Maximum Size of a Graph with Given Fractional Matching Number*

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Abstract

For three integers n, k, d, we determine the maximum size of a graph on n vertices with fractional matching number k and maximum degree at most d. As a consequence, we obtain the maximum size of a graph with given number of vertices and fractional matching number. This partially confirms a conjecture proposed by Alon et al. on the maximum size of r-uniform hypergraph with a fractional matching number for the special case when $r = 2$.

Mathematics Subject Classifications: 05C35, 05C70, 05C72

1 Introduction

For an integer $r \geqslant 2$, an *r-uniform hypergraph* or more simply, an *r-graph*, is a pair $H = (V(H), E(H))$ with vertex set $V(H)$ and edge set $E(H) \subseteq \binom{V(H)}{r}$. We call the number of the edges in H the *size* of H. A *matching* of H is a set of edges, no two of which are intersecting. The *matching number* of H, denoted by $\nu(H)$, is the size of a maximum matching in H. A matching M in H is *perfect* if every vertex of H is incident with an edge of M. An r-graph is called a *graph* if $r = 2$, denoted by G. We denote the maximum degree of the vertices of G by $\Delta(G)$. For a subset S of $V(G)$, we use $G[S]$ to denote the subgraph of G induced by S. For the terminologies and concepts not defined here, we refer the readers to [3, 5, 15].

Let $g_r(n, k)$ denote the maximum size of a r-graph H on n vertices with matching number k. In particular, we replace $g_2(n, k)$ by $g(n, k)$. In 1959, Erdős and Gallai [8]

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determined $g(n,k)$, that is, the maximum size of a graph G on n vertices with matching number k.

Theorem 1. *[8] For $n \geqslant 2k+1$,*

$$g(n,k) = \max\left\{\binom{2k+1}{2}, \frac{k(2n-k-1)}{2}\right\}.$$

Later in [7], Erdős further conjectured that, for $n \geqslant r(k+1)-1$,

$$g_r(n,k) = \max\left\{\binom{r(k+1)-1}{r}, \binom{n}{r} - \binom{n-k}{r}\right\}.$$

This is an important conjecture in hypergraph theory and abundant literatures devote to it. For the recent progresses on Erdős matching conjecture, we refer to [10, 11, 12] for details.

In addition to matching number, some other restrictions were also considered in the literature. Let $g(n,k,d)$ denote the maximum size of the graphs on n vertices with matching number k and the maximum degree at most d. In [6], Chvátal and Hanson determined $g(n,k,d)$, and hence, generalized Theorem 1.

Theorem 2. [6] *Let n,k,d be three positive integers with $n \geqslant 2k+1$. Set $d_0 = \lfloor \frac{d+1}{2} \rfloor$.*
(i) If $d \leqslant 2k$ and $n \leqslant 2k + \lfloor \frac{k}{d_0} \rfloor$, then

$$g(n,k,d) = \begin{cases} \min\left\{\lfloor \frac{nd}{2} \rfloor,\ dk + \frac{d-1}{2}\left\lfloor \frac{2(n-k)}{d+3} \right\rfloor\right\}, & \text{if } d \text{ is odd;} \\ \frac{nd}{2}, & \text{if } d \text{ is even.} \end{cases}$$

(ii) If $d \leqslant 2k$ and $n \geqslant 2k + \lfloor \frac{k}{d_0} \rfloor$, then

$$g(n,k,d) = dk + \left\lfloor \frac{d}{2} \right\rfloor \left\lfloor \frac{k}{d_0} \right\rfloor.$$

(iii) If $d \geqslant 2k+1$, then

$$g(n,k,d) = \begin{cases} \max\left\{\binom{2k+1}{2},\ \left\lfloor \frac{k(n+d-k)}{2} \right\rfloor\right\}, & \text{if } n \leqslant k+d; \\ dk, & \text{if } n \geqslant k+d. \end{cases}$$

We refer to [2, 4, 13] for more details on this topic.

Let us focus on the fractional version of Theorem 2 in the following.

A *fractional matching* of an r-graph H is a function f assigning each edge with a real number in $[0,1]$ so that $\sum_{e\in\Gamma(v)} f(e) \leqslant 1$ for each $v \in V(H)$, where $\Gamma(v)$ is the set of edges incident with v in H. The *fractional matching number* of H, denoted by $\nu_f(H)$, is the maximum value of $\sum_{e\in E(H)} f(e)$ over all fractional matchings f. A fractional matching of H is *perfect* if $\sum_{e\in\Gamma(H)} f(e) = 1$ for each $v \in V(H)$. Clearly, H has a fractional perfect matching if and only if $\nu_f(H) = \frac{|V(G)|}{r}$. It was shown that the fractional matching number

$\nu_f(G)$ of a graph G is either an integer or a semi-integer, that is, $2\nu_f(G)$ is integer (see [16], Theorem 2.1.5).

