On Gröbner Basis and Cohen-Macaulay Property of Closed Path Polyominoes

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Abstract

In this paper we introduce some monomial orders for the class of closed path polyominoes and we prove that the set of the generators of the polyomino ideal attached to a closed path forms the reduced Gröbner basis with respect to these monomial orders. It is known that the polyomino ideal attached to a closed path containing an L-configuration or a ladder of at least three steps, equivalently having no zig-zag walks, is prime. As a consequence, we obtain that the coordinate ring of a closed path having no zig-zag walks is a normal Cohen-Macaulay domain.

Mathematics Subject Classifications: 05B50, 05E40, 13C05, 13G05, 13C14

1 Introduction

Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates. An interesting topic in Commutative Algebra is the ideal of the t-minors of X for any integer $1 \leqslant t \leqslant \min\{m, n\}$. Many matematicians investigated the main algebraic properties of such ideals, called determinantal ideals. See for example [6], [7], [8], [13] about the ideals generated by all t-minors of a one or two sided ladder, [14], [20], [23] about the ideals of adjacent 2-minors, [15] about the ideals generated by an arbitrary set of 2-minors in a $2 \times n$ matrix, or also [11] about the ideals generated by 2-minors associated to a graph. For further references see [3].

In 1953 S.W. Golomb coined the term *polyomino* to indicate a finite collection of unitary squares joined edge by edge ([12]). These polygons have been studied in Combinatorial Mathematics, in particular in some tiling problems of the plane. In 2012 they have been linked to Commutative Algebra by A.A. Qureshi ([24]): if $\mathcal{P}$ is a polyomino, K is a field and S is the polynomial ring over K in the variables x_a with $a \in V(\mathcal{P})$, set of the vertices of $\mathcal{P}$, then we can associate to $\mathcal{P}$ the ideal generated by all inner 2-minors of $\mathcal{P}$. This ideal is called the *polyomino ideal* of $\mathcal{P}$ and it is denoted by $I_{\mathcal{P}}$. Many mathematicians have taken an interest in classifying all polyominoes, for which the quotient ring $K[\mathcal{P}] = S/I_{\mathcal{P}}$ is a normal Cohen-Macaulay domain.

The primality of $I_{\mathcal{P}}$ is studied in several papers, see [4], [5], [17], [18], [19], [21], [22], [25], [27], [28]. Moreover in [17] and [25] the authors prove that if $\mathcal{P}$ is a simple polyomino then $K[\mathcal{P}]$ is a normal Cohen-Macaulay domain. In other papers some new classes of non-simple polyominoes are examined. In [19] and [27], the authors show that the polyominoes obtained by removing a convex polyomino from a rectangle are prime, generalizing the same result for the rectangular polyominoes minus an internal rectangle proved in [27]. In [21] the authors introduce a particular sequence of inner intervals of $\mathcal{P}$, called a zig-zag walk, and they prove that $\mathcal{P}$ does not contain zig-zag walks if $I_{\mathcal{P}}$ is prime. It seems that the non-existence of zig-zag walks in a polyomino could characterize its primality [Conjecture 4.6, [21]]. In [4] the authors support this conjecture introducing a new class of polyominoes, called closed paths, and showing that having no zig-zag walks is a necessary and sufficient condition for their primality. An analogous result is proved in [5] for the weakly closed path polyominoes. In [22] the authors study the reduced Gröbner basis of polyomino ideals and introduce some conditions in order to the generators of $I_{\mathcal{P}}$ form the reduced Gröbner basis with respect to some suitable degree reverse lexicographic monomial orders. Eventually, for further references about several algebraic properties of polyomino ideals we report [1], [9], [10] and [26].

In this paper we study the Gröbner bases of the polyomino ideal attached to a closed path and we show that there exist some monomial orders such that the set of generators of the ideal forms the reduced Gröbner basis with respect to these orders. In Section 2 we introduce the notations about polyominoes and closed paths. In Section 3 we provide a class of monomial orderings that generalizes the class introduced in [24] by Qureshi and we give some conditions on such orderings so that the S-polynomial of two generators of $I_{\mathcal{P}}$ attached to a collection of cells reduces to 0 modulo the set of generators of $I_{\mathcal{P}}$. In Section 4 we introduce some new configurations in a closed path, in particular the W-pentominoes, the LD-horizontal and vertical skew tetrominoes and hexominoes and the RW-heptominoes. For each case in which one of the previous configurations is not in the closed path we provide a set of suitable vertices which allow us to define some particular suitable monomial orders. Moreover in Definition 4.1 we provide a pseudo-algorithm to deal the general case. Finally, we prove that the set of generators of the polyomino ideal attached to a closed path is the reduced Gröbner basis with respect to a suitable choice of the monomial order, so the coordinate ring of a closed path having no zig-zag walks is a normal Cohen-Macaulay domain. We conclude the paper giving an example of a non-simple polyomino whose universal Gröbner basis is not square-free and providing some related open questions.

2 Polyominoes, closed paths and polyomino ideals

Let $(i,j),(k,l)\in\mathbb{Z}^2$. We say that $(i,j)\leqslant(k,l)$ if $i\leqslant k$ and $j\leqslant l$. Consider $a=(i,j)$ and $b=(k,l)$ in $\mathbb{Z}^2$ with $a\leqslant b$. The set $[a,b]=\{(m,n)\in\mathbb{Z}^2 : i\leqslant m\leqslant k,\ j\leqslant n\leqslant l\}$ is called an *interval* of $\mathbb{Z}^2$. in addition, if $i<k$ and $j<l$ then $[a,b]$ is a *proper* interval. In such a case we say a,b the *diagonal corners* of $[a,b]$ and $c=(i,l)$, $d=(k,j)$ the *anti-diagonal corners* of $[a,b]$. If $j=l$ (or $i=k$) then a and b are in *horizontal* (or *vertical*) *position*. We denote by $]a,b[$ the set $\{(m,n)\in\mathbb{Z}^2 : i<m<k,\ j<n<l\}$. A proper interval $C=[a,b]$ with $b=a+(1,1)$ is called a *cell* of $\mathbb{Z}^2$; moreover, the elements a, b, c and d are called respectively the *lower left*, *upper right*, *upper left* and *lower right corner* of C. The sets $\{a,c\}$, $\{c,b\}$, $\{b,d\}$ and $\{a,d\}$ are the *edges* of C. We put $V(C)=\{a,b,c,d\}$ and $E(C)=\{\{a,c\},\{c,b\},\{b,d\},\{a,d\}\}$.
Let $\mathcal{S}$ be a non-empty collection of cells in $\mathbb{Z}^2$. The set of the vertices and the edges of $\mathcal{S}$ are respectively $V(\mathcal{S})=\bigcup_{C\in\mathcal{S}}V(C)$ and $E(\mathcal{S})=\bigcup_{C\in\mathcal{S}}E(C)$. If C and D are two distinct cells of $\mathcal{S}$, then a *walk* from C to D in $\mathcal{S}$ is a sequence $\mathcal{C}: C=C_1,\dots,C_m=D$ of cells of $\mathbb{Z}^2$ such that $C_i\cap C_{i+1}$ is an edge of C_i and C_{i+1} for $i=1,\dots,m-1$. In addition, if $C_i\neq C_j$ for all $i\neq j$, then $\mathcal{C}$ is called a *path* from C to D. We say that C and D are connected in $\mathcal{S}$ if there exists a path of cells in $\mathcal{S}$ from C to D. A *polyomino* $\mathcal{P}$ is a non-empty, finite collection of cells in $\mathbb{Z}^2$ where any two cells of $\mathcal{P}$ are connected in $\mathcal{P}$. For instance, see Figure 1.

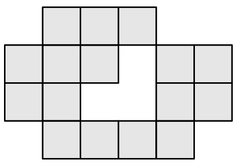


Figure 1: A polyomino.

We say that a polyomino $\mathcal{P}$ is *simple* if for any two cells C and D not in $\mathcal{P}$ there exists a path of cells not in $\mathcal{P}$ from C to D. A finite collection of cells $\mathcal{H}$ not in $\mathcal{P}$ is a *hole* of $\mathcal{P}$ if any two cells of $\mathcal{H}$ are connected in $\mathcal{H}$ and $\mathcal{H}$ is maximal with respect to set inclusion. For example, the polyomino in Figure 1 is not simple with an hole. Obviously, each hole of $\mathcal{P}$ is a simple polyomino and $\mathcal{P}$ is simple if and only if it has not any hole.
Consider two cells A and B of $\mathbb{Z}^2$ with $a=(i,j)$ and $b=(k,l)$ as the lower left corners of A and B and $a\leqslant b$. A *cell interval* $[A,B]$ is the set of the cells of $\mathbb{Z}^2$ with lower left corner (r,s) such that $i\leqslant r\leqslant k$ and $j\leqslant s\leqslant l$. If (i,j) and (k,l) are in horizontal (or vertical) position, we say that the cells A and B are in *horizontal* (or *vertical*) *position*.
Let $\mathcal{P}$ be a polyomino. Consider two cells A and B of $\mathcal{P}$ in vertical or horizontal position. The cell interval $[A,B]$, containing $n>1$ cells, is called a *block of $\mathcal{P}$ of rank n* if all cells of $[A,B]$ belong to $\mathcal{P}$. The cells A and B are called *extremal* cells of $[A,B]$. Moreover, a block $\mathcal{B}$ of $\mathcal{P}$ is *maximal* if there does not exist any block of $\mathcal{P}$ which contains properly

$\mathcal{B}$. It is clear that an interval of $\mathbb{Z}^2$ identifies a cell interval of $\mathbb{Z}^2$ and vice versa, hence we can associated to an interval I of $\mathbb{Z}^2$ the corresponding cell interval denoted by $\mathcal{P}_I$. A proper interval $[a,b]$ is called an *inner interval* of $\mathcal{P}$ if all cells of $\mathcal{P}_{[a,b]}$ belong to $\mathcal{P}$. An interval $[a,b]$ with $a=(i,j)$, $b=(k,j)$ and $i<k$ is called a *horizontal edge interval* of $\mathcal{P}$ if the sets $\{(\ell,j),(\ell+1,j)\}$ are edges of cells of $\mathcal{P}$ for all $\ell=i,\dots,k-1$. In addition, if $\{(i-1,j),(i,j)\}$ and $\{(k,j),(k+1,j)\}$ do not belong to $E(\mathcal{P})$, then $[a,b]$ is called a *maximal* horizontal edge interval of $\mathcal{P}$. We define similarly a *vertical edge interval* and a *maximal* vertical edge interval.
Following [21] we recall the definition of a *zig-zag walk* of $\mathcal{P}$. A zig-zag walk of $\mathcal{P}$ is a sequence $\mathcal{Z}: I_1,\dots,I_\ell$ of distinct inner intervals of $\mathcal{P}$ where, for all $i=1,\dots,\ell$, the interval I_i has either diagonal corners v_i, z_i and anti-diagonal corners u_i, v_{i+1} or anti-diagonal corners v_i, z_i and diagonal corners u_i, v_{i+1}, such that:

1. $I_1\cap I_\ell=\{v_1=v_{\ell+1}\}$ and $I_i\cap I_{i+1}=\{v_{i+1}\}$, for all $i=1,\dots,\ell-1$;
2. v_i and v_{i+1} are on the same edge interval of $\mathcal{P}$, for all $i=1,\dots,\ell$;
3. for all $i,j\in\{1,\dots,\ell\}$ with $i\neq j$, there exists no inner interval J of $\mathcal{P}$ such that z_i, z_j belong to $V(J)$.

