, L)$. \hfill □

**Proposition 8.** Let $M$ be a simplicial complex on $[m]$ and $(K, L) = \{(K_i, L_i)\}_{i \in [m]}$ be a family of pairs of simplicial complexes. Then $\phi \subset \bigsqcup_{i \in [m]} V(K_i)$ is a facet of $Z^*_M(K, L)$ if and only if

- $\phi_i \in F_{K_i} \cup F_{L_i}$ for any $i \in [m]$,
- $\overline{\phi} \in M$, and
- $\overline{\phi} \cup \{i\} \notin M$ for any $i \in [m]$ such that $\phi_i \in F_{L_i} \setminus F_{K_i}$.

**Proof.** Let $\phi$ be a facet of $Z^*_M(K, L)$. For $k \in [m]$ and $\sigma \in K_k$, define $\phi^{(k, \sigma)} \subset \bigsqcup_{i \in [m]} V(K_i)$ by

$$\phi_i^{(k, \sigma)} = \begin{cases} 
\phi_i & \text{(i \neq k)} \\
\sigma & \text{(i = k)}.
\end{cases}$$

First, assume that there exists $\sigma \in K_k$ such that $\phi_k \not\subset \sigma$ for some $k \in \overline{\phi}$. We have $\sigma \notin L_k$ since $\sigma \supset \phi_k \notin L_k$. Therefore, we get $\phi^{(k, \sigma)} = \overline{\phi} \in M$. So, $\phi^{(k, \sigma)}$ is a simplex of $Z^*_M(K, L)$ which satisfies $\phi^{(k, \sigma)} \supsetneq \phi$, a contradiction. Thus, $\phi_k$ is a facet of $K_k$.

Second, assume that there exists $\tau \in L_k$ such that $\phi_k \not\subset \tau$ for some $k \notin \overline{\phi}$. Since $\phi_k, \tau \in L_k$, we get $\phi^{(k, \tau)} = \overline{\phi} \in M$. So, $\phi^{(k, \tau)}$ is a simplex of $Z^*_M(K, L)$ which satisfies $\phi^{(k, \tau)} \supsetneq \phi$, a contradiction. Thus, $\phi_k$ is a facet of $L_k$.

Finally, assume that there exists $j \in [m]$ such that $\phi_j \in F_{L_j} \setminus F_{K_j}$ and $\overline{\phi} \cup \{j\} \notin M$. Then there must be $\rho \in K_j \setminus L_j$ such that $\phi_j \not\subset \rho$ since $\phi_j$ is a facet of $L_j$ and is not a
facet of $K_j$. We get that $\phi^{(j,\phi)}$ is a simplex of $Z^*_M(K,L)$ since $\overline{\phi^{(j,\phi)}} = \overline{\phi} \cup \{j\} \in M$. This is a contradiction to the maximality of $\phi$. By the above three arguments, we conclude that a facet $\phi$ of $Z^*_M(K,L)$ satisfies three conditions in the proposition.

Conversely, suppose that $\phi \subset \bigcup_{i \in [m]} V(K_i)$ satisfies three conditions in the proposition. Since $\phi_i \in F_{K_i} \cup F_{L_i} \subset K_i$ for any $i \in [m]$ and $\overline{\phi} \in M$, $\phi$ is a simplex of $Z^*_M(K,L)$. We assume that there exists $\psi \in Z^*_M(K,L)$ such that $\phi \subset \psi$ and deduce a contradiction. There must exists $k \in [m]$ such that $\phi_k \subset \psi_k$. This means that $\phi_k$ is not a facet of $K_k$, which implies $\phi_k \in F_{L_k} \setminus F_{K_k}$. It follows that $k \in \overline{\psi}$ since $\phi_k \subset \psi_k$ and $\phi_k$ is a facet of $L_k$. Thus, we get $\overline{\psi} \cup \{k\} \subset \overline{\psi} \in M$, which contradicts to the third condition in the proposition. Therefore, $\phi$ is a facet of $Z^*_M(K,L)$.

**Remark 9.** If $F_{L_i} \subset F_{K_i}$ for any $i \in [m]$, then by Proposition 8, $\phi \subset \bigcup_{i \in [m]} V(K_i)$ is a facet of $Z^*_M(K,L)$ if and only if $\phi_i \in F_{K_i}$ for any $i \in [m]$ and $\overline{\phi} \in M$.

Before investigating the shellability of $Z^*_M(K,L)$, we show a necessary and sufficient condition for $Z^*_M(K,L)$ being pure.

**Theorem 10.** Let $M \neq \emptyset$ be a simplicial complex which is not a simplex and $K \supseteq L$ be a pair of simplicial complexes. Then $Z^*_M(K,L)$ is pure if and only if

- $K$ is pure and $F_L \subset F_K$, or
- $K,L,M$ are pure.

**Proof.** We set $V(M) = [m]$.

Assume that $Z^*_M(K,L)$ is pure. For $S \in F_M$, $\sigma \in F_K$ and $\tau \in F_L$, define $\Phi_S(\sigma, \tau) \subset \bigcup_{i \in [m]} V(K)$ by

$$
\Phi_S(\sigma, \tau)_i = \begin{cases} 
\sigma & (i \in S) \\
\tau & (i \notin S).
\end{cases}
$$

Here, remark that $\Phi_S(\sigma, \tau)$ is a facet of $Z^*_M(K,L)$ if $\sigma \in F_K \setminus F_L$ or $\tau \in F_K \cap F_L$ by Proposition 8. If $\Phi_S(\sigma, \tau)$ is a facet of $Z^*_M(K,L)$, then we have

$$
|\Phi_S(\sigma, \tau)| = \sum_{i \in [m]} |\Phi_S(\sigma, \tau)_i| = |S||\sigma| + (m - |S|)|\tau|.
$$

Note that we have $0 < |S| < m$ since $M$ is neither $\{\emptyset\}$ nor a simplex.

Suppose that $F_K \cap F_L \neq \emptyset$. Take $S \in F_M$, $\sigma^0 \in F_K \setminus F_L$ and $\tau^0 \in F_K \cap F_L$. For any facets $\sigma, \sigma' \in F_K$ of $K$, $\Phi_S(\sigma, \tau^0)$ and $\Phi_S(\sigma', \tau^0)$ are facets of $Z^*_M(K,L)$. Since $Z^*_M(K,L)$ is pure, we have $|\Phi_S(\sigma, \tau^0)| = |\Phi_S(\sigma', \tau^0)|$, namely

$$
|S||\sigma| + (m - |S|)|\tau^0| = |S||\sigma'| + (m - |S|)|\tau^0|.
$$

It follows from $|S| > 0$ that

$$
|\sigma| = |\sigma'|.
$$
Therefore, $K$ is pure. For any facets $\tau, \tau' \in F_L$ of $L$, $\Phi_S(\sigma^0, \tau)$ and $\Phi_S(\sigma^0, \tau')$ are facets of $Z^*_M(K, L)$. Since $Z^*_M(K, L)$ is pure, we have $|\Phi_S(\sigma^0, \tau)| = |\Phi_S(\sigma^0, \tau')|$, namely

$$|S||\sigma^0| + (m - |S|)|\tau| = |S||\sigma^0| + (m - |S|)|\tau'|.$$  

It follows from $|S| < m$ that

$$|\tau| = |\tau'|.$$  

Therefore, $L$ is pure. Hence, we obtain

$$\dim K = |\tau^0| - 1 = \dim L.$$  

If there exists $\tau^1 \in F_L \setminus F_K$, then there must exist $\sigma^1 \in F_K \setminus F_L$ such that $\tau^1 \subseteq \sigma^1$. So, we get

$$\dim K = |\sigma^1| - 1 > |\tau^1| - 1 = \dim L,$$

which is a contradiction. Thus, we conclude that $F_L \setminus F_K = \emptyset$, namely $F_L \subset F_K$.