Let $\mathcal{F}(n,k,d)$ denote the class of graphs on n vertices with fractional matching number k and the maximum degree at most d. Further, let $f(n,k,d)$ denote the maximum size of the graphs in $\mathcal{F}(n,k,d)$, i.e.,

$$f(n,k,d) = \max\{|E(G)| : |V(G)| = n,\ \nu_f(G) = k,\ \Delta(G) \leqslant d\}.$$

It is clear that $f(n,k,d) = f(n,k,n-1)$ when $d \geqslant n-1$, and hence we always assume $d \leqslant n-1$.

In this paper, we determine $f(n,k,d)$ as follows.

Theorem 3. *Let $n, 2k, d$ be three positive integers with $n \geqslant 2k$. If $2k$ is even, then*

$$f(n,k,d) = \begin{cases} \max\left\{\binom{2k}{2}, \left\lfloor\frac{k(n+d-k)}{2}\right\rfloor\right\}, & \text{if } d \geqslant 2k-1, n \leqslant d+k; \\ dk, & \text{otherwise.} \end{cases}$$

If $2k$ is odd, then

$$f(n,k,d) = \begin{cases} \max\left\{\binom{2k}{2}, d(k-\frac{3}{2})+3\right\}, & \text{if } d \geqslant 2k-1, n \geqslant d+k-\frac{3}{2}; \\ \max\left\{\binom{2k}{2}, \left\lfloor\frac{(k-\frac{3}{2})(n+d-k+\frac{3}{2})}{2}\right\rfloor+3\right\}, & \text{if } d \geqslant 2k-1, n \leqslant d+k-\frac{3}{2}; \\ \lfloor dk \rfloor, & \text{if } d \leqslant 2k-1. \end{cases}$$

For a graph G, let $f(n,k) = \max\{|E(G)| : |V(G)| = n,\ \nu_f(G) = k\}$. Note that $f(n,k,n-1) = f(n,k)$. As a consequence of Theorem 3, we obtain the following fractional version of Theorem 1.

Theorem 4. *Let $n, 2k$ be two positive integers with $n \geqslant 2k$. If $2k$ is even, then*

$$f(n,k) = \max\left\{\binom{2k}{2},\ \frac{k(2n-k-1)}{2}\right\}.$$

If $2k$ is odd, then

$$f(n,k) = \max\left\{\binom{2k}{2},\ \frac{(k-\frac{3}{2})(2n-k+\frac{1}{2})}{2}+3\right\}.$$

In [1], Alon et al. formulated the fractional version of Erdős matching conjecture for r-uniform hypergraphs as follows. Let $f_r^*(n,s) = \max\{|E(H)| : |V(H)| = n,\ \nu_f(H) < s\}$.

Conjecture 5. [1] *For all integers $r \geqslant 2$ and an integer s with $0 \leqslant s \leqslant \frac{n}{r}$,*

$$f_r^*(n,s) = \max\left\{\binom{rs-1}{r},\ \binom{n}{r} - \binom{n-s+1}{r}\right\}.$$

In the same article, they also showed that Conjecture 5 asymptotically holds for $r \in \{3,4\}$ and $0 \leqslant s \leqslant \frac{n}{r+1}$ when n tends to infinity. For $r=2$, Theorem 1 implies that Conjecture 5 is asymptotically true when n goes to infinity. As far as we know, this conjecture still remains open even if $r=2$ in general. In fact, by Theorem 4, we obtain the following corollary, which confirms that Conjecture 5 is true for $r=2$.

Corollary 6. *For any nonnegative integer s with $n \geqslant 2s$,*

$$f_2^*(n,s) = \max\left\{\binom{2s-1}{2},\ \binom{n}{2} - \binom{n-s+1}{2}\right\}.$$

The rest of this article is organized as follows. In Section 2, we establish the necessary upper bounds on $f(n,k,d)$. In Section 3, we construct the corresponding extremal graphs attaining these upper bounds and, hence, give a proof of Theorem 3.

2 Upper bounds

To obtain the upper bounds of $f(n,k,d)$, we begin this section with the following known result, called fractional Tutte-Berge formula, which characterizes the fractional matching number of a graph.

Theorem 7. [16] *Let G be a graph G with n vertices. Then*

$$\nu_f(G) = \frac{1}{2}\left(n - \max_{S \subseteq V(G)}\{i(G-S) - |S|\}\right),$$

where $i(G-S)$ is the number of isolated vertices in $G-S$.

