Maximum Size of a Graph with Given Fractional Matching Number

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Abstract

For three integers \( n, k, d \), we determine the maximum size of a graph on \( n \) vertices with fractional matching number \( k \) and maximum degree at most \( d \). As a consequence, we obtain the maximum size of a graph with given number of vertices and fractional matching number. This partially confirms a conjecture proposed by Alon et al. on the maximum size of \( r \)-uniform hypergraph with a fractional matching number for the special case when \( r = 2 \).

Mathematics Subject Classifications: 05C35, 05C70, 05C72

1 Introduction

For an integer \( r \geq 2 \), an \( r \)-uniform hypergraph or more simply, an \( r \)-graph, is a pair \( H = (V(H), E(H)) \) with vertex set \( V(H) \) and edge set \( E(H) \subseteq \binom{V(H)}{r} \). We call the number of the edges in \( H \) the size of \( H \). A matching of \( H \) is a set of edges, no two of which are intersecting. The matching number of \( H \), denoted by \( \nu(H) \), is the size of a maximum matching in \( H \). A matching \( M \) in \( H \) is perfect if every vertex of \( H \) is incident with an edge of \( M \). An \( r \)-graph is called a graph if \( r = 2 \), denoted by \( G \). We denote the maximum degree of the vertices of \( G \) by \( \Delta(G) \). For a subset \( S \) of \( V(G) \), we use \( G[S] \) to denote the subgraph of \( G \) induced by \( S \). For the terminologies and concepts not defined here, we refer the readers to [3, 5, 15].

Let \( g_r(n, k) \) denote the maximum size of a \( r \)-graph \( H \) on \( n \) vertices with matching number \( k \). In particular, we replace \( g_2(n, k) \) by \( g(n, k) \). In 1959, Erdős and Gallai [8]
determined \( g(n, k) \), that is, the maximum size of a graph \( G \) on \( n \) vertices with matching number \( k \).

**Theorem 1.** [8] For \( n \geq 2k + 1 \),

\[
g(n, k) = \max \left\{ \binom{2k + 1}{2}, \frac{k(2n - k - 1)}{2} \right\}.
\]

Later in [7], Erdős further conjectured that, for \( n \geq r(k + 1) - 1 \),

\[
g_r(n, k) = \max \left\{ \left( \binom{r(k + 1) - 1}{r} \right), \frac{n}{r} - \binom{n - k}{r} \right\}.
\]

This is an important conjecture in hypergraph theory and abundant literatures devote to it. For the recent progresses on Erdős matching conjecture, we refer to [10, 11, 12] for details.

In addition to matching number, some other restrictions were also considered in the literature. Let \( g(n, k, d) \) denote the maximum size of the graphs on \( n \) vertices with matching number \( k \) and the maximum degree at most \( d \). In [6], Chvátal and Hanson determined \( g(n, k, d) \), and hence, generalized Theorem 1.

**Theorem 2.** [6] Let \( n, k, d \) be three positive integers with \( n \geq 2k + 1 \). Set \( d_0 = \lfloor \frac{d+1}{2} \rfloor \).

(i) If \( d \leq 2k \) and \( n \leq 2k + \lfloor \frac{k}{d_0} \rfloor \), then

\[
g(n, k, d) = \min \left\{ \left\lfloor \frac{nd}{2} \right\rfloor, dk + \frac{d-1}{2} \left\lfloor \frac{2(n-k)}{d+1} \right\rfloor \right\}.
\]

if \( d \) is odd;

\[
g(n, k, d) = \frac{d}{2} \lfloor \frac{k}{d_0} \rfloor.
\]

if \( d \) is even.

(ii) If \( d \leq 2k \) and \( n \geq 2k + \lfloor \frac{k}{d_0} \rfloor \), then

\[
g(n, k, d) = dk + \frac{d}{2} \lfloor \frac{k}{d_0} \rfloor.
\]

(iii) If \( d \geq 2k + 1 \), then

\[
g(n, k, d) = \left\lfloor \frac{(2k+1)}{2} \right\rfloor, \frac{k(n+d-k)}{2} \right\}, \text{ if } n \leq k + d;
\]

\[
g(n, k, d) = \frac{dk}{2}, \text{ if } n \geq k + d.
\]

We refer to [2, 4, 13] for more details on this topic.

Let us focus on the fractional version of Theorem 2 in the following.