Next, suppose that $F_K \cap F_L = \emptyset$. Take $S \in F_M$ and $\tau^0 \in F_L$. Since $\tau^0$ is not a facet of $K$, there exist $\sigma^0 \in F_K$ such that $\tau^0 \subseteq \sigma^0$. For any facet $\sigma$ of $K$, $\Phi_S(\sigma, \tau^0)$ is a facet of $Z^*_M(K, L)$ since $F_K = F_K \setminus F_L$. Therefore, by the same argument as above, we obtain that both $K$ and $L$ are pure. For any facet $T, T' \in F_M$ of $M$, $\Phi_T(\sigma^0, \tau^0)$ and $\Phi_T(\sigma^0, \tau^0)$ are facets of $Z^*_M(K, L)$. Since $Z^*_M(K, L)$ is pure, we have $|\Phi_T(\sigma^0, \tau^0)| = |\Phi_T(\sigma^0, \tau^0)|$, namely

$$|T||\sigma^0| + (m - |T|)|\tau^0| = |T'||\sigma^0| + (m - |T'|)|\tau^0|.$$  

It follows that

$$|T| = |T'|$$  

since $\sigma^0 \supseteq \tau^0$. Therefore, we conclude that $M$ is pure.

Conversely, assume that $K$ is pure and $F_L \subset F_K$. Then for any facet $\phi \in F_{Z^*_M(K, L)}$, we get

$$|\phi| = \sum_{i=1}^{m} |\phi_i| = m(\dim K + 1).$$  

Therefore, $Z^*_M(K, L)$ is pure.

Finally, assume that $K, L, M$ are pure and that there exists $\tau \in F_L \setminus F_K$. Since there exists $\sigma \in F_K \setminus F_L$ such that $\tau \subset \sigma$, we get

$$\dim K = |\sigma| - 1 > |\tau| - 1 = \dim L.$$
So, we obtain $F_K \cap F_L = \emptyset$ since both $K$ and $L$ are pure. Thus, for any $\phi \in F_{Z^*_M(K, L)}$, $\bar{\phi}$ is a facet of $M$. This is because we have $\bar{\phi} \cup \{i\} \notin M$ for any $i \notin \bar{\phi}$ since $F_L = F_L \setminus F_K$. Hence, we get

$$|\phi| = \sum_{i \in \bar{\phi}} |\phi_i| + \sum_{i \notin \bar{\phi}} |\phi_i|$$

$$= |\bar{\phi}|(\dim K + 1) + (m - |\bar{\phi}|)(\dim L + 1)$$

$$= (\dim M + 1)(\dim K + 1) + (m - \dim M - 1)(\dim L + 1)$$

for any facet $\phi \in F_{Z^*_M(K, L)}$ since $M$ is pure. Therefore, we conclude that $Z^*_M(K, L)$ is pure. □

4 Shellability of Polyhedral Joins

We first present two sufficient conditions for $Z^*_M(K, L)$ being shellable.

**Theorem 11.** Let $M$ be an arbitrary simplicial complex and $K \supset L$ be a pair of simplicial complexes. Suppose that $K, L$ satisfy the following three conditions.

1. $K$ is shellable,

2. $F_L \subset F_K$, and

3. there exists a shelling $<_K$ on $K$ such that $\tau <_K \sigma$ for any $\tau \in F_L$ and any $\sigma \in F_K \setminus F_L$.

Then $Z^*_M(K, L)$ is shellable.

**Proof.** We set $V(M) = [m]$. Consider a linear order $<$ on $F_{Z^*_M(K, L)}$ such that $\phi < \psi$ if and only if

$$\phi_1 = \psi_1, \ldots, \phi_{j-1} = \psi_{j-1}, \phi_j <_K \psi_j$$

for some $j \in [m]$. We show that this order is a shelling on $Z^*_M(K, L)$.

Let $\phi, \psi$ be facets of $Z^*_M(K, L)$ such that $\phi_1 = \psi_1, \ldots, \phi_{j-1} = \psi_{j-1}, \phi_j <_K \psi_j$. Since $<_K$ is a shelling on $K$, there exists $\sigma \in F_K$ and $x \in \psi_j \setminus \phi_j$ such that $\sigma <_K \psi_j$ and $\sigma \cap \psi_j = \psi_j \setminus \{x\}$ by Lemma 3. Define $\chi \subset \bigsqcup_{i \in [m]} V(K)$ by

$$\chi_i = \begin{cases} 
\psi_i & (i \neq j) \\
\sigma & (i = j)
\end{cases}$$

If $\psi_j \in F_K \setminus F_L$, then we have $\chi \subset \psi \notin M$. If $\psi_j \in F_L$, then by condition (3), $\sigma$ must be a facet of $L$ since $\sigma <_K \psi_j$. So, we have $\chi = \psi \notin M$. In both cases, we get that $\chi \in M$, which implies that $\chi$ is a simplex of $Z^*_M(K, L)$. Furthermore, $\chi$ is a facet of $Z^*_M(K, L)$ by Remark 9.
We have \( \chi < \psi \) because
\[
\chi_1 = \psi_1, \ldots, \chi_{j-1} = \psi_{j-1}, \chi_j = \sigma <_K \psi_j.
\]
Moreover, we have
\[
\chi_i \cap \psi_i = \begin{cases} 
\psi_i & (i \neq j) \\
\sigma \cap \psi_j & (i = j)
\end{cases} = \begin{cases} 
\psi_i & (i \neq j) \\
\psi_j \setminus \{x\} & (i = j)
\end{cases}.
\]
Hence, we obtain
\[
\chi \cap \psi = \psi \setminus \{x\}.
\]
Finally, we have \( x \in \psi \setminus \phi \) since \( x \in \psi_j \setminus \phi_j \). Therefore, by the above argument, we get \( \chi \in F_{Z_M(K, L)} \) and \( x \in \psi \setminus \phi \) such that \( \chi < \psi \) and \( \chi \cap \psi = \psi \setminus \{x\} \). By Lemma 3, we conclude that \( Z^+_M(K, L) \) is shellable.

The following claim indicates that we cannot drop condition (2) in Theorem 11 for some \( M \).

**Claim 12.** Let \( (K, L) \) be a pair of simplicial complexes such that \( F_L \setminus F_K \neq \emptyset \). Then for \( M = ([1, 2], \{3, 4\}) \), \( Z^+_M(K, L) \) is not shellable.