Before the proof of upper bounds of $f(n,k,d)$, we also need a result with regard to the maximum value of a function, which plays a vital role in our proof. Let $n, 2k, d$ be three positive integers with $n \geqslant 2k$ and $d \leqslant n-1$. We now define a function

$$F(x) = \min\left\{dx,\ \frac{x(n+d-x)}{2}\right\} + \min\left\{d(k-x),\ \binom{2k-2x}{2}\right\}$$

on nonnegative real number x, and its maximum value in given intervals can be obtained.

Lemma 8. *If $2k$ is even and $0 \leqslant x \leqslant k$, then*

$$F(x) \leqslant \begin{cases} \max\left\{\binom{2k}{2}, \frac{k(n+d-k)}{2}\right\}, & \text{if } d \geqslant 2k-1, n \leqslant d+k; \\ dk, & \text{otherwise.} \end{cases}$$

If $2k$ is odd and $0 \leqslant x \leqslant k - \frac{3}{2}$, then

$$F(x) \leqslant \begin{cases} \max\left\{\binom{2k}{2}, d(k-\frac{3}{2})+3\right\}, & \text{if } d \geqslant 2k-1, n \geqslant d+k-\frac{3}{2}; \\ \max\left\{\binom{2k}{2}, \frac{(k-\frac{3}{2})(n+d-k+\frac{3}{2})}{2}+3\right\}, & \text{if } d \geqslant 2k-1, n \leqslant d+k-\frac{3}{2}; \\ dk, & \text{if } d \leqslant 2k-1. \end{cases}$$

Lemma 8 can be proved trivially by the convexity of the function, and for the coherence and completeness we will prove it in Appendix.

Now we present the upper bounds of $f(n,k,d)$.

Lemma 9. *Let $n, 2k, d$ be three positive integers with $n \geqslant 2k$. If $2k$ is even, then*

$$f(n,k,d) \leqslant \begin{cases} \max\left\{\binom{2k}{2},\ \left\lfloor\frac{k(n+d-k)}{2}\right\rfloor\right\}, & \text{if } d \geqslant 2k-1, n \leqslant d+k; \\ dk, & \text{otherwise.} \end{cases}$$

If $2k$ is odd, then

$$f(n,k,d) \leqslant \begin{cases} \max\left\{\binom{2k}{2}, d(k-\frac{3}{2})+3\right\}, & \text{if } d \geqslant 2k-1, n \geqslant d+k-\frac{3}{2}; \\ \max\left\{\binom{2k}{2}, \left\lfloor\frac{(k-\frac{3}{2})(n+d-k+\frac{3}{2})}{2}\right\rfloor+3\right\}, & \text{if } d \geqslant 2k-1, n \leqslant d+k-\frac{3}{2}; \\ \lfloor dk \rfloor, & \text{if } d \leqslant 2k-1. \end{cases}$$

Proof. Let G be a graph with n vertices satisfying $\nu_f(G) = k$ and $\Delta(G) \leqslant d$. By Theorem 7, there exists a subset S of $V(G)$ such that $i(G-S)-|S| = n-2k$. Let $C \subseteq V(G)$ be the set of vertices that are neither in S nor isolated in $G-S$. Then $|C| = n-|S|-i(G-S) \neq 1$. We set $s = |S|$ and $c = |C|$ for convenience. Then we have $0 \leqslant c = 2k-2s \neq 1$, that is, the range of s is $D = \{s \in N : s \leqslant k, s \neq k - \frac{1}{2}\}$.

Let α be the number of edges in $G[S]$, let β be the number of edges in G having exactly one vertex in S, and let γ be the number of edges in $G[C]$. Since $\Delta(G) \leqslant d$, we have $2\alpha + \beta \leqslant ds$. On the one hand, $\alpha + \beta \leqslant 2\alpha + \beta \leqslant ds$. On the other hand, since $\beta \leqslant s(n-s)$, we obtain $\alpha + \beta \leqslant \frac{s(n+d-s)}{2}$. Consequently,

$$\alpha + \beta \leqslant \min\left\{ds,\ \frac{s(n+d-s)}{2}\right\}.$$

Moreover, for the subgraph $G[C]$ of G, if $d \leqslant c-1$ then $\gamma \leqslant \frac{dc}{2}$; otherwise $\gamma \leqslant \binom{c}{2}$. Clearly, we have

$$\gamma \leqslant \min\left\{\frac{dc}{2},\ \binom{c}{2}\right\} = \min\left\{d(k-s),\ \binom{2k-2s}{2}\right\}.$$