According to [4], we recall the definition of a *closed path polyomino*, and the configuration of cells characterizing its primality. We say that a polyomino $\mathcal{P}$ is a *closed path* if it is a sequence of cells $A_1,\dots,A_n,A_{n+1}$, $n>5$, such that:

1. $A_1=A_{n+1}$;
2. $A_i\cap A_{i+1}$ is a common edge, for all $i=1,\dots,n$;
3. $A_i\neq A_j$, for all $i\neq j$ and $i,j\in\{1,\dots,n\}$;
4. For all $i\in\{1,\dots,n\}$ and for all $j\notin\{i-2,i-1,i,i+1,i+2\}$ then $V(A_i)\cap V(A_j)=\varnothing$, where $A_{-1}=A_{n-1}$, $A_0=A_n$, $A_{n+1}=A_1$ and $A_{n+2}=A_2$.

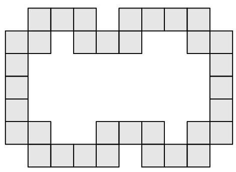


Figure 2: An example of a closed path.

A path of five cells C_1,C_2,C_3,C_4,C_5 of $\mathcal{P}$ is called an *L-configuration* if the two sequences C_1,C_2,C_3 and C_3,C_4,C_5 go in two orthogonal directions. A set $\mathcal{B}=\{\mathcal{B}_i\}_{i=1,\dots,n}$ of maximal horizontal (or vertical) blocks of rank at least two, with $V(\mathcal{B}_i)\cap V(\mathcal{B}_{i+1})=\{a_i,b_i\}$ and $a_i\neq b_i$ for all $i=1,\dots,n-1$, is called a *ladder of n steps* if $[a_i,b_i]$ is not on the same edge interval of $[a_{i+1},b_{i+1}]$ for all $i=1,\dots,n-2$. For instance, in Figure 3 there

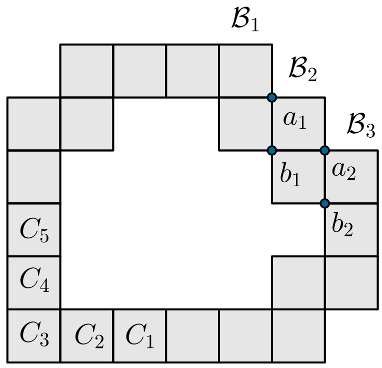


Figure 3

is a closed path having an L-configuration and a ladder of three steps. We recall that a closed path has no zig-zag walks if and only if it contains an L-configuration or a ladder of at least three steps (see [4]).
Let $\mathcal{P}$ be a polyomino. We set $S = K[x_v|v \in V(\mathcal{P})]$, where K is a field. If $[a,b]$ is an inner interval of $\mathcal{P}$, with a,b and c,d respectively diagonal and anti-diagonal corners, then the binomial $x_a x_b - x_c x_d$ is called an *inner 2-minor* of $\mathcal{P}$. We define $I_{\mathcal{P}}$ as the ideal in S generated by all the inner 2-minors of $\mathcal{P}$ and we call it the *polyomino ideal* of $\mathcal{P}$. We set also $K[\mathcal{P}] = S/I_{\mathcal{P}}$, which is the *coordinate ring* of $\mathcal{P}$. Eventually, we recall that if $\mathcal{P}$ is a closed path then having no zig-zag walks, equivalently $\mathcal{P}$ contains an L-configuration or a ladder of at least three steps, is a necessary and sufficient condition in order to $K[\mathcal{P}]$ is a domain by [4, Theorem 6.2].

3 Preliminary results

Let $\mathcal{P}$ be a non-empty collection of cells with $V(\mathcal{P}) = \{a_1, \dots, a_n\}$. We define a P-*order* to be a total order on the set $V(\mathcal{P})$. Observe that the monomial orderings defined in [22] and [24] are induced by specific P-orders, in particular we recall the monomial order $<^1$ introduced in [24], which will be useful for this paper: we say that $a <^1 b$ if and only if, for $a = (i,j)$ and $b = (k,l)$, $i < k$, or $i = k$ and $j < l$.
If $<^{\mathsf{P}}$ is a P-order, we denote by $<^{\mathsf{P}}_{\text{lex}}$ the lexicographic order induced by $<^{\mathsf{P}}$ on $S = K[x_v \mid v \in V(\mathcal{P})]$, that is the lexicographic order induced by the total order on the variables defined in the following way: $x_{a_i} <^{\mathsf{P}}_{\text{lex}} x_{a_j}$ if and only if $a_i <^{\mathsf{P}} a_j$ for $i,j \in \{1,\dots,n\}$. If $f \in S$, we denote by $\text{in}(f)$ the leading term of f with respect to $<^{\mathsf{P}}_{\text{lex}}$.
Let $f,g \in I_{\mathcal{P}}$, we denote by $S(f,g)$ the S-polynomial of f,g with respect to $<^{\mathsf{P}}_{\text{lex}}$. Let $\mathcal{G}$ be the set of all inner 2-minors of $\mathcal{P}$ (that is the set of generators of $I_{\mathcal{P}}$). We want to study some conditions on $<^{\mathsf{P}}$ in order to $S(f,g)$ reduces to 0 modulo $\mathcal{G}$.
First of all observe that if $[a,b]$ and $[\alpha,\beta]$ are two inner intervals and $[a,b] \cap [\alpha,\beta]$ does not contain any corner of $[a,b]$ and $[\alpha,\beta]$, then $\gcd(\text{in}(f_{a,b}), \text{in}(f_{\alpha,\beta})) = 1$ so $S(f_{a,b}, f_{\alpha,\beta})$ reduces to 0. So it suffices to study the remaining cases.
In the remainder of this section the inner intervals $[a,b]$ and $[\alpha,\beta]$ have respectively c,d

and γ, δ as anti-diagonal corners, as in figure 4.

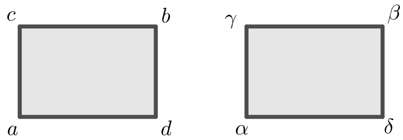


Figure 4

In the following results we examine all possible cases in which $|\{a,b,c,d\} \cap \{\alpha,\beta,\gamma,\delta\}|$ is equal to 1 or 2.

Remark 1. Let $\mathcal{P}$ be a collection of cells and $[a,b]$ and $[\alpha,\beta]$ be two inner intervals such that $|\{a,b,c,d\} \cap \{\alpha,\beta,\gamma,\delta\}| = 2$. Then $S(f_{a,b}, f_{\alpha,\beta})$ reduces to 0 modulo $\mathcal{G}$ with respect to $<^{\mathsf{P}}_{\text{lex}}$ for any P-order. In fact, we may assume that $\alpha = d$ and $\gamma = b$. Consider the non-trivial case when $\gcd(\text{in}(f_{a,b}), \text{in}(f_{\alpha,\beta})) \neq 1$. If $\text{in}(f_{a,b}) = x_a x_b$ and $\text{in}(f_{\alpha,\beta}) = -x_b x_\delta$, then $S(f_{a,b}, f_{\alpha,\beta}) = -x_\delta x_d x_c + x_a x_\beta x_d = x_d(x_a x_\beta - x_c x_\delta)$. The desired claim follows, because $f_{a,\beta} \in I_{\mathcal{P}}$ and $S(f_{a,b}, f_{\alpha,\beta})$ reduces to 0 with respect to $f_{a,\beta}$. If $\text{in}(f_{a,b}) = -x_d x_c$ and $\text{in}(f_{\alpha,\beta}) = x_\beta x_d$, then we obtain the desired conclusion by arguing as before. All other cases can be discussed similarly.

Lemma 2. *Let $\mathcal{P}$ be a collection of cells and $[a,b]$ and $[\alpha,\beta]$ be two inner intervals with $\beta = b$ and $\gamma \in]c,b[$ (see Figure 5(a)). Let h be the vertex such that $[h,b]$ is the inner interval having d,γ as anti-diagonal corner and r be the vertex such that $[r,h]$ is the interval having a,α as anti-diagonal corner. Let $<^{\mathsf{P}}$ be a P-order on $V(\mathcal{P})$ and suppose that* $\gcd(\text{in}(f_{a,b}), \text{in}(f_{\alpha,\beta})) \neq 1$*. Then $S(f_{a,b}, f_{\alpha,\beta})$ reduces to 0 modulo $\mathcal{G}$ with respect to $<^{\mathsf{P}}_{\text{lex}}$ if and only if one of the following conditions occurs:*

1. *$x_a x_\gamma x_\delta <^{\mathsf{P}}_{\text{lex}} x_\alpha x_c x_d$ and in addition $h, \delta <^{\mathsf{P}} \alpha$ or $h, \delta <^{\mathsf{P}} d$;*
2. *$x_a x_\gamma x_\delta <^{\mathsf{P}}_{\text{lex}} x_\alpha x_c x_d$, $\{r,h,a,\alpha\}$ is the set of vertices of an inner interval of $\mathcal{P}$ and in addition $r, \gamma <^{\mathsf{P}} \alpha$ or $r, \gamma <^{\mathsf{P}} c$;*
3. *$x_\alpha x_c x_d <^{\mathsf{P}}_{\text{lex}} x_a x_\gamma x_\delta$ and in addition $h, c <^{\mathsf{P}} a$ or $h, c <^{\mathsf{P}} \gamma$;*
4. *$x_\alpha x_c x_d <^{\mathsf{P}}_{\text{lex}} x_a x_\gamma x_\delta$, $\{r,h,a,\alpha\}$ is the set of vertices of an inner interval of $\mathcal{P}$ and in addition $r, d <^{\mathsf{P}} a$ or $r, d <^{\mathsf{P}} \delta$.*

The same characterization holds for $S(f_{c,d}, f_{\gamma,\delta})$, $S(f_{b,a}, f_{b,\alpha})$ and $S(f_{d,c}, f_{\delta,\gamma})$ considering all the rotations of the described configuration (see respectively Figure 5(b), Figure 5(c) and Figure 5(d)).