A fractional matching of an \( r \)-graph \( H \) is a function \( f \) assigning each edge with a real number in \([0, 1]\) so that \( \sum_{e \in \Gamma(v)} f(e) \leq 1 \) for each \( v \in V(H) \), where \( \Gamma(v) \) is the set of edges incident with \( v \) in \( H \). The fractional matching number of \( H \), denoted by \( \nu_f(H) \), is the maximum value of \( \sum_{e \in \Gamma(V)} f(e) \) over all fractional matchings \( f \). A fractional matching of \( H \) is perfect if \( \sum_{e \in \Gamma(V)} f(e) = 1 \) for each \( v \in V(H) \). Clearly, \( H \) has a fractional perfect matching if and only if \( \nu_f(H) = \frac{|V(G)|}{r} \). It was shown that the fractional matching number
\( \nu_f(G) \) of a graph \( G \) is either an integer or a semi-integer, that is, \( 2\nu_f(G) \) is integer (see [16], Theorem 2.1.5).

Let \( \mathcal{F}(n, k, d) \) denote the class of graphs on \( n \) vertices with fractional matching number \( k \) and the maximum degree at most \( d \). Further, let \( f(n, k, d) \) denote the maximum size of the graphs in \( \mathcal{F}(n, k, d) \), i.e.,

\[
f(n, k, d) = \max\{|E(G)| : |V(G)| = n, \ \nu_f(G) = k, \ \Delta(G) \leq d \}.
\]

It is clear that \( f(n, k, d) = f(n, k, n-1) \) when \( d \geq n-1 \), and hence we always assume \( d \leq n-1 \).

In this paper, we determine \( f(n, k, d) \) as follows.

**Theorem 3.** Let \( n, 2k, d \) be three positive integers with \( n \geq 2k \). If \( 2k \) is even, then

\[
f(n, k, d) = \max\left\{ \binom{2k}{2}, \frac{k(n+d-k)}{2} \right\}, \quad \text{if } d \geq 2k-1, n \leq d+k;
\]

\[
dk, \quad \text{otherwise.}
\]

If \( 2k \) is odd, then

\[
f(n, k, d) = \begin{cases} 
\max\left\{ \binom{2k}{2}, d(k - \frac{3}{2} + 3) \right\}, & \text{if } d \geq 2k-1, n \geq d+k - \frac{3}{2}, \\
\max\left\{ \binom{2k}{2}, \left[ \frac{(k-3)(n+d-k+2)}{2} \right] + 3 \right\}, & \text{if } d \geq 2k-1, n \leq d+k - \frac{3}{2}, \\
dk, & \text{if } d \leq 2k-1.
\end{cases}
\]

For a graph \( G \), let \( f(n, k) = \max\{|E(G)| : |V(G)| = n, \ \nu_f(G) = k \} \). Note that \( f(n, k, n-1) = f(n, k) \). As a consequence of Theorem 3, we obtain the following fractional version of Theorem 1.

**Theorem 4.** Let \( n, 2k \) be two positive integers with \( n \geq 2k \). If \( 2k \) is even, then

\[
f(n, k) = \max\left\{ \binom{2k}{2}, \frac{k(2n-k-1)}{2} \right\}.
\]

If \( 2k \) is odd, then

\[
f(n, k) = \max\left\{ \binom{2k}{2}, \frac{(k-\frac{3}{2})(2n-k+\frac{1}{2})}{2} + 3 \right\}.
\]

In [1], Alon et al. formulated the fractional version of Erdős matching conjecture for \( r \)-uniform hypergraphs as follows. Let \( f^*(r, n, s) = \max\{|E(H)| : |V(H)| = n, \ \nu_f(H) < s \} \).

**Conjecture 5.** [1] For all integers \( r \geq 2 \) and an integer \( s \) with \( 0 \leq s \leq \frac{n}{r} \),

\[
f^*(r, n, s) = \max\left\{ \binom{rs-1}{r}, \binom{n}{r} - \binom{n-s+1}{r} \right\}.
\]
In the same article, they also showed that Conjecture 5 asymptotically holds for \( r \in \{3,4\} \) and \( 0 \leq s \leq \frac{n}{r+1} \) when \( n \) tends to infinity. For \( r = 2 \), Theorem 1 implies that Conjecture 5 is asymptotically true when \( n \) goes to infinity. As far as we know, this conjecture still remains open even if \( r = 2 \) in general. In fact, by Theorem 4, we obtain the following corollary, which confirms that Conjecture 5 is true for \( r = 2 \).