As a result,

$$\begin{aligned} |E(G)| &= \alpha + \beta + \gamma \\ &\leqslant \min\left\{ds,\ \frac{s(n+d-s)}{2}\right\} + \min\left\{d(k-s),\ \binom{2k-2s}{2}\right\} \\ &= F(s). \end{aligned}$$

Recall that $D = \{s \in N : s \leqslant k, s \neq k - \frac{1}{2}\}$. Therefore,

$$f(n,k,d) \leqslant \max_{s \in D} F(s).$$

Since $f(n,k,d)$ is an integer, it suffices to obtain the maximum integer which is no more than the maximum value of $F(s)$ for any $s \in D$. Moreover, if $2k$ is odd, then the fact $s \in D$ implies $s \leqslant k - \frac{3}{2}$. By Lemma 8, we complete the proof. □

3 Constructions of extremal graphs

In this section, we construct several extremal graphs satisfying conditions to attain $f(n,k,d)$. We begin with some useful results.

For a k-regular graph G, if we assign to each edge a number $\frac{1}{k}$, then we obtain a fractional perfect matching of G. Therefore we have the following proposition.

Proposition 10. *Let G be a k-regular graph. Then G has a fractional perfect matching.*

We call that a sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers is *graphic* if it is a degree sequence of a simple graph G, and the graph G is said to *realize* the sequence $d_1, d_2, \ldots, d_n$. The following characterization of graphic sequence is due to Erdős and Gallai.

Theorem 11. [9] *A sequence $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$ is graphic if and only if $\sum_{i=1}^n d_i$ is even and*

$$\sum_{i=1}^{t} d_i \leqslant t(t-1) + \sum_{i=t+1}^{n} \min\{t, d_i\}$$

for $t = 1, 2, \ldots, n$.

A spanning subgraph of a graph G is said to k-factor if the degree of its each vertex is equal to k. The following result gives a sufficient condition with regard to degree sequences such that the graph contains a k-factor.

Theorem 12. [14] *If the sequences $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$ and $d_1 - k \geqslant d_2 - k \geqslant \cdots \geqslant d_n - k$ are graphic, then the sequence $d_1, d_2, \ldots, d_n$ can be realized by a graph G which contains a k-factor.*

Let d and n be two odd numbers with $3 \leqslant d \leqslant n-2$. Theorem 11 ensures that two sequences $d, d, \ldots, d, d-1$ and $d-2, d-2, \ldots, d-2, d-3$ of length n are both graphic. By Theorem 12, the sequence $d, d, \ldots, d-1$ can be realized by a graph G which contains a 2-factor. We know that if a graph G contains a 2-factor, then it has a fractional perfect matching. Therefore we have the following result, which is useful in our forthcoming argument.

Corollary 13. *Let d and n be two odd numbers with $3 \leqslant d \leqslant n-2$. Then the sequence $d, d, \ldots, d, d-1$ of length n can be realized by a graph which has a fractional perfect matching.*

Now we construct extremal graphs to attain $f(n,k,d)$. Several notations are used in our constructions. We use K_n and $\overline{K}_n$ to denote a complete graph and an empty graph with n vertices, respectively. For two graphs G_1 and G_2, we denote the disjoint union of G_1 and G_2 by $G_1 \cup G_2$. Let us first consider the case that $2k$ is even.

Lemma 14. *Let $n, 2k, d$ be three positive integers with $n \geqslant 2k$. If $2k$ is even, then*

$$f(n,k,d) \geqslant \begin{cases} \max\left\{\binom{2k}{2}, \left\lfloor \frac{k(n+d-k)}{2} \right\rfloor\right\}, & \text{if } d \geqslant 2k-1, n \leqslant d+k; \\ dk, & \text{otherwise.} \end{cases}$$

Proof. Let us divide into two cases to complete the proof.

Case 1. $d \geqslant 2k-1$ and $n \leqslant d+k$.

If $d \leqslant 5k-n-2$, then

$$\max\left\{\binom{2k}{2}, \left\lfloor \frac{k(n-k+d)}{2} \right\rfloor\right\} = \binom{2k}{2}.$$

We can check that $K_{2k} \cup \overline{K}_{n-2k} \in \mathcal{F}(n,k,d)$ and its size is $\binom{2k}{2}$. Thus $f(n,k,d) \geqslant \binom{2k}{2}$.