Proof. Observe that $\gcd(\text{in}(f_{a,b}), \text{in}(f_{\alpha,\beta})) \neq 1$ if and only if $\text{in}(f_{a,b}) = x_a x_b$ and $\text{in}(f_{\alpha,\beta}) = x_\alpha x_b$. Since $S(f_{a,b}, f_{\alpha,\beta}) = -x_\alpha x_c x_d + x_a x_\gamma x_\delta$, we have two possibilities:
1) $\text{in}(S(f_{a,b}, f_{\alpha,\beta})) = -x_\alpha x_c x_d$, in particular $x_a x_\gamma x_\delta <^{\mathsf{P}}_{\text{lex}} x_\alpha x_c x_d$. Observe that, since $x_c x_d$

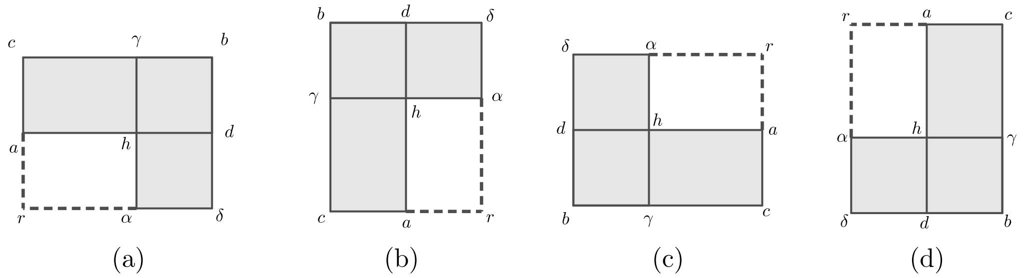


Figure 5

is not the leading term of $f_{a,b}$, in such a case the only possibilities for the reduction of $S(f_{a,b}, f_{\alpha,\beta})$ is through a first division by $f_{\alpha,d}$ if $\mathrm{in}(f_{\alpha,d}) = x_\alpha x_d$ or by $f_{r\gamma}$ if $\mathrm{in}(f_{r,\gamma}) = -x_\alpha x_c$. The first case is possible if and only if $(h, \delta <^{\mathsf{P}} \alpha) \vee (h, \delta <^{\mathsf{P}} d)$, and in such case indeed, after a little computation, $S(f_{a,b}, f_{\alpha,\beta})$ reduces by $f_{\alpha,d}$ to $x_\delta(x_a x_\gamma - x_c x_h) = x_\delta f_{a\gamma}$ and this one reduces to 0. For this case we obtain the condition (1) of this lemma. The second case is possible if and only if the condition (2) is satisfied, that is if $[r, h]$ is an inner interval of $\mathcal{P}$ and $(r, \gamma <^{\mathsf{P}} \alpha) \vee (r, \gamma <^{\mathsf{P}} c)$. In such a case in fact $S(f_{a,b}, f_{\alpha,\beta})$ reduces through $f_{r,\gamma}$ to $x_\gamma(x_a x_\delta - x_r x_d) = x_\gamma f_{a\delta}$ and this one reduces to 0.
2) $\mathrm{in}(S(f_{a,b}, f_{\alpha,\beta})) = x_a x_\gamma x_\delta$, in particular $x_\alpha x_c x_d <^{\mathsf{P}}_{\mathrm{lex}} x_a x_\gamma x_\delta$. We can argue as in the first part of this proof observing that, since $x_\gamma x_\delta$ is not the leading term of $f_{\alpha,\beta}$, in such a case the only possibilities for the reduction of $S(f_{a,b}, f_{\alpha,\beta})$ is through $f_{a,\gamma}$ if $\mathrm{in}(f_{a,\gamma}) = x_a x_\gamma$ or by $f_{r,d}$ if $\mathrm{in}(f_{r,d}) = -x_a x_\delta$. The first case is possible if and only if $(h, c <^{\mathsf{P}} a) \vee (h, c <^{\mathsf{P}} \gamma)$, that is the condition (3) holds, while the second is possible if and only if $[r, h]$ is an inner interval of $\mathcal{P}$ and $(r, d <^{\mathsf{P}} a) \vee (r, d <^{\mathsf{P}} \delta)$, that is the condition (4) is satisfied. In both cases $S(f_{a,b}, f_{\alpha,\beta})$ reduces to 0.
The last statement of this lemma is verified since the only effects of the rotation of a configuration are the different notations for the same intervals (for instance $[a, b]$ becomes $[c, d]$, $[b, a]$ or $[d, c]$) or the change of the sign of the binomials in the generators of $I_{\mathcal{P}}$. □

The following four lemmas can be proved by the same arguments of Lemma 2, so we omit their proofs.

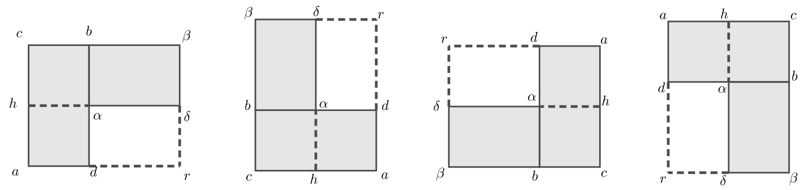
Lemma 3. *Let $\mathcal{P}$ be a collection of cells and $[a, b]$ and $[\alpha, \beta]$ be two inner intervals with $\gamma = b$ and $\alpha \in]d, b[$ (see Figure 6(a)). Let h be the vertex such that $[h, b]$ is the inner interval having c, α as anti-diagonal corner and r be the vertex such that r, α are the anti-diagonal corners of the interval $[d, \delta]$. Let $<^{\mathsf{P}}$ be a* P*-order on $V(\mathcal{P})$ and suppose that $\gcd(\mathrm{in}(f_{a,b}), \mathrm{in}(f_{\alpha,\beta})) \neq 1$. Then $S(f_{a,b}, f_{\alpha,\beta})$ reduces to 0 modulo $\mathcal{G}$ with respect to $<^{\mathsf{P}}_{\mathrm{lex}}$ if and only if one of the following conditions occurs:*

1. *$x_a x_\alpha x_\beta <^{\mathsf{P}}_{\mathrm{lex}} x_\delta x_c x_d$ and in addition $h, \beta <^{\mathsf{P}} c$ or $h, \beta <^{\mathsf{P}} \delta$;*
2. *$x_a x_\alpha x_\beta <^{\mathsf{P}}_{\mathrm{lex}} x_\delta x_c x_d$, $\{d, \delta, \alpha, r\}$ is the set of vertices of an inner interval of $\mathcal{P}$ and in addition $r, \alpha <^{\mathsf{P}} \delta$ or $r, \alpha <^{\mathsf{P}} d$;*

3. $x_\delta x_c x_d <^{\mathsf{P}}_{\text{lex}} x_a x_\alpha x_\beta$ *and in addition* $h, d <^{\mathsf{P}} a$ *or* $h, d <^{\mathsf{P}} \alpha$*;*

4. $x_\delta x_c x_d <^{\mathsf{P}}_{\text{lex}} x_a x_\alpha x_\beta$*,* $\{d, \delta, \alpha, r\}$ *is the set of vertices of an inner interval of* $\mathcal{P}$ *and in addition* $r, c <^{\mathsf{P}} a$ *or* $r, c <^{\mathsf{P}} \beta$*.*

The same characterization holds for $S(f_{c,d}, f_{\gamma,\delta})$*,* $S(f_{b,a}, f_{\beta,\alpha})$ *and* $S(f_{d,c}, f_{\delta,\gamma})$ *considering all the rotations of the described configuration (see respectively Figure 6(b), Figure 6(c) and Figure 6(d)).*



(a) (b) (c) (d)

Figure 6

Lemma 4. *Let* $\mathcal{P}$ *be a collection of cells and* $[a, b]$ *and* $[\alpha, \beta]$ *be two inner intervals with* $\alpha = c$ *and* $b \in]\alpha, \delta[$ *(see Figure 7(a)). Let* h *be the vertex such that* h, δ *are the diagonal corners of the inner interval* $[b, \beta]$ *and* r *be the vertex such that* r, b *are the anti-diagonal corners of the interval* $[d, \delta]$*. Let* $<^{\mathsf{P}}$ *be a* P*-order on* $V(\mathcal{P})$ *and suppose that* $\gcd(\text{in}(f_{a,b}), \text{in}(f_{\alpha,\beta})) \neq 1$*. Then* $S(f_{a,b}, f_{\alpha,\beta})$ *reduces to 0 modulo* $\mathcal{G}$ *with respect to* $<^{\mathsf{P}}_{\text{lex}}$ *if and only if one of the following conditions occurs:*

1. $x_d x_\delta x_\gamma <^{\mathsf{P}}_{\text{lex}} x_\beta x_a x_b$ *and in addition* $h, \delta <^{\mathsf{P}} b$ *or* $h, \delta <^{\mathsf{P}} \beta$*;*

2. $x_d x_\delta x_\gamma <^{\mathsf{P}}_{\text{lex}} x_\beta x_a x_b$*,* $\{d, \delta, b, r\}$ *is the set of vertices of an inner interval of* $\mathcal{P}$ *and in addition* $r, \gamma <^{\mathsf{P}} a$ *or* $r, \gamma <^{\mathsf{P}} \beta$*;*

3. $x_\beta x_a x_b <^{\mathsf{P}}_{\text{lex}} x_d x_\delta x_\gamma$ *and in addition* $h, a <^{\mathsf{P}} d$ *or* $h, a <^{\mathsf{P}} \gamma$*;*