**Proof.** We assume that \( Z^+_M(K, L) \) is shellable and deduce a contradiction. Take \( \tau^0 \in F_L \setminus F_K \). Fix a shelling on \( Z^+_M(K, L) \) and define \( \Phi, \Psi \in F_{Z_M(K, L)} \) by
\[
\Phi = \min \left\{ \phi \in F_{Z_M(K, L)} \mid \phi_1 = \phi_2 = \tau^0, \phi_3 \supseteq \tau^0, \phi_4 \supseteq \tau^0 \right\},
\]
\[
\Psi = \min \left\{ \psi \in F_{Z_M(K, L)} \mid \psi_1 \supseteq \tau^0, \psi_2 \supseteq \tau^0, \psi_3 = \psi_4 = \tau^0 \right\}.
\]
Note that \( \Phi, \Psi \) are well-defined because there exists \( \sigma \in F_K \setminus F_L \) such that \( \sigma \supseteq \tau^0 \) since \( \tau^0 \in F_L \setminus F_K \).

We may suppose that \( \Phi < \Psi \). Then there exists \( \chi < \Psi, j \in [m] \) and \( x \in \Psi_j \setminus \Phi_j \) such that \( \chi \cap \Psi = \Psi \setminus \{x\} \). Since \( \Psi_j \setminus \Phi_j \neq \emptyset \), we obtain \( j \in \{1, 2\} \). Here, we set \( j = 1 \). Then for \( i = 2, 3, 4 \), we have \( \chi_i \cap \Psi_i = \Psi_i \), namely \( \chi_i \supseteq \Psi_i \). So, we get \( \chi_2 = \Psi_2 \in F_K \setminus F_L \) since \( \Psi_2 \) is a facet of \( K \). Thus, \( 2 \in \chi \). This implies that \( \chi_i \in F_L \) for \( i = 3, 4 \). Therefore, we obtain \( \chi_i = \tau^0 \) for \( i = 3, 4 \) since \( \chi_i \supseteq \Psi_i = \tau^0 \).

On the other hand, we have \( x \in \Psi_1 \setminus \Phi_1 = \Psi_1 \setminus \tau^0 \) since \( \Phi_1 = \tau^0 \). Then it follows from \( \tau^0 \subset \Psi_1 \) that \( \tau^0 \subset \Psi_1 \setminus \{x\} = \chi_1 \cap \Psi_1 \subset \chi_1 \). If \( \chi_1 = \tau^0 \), then we have \( \chi = \{2\} \) and \( \chi_1 \in F_L \setminus F_K \). This contradicts to Proposition 8 since \( \chi \) is a facet of \( Z^+_M(K, L) \). So, we get \( \chi_1 \not\supseteq \tau^0 \). However, this is also a contradiction to the minimality of \( \Psi \). Therefore, we conclude that \( Z^+_M(K, L) \) is not shellable.

**Theorem 13.** Let \( M \) be a simplicial complex on \( [m] \) and \( (K, L) = \{(K_i, L_i)\}_{i \in [m]} \) be a family of pairs of simplicial complexes. Suppose that \( M \) is shellable and that for any \( i \in [m] \),
there exists $\alpha_i \in F_{K_i}$ such that $F_{L_i} \subset \{ \alpha_i \setminus \{ x \} \mid x \in \alpha_i \}$, and

- $K_i$ is shellable with a shelling $<_i$ such that $\alpha_i$ is the minimum element with respect to $<_i$.

Then $Z^*_M(K, L)$ is shellable.

Proof. We fix a shelling on $M$ and denote by $<_M$ on $F_{L_i}$. Take an arbitrary linear order $<_i'$ on $F_{L_i}$. Here we remark that we have $F_{K_i} \cap F_{L_i} = \emptyset$ for any $i \in [m]$ since we have $\tau \subseteq \alpha_i$ for any $\tau \in F_{L_i}$.

Let $\phi, \psi$ be facets of $Z^*_M(K, L)$. Consider a linear order $<_i$ on $F_{Z^*_M(K, L)}$ such that $\phi < \psi$ if and only if

- $\phi <_M \psi$, or
- $\phi = \psi$ and $\phi_1 = \psi_1, \ldots, \phi_{j-1} = \psi_{j-1}, \phi_j <_j \psi_j$ for some $j \in \phi$, or
- $\phi = \psi$ and $\phi_1 = \psi_1, \ldots, \phi_{j-1} = \psi_{j-1}, \phi_j <'_j \psi_j$ for some $j \notin \phi$.

We show that this order is a shelling on $Z^*_M(K, L)$.

First, suppose that $\phi <_M \psi$. Then, by Lemma 3, there exists $T \in F_M$ and $j \in \psi \setminus \phi$ such that $T <_M \psi$ and $T \cap \psi = \psi \setminus \{ j \}$. It follows from $j \in \psi \setminus \phi$ that $\psi_j \in F_{K_j}$ and $\phi_j \in F_{L_j}$.

If $\psi_j$ is not minimum with respect to $<_j$, namely $\alpha_j <_j \psi_j$ in $F_{K_j}$, then there exists $\sigma \in F_{K_j}$ and $x \in \psi_j \setminus \alpha_j$ such that $\sigma <_j \psi_j$ and $\sigma \cap \psi_j = \psi_j \setminus \{ x \}$. By the assumption of the theorem, $\phi_j \in F_{L_j}$ is a face of $\alpha_j$. So, we get $x \in \psi_j \setminus \phi_j \subset \psi \setminus \phi$. Now we define $\chi \subset \bigsqcup_{i \in [m]} V(K_i)$ by

$$
\chi_i = \begin{cases} 
\psi_i & (i \neq j) \\
\sigma & (i = j).
\end{cases}
$$

Since $\psi_j, \sigma \in F_{K_j}$, we have $\chi = \overline{\psi} \in F_M$. Thus, by Proposition 8, $\chi$ is a facet of $Z^*_M(K, L)$. Furthermore, we have $\chi < \psi$ since

$$
\chi_1 = \psi_1, \ldots, \chi_{j-1} = \psi_{j-1}, \chi_j = \sigma <_j \psi_j.
$$

Moreover, we have

$$
\chi_i \cap \psi_i = \begin{cases} 
\psi_i & (i \neq j) \\
\sigma \cap \psi_j & (i = j)
\end{cases} = \begin{cases} 
\psi_i & (i \neq j) \\
\psi_j \setminus \{ x \} & (i = j),
\end{cases}
$$

namely

$$
\chi \cap \psi = \psi \setminus \{ x \}.
$$
By the above argument, we get \( \chi \in \mathbf{F}_{Z^*_{M}(K,L)} \) and \( x \in \psi \setminus \phi \) such that \( \chi < \psi \) and \( \chi \cap \psi = \psi \setminus \{x\} \).