If $d \geqslant 5k-n-2$, then

$$\max\left\{\binom{2k}{2}, \left\lfloor \frac{k(n-k+d)}{2} \right\rfloor\right\} = \left\lfloor \frac{k(n-k+d)}{2} \right\rfloor.$$

Consider the parity of $k(n-k+d)$. When $k(n-k+d)$ is even, we take a $(k-n+d)$-regular graph G_0 with k vertices. When $k(n-k+d)$ is odd, we take a graph G_0 with $k-1$ vertices of degree $k-n+d$ and a vertex of degree $k-n+d-1$. Add $n-k$ independent vertices to G_0 such that they are adjacent to each vertex of G_0, which contains a matching covering each vertex of G_0 between G_0 and these independent vertices, and denote the resulting graph by G. Clearly, $\nu_f(G) = k$ and $\Delta(G) = d$, and then $G \in \mathcal{F}(n,k,d)$. By the construction of G, we have

$$|E(G)| = \left\lfloor \frac{k(k-n+d)}{2} \right\rfloor + k(n-k) = \left\lfloor \frac{k(n-k+d)}{2} \right\rfloor.$$

Thus $f(n,k,d) \geqslant \left\lfloor \frac{k(n-k+d)}{2} \right\rfloor$.

Case 2. $d \leqslant 2k-1$ or $n \geqslant d+k$.

If $d \leqslant 2k-1$ and $n \leqslant d+k$, we take a d-regular graph G_0 with $2k$ vertices. Proposition 10 ensures that G_0 has a fractional perfect matching, and then $\nu_f(G_0) = k$. Clearly, $G = G_0 \cup \overline{K}_{n-2k} \in \mathcal{F}(n,k,d)$ and $|E(G)| = dk$. Thus $f(n,k,d) \geqslant dk$.

If $n \geqslant d+k$, let $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_h\}$ be two sets of vertices, where $h = n-k \geqslant k$. For $i = 1, 2, \ldots, k$, joining the vertex x_i with every vertex y_j by an edge, where $j = i+t \pmod h$ for $t = 0, 1, \ldots, d-1$, we call the resulting graph G. Clearly, $\nu_f(G) = k$, $\Delta(G) = d$ and $|E(G)| = dk$. Thus $f(n,k,d) \geqslant dk$. □

Next we give the lower bounds of $f(n,k,d)$ when $2k$ is odd.

Lemma 15. *Let $n, 2k, d$ be three positive integers with $n \geqslant 2k$. If $2k$ is odd, then*

$$f(n,k,d) \geqslant \begin{cases} \max\left\{\binom{2k}{2}, d(k-\frac{3}{2})+3\right\}, & \text{if } d \geqslant 2k-1, n \geqslant d+k-\frac{3}{2}; \\ \max\left\{\binom{2k}{2}, \left\lfloor \frac{(k-\frac{3}{2})(n+d-k+\frac{3}{2})}{2} \right\rfloor + 3\right\}, & \text{if } d \geqslant 2k-1, n \leqslant d+k-\frac{3}{2}; \\ \lfloor dk \rfloor, & \text{if } d \leqslant 2k-1. \end{cases}$$

Proof. The fact that $2k$ is odd implies $d \geqslant 2$; otherwise $d = 1$, which implies fractional matching number k must be an integer, a contradiction. If $k = \frac{3}{2}$, then we have $G =$

$K_3 \cup \overline{K}_{n-3}$, and hence $f(n, \frac{3}{2}, d) = 3$. So we may assume $k \geqslant \frac{5}{2}$ in the following discussion. Now let us discuss three cases.

Case 1. $d \geqslant 2k - 1$ and $n \geqslant d + k - \frac{3}{2}$.

If $d \leqslant \frac{k(2k-1)-3}{k-\frac{3}{2}}$, then

$$\max\left\{\binom{2k}{2}, d(k - \frac{3}{2}) + 3\right\} = \binom{2k}{2}.$$

Clearly, $K_{2k} \cup \overline{K}_{n-2k} \in \mathcal{F}(n, k, d)$. Thus $f(n, k, d) \geqslant \binom{2k}{2}$.

If $d \geqslant \frac{k(2k-1)-3}{k-\frac{3}{2}}$, then

$$\max\left\{\binom{2k}{2}, d(k - \frac{3}{2}) + 3\right\} = d(k - \frac{3}{2}) + 3.$$