4. $x_\beta x_a x_b <^{\mathsf{P}}_{\text{lex}} x_d x_\delta x_\gamma$*,* $\{d, \delta, b, r\}$ *is the set of vertices of an inner interval of* $\mathcal{P}$ *and in addition* $r, b <^{\mathsf{P}} d$ *or* $r, b <^{\mathsf{P}} \delta$*.*

The same characterization holds for $S(f_{c,d}, f_{\gamma,\delta})$*,* $S(f_{b,a}, f_{\beta,\alpha})$ *and* $S(f_{d,c}, f_{\delta,\gamma})$ *considering all the rotations of the described configuration (see respectively Figure 7(b), Figure 7(c) and Figure 7(d)).*

Lemma 5. *Let* $\mathcal{P}$ *be a collection of cells and* $[a, b]$ *and* $[\alpha, \beta]$ *be two inner intervals with* $\gamma = c$ *and* $\delta \in]a, b[$ *(see Figure 8(a)). Let* $[h, r]$ *be the inner interval having* d, δ *as anti-diagonal corners. Let* $<^{\mathsf{P}}$ *be a* P*-order on* $V(\mathcal{P})$ *and suppose that* $\gcd(\text{in}(f_{a,b}), \text{in}(f_{\alpha,\beta})) \neq 1$*. Then* $S(f_{a,b}, f_{\alpha,\beta})$ *reduces to 0 modulo* $\mathcal{G}$ *with respect to* $<^{\mathsf{P}}_{\text{lex}}$ *if and only if one of the following conditions occurs:*

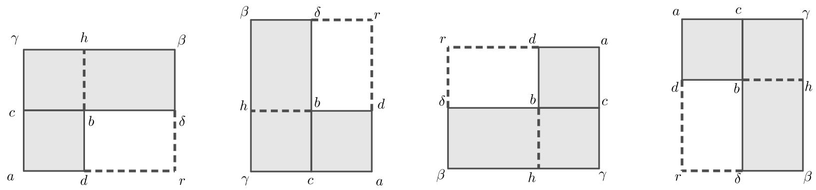
1. $x_d x_\alpha x_\beta <^{\mathsf{P}}_{\text{lex}} x_\delta x_a x_b$ *and in addition* $h, \alpha <^{\mathsf{P}} a$ *or* $h, \alpha <^{\mathsf{P}} \delta$*;*

2. $x_d x_\alpha x_\beta <^{\mathsf{P}}_{\text{lex}} x_\delta x_a x_b$ *and in addition* $r, \beta <^{\mathsf{P}} \delta$ *or* $r, \beta <^{\mathsf{P}} b$*;*

3. $x_\delta x_a x_b <^{\mathsf{P}}_{\text{lex}} x_d x_\alpha x_\beta$ *and in addition* $r, a <^{\mathsf{P}} \alpha$ *or* $r, a <^{\mathsf{P}} d$*;*

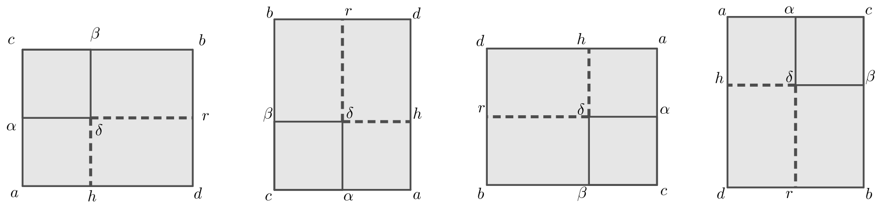
4. $x_\delta x_a x_b <^{\mathsf{P}}_{\text{lex}} x_d x_\alpha x_\beta$ *and in addition* $h, b <^{\mathsf{P}} d$ *or* $h, b <^{\mathsf{P}} \beta$*.*

The same characterization holds for $S(f_{c,d}, f_{\gamma,\delta})$*,* $S(f_{b,a}, f_{\beta,\alpha})$ *and* $S(f_{d,c}, f_{\delta,\gamma})$ *considering all the rotations of the described configuration (see respectively Figure 8(b), Figure 8(c) and Figure 8(d)).*



(a) (b) (c) (d)

Figure 7



(a) (b) (c) (d)

Figure 8

Lemma 6. *Let* $\mathcal{P}$ *be a collection of cells and* $[a, b]$ *and* $[\alpha, \beta]$ *be two inner intervals with* $\alpha = b$ *and* $\beta \notin [a, b]$ *(see Figure 9(a)). Let* h, r *be the anti-diagonal corners, different to* b*, respectively of the intervals* $[d, \delta]$ *and* $[c, \gamma]$*. Let* $<^{\mathsf{P}}$ *be a* P*-order on* $V(\mathcal{P})$ *and suppose that* $\gcd(\text{in}(f_{a,b}), \text{in}(f_{\alpha,\beta})) \neq 1$*. Then* $S(f_{a,b}, f_{\alpha,\beta})$ *reduces to 0 modulo* $\mathcal{G}$ *with respect to* $<^{\mathsf{P}}_{\text{lex}}$ *if and only if one of the following conditions occurs:*

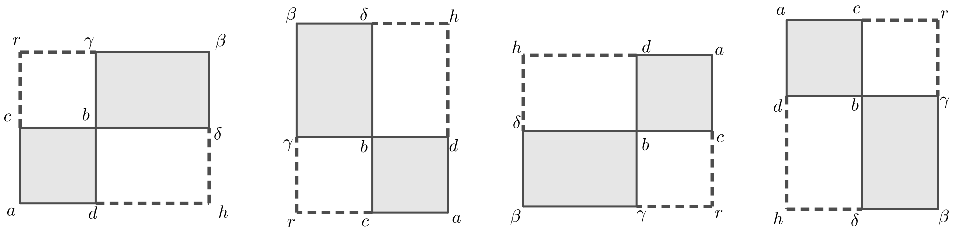
1. $x_a x_\gamma x_\delta <^{\mathsf{P}}_{\text{lex}} x_\beta x_d x_c$*,* $\{d, \delta, b, h\}$ *is the set of vertices of an inner interval of* $\mathcal{P}$ *and in addition* $h, \gamma <^{\mathsf{P}} \beta$ *or* $h, \gamma <^{\mathsf{P}} d$*;*

2. $x_a x_\gamma x_\delta <^{\mathsf{P}}_{\text{lex}} x_\beta x_d x_c$*,* $\{c, \gamma, b, r\}$ *is the set of vertices of an inner interval of* $\mathcal{P}$ *and in addition* $r, \delta <^{\mathsf{P}} \beta$ *or* $r, \delta <^{\mathsf{P}} c$*;*

3. $x_\beta x_d x_c <^{\mathsf{P}}_{\text{lex}} x_a x_\gamma x_\delta$, $\{c,\gamma,b,r\}$ *is the set of vertices of an inner interval of* $\mathcal{P}$ *and in addition* $r,d <^{\mathsf{P}} a$ *or* $r,d <^{\mathsf{P}} \gamma$;

4. $x_\beta x_d x_c <^{\mathsf{P}}_{\text{lex}} x_a x_\gamma x_\delta$, $\{d,\delta,b,h\}$ *is the set of vertices of an inner interval of* $\mathcal{P}$ *and in addition* $c,h <^{\mathsf{P}} a$ *or* $c,h <^{\mathsf{P}} \delta$.

The same characterization holds for $S(f_{c,d}, f_{\gamma,\delta})$, $S(f_{b,a}, f_{\beta,\alpha})$ *and* $S(f_{d,c}, f_{\delta,\gamma})$ *considering all the rotations of the described configuration (see respectively Figure 9(b), Figure 9(c) and Figure 9(d)).*



(a) (b) (c) (d)

Figure 9

4 Gröbner basis of the polyomino ideal of a closed path.

Let $\mathcal{P}$ be a polyomino. In this section we examine four special configurations of cells of a polyomino, that permit us when $\mathcal{P}$ is a closed path to provide some particular subsets $Y \subset V(\mathcal{P})$ for which we can define the following P-order.

Definition 7. Let $Y \subset V(\mathcal{P})$. We define the P-order $<^Y$ in the following way:

$$a <^Y b \Leftrightarrow \begin{cases} a \notin Y \text{ and } b \in Y \\ a,b \notin Y \text{ and } a <^1 b \\ a,b \in Y \text{ and } a <^1 b \end{cases}$$

for $a,b \in V(\mathcal{P})$.

We call a *W-pentomino with middle cell* A a subpolyomino of $\mathcal{P}$ consisting of an horizontal block $\mathcal{B}_1 = [A_1, B_1]$ of rank two, a vertical block $\mathcal{B}_2 = [A_2, B_2]$ of rank two and a cell A not belonging to $\mathcal{B}_1 \cup \mathcal{B}_2$, such that $V(\mathcal{B}_1) \cap V(\mathcal{B}_2) = \{w\}$ and where w is the lower right corner of A. Moreover, if $\mathcal{W}$ is a W-pentomino with middle cell A, we denote by x_W the left upper corner of A, with y_W the lower right corner of B_1 and with z_W the lower right corner of A_2. See Figure 10.
We call an *LD-horizontal (vertical) skew tetromino* a subpolyomino of $\mathcal{P}$ consisting of two horizontal (vertical) blocks of rank two $\mathcal{B}_1 = [A_1, B_1]$ and $\mathcal{B}_2 = [A_2, B_2]$ such that $V(B_1) \cap V(A_2) = \{w_1, w_2\}$ and w_1, w_2 are right and left upper (lower and upper right)

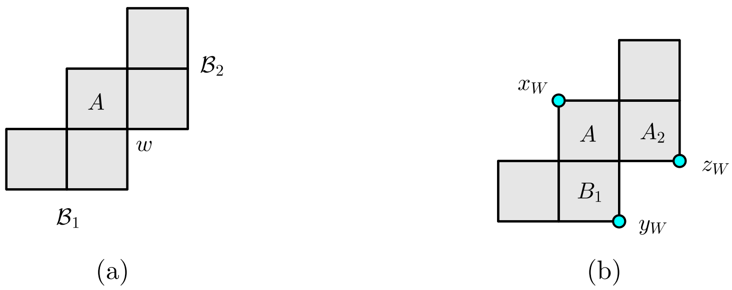


Figure 10: W-pentomino.

corners of B_1. Moreover, if $\mathcal{C}$ is an LD-horizontal (vertical) skew tetromino, we denote by $x_\mathcal{C}, y_\mathcal{C}$ the left and right upper corners of A_2 (the upper and lower left corners of B_1), and with $a_\mathcal{C}, b_\mathcal{C}$ the left and right lower corners of B_1 (the upper and lower right corners of A_2). See Figure 11.