If \( \psi_j = \alpha_j \), then it follows from the assumption of the theorem that there exists \( x \in \alpha_j \) such that \( \phi_j = \alpha_j \setminus \{x\} \). In this case, we define \( \chi \subset \bigcup_{i \in [m]} V(K_i) \) by

\[
\chi_i = \begin{cases} 
\psi_i \in \mathbf{F}_{K_i} (i \in T \cap \overline{\psi}) \\
\phi_j \in \mathbf{F}_{L_j} (i = j), \\
\alpha_i \in \mathbf{F}_{K_i} (i \in T \setminus \overline{\psi}), \\
\psi_i \in \mathbf{F}_{L_i} (i \in [m] \setminus (T \cup \overline{\psi})). 
\end{cases}
\]

Then we have

\[
\chi = (T \cap \overline{\psi}) \cup (T \setminus \overline{\psi}) = T.
\]

Hence, \( \chi \) is a facet of \( Z^*_{M}(K,L) \). It follows from \( T <_M \overline{\psi} \) that \( \chi <_M \overline{\psi} \). Furthermore, since \( \psi_i \) is a face of \( \alpha_i \) if \( \psi_i \in \mathbf{F}_{L_i} \), we have

\[
\chi_i \cap \psi_i = \begin{cases} 
\psi_i (i \in (T \cap \overline{\psi}) \cup ([m] \setminus (T \cup \overline{\psi}))), \\
\phi_j \cap \psi_j (i = j), \\
\alpha_i \cap \psi_i (i \in T \setminus \overline{\psi}), \\
\psi_i (i \neq j), \\
\psi_j \setminus \{x\} (i = j).
\end{cases}
\]

The last equality follows from

\[
\phi_j \cap \psi_j = (\alpha_j \setminus \{x\}) \cap \alpha_j = \alpha_j \setminus \{x\} = \psi_j \setminus \{x\}.
\]

By the above argument, we get \( \chi \in \mathbf{F}_{Z^*_{M}(K,L)} \) and \( x \in \psi \setminus \phi \) such that \( \chi < \psi \) and \( \chi \cap \psi = \psi \setminus \{x\} \).

Next, suppose that \( \overline{\phi} = \overline{\psi} \) and \( \phi_1 = \psi_1, \ldots, \phi_{j-1} = \psi_{j-1}, \phi_j <_j \psi_j \) for some \( j \in \overline{\phi} \).

There exists \( \sigma \in \mathbf{F}_{K_j} \) and \( x \in \psi_j \setminus \phi_j \) such that \( \sigma <_j \psi_j \) and \( \sigma \cap \psi_j = \psi_j \setminus \{x\} \). We define \( \chi \subset \bigcup_{i \in [m]} V(K_i) \) by

\[
\chi_i = \begin{cases} 
\psi_i (i \neq j), \\
\sigma (i = j).
\end{cases}
\]

We have \( \psi_j \in \mathbf{F}_{K_j} \) since \( j \in \overline{\phi} = \overline{\psi} \). So, it follows from \( \sigma \in \mathbf{F}_{K_j} \) that \( \overline{\chi} = \overline{\psi} \). Hence, \( \chi \) is a facet of \( Z^*_{M}(K,L) \).

By the same argument as the first case, we see that \( \chi \in \mathbf{F}_{Z^*_{M}(K,L)} \) and \( x \in \psi \setminus \phi \) satisfy \( \chi < \psi \) and \( \chi \cap \psi = \psi \setminus \{x\} \).

Finally, suppose that \( \overline{\phi} = \overline{\psi} \) and \( \phi_1 = \psi_1, \ldots, \phi_{j-1} = \psi_{j-1}, \phi_j <_j \psi_j \) for some \( j \notin \overline{\phi} \).

By the assumption of the theorem, there exists \( x, y \in \alpha_j (x \neq y) \) such that \( \phi_j = \alpha_j \setminus \{x\}, \)
\[ \psi_j = \alpha_j \setminus \{y\} \]. So, we have \( \phi_j \cap \psi_j = \alpha_j \setminus \{x, y\} = \psi_j \setminus \{x\} \) since \( x \neq y \). Here we define 
\[ \chi \subset \bigsqcup_{i \in [m]} V(K_i) \] by

\[ \chi_i = \begin{cases} \psi_i & (i \neq j) \\ \phi_j & (i = j). \end{cases} \]

We have \( \phi_j, \psi_j \in F_{L_j} \) since \( j \in \phi = \psi \). So, it follows that \( \chi = \psi \). Hence, \( \chi \) is a facet of \( \mathcal{Z}_m^*(K, L) \). By the same argument as the first case, we see that \( \chi \in \mathcal{F}_{\mathcal{Z}_m^*(K, L)} \) and \( x \in \psi \setminus \phi \) which satisfy \( \chi < \psi \) and \( \chi \cap \psi = \psi \setminus \{x\} \).

In all four cases above, we obtain \( \chi \in \mathcal{F}_{\mathcal{Z}_m^*(K, L)} \) and \( x \in \psi \setminus \phi \) which satisfy \( \chi < \psi \) and \( \chi \cap \psi = \psi \setminus \{x\} \). By Lemma 3, we conclude that \( \mathcal{Z}_m^*(K, L) \) is shellable. \( \square \)

**Remark 14.** Consider \( M = \langle \{1\}, \{2\} \rangle \) and \( (K, L) = \{(K_i, L_i)\}_{i=1,2} \) such that

\[ K_i = \langle \{a_i, b_i\}, \{b_i, c_i\} \rangle, \quad L_i = \langle \{a_i\}, \{c_i\} \rangle. \]

Then we see that

\[ \mathcal{Z}_m^*(K, L) = \langle \{a_1, b_1, a_2\}, \{a_1, b_1, c_2\}, \{b_1, c_1, a_2\}, \{b_1, c_1, c_2\}, \{a_1, a_2, b_2\}, \{a_1, b_2, c_2\}, \{c_1, a_2, b_2\}, \{c_1, b_2, c_2\} \rangle \]

(see Figure 1) is shellable. However, there is no \( \alpha_i \in \{\{a_i, b_i\}, \{b_i, c_i\}\} \) such that \( \{a_i\}, \{c_i\} \subset \alpha_i \). So, the condition that \( \mathcal{F}_{L_i} \subset \{\alpha_i \setminus \{x\} \in K \mid x \in \alpha_i \} \) for some \( \alpha_i \in \mathcal{F}_{K_i} \) is not necessary for \( \mathcal{Z}_m^*(K, L) \) being shellable.

![Figure 1: \( \mathcal{Z}_m^*(K, L) \) in Remark 14.](image)

**Example 15.** Let \( K \) be a simplicial complex on \([m]\) and \( J = (j_1, \ldots, j_m) \) be a tuple of positive integers. Define families of pairs of simplicial complexes as follows:

\[ (\Delta^{j_1-1}, \partial \Delta^{j_1-1}) = \{(\Delta^{j_1-1}, \partial \Delta^{j_1-1})\}_{i \in [m]}, \]
\[ (pt^J, \emptyset) = \left\{ \left( \bigsqcup_{j_i} pt, \emptyset \right) \right\}_{i \in [m]} \]

By Theorem 13, \( \mathcal{Z}_m^*(\Delta^{j_1-1}, \partial \Delta^{j_1-1}) \) and \( \mathcal{Z}_m^*(pt^J, \emptyset) \) are shellable if \( K \) is shellable.

Next, we state a necessary condition for \( \mathcal{Z}_m^*(K, L) \) being shellable under a certain assumption.
Theorem 16. Let $M$ be a simplicial complex which is not a simplex and $K \supseteq L$ be a pair of simplicial complexes such that $F_L \subset F_K$. If $Z'_M(K, L)$ is shellable, then both $K$ and $L$ are shellable.

Proof. We set $V(M) = [m]$. Let $(K_i, L_i) = \{(K_i, L_i)\}_{i \in [m]}$ be a family of pairs of simplicial complexes such that $K_i = K$, $L_i = L$ for any $i \in [m]$. Fix a shelling $\varphi$ on $Z'_M(K, L)$. Take $S \in F_M$ and $k \in [m]$ such that $k \notin S$. This is possible since $M$ is not a simplex. We prove that both $K_k$ and $L_k$ are shellable.