Let $X = \{x_1, x_2, \ldots, x_l\}$ and $Y = \{y_1, y_2, \ldots, y_h\}$ be two sets of vertices, where $l = k - \frac{3}{2}$ and $h = n - k + \frac{3}{2}$. Note that $h - l \geqslant 3$ as $n \geqslant 2k$. For $i = 1, 2, \ldots, l$, joining the vertex x_i with every vertex y_j by an edge, where $j = i + t \pmod h$ for $t = 0, 1, \ldots, d-1$, we call the resulting graph G_0. Clearly, $M = \{x_i y_i \in E(G_0) : i = 1, 2, \ldots, l\}$ forms a matching covering X, and there exist at least three independent vertices y_{l+1}, y_{l+2} and y_{l+3} in Y which are not covered by M as $h - l \geqslant 3$. Add three edges in $\{y_{l+1}, y_{l+2}, y_{l+3}\}$ such that they form a K_3 to G_0, and call the resulting graph G. Clearly, $\nu_f(G) = l + \frac{3}{2} = k$ and $\Delta(G) = d$, and hence $G \in \mathcal{F}(n, k, d)$. Thus $f(n, k, d) \geqslant d(k - \frac{3}{2}) + 3$.

Case 2. $d \geqslant 2k - 1$ and $n \leqslant d + k - \frac{3}{2}$.

If $n \geqslant \frac{4k^2-2k-6}{k-\frac{3}{2}} + k - d - \frac{3}{2}$, then

$$\max\left\{\binom{2k}{2}, \left\lfloor \frac{(k-\frac{3}{2})(n-k+d+\frac{3}{2})}{2} \right\rfloor + 3\right\} = \binom{2k}{2}.$$

Clearly, $K_{2k} \cup \overline{K}_{n-2k} \in \mathcal{F}(n, k, d)$. Thus $f(n, k, d) \geqslant \binom{2k}{2}$.

If $n \leqslant \frac{4k^2-2k-6}{k-\frac{3}{2}} + k - d - \frac{3}{2}$, then

$$\max\left\{\binom{2k}{2}, \left\lfloor \frac{(k-\frac{3}{2})(n-k+d+\frac{3}{2})}{2} \right\rfloor + 3\right\} = \left\lfloor \frac{(k-\frac{3}{2})(n-k+d+\frac{3}{2})}{2} \right\rfloor + 3.$$

Consider the parity of $(k-\frac{3}{2})(n-k+d+\frac{3}{2})$. When $(k-\frac{3}{2})(n-k+d+\frac{3}{2})$ is even, we take a graph G_0 with $k - \frac{3}{2}$ vertices of degree $k - n + d - \frac{3}{2}$. When $(k-\frac{3}{2})(n-k+d+\frac{3}{2})$ is odd, we take a graph G_0 with $k - \frac{5}{2}$ vertices of degree $k - n + d - \frac{3}{2}$ and a vertex of degree $k - n + d - \frac{5}{2}$. Add $n - k - \frac{3}{2}$ independent vertices to G_0 such that they are adjacent to each vertex of G_0, which contains a matching covering each vertex of G_0 between G_0 and these independent vertices, and denote the resulting graph by G_1. Therefore $\nu_f(G_1) = k - \frac{3}{2}$. Furthermore, add a K_3 formed by remainder three vertices to G_1 such that each vertex of K_3 are adjacent to each vertex of G_0, and denote the resulting graph by G. Then we have $\nu_f(G) = \nu_f(G_1) + \frac{3}{2} = k$ and $\Delta(G) = d$. Then $G \in \mathcal{F}(n,k,d)$. By the construction of G, we have

$$
\begin{aligned}
|E(G)| &= \left\lfloor \frac{(k-\frac{3}{2})(k-n+d-\frac{3}{2})}{2} \right\rfloor + (k-\frac{3}{2})(n-k+\frac{3}{2}) + 3 \\
&= \left\lfloor \frac{(k-\frac{3}{2})(n-k+d+\frac{3}{2})}{2} \right\rfloor + 3.
\end{aligned}
$$

Thus $f(n,k,d) \geqslant \left\lfloor \frac{(k-\frac{3}{2})(n-k+d+\frac{3}{2})}{2} \right\rfloor + 3$.

Case 3. $d \leqslant 2k-1$.

When d is even, we can take a d-regular graph G_0 with $2k$ vertices. By Proposition 10, G_0 has a fractional perfect matching. When d is odd, Corollary 13 ensures that we can find a graph G_0 with $2k-1$ vertices of degree d and one vertex of degree $d-1$ such that G_0 contains a fractional perfect matching. Therefore, $\nu_f(G_0) = k$. Clearly, $G = G_0 \cup \overline{K}_{n-2k} \in \mathcal{F}(n,k,d)$ and $|E(G)| = \lfloor dk \rfloor$. Thus $f(n,k,d) \geqslant \lfloor dk \rfloor$. □

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Appendix. Proof of Lemma 8

Let us recall that $n \geqslant 2k$, $d \leqslant n-1$ and the function

$$F(x) = \min\left\{dx,\ \frac{x(n+d-x)}{2}\right\} + \min\left\{d(k-x),\ \binom{2k-2x}{2}\right\},$$

where $x \geqslant 0$.