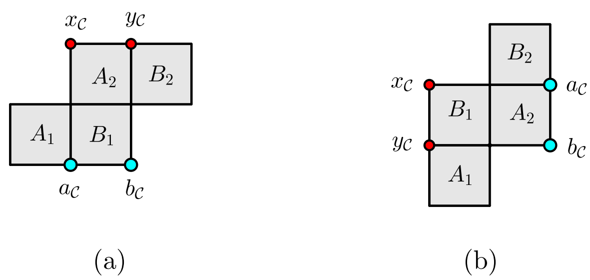
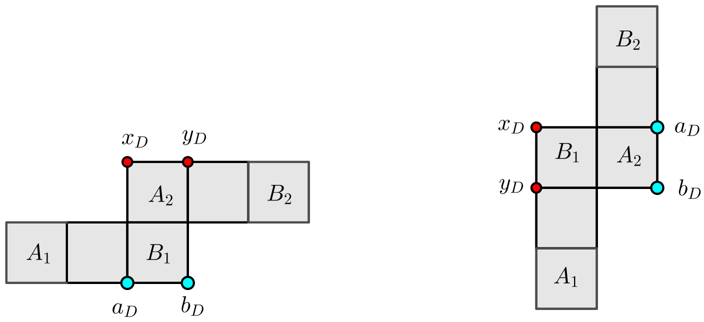


Figure 11: LD-horizontal skew tetromino (a) and LD-vertical skew tetromino (b).

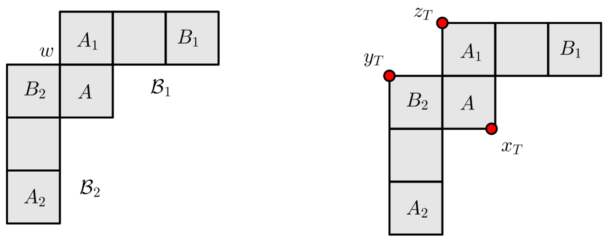
We call an *LD-horizontal (vertical) skew hexomino* a subpolyomino of $\mathcal{P}$ consisting of two horizontal (vertical) blocks of rank three $\mathcal{B}_1 = [A_1, B_1]$ and $\mathcal{B}_2 = [A_2, B_2]$ such that $V(B_1) \cap V(A_2) = \{w_1, w_2\}$ and w_1, w_2 are respectively the right and left upper (lower and upper right) corners of B_1. Moreover, if $\mathcal{D}$ is an LD-horizontal (vertical) skew tetromino, we denote by x_D, y_D the left and right upper corners of A_2 (the upper and lower left corners of B_1), and by a_D, b_D the the left and right lower corners of B_1 (the upper and lower right corners of A_2). See Figure 12.
We call an *RW-heptomino with middle cell* A a subpolyomino of $\mathcal{P}$ consisting of an horizontal block $\mathcal{B}_1 = [A_1, B_1]$ of rank three, a vertical block $\mathcal{B}_2 = [A_2, B_2]$ of rank three and a cell A not belonging to $\mathcal{B}_1 \cup \mathcal{B}_2$, such that $V(\mathcal{B}_1) \cap V(\mathcal{B}_2) = \{w\}$ and where w is the upper left corner of A. Moreover, if $\mathcal{T}$ is an RW-pentomino with middle cell A, we denote by x_T the right lower corner of A, with y_T the left upper corner of B_2 and by z_T the left upper corner of A_1. See Figure 13.

Theorem 8. *Let $\mathcal{P}$ be a closed path polyomino not containing any W-pentomino. Let $\mathcal{R}$ be the set of all LD-horizontal and vertical skew tetrominoes contained in $\mathcal{P}$ and let $Y = \bigcup_{\mathcal{C} \in \mathcal{R}} \{x_\mathcal{C}, y_\mathcal{C}\}$. Then $\mathcal{G}$ is the reduced Gröbner basis of $I_\mathcal{P}$ with respect to the monomial order $<^Y_{\text{lex}}$.*



(a) (b)

Figure 12: LD-horizontal skew hexomino (a) and LD-vertical skew hexomino (b).



(a) (b)

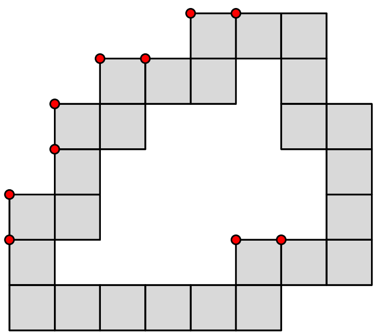
Figure 13: RW-heptomino.

Proof. Let $f = x_p x_q - x_r x_s$ and $g = x_u x_v - x_w x_z$ be the two binomials attached respectively to the inner intervals $[p, q]$ and $[u, v]$ of $\mathcal{P}$. We prove that $S(f, g)$ reduces to 0 modulo $\mathcal{G}$ with respect to $<^Y_{\text{lex}}$, examining all possible cases on $\{p, q, r, s\} \cap \{u, v, w, z\}$.
The case $\{p, q, r, s\} \cap \{u, v, w, z\} = \varnothing$ is trivial. If $|\{p, q, r, s\} \cap \{u, v, w, z\}| = 2$, then the claim follows from Remark 1. Assume that $|\{p, q, r, s\} \cap \{u, v, w, z\}| = 1$ and that $[p, q]$ is not contained in $[u, v]$ or vice versa. Suppose that $q = v$. For the structure of $\mathcal{P}$ we may assume that $s \in]z, v[$ and $w \in]r, q[$, so there exists $k \in \{1, \dots, n\}$ such that $A_k = [p, q] \cap [u, v]$. Let A_{k-1} be the cell of $\mathcal{P}_{[p,q]}$ adjacent to A_k. If A_{k-2} is at North of A_{k-1} then none of the vertices in $[p, q]$ and $[u, v]$ belongs to Y and the assertion follows from (1) of Lemma 2, since $x_p x_w x_z <^Y_{lex} x_r x_s x_u$ and $h, z <^Y s$, where $\{h\} = [p, s] \cap [u, w]$. If A_{k-2} is at West of A_{k-1} then we have two possibilities. Firstly, if A_{k-2} is not a cell of an LD-horizontal skew tetromino then none of the vertices in $[p, q]$ and $[u, v]$ belongs to Y, so the claim follows as before by applying (1) of Lemma 2. Secondly, if A_{k-2} is a cell of an LD-horizontal skew tetromino, then among the vertices of $[p, q]$ and $[u, v]$ only r belongs to Y, so the claim follows since $\gcd(\text{in}(f), \text{in}(g)) = 1$. The cases $r = w$, $p = u$ and $s = z$ can be proved similarly to the previous ones. Suppose that $q = w$. We may assume that $u \in]s, q[$, because the arguments are similar when $s \in]u, w[$. Let A_k be the cell of $\mathcal{P}$ having r, u as anti-diagonal corners and we

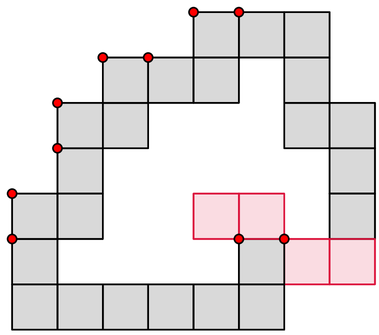
denote by A_{k-1} and A_{k+1} respectively the cells of $\mathcal{P}_{[p,q]}$ and $\mathcal{P}_{[u,v]}$ adjacent to A_k. If $\{A_{k-2}, A_{k-1}, A_k, A_{k+1}\}$ is an LD-vertical skew tetromino or $\{A_{k-2}, A_{k-1}, A_k, A_{k+1}, A_{k+2}\}$ is an L-configuration then $\gcd(\mathrm{in}(f), \mathrm{in}(g)) = 1$. If $\{A_{k-1}, A_k, A_{k+1}, A_{k+2}\}$ is an LD-horizontal skew tetromino, then $\gcd(\mathrm{in}(f), \mathrm{in}(g)) = x_w$ and applying (1) of Lemma 3 we have the desired conclusion. Similar arguments hold in the cases $s = u$, $v = r$ and $z = p$. Suppose $q = u$ and let A_k and A_{k+2} be the cells of $\mathcal{P}$ having respectively q as upper right and lower left corner. If $\{A_{k-1}, A_k, A_{k+1}, A_{k+2}\}$ or $\{A_k, A_{k+1}, A_{k+2}, A_{k+3}\}$ is an LD-vertical skew tetromino, then $\gcd(\mathrm{in}(f), \mathrm{in}(g)) = 1$. If $\{A_{k-1}, A_k, A_{k+1}, A_{k+2}, A_{k+3}\}$ is an L-configuration, the claim follows either by $\gcd(\mathrm{in}(f), \mathrm{in}(g)) = 1$ or by applying Lemma 6 if $\gcd(\mathrm{in}(f), \mathrm{in}(g)) \neq 1$. If $\{A_{k-1}, A_k, A_{k+1}, A_{k+2}, A_{k+3}\}$ is not an L-configuration and does not contain an LD-vertical skew tetromino, then the only two possibilities are that either $\{A_{k-1}, A_k, A_{k+1}, A_{k+2}\}$ or $\{A_k, A_{k+1}, A_{k+2}, A_{k+3}\}$ is an LD-horizontal skew tetromino. In both cases $\gcd(\mathrm{in}(f), \mathrm{in}(g)) = 1$, in particular in the first case the claim follows since $\mathcal{P}$ has not any W-pentomino, so A_{k+3} is at East of A_{k+2}. The other cases $s = w$, $z = r$ or $v = p$ can be proved by similar arguments. Finally, it is easy to see that in such cases $\mathcal{G}$ is also the reduced Gröbner basis of $I_{\mathcal{P}}$. ☐

In Figure 14(a) is shown an example of polyomino satisfying Theorem 8.