**Corollary 6.** For any nonnegative integer \( s \) with \( n \geq 2s \),

\[
\begin{align*}
f_2^*(n,s) &= \max \left\{ \binom{2s-1}{2}, \binom{n}{2} - \binom{n-s+1}{2} \right\}.
\end{align*}
\]

The rest of this article is organized as follows. In Section 2, we establish the necessary upper bounds on \( f(n,k,d) \). In Section 3, we construct the corresponding extremal graphs attaining these upper bounds and, hence, give a proof of Theorem 3.

### 2 Upper bounds

To obtain the upper bounds of \( f(n,k,d) \), we begin this section with the following known result, called fractional Tutte-Berge formula, which characterizes the fractional matching number of a graph.

**Theorem 7.** [16] Let \( G \) be a graph \( G \) with \( n \) vertices. Then

\[
\nu_f(G) = \frac{1}{2} \left( n - \max_{S \subseteq V(G)} \{ i(G - S) - |S| \} \right),
\]

where \( i(G - S) \) is the number of isolated vertices in \( G - S \).

Before the proof of upper bounds of \( f(n,k,d) \), we also need a result with regard to the maximum value of a function, which plays a vital role in our proof. Let \( n,2k,d \) be three positive integers with \( n \geq 2k \) and \( d \leq n - 1 \). We now define a function

\[
F(x) = \min \left\{ dx, \frac{x(n+d-x)}{2} \right\} + \min \left\{ d(k-x), \frac{(2k-2x)}{2} \right\}
\]

on nonnegative real number \( x \), and its maximum value in given intervals can be obtained.

**Lemma 8.** If \( 2k \) is even and \( 0 \leq x \leq k \), then

\[
F(x) \leq \begin{cases} 
\max \left\{ \binom{2k}{2}, \frac{k(n+d-k)}{2} \right\}, & \text{if } d \geq 2k-1, n \leq d + k; \\
dk, & \text{otherwise}.
\end{cases}
\]

If \( 2k \) is odd and \( 0 \leq x \leq k - \frac{3}{2} \), then

\[
F(x) \leq \begin{cases} 
\max \left\{ \binom{2k}{2}, \frac{(k-d) + 3}{2} \right\}, & \text{if } d \geq 2k-1, n \geq d + k - \frac{3}{2}; \\
\max \left\{ \binom{2k}{2}, \frac{k-d + 3}{2} \right\} + 3, & \text{if } d \geq 2k-1, n \leq d + k - \frac{3}{2}; \\
dk, & \text{if } d \leq 2k-1.
\end{cases}
\]
Lemma 8 can be proved trivially by the convexity of the function, and for the coherence and completeness we will prove it in Appendix.

Now we present the upper bounds of \( f(n, k, d) \).

**Lemma 9.** Let \( n, 2k, d \) be three positive integers with \( n \geq 2k \). If \( 2k \) is even, then

\[
f(n, k, d) \leq \max \left\{ \frac{(2k)}{2}, \frac{k(n+d-k)}{2} \right\}, \quad \text{if } d \geq 2k - 1, n \leq d + k;
\]

\[
\text{otherwise.}
\]

If \( 2k \) is odd, then

\[
f(n, k, d) \leq \max \left\{ \frac{(2k)}{2}, \frac{d(k - \frac{3}{2}) + 3}{2} \right\}, \quad \text{if } d \geq 2k - 1, n \geq d + k - \frac{3}{2};
\]

\[
\max \left\{ \frac{(2k)}{2}, \frac{(k - \frac{3}{2})(n+d-k+\frac{1}{2})}{2} + 3 \right\}, \quad \text{if } d \geq 2k - 1, n \leq d + k - \frac{3}{2};
\]

\[
\left\lfloor dk \right\rfloor, \quad \text{if } d \leq 2k - 1.
\]

**Proof.** Let \( G \) be a graph with \( n \) vertices satisfying \( \nu_f(G) = k \) and \( \Delta(G) \leq d \). By Theorem 7, there exists a subset \( S \) of \( V(G) \) such that \( i(G - S) - |S| = n - 2k \). Let \( C \subseteq V(G) \) be the set of vertices that are neither in \( S \) nor isolated in \( G - S \). Then \( |C| = n - |S| - i(G - S) \neq 1 \).

We set \( s = |S| \) and \( c = |C| \) for convenience. Then we have \( 0 \leq c = 2k - 2s \neq 1 \), that is, the range of \( s \) is \( D = \{s \in N : s \leq k, s \neq k - \frac{1}{2}\} \).