First, we prove that $K_k$ is shellable. For a facet $\sigma \in F_{K_k}$, define $\Phi^\sigma \in F_{Z'_M(K, L)}$ by

$$\Phi^\sigma = \min \{ \phi \in F_{Z'_M(K, L)} \mid \phi_k = \sigma \}.$$ 

Note that $\Phi^\sigma$ is well-defined since by Remark 9, there exists a facet $\phi \in F_{Z'_M(K, L)}$ such that

$$\phi_k = \begin{cases} \sigma & (i = k) \\ \tau & (i \neq k), \end{cases}$$

where $\tau$ is an arbitrary facet of $L$. For $\sigma, \sigma' \in F_{K_k}$, we define a relation $\sigma <_K \sigma'$ by $\Phi^\sigma < \Phi^{\sigma'}$. It is obvious that $<_K$ defines a linear order on $F_{K_k}$. We prove that $<_K$ is a shelling on $K_k$.

For $\sigma', \sigma \in F_{K_k}$ such that $\Phi^{\sigma'} < \Phi^{\sigma}$, there exists $\chi \in F_{Z'_M(K, L)}$ and $x \in \Phi^{\sigma} \setminus \Phi^{\sigma'}$ such that $\chi < \Phi^{\sigma}$ and $\chi \cap \Phi^{\sigma} = \Phi^{\sigma'} \setminus \{x\}$. If $x \notin \Phi^{\sigma'}$, then we have $\chi_k \cap \Phi^{\sigma} = \Phi^{\sigma}$, which means that $\Phi^{\sigma} \setminus \chi_k$. By Proposition 8, $\Phi^{\sigma}$ and $\chi_k$ are facets of $K_k$ since $F_{K_k} \cup F_{L_k} = F_{K_k}$. So, we obtain $\chi_k = \Phi^{\sigma} = \sigma$. This contradicts to the minimality of $\Phi^{\sigma}$. Hence, we get $x \in \Phi^{\sigma}$. Moreover, it follows from $x \notin \Phi^{\sigma'} \setminus \Phi^{\sigma}$ that $x \in \Phi^{\sigma} \setminus \Phi^{\sigma'} = \sigma \setminus \sigma'$. By $\chi' \cap \Phi^\sigma = \Phi^\sigma \setminus \{x\}$ and $x \in \Phi^\sigma$, we obtain $\chi_k \cap \sigma = \chi_k \cap \Phi^\sigma = \Phi^\sigma \setminus \{x\} = \sigma \setminus \{x\}$. Furthermore, we get $\Phi^{\chi_k} < \chi < \Phi^{\sigma}$. Therefore, $\chi_k \in F_{K_k}$ and $x \in \sigma \setminus \sigma'$ satisfy $\chi_k <_K \sigma$ and $\chi_k \cap \sigma = \sigma \setminus \{x\}$. By Lemma 3, $<_K$ is a shelling on $K_k$.

Next, we prove that $L_k$ is shellable. There exists $\sigma^0 \in F_{K_k} \setminus F_{L_k}$ since $K \neq L$. For a facet $\tau \in F_{L_k}$, define $\Psi^\tau \in F_{Z'_M(K, L)}$ by

$$\Psi^\tau = \min \{ \psi \in F_{Z'_M(K, L)} \mid \psi_i = \sigma^0 \text{ for any } i \in S, \text{ and } \psi_k = \tau \}.$$ 

Note that $\Psi^\tau$ is well-defined. This is because there exists a facet $\psi \in F_{Z'_M(K, L)}$ defined by

$$\psi_i = \begin{cases} \sigma^0 & (i \in S) \\ \tau & (i \notin S). \end{cases}$$

For $\tau, \tau' \in F_{L_k}$, we define a relation $\tau <_L \tau'$ by $\Psi^\tau < \Psi^{\tau'}$. It is obvious that $<_L$ defines a linear order on $F_{L_k}$. We prove that $<_L$ is a shelling on $L_k$.

For $\tau', \tau \in F_{L_k}$ such that $\Psi^{\tau'} < \Psi^\tau$, there exists $\chi \in F_{Z'_M(K, L)}$ and $x \in \Psi^\tau \setminus \Psi^{\tau'}$ such that $\chi < \Psi^\tau$ and $\chi \cap \Psi^\tau = \Psi^{\tau'} \setminus \{x\}$. Let $j \in [m]$ be a vertex of $M$ such that $x \in \Psi^\tau_j \setminus \Psi^{\tau'}_j$. Then we have $j \notin S$ since $\Psi^\tau_j \neq \Psi^{\tau'}_j$. So, for any $i \in S$, we get $\chi_i \cap \Psi^\tau_i = \Psi^{\tau'}_i$, namely...
Then we see that $K$ (see Figure 2) is shellable. However, $\chi_i \supset \Psi^i = \sigma^0$. It follows from $\sigma^0 \in F_{K_k}$ that $\chi_i = \sigma^0$ for any $i \in S$. Hence, by the minimality of $\Psi^i$, we get $\chi_k \not= \tau$. On the other hand, we get $\chi = S$ since $S$ is a facet of $M$ and $S \subset \chi$. So, we have $k \not\in S = \chi$, which means that $\chi_k \in F_{L_k}$. Therefore, if $j \neq k$, we obtain $\chi_k \supset \tau$ and $\chi_k, \tau \in F_{L_k}$, a contradiction. Thus, we conclude that $j = k$. Hence, we get $x \in \Psi^i \setminus \Psi^j$ and $\chi_k \cap \Psi^j = \Psi^j \setminus \{x\}$, namely $x \in \tau \setminus \tau'$ and $\chi_k \cap \tau = \tau \setminus \{x\}$. Moreover, we have $\Psi^\chi < \chi$ since $\chi_i = \sigma^0$ for any $i \in S$ and $\chi_k \in F_{L_k}$. Hence, by $\chi < \Psi^j$, we obtain $\Psi^\chi < \Psi^j$, namely $\chi_k < L \tau$. By Lemma 3, $<_L$ is a shelling on $L_k$. \hfill $\square$

**Remark 17.** In general, the shellability of $Z_M^*(K, L)$ does not imply the shellability of each $K_i$. For example, consider $M = \langle \{1\}, \{2\} \rangle$ and $(K, L) = \{(K_i, L_i)\}_{i=1,2}$ such that

$$K_1 = \langle \{a, b\}, \{c, d\} \rangle, \quad L_1 = \langle \{b\}, \{c, d\} \rangle, \quad K_2 = \langle \{e, f\} \rangle, \quad L_2 = \langle \{f\} \rangle.$$ 

Then we see that

$$Z_M^*(K, L) = \langle \{c, d, e, f\}, \{b, e, f\}, \{a, b, f\} \rangle$$

(see Figure 2) is shellable. However, $K_1$ is not shellable.

![Figure 2: $Z_M^*(K, L)$ in Remark 17.](image)

**Remark 18.** The shellability of $Z_M^*(K, L)$ does not imply $F_L \subset F_K$. For example, consider $M = \langle \{1\}, \{2\} \rangle$ and $(K, L) = \{(K_i, L_i)\}_{i=1,2}$ such that

$$K_i = \langle \{a_i, b_i\} \rangle, \quad L_i = \langle \{b_i\} \rangle.$$ 

Then we see that

$$Z_M^*(K, L) = \langle \{a_1, b_1, b_2\}, \{b_1, a_2, b_2\} \rangle$$

is shellable. However, $\{b_i\} \in F_{L_i} \setminus F_{K_i}$ for $i = 1, 2$.