Remark 9. In [22] the authors introduced the class of *thin* polyominoes, that consists of all polyominoes not containing the configuration whose shape is a square made up of four cells. Such class can be viewed as a generalization of closed paths. We observe that the conclusion of the previous theorem does not hold in general for thin polyominoes, using the same monomial order. In fact, we can consider the thin polyomino in Figure 14(b) and it is not difficult to show that the S-polynomial associated to the marked intervals does not reduce to 0.



(a)



(b)

Figure 14: The highlighted points belong to Y.

Remark 10. By the same arguments, the statement of Theorem 8 holds also for $Y = \bigcup_{\mathcal{C}\in\mathcal{R}}\{a_{\mathcal{C}}, b_{\mathcal{C}}\}$.

Theorem 11. *Let $\mathcal{P}$ be a closed path polyomino not containing any RW-heptomino. Let $\mathcal{R}_1$ be the set of all LD-horizontal and vertical skew hexominoes contained in $\mathcal{P}$ and* *let $\mathcal{R}_2$ be the set of all W-pentominoes contained in $\mathcal{P}$. Let $Y = (\bigcup_{\mathcal{D}\in\mathcal{R}_1}\{a_D, b_D\}) \cup (\bigcup_{\mathcal{W}\in\mathcal{R}_2}\{x_W, y_W\})$. Then $\mathcal{G}$ is the reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to the monomial order $<^Y_{\text{lex}}$.*

Proof. Let $f = x_p x_q - x_r x_s$ and $g = x_u x_v - x_w x_z$ be the two binomials attached respectively to the inner intervals $[p, q]$ and $[u, v]$ of $\mathcal{P}$. We discuss the case $|\{p, q, r, s\} \cap \{u, v, w, z\}| = 1$, where $[p, q]$ is not contained in $[u, v]$ or vice versa. The cases $q = v$, $r = w$, $p = u$ and $s = z$, as well as $q = w$, $s = u$, $v = r$ and $z = p$, can be proved as in Theorem 8. Suppose $q = u$ and let A_k and A_{k+2} be the cells of $\mathcal{P}$ having respectively q as upper right and lower left corner. If $\{A_{k-1}, A_k, A_{k+1}, A_{k+2}, A_{k+3}\}$ is an L-configuration the claim follows either if $\gcd(\text{in}(f), \text{in}(g)) = 1$ or by applying Lemma 6 if $\gcd(\text{in}(f), \text{in}(g)) \neq 1$. If $\{A_{k-2}, A_{k-1}, A_k, A_{k+1}, A_{k+2}, A_{k+3}\}$ or $\{A_{k-1}, A_k, A_{k+1}, A_{k+2}, A_{k+3}, A_{k+4}\}$ is an LD-horizontal or vertical skew hexomino, then $\gcd(\text{in}(f), \text{in}(g)) = 1$. Since there does not exist any RW-heptomino, the last possibilities consist in being A_{k-1}, A_k, A_{k+1} or A_{k+2} the middle cell of a W-pentomino. In all these cases we have the desired conclusion either if $\gcd(\text{in}(f), \text{in}(g)) = 1$ or by applying Lemma 6 if $\gcd(\text{in}(f), \text{in}(g)) \neq 1$. The cases $s = w$, $z = r$ or $v = p$ can be proved by similar arguments. ☐

Remark 12. With the same arguments, the statement of Theorem 11 holds also considering $Y = (\bigcup_{\mathcal{D}\in\mathcal{R}_1}\{a_D, b_D\}) \cup (\bigcup_{\mathcal{W}\in\mathcal{R}_2}\{x_W, z_W\})$.

Given a closed path polyomino $\mathcal{P}$ containing both W-pentominoes and RW-heptominoes, our aim is to find a P-order $<^Y$, for a suitable set $Y \subset V(\mathcal{P})$, such that $\mathcal{G}$ is the Gröbner basis of $I_{\mathcal{P}}$ with respect to the monomial order $<^Y_{\text{lex}}$. We are going to define the set Y by combining the previous construction and the highlighted points in Figures 10, 12, 13, and proceeding iteratively from the structure of the polyomino and the arrangement of the cells. In order to simplify notations and writings, we summarize in the table in Figure 15 the arrangements with highlighted points already introduced in the previous definitions that are useful to define the new set Y. We build up the set Y using the algorithm explained below, for which it is also important to consider the configurations described in Figure 16 and Figure 17.

Algorithm 4.1. Let $\mathcal{P}$ be a closed path polyomino, whose sequence of cells is $A_1, A_2, \dots, A_n$, A_{n+1} (with $A_1 = A_{n+1}$) and containing both W-pentominoes and RW-heptominoes. Let $i, j \in \{1, 2, \dots, n, n+1\}$ with $i < j$. We define $Y_{i,j} \subset V(\mathcal{P})$ be the set provided by the algorithmic scheme described below:

1. Start with $Y_{i,j} = \varnothing$.
2. Define $\mathcal{Q} = \{q \in \{i, \dots, j\} \mid A_q \text{ is the middle cell of a RW-heptomino}\}$.
3. If $\mathcal{Q} \neq \varnothing$ define $q_1 = \min \mathcal{Q}$, otherwise define $q_1 = j$.
4. FOR $k \in \{i, \dots, q_1\}$ DO:
   (a) IF A_k is the middle cell of a W-pentomino THEN $Y_{i,j} = Y_{i,j} \cup \{x_W, z_W\}$ with reference to II-A of Figure 15.

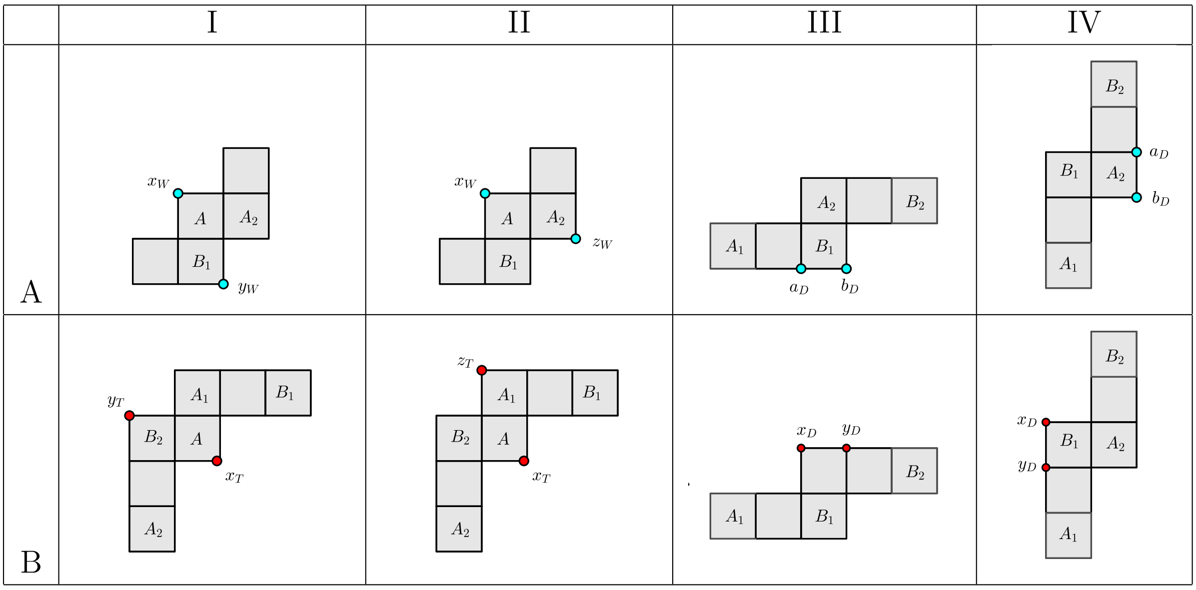


Figure 15

   (b) IF $A_k, A_{k+1}, \ldots, A_{k+6}$ is a sequence of cells of an LD-horizontal skew hexomino THEN $Y_{i,j} = Y_{i,j} \cup \{a_D, b_D\}$ with reference to III-A of Figure 15.

   (c) IF $A_k, A_{k+1}, \ldots, A_{k+6}$ is a sequence of cells of an LD-vertical skew hexomino THEN $Y_{i,j} = Y_{i,j} \cup \{a_D, b_D\}$ with reference to IV-A of Figure 15.

5. Define $\mathcal{R} = \{r \in \{q_1 + 1, \ldots, j\} \mid A_r$ is the middle cell of a W-pentomino$\}$.

6. If $\mathcal{R} \neq \varnothing$ define $r_1 = \min \mathcal{R}$, otherwise define $r_1 = j$.

7. Define $Q = q_1$ and $R = r_1$.

8. Consider the RW-heptomino with middle cell A_Q and let $M = \max\{m \in \{i, \ldots, Q\} \mid A_m \cap Y_{i,j} \neq \varnothing\}$.

9. FOR $k \in \{Q, \ldots, R\}$ DO:

   (a) IF A_k is the middle cell of an RW-heptomino THEN

   IF A_M and A_Q do not occur as in the configurations of Figure 16

   THEN $Y_{i,j} = Y_{i,j} \cup \{x_T, y_T\}$ with reference to I-B of Figure 15

   ELSE $Y_{i,j} = Y_{i,j} \cup \{x_T, z_T\}$ with reference to II-B of Figure 15.

   (b) IF $A_k, A_{k+1}, \ldots, A_{k+6}$ is a sequence of cells of an LD-horizontal skew hexomino THEN $Y_{i,j} = Y_{i,j} \cup \{x_D, y_D\}$ with reference to III-B of Figure 15.

   (c) IF $A_k, A_{k+1}, \ldots, A_{k+6}$ is a sequence of cells of an LD-vertical skew hexomino THEN $Y_{i,j} = Y_{i,j} \cup \{x_D, y_D\}$ with reference to IV-B of Figure 15.

   (d) IF $R = j$ THEN RETURN $Y_{i,j}$.