It is easy to see that $dx = \frac{x(n+d-x)}{2}$ iff $x = n-d$, and $d(k-x) = \binom{2k-2x}{2}$ iff $x = k - \frac{d+1}{2}$. If $k - \frac{d+1}{2} \geqslant 0$, then $k - \frac{d+1}{2} < k - \frac{d}{2} \leqslant 2k-d \leqslant n-d$; otherwise $k - \frac{d+1}{2} \leqslant 0 < n-d$. So $k - \frac{d+1}{2} < n-d$ always holds. We can rewrite the function

$$F(x) = \begin{cases} dk, & \text{if } x \leqslant k - \frac{d+1}{2}; \\ dx + \binom{2k-2x}{2}, & \text{if } k - \frac{d+1}{2} \leqslant x \leqslant n-d; \\ \frac{x(n+d-x)}{2} + \binom{2k-2x}{2}, & \text{if } x \geqslant n-d. \end{cases}$$

Let $F_1(x) = dk$, $F_2(x) = dx + \binom{2k-2x}{2}$ and $F_3(x) = \frac{x(n+d-x)}{2} + \binom{2k-2x}{2}$. Note that $F_i(x)$ is convex for $i = 1,2,3$. To obtain the maximum value of $F(x)$, it suffices to discuss maximum values of $F_i(x)$'s in corresponding intervals.

Case 1. $2k$ is even and $0 \leqslant x \leqslant k$.

(i) If $d \geqslant 2k-1$ and $n \geqslant d+k$, then $k - \frac{d+1}{2} \leqslant 0 < k \leqslant n-d$. Clearly, $F(x) = F_2(x)$ for any x with $0 \leqslant x \leqslant k$. By the convexity of $F_2(x)$, we have

$$\begin{aligned} F(x) &\leqslant \max\{F_2(0), F_2(k)\} \\ &= \max\left\{\binom{2k}{2},\ dk\right\} \\ &= dk. \end{aligned}$$

(ii) If $d \geqslant 2k-1$ and $n \leqslant d+k$, then $k - \frac{d+1}{2} \leqslant 0 < n-d \leqslant k$. Clearly, $F(x) = F_2(x)$ when $0 \leqslant x \leqslant n-d$, and $F(x) = F_3(x)$ when $n-d \leqslant x \leqslant k$. By the convexity of $F_3(x)$, we have $F_3(n-d) \leqslant \max\{F_3(0), F_3(k)\}$. Since $F_2(0) = F_3(0)$, it follows that

$$\begin{aligned} F(x) &\leqslant \max\{F_2(0), F_3(n-d), F_3(k)\} \\ &= \max\left\{\binom{2k}{2},\ \frac{k(n+d-k)}{2}\right\}. \end{aligned}$$

(iii) If $d \leqslant 2k-1$ and $n \geqslant d+k$, then $0 \leqslant k - \frac{d+1}{2} < k \leqslant n-d$. Clearly, $F(x) = F_1(x)$ when $0 \leqslant x \leqslant k - \frac{d+1}{2}$, and $F(x) = F_2(x)$ when $k - \frac{d+1}{2} \leqslant x \leqslant k$. Then we obtain

$$\begin{aligned} F(s) &\leqslant \max\left\{F_1(0), F_1(k - \frac{d+1}{2}), F_2(k)\right\} \\ &= dk. \end{aligned}$$

(iv) If $d \leqslant 2k-1$ and $n \leqslant d+k$, then $0 \leqslant k - \frac{d+1}{2} < n-d \leqslant k$. Clearly, $F(x) = F_1(x)$ when $0 \leqslant x \leqslant k - \frac{d+1}{2}$, and $F(x) = F_2(x)$ when $k - \frac{d+1}{2} \leqslant x \leqslant n-d$, and $F(x)=F_3(x)$ when $n-d\leqslant x\leqslant k$. By the convexity of $F_2(x)$, we have $F_2(n-d)\leqslant \max\{F_2(k-\frac{d+1}{2}),F_2(k)\}=dk$. This implies that

$$\begin{aligned}F(x)&\leqslant \max\left\{F_1(0),F_2(k-\frac{d+1}{2}),F_2(n-d),F_3(k)\right\}\\&=\max\left\{dk,\ \frac{k(n+d-k)}{2}\right\}\\&=dk.\end{aligned}$$

Consequently, if $2k$ is even and $0\leqslant x\leqslant k$, then

$$F(x)\leqslant\begin{cases}\max\left\{\binom{2k}{2},\frac{k(n+d-k)}{2}\right\}, & \text{if } d\geqslant 2k-1, n\leqslant d+k;\\ dk, & \text{otherwise}.\end{cases}$$

Case 2. $2k$ is odd and $0\leqslant x\leqslant k-\frac{3}{2}$.