Even though $(K, L)$ satisfies that $K$ is shellable, $L$ is shellable and $F_L \subset F_K$, $Z_M^*(K, L)$ is not necessarily shellable. In order to show this, we prove another necessary condition.

**Theorem 19.** Let $M$ be a simplicial complex which is not a simplex and $K \supseteq L$ be a pair of simplicial complexes. Suppose that $Z_M^*(K, L)$ is shellable. Then there exists a pair $(\sigma, \tau)$ of $\sigma \in F_K \setminus F_L$ and $\tau \in F_L$ such that

$$|\sigma \cap \tau| = \max_{\rho \in F_K \setminus F_L} |\rho| - 1.$$
Proof. Fix a shelling $\phi^1 < \cdots < \phi^t$ on $Z^*_M(K, L)$ such that $|\phi^p| \geq |\phi^q|$ for any $1 \leq p < q \leq t$, which exists by Lemma 4.

We first show that there exists an index $s$ such that $\overline{\phi^q} \not\subseteq \overline{\phi^s}$ for any $p \in [s-1]$. Since $M$ is not a simplex, there exists $j \in [m]$ such that $j \notin \overline{\phi^j}$. Take a facet $S \ni j$ of $M$. Here, it follows from $K \neq L$ that there exists a facet $\rho \in F_K \setminus F_L$ of $K$. Now, consider a subset $\psi \subset \bigcup_{i \in [m]} V(\Phi)$ which satisfies

- if $i \in S$, then $\psi_i = \rho$,
- if $i \notin S$, then $\psi_i \in F_L$.

We have $\psi_i \in F_K \cup F_L$ for any $i \in [m]$, $\overline{\psi} = S \in M$ and $\overline{\psi} \cup \{i\} \notin M$ for any $i \notin S$ since $S$ is a facet of $M$. By Proposition 8, $\psi$ is a facet of $Z^*_M(K, L)$. Therefore, we have $\psi = \phi^q$ for some $q \in [t]$. Since $j \in S \setminus \overline{\phi^j} = \overline{\phi^q} \setminus \overline{\phi^j}$, we get $\overline{\phi^q} \subset \overline{\phi^j}$. Now we set $s = \min\{r \in [t] \mid \overline{\phi^q} \not\subseteq \overline{\phi^r}\}$. It follows from the minimality of $s$ that $\overline{\phi^p} \subset \overline{\phi^s}$ for any $p \in [s-1]$. Thus, we get $\overline{\phi^p} \not\subseteq \overline{\phi^s}$ for any $p \in [s-1]$ as desired.

Now, take an index $s$ as above. For an index $p'$ such that $\overline{\phi^p} = \overline{\phi^s}$, $p'$ must be larger than $s$, which implies $|\phi^p| \leq |\phi^s|$. So, $\phi^s$ is the largest facet among facets $\phi$ such that $\overline{\phi} = \overline{\phi^s}$. By the definition of a shelling, there must exist $p$ with $1 \leq p < s$ such that $|\phi^p \cap \phi^s| = |\phi^s|-1$. This equality implies that we have

$$|\phi^p_i \cap \phi^s_i| \geq |\phi^s_i| - 1$$

for any $i \in [m]$. On the other hand, it follows from $\overline{\phi^p} \not\subseteq \overline{\phi^s}$ that there exists $j \in [m]$ such that $\overline{\phi^j} \notin \overline{\phi^p}$, namely $j \in [m]$ such that $\phi^p_j \notin F_L$ and $\phi^s_j \in F_K \setminus F_L$. Since $\phi^s_j$ is a facet of $K$, we have $\phi^p_j \cap \phi^s_j \subseteq \phi^s_j$. Thus, we get

$$|\phi^p_j \cap \phi^s_j| = |\phi^s_j| - 1.$$ 

Assume that there exists $\mu \in F_K \setminus F_L$ such that $|\mu| > |\phi^s_i|$. Then, a subset $\chi \subset \bigcup_{i \in [m]} V(\Phi)$ defined by

$$\chi_i = \begin{cases} 
\phi^i_i & (i \neq j) \\
\mu & (i = j)
\end{cases}$$

is a facet of $Z^*_M(K, L)$. Since $|\chi| > |\phi^s|$ and $\overline{\chi} = \overline{\phi^s}$, this contradicts to the maximality of $|\phi^s|$. Hence, we obtain $|\phi^s_j| = \max_{\rho \in F_K \setminus F_L} |\rho|$. Then, $\sigma = \phi^s_j \in F_K \setminus F_L$ and $\tau = \phi^s_j \in F_L$ satisfy

$$|\sigma \cap \tau| = |\sigma| - 1 = \max_{\rho \in F_K \setminus F_L} |\rho| - 1,$$

which is the desired conclusion. \qed
Example 20. Consider $M = \{\{1\}, \{2\}\}$ and $(K, L) = \{(K_i, L_i)\}_{i=1,2}$ such that

$$K_i = \langle\{a_i, b_i\}, \{b_i, c_i\}, \{d_i\}\rangle, \quad L_i = \langle\{c_i\}, \{d_i\}\rangle.$$  

For $\{a_i, b_i\} \in F_{K_i} \setminus F_{L_i}$, there exists no $\tau \in F_{L_i}$ such that $|\{a_i, b_i\} \cap \tau| = 1$. For $\{d_i\} \in F_{L_i}$, there exists no $\sigma \in F_{K_i} \setminus F_{L_i}$ such that $|\sigma \cap \{d_i\}| = 1$. On the other hand, we see that

$$Z_M^*(K, L) = \langle\{a_1, b_1, c_2\}, \{b_1, c_1, c_2\}, \{b_1, c_1, d_2\}, \{a_1, b_1, d_2\},$$

$$\{c_1, b_2, c_2\}, \{d_1, b_2, c_2\}, \{d_1, a_2, b_2\}, \{c_1, a_2, b_2\}, \{d_1, d_2\}\rangle$$

(see Figure 3) is shellable. So, Theorem 19 only guarantees the existence of at least one pair $(\sigma, \tau)$, in this example, $\sigma = \{b_i, c_i\}$ and $\tau = \{c_i\}$.

\[
\begin{array}{ccc}
  & a_1 & \\
  a_2 & \\
  b_1 & \\
  c_1 & c_2 & \\
  d_1 & d_2 & \\
\end{array}
\]

Figure 3: $Z_M^*(K, L)$ in Example 20.