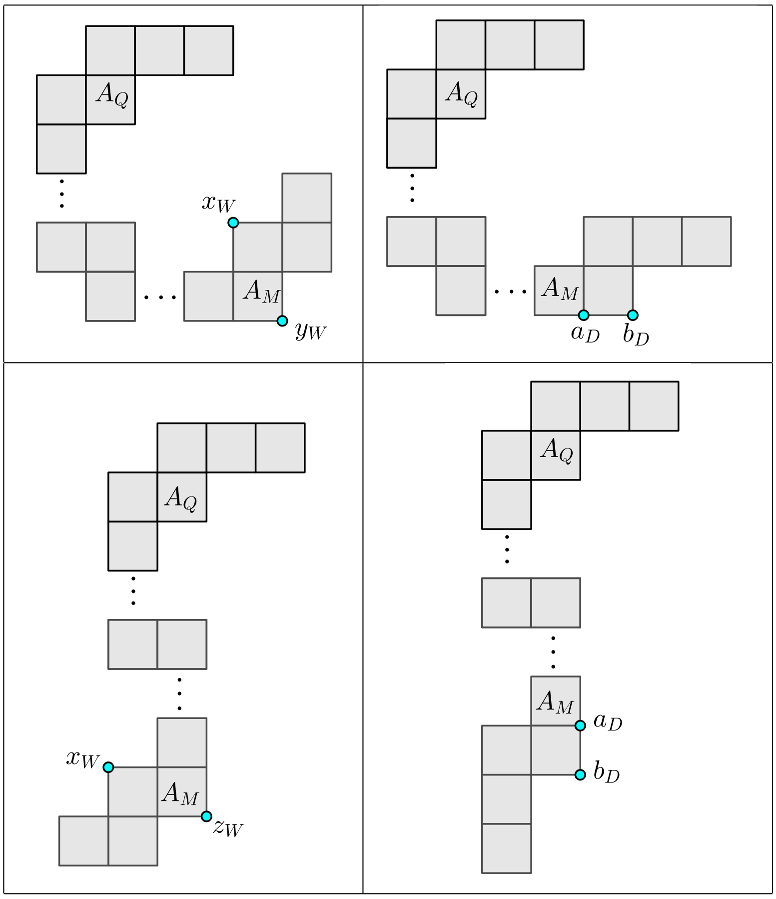


Figure 16: Conflicting configurations with I-B.

10. Define $\mathcal{Q} = \{q \in \{r_1+1, \ldots, j\} \mid A_q$ is the middle cell of a RW-heptomino$\}$.

11. If $\mathcal{Q} \neq \varnothing$ define $q_2 = \min \mathcal{Q}$, otherwise define $q_2 = j$.

12. Define $R = r_1$ and $Q = q_2$.

13. Consider the W-pentomino with middle cell A_R and let $M = \max\{m \in \{i, \ldots, R\} \mid A_m \cap Y_{i,j} \neq \varnothing\}$.

14. FOR $k \in \{R, \ldots, Q\}$ DO:

    (a) IF A_k is the middle cell of a W-pentomino THEN

    IF A_M and A_R do not occur as in the configurations of Figure 17
    THEN $Y_{i,j} = Y_{i,j} \cup \{x_W, y_W\}$ with reference to I-A of Figure 15
    ELSE $Y_{i,j} = Y_{i,j} \cup \{x_W, z_W\}$ with reference to II-A of Figure 15.

    (b) IF $A_k, A_{k+1}, \ldots, A_{k+6}$ is a sequence of cells of an LD-horizontal skew hexomino THEN $Y_{i,j} = Y_{i,j} \cup \{a_D, b_D\}$ with reference to III-A of Figure 15.

(c) IF $A_k, A_{k+1}, \ldots, A_{k+6}$ is a sequence of cells of an LD-vertical skew hexomino THEN $Y_{i,j} = Y_{i,j} \cup \{a_D, b_D\}$ with reference to IV-A of Figure 15.

(d) IF $Q = j$ THEN RETURN $Y_{i,j}$.

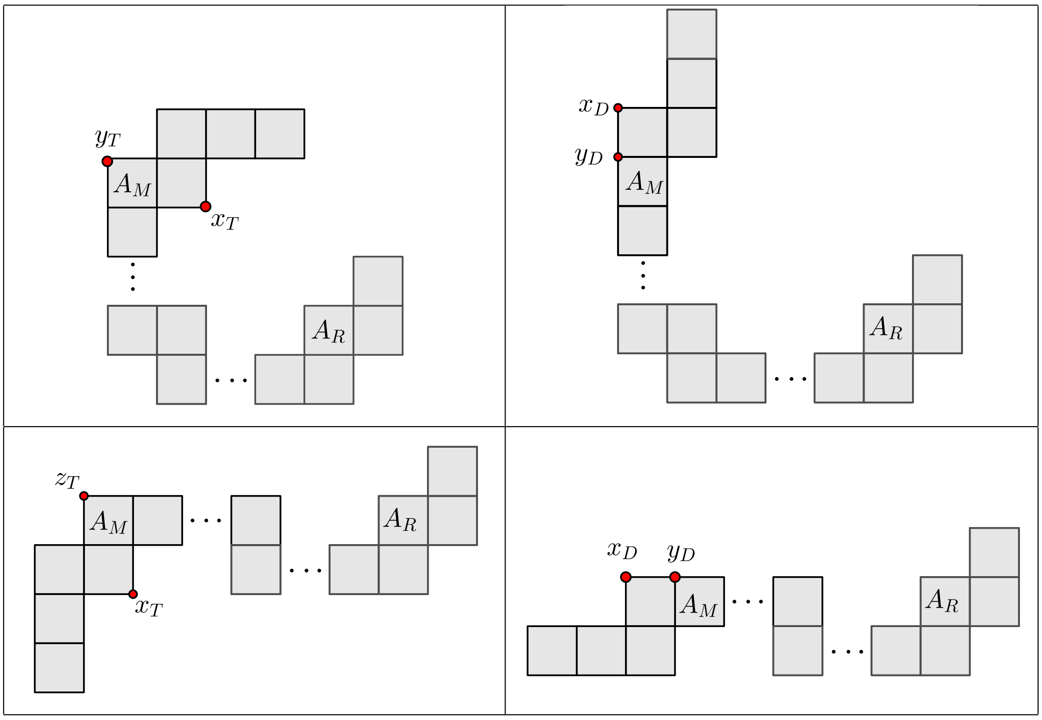


Figure 17: Conflicting configurations with I-A.

15. $\ell = 2$.

16. WHILE $\ell > 1$ DO

(a) Define $\mathcal{R} = \{r \in \{q_\ell + 1, \ldots, j\} \mid A_r$ is the middle cell of a W-pentomino$\}$.

(b) If $\mathcal{R} \neq \varnothing$ define $r_\ell = \min \mathcal{R}$, otherwise define $r_\ell = j$.

(c) Define $Q = q_\ell$ and $R = r_\ell$.

(d) $M = \max\{m \in \{i, \ldots, Q\} \mid A_m \cap Y_{i,j} \neq \varnothing\}$.

(e) Execute the instructions in (9).

(f) Define $\mathcal{Q} = \{q \in \{r_\ell + 1, \ldots, j\} \mid A_q$ is the middle cell of a RW-heptomino$\}$.

(g) If $\mathcal{Q} \neq \varnothing$ define $q_{\ell+1} = \min \mathcal{Q}$, otherwise define $q_{\ell+1} = j$.

(h) Define $R = r_\ell$ and $Q = q_{\ell+1}$.

(i) $M = \max\{m \in \{i, \ldots, R\} \mid A_m \cap Y_{i,j} \neq \varnothing\}$.

(j) Execute the instructions in (14).

(k) $\ell = \ell + 1$.

17. END

Observe that, since $r_\ell < r_{\ell+1}$ and $q_\ell < q_{\ell+1}$ for all $\ell \in \mathbb{N}$ then there exists $\overline{\ell}$ such that $r_{\overline{\ell}} = j$ or $q_{\overline{\ell}} = j$, so the procedure stops and the set $Y_{i,j}$ is returned.

Definition 13. Let $\mathcal{P}$ be a closed path polyomino containing both W-pentominoes and RW-heptominoes. Consider a W-pentomino $\mathcal{W}$ of $\mathcal{P}$ and suppose that $\mathcal{W}$ contains the cells A_1, A_2, A_3, A_4 and A_5, labelled bottom up as in Figure 18. We put $L = Y_{2,n+1}$. In Figure 19 we make in evidence, for instance, the points belonging to L.

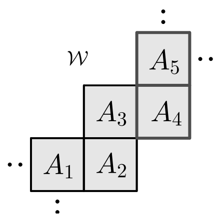


Figure 18

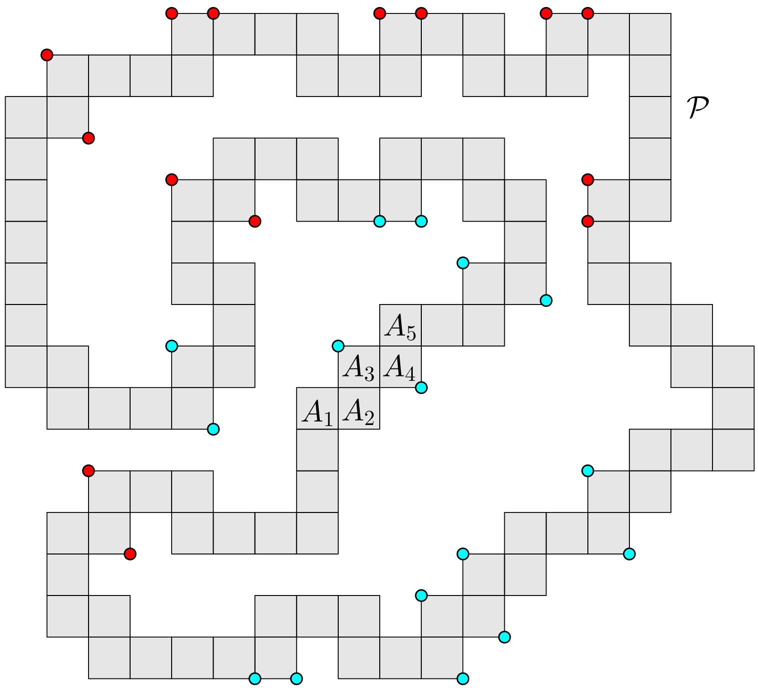


Figure 19: The set $Y_{2,n+1} \subset V(\mathcal{P})$ consists of the highlighted points.