Note that $d\geqslant 2$. Otherwise $d=1$ implies that k is an integer, which contradicts that $2k$ is odd.

(i) If $d\geqslant 2k-1$ and $n\geqslant d+k-\frac{3}{2}$, then $k-\frac{d+1}{2}\leqslant 0\leqslant k-\frac{3}{2}\leqslant n-d$. Clearly, $F(x)=F_2(x)$ for any x with $0\leqslant x\leqslant k-\frac{3}{2}$. By the convexity of $F_2(x)$, we have

$$\begin{aligned}F(x)&\leqslant\max\left\{F_2(0),F_2(k-\frac{3}{2})\right\}\\&=\max\left\{\binom{2k}{2},\ d(k-\frac{3}{2})+3\right\}.\end{aligned}$$

(ii) If $d\geqslant 2k-1$ and $n\leqslant d+k-\frac{3}{2}$, then $k-\frac{d+1}{2}\leqslant 0<n-d\leqslant k-\frac{3}{2}$. Clearly, $F(x)=F_2(x)$ when $0\leqslant x\leqslant n-d$, and $F(x)=F_3(x)$ when $n-d\leqslant x\leqslant k-\frac{3}{2}$. By the convexity of $F_3(x)$, we have $F_3(n-d)\leqslant\max\{F_3(0),F_3(k-\frac{3}{2})\}$. Since $F_2(0)=F_3(0)$, it follows that

$$\begin{aligned}F(x)&\leqslant\max\left\{F_2(0),F_3(n-d),F_3(k-\frac{3}{2})\right\}\\&=\max\left\{\binom{2k}{2},\ \frac{(k-\frac{3}{2})(n+d-k+\frac{3}{2})}{2}+3\right\}.\end{aligned}$$

(iii) If $d\leqslant 2k-1$ and $n\geqslant d+k-\frac{3}{2}$, then $0\leqslant k-\frac{d+1}{2}\leqslant k-\frac{3}{2}\leqslant n-d$. Clearly, $F(x)=F_1(x)$ when $0\leqslant x\leqslant k-\frac{d+1}{2}$, and $F(x)=F_2(x)$ when $k-\frac{d+1}{2}\leqslant x\leqslant k-\frac{3}{2}$. Then we obtain

$$\begin{aligned}F(x)&\leqslant\max\left\{F_1(0),F_1(k-\frac{d+1}{2}),F_2(k-\frac{3}{2})\right\}\\&=dk.\end{aligned}$$

(iv) If $d \leqslant 2k-1$ and $n \leqslant d+k-\frac{3}{2}$, then $0 \leqslant k-\frac{d+1}{2} < n-d \leqslant k-\frac{3}{2}$. Clearly, $F(x)=F_1(x)$ when $0 \leqslant x \leqslant k-\frac{d+1}{2}$, and $F(x)=F_2(x)$ when $k-\frac{d+1}{2} \leqslant x \leqslant n-d$, and $F(x)=F_3(x)$ when $n-d \leqslant x \leqslant k-\frac{3}{2}$. By the convexity of $F_2(x)$, we have $F_2(n-d) \leqslant \max\{F_2(k-\frac{d+1}{2}), F_2(k-\frac{3}{2})\} = dk$. This implies that

$$\begin{aligned} F(x) &\leqslant \max\left\{F_1(0), F_2(k-\frac{d+1}{2}), F_2(n-d), F_3(k-\frac{3}{2})\right\} \\ &= \max\left\{dk,\ \frac{(k-\frac{3}{2})(n+d-k+\frac{3}{2})}{2}\right\} \\ &= dk. \end{aligned}$$

Consequently, if $2k$ is odd and $0 \leqslant x \leqslant k-\frac{3}{2}$, then

$$F(x) \leqslant \begin{cases} \max\left\{\binom{2k}{2}, d(k-\frac{3}{2})+3\right\}, & \text{if } d \geqslant 2k-1, n \geqslant d+k-\frac{3}{2}; \\ \max\left\{\binom{2k}{2}, \frac{(k-\frac{3}{2})(n+d-k+\frac{3}{2})}{2}+3\right\}, & \text{if } d \geqslant 2k-1, n \leqslant d+k-\frac{3}{2}; \\ dk, & \text{if } d \leqslant 2k-1. \end{cases}$$

We complete the proof of Lemma 8.