Remark 21. The converse of Theorem 16 and the converse of Theorem 19 do not hold. Consider $M = \{\{1\}, \{2\}\}$ and $(K, L) = \{(K_i, L_i)\}_{i=1,2}$ such that

$$K_i = \langle\{a_i, b_i, c_i\}, \{a_i, c_i, d_i\}, \{b_i, d_i\}\rangle, \quad L_i = \langle\{a_i, b_i, c_i\}, \{b_i, d_i\}\rangle.$$  

Two facets $\{a_i, c_i, d_i\} \in F_{K_i} \setminus F_{L_i}$ and $\{a_i, b_i, c_i\} \in F_{L_i}$ satisfy the condition in Theorem 19. Moreover, both $K = K_1 = K_2$ and $L = L_1 = L_2$ are shellable and $F_L \subset F_K$. However,

$$Z_M^*(K, L) = \langle\{a_1, b_1, c_1, a_2, b_2, c_2\}, \{a_1, c_1, d_1, a_2, b_2, c_2\},$$

$$\{a_1, b_1, a_2, c_2, d_2\}, \{a_1, b_1, c_1, b_2, d_2\}, \{a_1, c_1, d_1, b_2, d_2\},$$

$$\{b_1, d_1, a_2, c_2, d_2\}, \{b_1, d_1, a_2, c_2, d_2\}\rangle$$

(see Figure 4) is not shellable. To see this, assume that $Z_M^*(K, L)$ is shellable and

$$\{a_1, c_1, d_1, b_2, d_2\} < \{b_1, d_1, a_2, c_2, d_2\}$$

in a shelling. There must exists a facet $\phi \neq \{b_1, d_1, a_2, c_2, d_2\}$ of $Z_M^*(K, L)$ and $x \in \{b_1, d_1, a_2, c_2, d_2\} \setminus \{a_1, c_1, d_1, b_2, d_2\} = \{b_1, a_2, c_2\}$ such that $\{b_1, d_1, a_2, c_2, d_2\} \setminus \{x\} \subset \phi$. However, there is no facet $\phi \neq \{b_1, d_1, a_2, c_2, d_2\}$ such that $\phi$ includes $\{d_1, a_2, c_2, d_2\}, \{b_1, d_1, a_2, d_2\}$ or $\{b_1, d_1, a_2, d_2\}$. This is a contradiction. In the same way, we also get $\{b_1, d_1, a_2, c_2, d_2\} \not< \{a_1, c_1, d_1, b_2, d_2\}$. So, $Z_M^*(K, L)$ has no shelling.

Corollary 22. Let $M$ be a simplicial complex and $K \supset L$ be a pair of simplicial complexes. If $\dim K - \dim L \geq 2$, then $Z_M^*(K, L)$ is not shellable.

In particular, for any simplicial complex $M$ and $K$ such that $\dim K \geq 1$, $Z_M^*(K, \emptyset)$ is not shellable.
For any $\sigma, \tau \in F_K \setminus F_L$, we have
\[ |\sigma \cap \tau| \leq |\tau| \leq \dim L + 1 \leq \dim K - 1 = \max_{\rho \in F_K \setminus F_L} |\rho| - 2. \]
Therefore, there is no pair $(\sigma, \tau)$ of $\sigma \in F_K \setminus F_L$ and $\tau \in F_L$ such that
\[ |\sigma \cap \tau| = \max_{\rho \in F_K \setminus F_L} |\rho| - 1. \]

By Theorem 19, we conclude that $Z^*_M(K, L)$ is not shellable. \hfill \Box

As the last result in this section, we prove that under a certain condition, the shellability of both $K$ and $L$ is equivalent to the shellability of $Z^*_M(K, L)$.

**Theorem 23.** Let $M$ be a simplicial complex which is not a simplex and $K$ be a simplicial complex. For a vertex $v_0$ of $K$, suppose that $F_{dL_{K}(v_0)} \subset F_K$. Then $Z^*_M(K, dL_{K}(v_0))$ is shellable if and only if both $K$ and $dL_{K}(v_0)$ are shellable.

**Proof.** We set $L = dL_{K}(v_0)$. If $Z^*_M(K, L)$ is shellable and $F_L \subset F_K$, then by Theorem 16, $K$ and $L$ are shellable. In order to prove the converse, it is sufficient to show that there is a shelling $<_K$ on $K$ such that $\tau <_K \sigma$ for any $\tau \in F_L$ and any $\sigma \in F_K \setminus F_L$. Then the proof is completed by Theorem 11.

Let $<$ be a shelling on $K$ and $<'$ be a shelling on $L$. Define a relation $<_K$ on $F_K$ by
\begin{itemize}
  \item $\tau <_K \sigma$ for any $\tau \in F_L$ and $\sigma \in F_K \setminus F_L$,
  \item for any $\tau, \tau' \in F_L$, $\tau <_K \tau'$ if and only if $\tau < <' \tau'$, and
  \item for any $\sigma, \sigma' \in F_K \setminus F_L$, $\sigma <_K \sigma'$ if and only if $\sigma < \sigma'$.
\end{itemize}

It is obvious that $<_K$ is a linear order. We prove that $<_K$ is a shelling on $K$. The goal of the proof is to show that for any $\rho, \rho' \in F_K$ such that $\rho <_K \rho'$, there exists $\rho'' \in F_K$ and $u \in \rho' \setminus \rho$ which satisfy $\rho'' <_K \rho'$ and $\rho'' \cap \rho' = \rho' \setminus \{u\}$.

For any $\tau, \tau' \in F_L$ such that $\tau < <' \tau'$, there exists $\tau'' \in F_L$ and $x \in \tau' \setminus \tau$ such that $\tau'' < <' \tau'$ and $\tau'' \cap \tau' = \tau' \setminus \{x\}$. Since $\tau'' \in F_L$ and $\tau'' < <' \tau'$, we get $\tau'' <_K \tau'$. 

\[ \text{Figure 4: } Z^*_M(K, L) \text{ in Remark 21.} \]
For any $\sigma, \sigma' \in F_K \setminus F_L$ such that $\sigma < \sigma'$, there exists $\rho \in F_K$ and $x \in \sigma' \setminus \sigma$ such that $\rho < \sigma'$ and $\rho \cap \sigma' = \sigma' \setminus \{x\}$. If $\rho \in F_L$, then we obtain $\rho <_K \sigma'$ since $\sigma' \in F_K \setminus F_L$. If $\rho \in F_K \setminus F_L$, then we obtain $\rho <_K \sigma'$ again since $\rho < \sigma'$. In both cases, we have $\rho <_K \sigma'$.

For $\tau \in F_L$ and $\sigma \in F_K \setminus F_L$, we have $v_0 \in \tau \setminus \sigma$ since $\sigma / \in \text{dl}_K(v_0)$ and $\tau \in \text{dl}_K(v_0)$. A simplex $\sigma \setminus \{v_0\} \in L$ is not a facet of $L$ since $\sigma \setminus \{v_0\}$ is not a facet of $K$ and we have $F_L \subset F_K$ by the assumption. Therefore, there exists a facet $\rho \in F_L$ such that $\rho \supseteq \sigma \setminus \{v_0\}$. Since $\rho \in \text{dl}_K(v_0)$, we obtain $\rho \cap \sigma = \sigma \setminus \{v_0\}$. We also get $\rho <_K \sigma$ because $\rho \in F_L$ and $\sigma \in F_K \setminus F_L$.

5 Applications to the shellability of graphs

In this section, we prove Theorem 2. Here we introduce a class of graphs which are constructed from two graphs $G, H$ and a subset $U \subset V(H)$. In the following, a set $\{u, v\} \subset V(G)$ of two vertices of a graph $G$ is denoted by $uv$.

Definition 24. Let $G, H$ be graphs and $U \subset V(H)$ be a subset of $V(H)$. We define a graph $G[H; U]$ by

$$V(G[H; U]) = V(G) \times V(H),$$

$$E(G[H; U]) = \left\{ (u_1, v_1)(u_2, v_2) \Bigg| \begin{array}{ll}
    u_1 = u_2 \text{ and } v_1, v_2 \in E(H), \\
    \text{or} \\
    u_1u_2 \in E(G) \text{ and } v_1, v_2 \in U
\end{array} \right\}.$$

Example 25. For graphs $G$ and $H$, $G[H; V(H)]$ is the lexicographic product $G[H]$. The definition of the lexicographic product of two graphs is given in, for example, [5].