Theorem 14. *Let $\mathcal{P}$ be a closed path. Suppose that $\mathcal{P}$ contains a W-pentomino and an RW-heptomino and let L be the set given in Definition 13. Then $\mathcal{G}$ is the reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^L_{\text{lex}}$.*

Proof. Let f and g be the two binomials attached respectively to the inner intervals $[p,q]$ and $[u,v]$ of $\mathcal{P}$. It suffices to show that $S(f,g)$ reduces to 0 modulo $\mathcal{G}$ in every case. Observe that the desired claim follows from Definitions 4.1 and 13, arguing as in Theorem 11. In fact, we always have that either $\gcd(\mathrm{in}(f),\mathrm{in}(g))=1$ or, if $\gcd(\mathrm{in}(f),\mathrm{in}(g))\neq 1$, it is sufficient to apply the lemmas of Section 3. □

Theorem 15. *Let $\mathcal{P}$ be a closed path polyomino having an L-configuration or a ladder of at least three steps, or equivalently having no zig-zag walks. Then $K[\mathcal{P}]$ is a normal Cohen-Macaulay domain.*

Proof. From Theorem 14 we obtain that there exists a monomial order $\prec$ such that $\mathcal{G}$ is the Gröbner basis of $I_{\mathcal{P}}$ with respect to $\prec$, in particular $I_{\mathcal{P}}$ admits a squarefree initial ideal with respect to some monomial order. Since $\mathcal{P}$ has an L-configuration or a ladder of three steps, from [4] we have that $I_{\mathcal{P}}$ is a toric ideal. By [16, Corollary 4.26] we obtain that $K[\mathcal{P}]$ is normal and by [2, Theorem 6.3.5] we obtain that $K[\mathcal{P}]$ is Cohen-Macaulay. □

Remark 16. In [18] the authors proved that if $\mathcal{P}$ is a balanced polyomino, equivalently $\mathcal{P}$ is simple, then the universal Gröbner basis is squarefree. In general this fact does not hold for a non-simple polyomino. Consider the closed path $\mathcal{P}$ in Figure 20. Let $\{V_1,V_2,V_3,V_4\}$

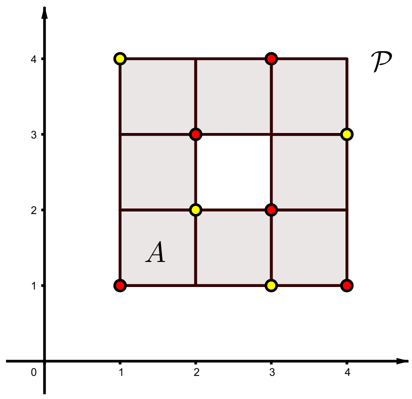


Figure 20

and $\{H_1,H_2,H_3,H_4\}$ be respectively the sets of the maximal vertical and horizontal edge intervals of $\mathcal{P}$ such that $r=(i,j)\in V_i\cap H_j$, and let $\{v_1,v_2,v_3,v_4\}$ and $\{h_1,h_2,h_3,h_4\}$ be the associated sets of the variables. Let w be another variable different from v_i and h_j. We recall from [4] that $I_{\mathcal{P}}=J_{\mathcal{P}}$, where $J_{\mathcal{P}}$ is the kernel of ϕ, defined as

$$\phi : K[x_{ij} : (i,j)\in V(\mathcal{P})] \longrightarrow K[\{v_i,h_j,w\} : i,j\in\{1,2,3,4\}]$$
$$\phi(x_{ij}) = v_i h_j w^k$$

where $k=0$ if $(i,j)\notin A$, and $k=1$, if $(i,j)\in A$.

Consider the binomial $f=x_{11}x_{23}x_{32}x_{34}x_{41}-x_{14}x_{22}x_{31}^2x_{43}$ attached to the vertices in red and yellow. Observe that $f\in I_{\mathcal{P}}$ because $\phi(x_{11}x_{23}x_{32}x_{34}x_{41})=\phi(x_{14}x_{22}x_{31}^2x_{43})$. We show that f is primitive, that is there does not exist any binomial $g=g^+-g^-$ in $I_{\mathcal{P}}$ with

$g \neq f$ such that $g^+|x_{11}x_{23}x_{32}x_{34}x_{41}$ and $g^-|x_{14}x_{22}x_{31}^2x_{43}$. Suppose by contradiction that there exists such a binomial. Observe that $2 < \deg(g) < 5$, since $f \neq g$ and all binomials of degree two satisfying the primitive conditions are not inner 2-minors. It is sufficient to prove that x_{11} (resp. x_{22}) cannot divide g^+ (resp. g^-). If that happens, then w divides $\phi(g^+)$, which is equal to $\phi(g^-)$, so x_{22} divides g^-. Since $g \in I_{\mathcal{P}} = J_{\mathcal{P}}$, in particular $\phi(g^+) = \phi(g^-)$, we obtain that $g^+ = x_{11}x_{23}x_{32}x_{34}x_{41}$ and $g^- = x_{14}x_{22}x_{31}^2x_{43}$ from easy calculations. Hence $f = g$, a contradiction. In conclusion we have that f is a primitive binomial of $I_{\mathcal{P}}$. Since for a toric ideal the universal Gröbner basis coincides with the Graver basis (see [29]), the primitive binomials of $I_{\mathcal{P}}$ form the universal Gröbner basis $\mathcal{G}$ of $I_{\mathcal{P}}$. Since f is a primitive binomial of $I_{\mathcal{P}}$, it follows that $\mathcal{G}$ is not squarefree. Anyway $I_{\mathcal{P}}$ is a radical ideal which admits a squarefree initial ideal with a different monomial ordering, for instance with respect to $<^1_{\text{lex}}$, since the set of generator of $I_{\mathcal{P}}$ is the reduced Gröbner basis by [24, Theorem 4.1].

We conclude providing some questions which follow immediately from the results of this paper.

- We ask if the initial ideal of $I_{\mathcal{P}}$, attached to a (weakly) closed path, with respect to the monomial orders $<^1_{\text{lex}}$ and $<^2_{\text{lex}}$ defined in [24] is squarefree.
- With reference to [5], we ask if also for all weakly closed path polyominoes there exist some monomial orders such that the set of the generators of the ideal is the reduced Gröbner basis.

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References

[1] C. Andrei. Algebraic Properties of the Coordinate Ring of a Convex Polyomino. *Electron. J. Comb.*, 28(1):#P1.45, 2021.

[2] W. Bruns and J. Herzog. Cohen-Macaulay Rings. Cambridge University Press, London, Cambridge, New York, 1993.

[3] W. Bruns and U. Vetter. Determinantal rings. Lecture Note in Mathematics 1327, 1988.

[4] C. Cisto and F. Navarra. Primality of closed path polyominoes. *J. Algebra Appl.*, in press, 2021.

[5] C. Cisto, F. Navarra and R. Utano. Primality of weakly closed path polyominoes. *Illinois J. Math.*, in press, 2022.

[6] A. Conca. Ladder Determinantal Rings. *J. Pure Appl. Algebra*, 98:119–134, 1995.

[7] A. Conca. Gorenstein ladder determinantal rings. *J. London Math. Soc.*, 54(3):453–474, 1996.

[8] A. Conca and J. Herzog. Ladder Determinantal Rings Have Rational Singularities. *Adv. Math.*, 132:120–147, 1997.

[9] V. Ene, J. Herzog and T. Hibi. Linearly related polyominoes. *J. Algebr. Comb.*, 41:949–968, 2015.

[10] V. Ene, J. Herzog, A. A. Qureshi and F. Romeo. Regularity and Gorenstein property of the L-convex Polyominoes. *Electron. J. Comb.*, 28(1):#P1.50, 2021.

[11] V. Ene and A. A. Qureshi. Ideals generated by diagonal 2-minors. *Commun. Algebra*, 41(8):3058–3066, 2013.

[12] S. W. Golomb. Polyominoes, puzzles, patterns, problems, and packagings. Second edition. Princeton University press, 1994.

[13] E. Gorla. Mixed ladder determinantal varieties from two-sided ladders. *J. Pure and Appl. Alg.*, 211(2):433–444, 2007.

[14] J. Herzog and T. Hibi. Ideals generated by adjacent 2-minors. *J. Commut. Algebra*, 4(4):525–549, 2012.

[15] J. Herzog, T. Hibi, F. Hreinsdoottir, T. Kahle and J. Rauh. Binomial edge ideals and conditional independence statements. *Adv. Appl. Math.*, 45(3):317–333, 2010.

[16] J. Herzog, T. Hibi and H. Ohsugi. Binomial Ideals. Graduate Texts in Mathematics, v. 279, Springer, 2018.

[17] J. Herzog and S. S. Madani. The coordinate ring of a simple polyomino. *Illinois J. Math.*, 58(4):981–995, 2014.

[18] J. Herzog, A. A. Qureshi and A. Shikama. Gröbner basis of balanced polyominoes. *Math. Nachr.*, 288(7):775–783, 2015.

[19] T. Hibi and A. A. Qureshi. Nonsimple polyominoes and prime ideals. *Illinois J. Math.*, 59(2):391–398, 2015.

[20] S. Hosten and S. Sullivant. Ideals of adjacent minors. *J. Algebra*, 277(2):615–642, 2004.

[21] C. Mascia, G. Rinaldo and F. Romeo. Primality of multiply connected polyominoes. *Illinois J. Math.*, 64(3):291–304, 2020.

[22] C. Mascia, G. Rinaldo and F. Romeo. Primality of polyomino ideals by quadratic Gröbner basis. *Math. Nachr.*, 295(3):593–606, 2022.

[23] H. Ohsugi and T. Hibi. Special simplices and Gorenstein toric rings. *J. Comb. Theory Ser. A.*, 113(4):718–725, 2006.

[24] A. A. Qureshi. Ideals generated by 2-minors, collections of cells and stack polyominoes. *J. Algebra*, 357:279–303, 2012.

[25] A. A. Qureshi, T. Shibuta and A. Shikama. Simple polyominoes are prime. *J. Commut. Algebra*, 9(3):413–422, 2017.

[26] G. Rinaldo and F. Romeo. Hilbert Series of simple thin polyominoes. *J. Algebr. Comb.*, 54:607–624, 2021.

[27] A. Shikama. Toric representation of algebras defined by certain nonsimple polyominoes. *J. Commut. Algebra*, 10(2):265–274, 2018.

[28] A. Shikama. Toric rings of nonsimple polyominoes. *Int. J. Algebra*, 9:195-201, 2015.

[29] B. Sturmfels and R. R. Thomas. Variation of cost functions in integer programming. *Math. Program.*, 77:357–387, 1997.