Example 26. Let $G, H$ be graphs and $v_0$ be a vertex of $H$. As defined in Section 1, $G[H; \{v_0\}]$ is the graph obtained from $G \cup \bigcup_{u \in V(G)} H_u$ by identifying $u \in V(G)$ with $v_0 \in H_u$, where $H_u$ is a copy of $H$.

The independence complex of $G[H; U]$ is described as a polyhedral join, stated in the following proposition.
Proposition 27. Let $G, H$ be graphs and $U \subset V(H)$ be a subset of $V(H)$. Then we have

$$I(G[H; U]) = Z_{I(G)}^*(I(H), I(H \setminus U)),$$

where $H \setminus U$ is a graph defined by $V(H \setminus U) = V(H) \setminus U$ and $E(H \setminus U) = \{uv \in E(H) \mid u, v \notin U\}$.

Proof. Remark that for $\phi \subset V(G) \times V(H) = V(G) \setminus I(H)$, $\phi_u (u \in V(G))$ and $\phi$ in Proposition 7 are reformulated as follows:

$$\phi_u = \{v \in V(H) \mid (u, v) \in \phi\},$$

$$\phi = \{u \in V(G) \mid \phi_u \cap U \neq \emptyset\}.$$

Let $\phi \in I(G[H; U])$ be an independent set of $G[H; U]$. Then, we have the followings.

- For any $u \in V(G)$ and $v_1, v_2 \in V(H)$, suppose that $v_1, v_2 \in \phi_u$. Then we have $(u, v_1)(u, v_2) \notin E(G[H; U])$ since $(u, v_1), (u, v_2) \in \phi$ and $\phi$ is an independent set of $G[H; U]$. By the definition of $E(G[H; U])$, we obtain $v_1v_2 \notin E(H)$. Therefore, $\phi_u$ is an independent set of $H$, namely $\phi_u \in I(H)$.

- For any $u_1, u_2 \in \overline{\phi}$, there exist $w_1, w_2 \in U$ such that $(u_1, w_1), (u_2, w_2) \in \phi$. Then $(u_1, w_1)(u_2, w_2) \notin E(G[H; U])$ since $\phi$ is an independent set of $G[H; U]$. By the definition of $E(G[H; U])$, we obtain $u_1u_2 \notin E(G)$. Therefore, $\overline{\phi}$ is an independent set of $G$, namely $\overline{\phi} \in I(G)$.

So, by Proposition 7, $\phi$ is a simplex of $Z_{I(G)}^*(I(H), I(H \setminus U))$.

Conversely, let $\psi$ be a simplex of $Z_{I(G)}^*(I(H), I(H \setminus U))$. It follows from Proposition 7 that we have $\psi_u \in I(H)$ for any $u \in V(G)$ and $\overline{\psi} \in I(G)$. Then, for any $(u_1, v_1), (u_2, v_2) \in \psi$, we have the followings.

- If $u_1 = u_2$, then $v_1v_2 \notin E(H)$ since $v_1, v_2 \in \psi_{u_1}$ and $\psi_{u_1}$ is an independent set of $H$. Furthermore, $u_1 = u_2$ implies that $u_1u_2 \notin E(G)$ since $G$ has no loops. Therefore, we get $(u_1, v_1)(u_2, v_2) \notin E(G[H; U])$.

- If $u_1 \neq u_2$ and $v_1, v_2 \in U$, then $u_1u_2 \notin E(G)$ since $u_1, u_2 \in \overline{\psi}$ and $\overline{\psi}$ is an independent set of $G$. Therefore, we get $(u_1, v_1)(u_2, v_2) \notin E(G[H; U])$.

- If $u_1 \neq u_2$ and $v_1 \notin U$ or $v_2 \notin U$, then $(u_1, v_1)(u_2, v_2) \notin E(G[H; U])$.

So, we conclude that $(u_1, v_1)(u_2, v_2) \notin E(G[H; U])$. Thus, $\psi$ is an independent set of $G[H; U]$, namely $\psi \in I(G[H; U])$. □

Example 28. Let $G$ be a graph with at least one edge and $H$ be a graph which is not a complete graph. Vander Meulen and Van Tuyyl [9, Theorem 2.3] proved that $I(G[H])$ is not shellable. We can deduce this result from Example 25, Proposition 27 and Corollary 22 since $I(G)$ is not a simplex and $I(H)$ is not 0-dimensional.
Now we are ready to prove Theorem 2.

Proof of Theorem 2. First, we show that (1) implies (2). By Proposition 27, (1) is equivalent to the condition that $Z_{I(G)}^*(I(H), I(H \setminus \{v_0\}))$ is pure and shellable for any graph $G$. It follows from Theorem 10 that $I(H)$ is pure, namely $H$ is well-covered. Now consider the cycle on 4 vertices $C_4$, namely the graph defined by

$$V(C_4) = \{1, 2, 3, 4\}, \quad E(C_4) = \{12, 23, 34, 41\}.$$  

Since $I(C_4) = \langle\{1, 3\}, \{2, 4\}\rangle$ and $I(C_4[H, H \setminus \{v_0\}])$ is shellable, Claim 12 indicates that $F_{I(H \setminus \{v_0\})} \subseteq F_{I(H)}$. Hence, for any maximal independent set $\tau$ of $H \setminus \{v_0\}$, we get that $\tau \cup \{v_0\}$ is not an independent set of $H$. Namely there exists $v \in \tau$ such that $v_0v \in E(H)$. Therefore, by Theorem 16, both $I(H)$ and $I(H \setminus \{v_0\})$ are shellable.

Next, we deduce (1) from (2). By the conditions in (2), $I(H)$ is pure, both $I(H)$ and $I(H \setminus \{v_0\})$ are shellable, and $F_{I(H \setminus \{v_0\})} \subseteq F_{I(H)}$. Therefore, it follows from Theorem 10 that $I(G[H; H \setminus \{v_0\}])$ is shellable for any graph $G$ which has at least one edge. For graph $G$ which has no edges, it follows from Lemma 5 that

$$Z_{I(G)}^*(I(H), I(H \setminus \{v_0\})) = I(H) \ast \cdots \ast I(H)$$  

is shellable since $I(H)$ is shellable. Therefore, we conclude that $Z_{I(G)}^*(I(H), I(H \setminus \{v_0\}))$ is pure and shellable, namely $G[H; \{v_0\}]$ is well-covered and shellable, for any graph $G$. \qed

Example 29. If $H$ is a complete graph, then $H$ satisfies the condition (2) in Theorem 2.

An example of $H$ which is not a complete graph is $C_5$, a cycle of length 5. Let $V(C_5) = \{a, b, c, d, e\}$ and $E(C_5) = \{ab, bc, cd, de, ea\}$. Then

$$I(C_5) = \langle\{a, c\}, \{b, d\}, \{c, e\}, \{d, a\}, \{e, b\}\rangle$$

is pure and shellable. Furthermore,

$$dl_{C_5}(a) = \langle\{b, d\}, \{e, b\}, \{c, e\}\rangle$$

is shellable and each facet of $dl_{C_5}(a)$ contains $b$ or $e$, which are adjacent to $a$.

References


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