Let \( \alpha \) be the number of edges in \( G[S] \), let \( \beta \) be the number of edges in \( G \) having exactly one vertex in \( S \), and let \( \gamma \) be the number of edges in \( G[C] \). Since \( \Delta(G) \leq d \), we have \( 2\alpha + \beta \leq ds \). On the one hand, \( \alpha + \beta \leq 2\alpha + \beta \leq ds \). On the other hand, since \( \beta \leq s(n - s) \), we obtain \( \alpha + \beta \leq \frac{s(n+d-s)}{2} \). Consequently,

\[
\alpha + \beta \leq \min \left\{ ds, \frac{s(n+d-s)}{2} \right\}.
\]

Moreover, for the subgraph \( G[C] \) of \( G \), if \( d \leq c - 1 \) then \( \gamma \leq \frac{dc}{2} \); otherwise \( \gamma \leq \binom{c}{2} \). Clearly, we have

\[
\gamma \leq \min \left\{ \frac{dc}{2}, \binom{c}{2} \right\} = \min \left\{ d(k - s), \binom{2k-2s}{2} \right\}.
\]

As a result,

\[
|E(G)| = \alpha + \beta + \gamma
\]

\[
\leq \min \left\{ ds, \frac{s(n+d-s)}{2} \right\} + \min \left\{ d(k - s), \binom{2k-2s}{2} \right\}
\]

\[
= F(s).
\]

Recall that \( D = \{s \in N : s \leq k, s \neq k - \frac{1}{2}\} \). Therefore,

\[
f(n, k, d) \leq \max_{s \in D} F(s).
\]

Since \( f(n, k, d) \) is an integer, it suffices to obtain the maximum integer which is no more than the maximum value of \( F(s) \) for any \( s \in D \). Moreover, if \( 2k \) is odd, then the fact \( s \in D \) implies \( s \leq k - \frac{3}{2} \). By Lemma 8, we complete the proof. \( \square \)
3 Constructions of extremal graphs

In this section, we construct several extremal graphs satisfying conditions to attain $f(n,k,d)$. We begin with some useful results.

For a $k$-regular graph $G$, if we assign to each edge a number $\frac{1}{k}$, then we obtain a fractional perfect matching of $G$. Therefore we have the following proposition.

**Proposition 10.** Let $G$ be a $k$-regular graph. Then $G$ has a fractional perfect matching.

We call that a sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers is graphic if it is a degree sequence of a simple graph $G$, and the graph $G$ is said to realize the sequence $d_1, d_2, \ldots, d_n$. The following characterization of graphic sequence is due to Erdős and Gallai.

**Theorem 11.** [9] A sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ is graphic if and only if
\[
\sum_{i=1}^n d_i \text{ is even and }
\sum_{i=t+1}^n \min\{t, d_i\}
\]
for $t = 1, 2, \ldots, n$.

A spanning subgraph of a graph $G$ is said to $k$-factor if the degree of its each vertex is equal to $k$. The following result gives a sufficient condition with regard to degree sequences such that the graph contains a $k$-factor.

**Theorem 12.** [14] If the sequences $d_1 \geq d_2 \geq \cdots \geq d_n$ and $d_1 - k \geq d_2 - k \geq \cdots \geq d_n - k$ are graphic, then the sequence $d_1, d_2, \ldots, d_n$ can be realized by a graph $G$ which contains a $k$-factor.

Let $d$ and $n$ be two odd numbers with $3 \leq d \leq n - 2$. Theorem 11 ensures that two sequences $d, d, \ldots, d, d - 1$ and $d - 2, d - 2, \ldots, d - 2, d - 3$ of length $n$ are both graphic. By Theorem 12, the sequence $d, d, \ldots, d - 1$ can be realized by a graph $G$ which contains a 2-factor. We know that if a graph $G$ contains a 2-factor, then it has a fractional perfect matching. Therefore we have the following result, which is useful in our forthcoming argument.

**Corollary 13.** Let $d$ and $n$ be two odd numbers with $3 \leq d \leq n - 2$. Then the sequence $d, d, \ldots, d, d - 1$ of length $n$ can be realized by a graph which has a fractional perfect matching.

Now we construct extremal graphs to attain $f(n, k, d)$. Several notations are used in our constructions. We use $K_n$ and $\overline{K}_n$ to denote a complete graph and an empty graph with $n$ vertices, respectively. For two graphs $G_1$ and $G_2$, we denote the disjoint union of $G_1$ and $G_2$ by $G_1 \cup G_2$. Let us first consider the case that $2k$ is even.

**Lemma 14.** Let $n, 2k, d$ be three positive integers with $n \geq 2k$. If $2k$ is even, then
\[
f(n, k, d) \geq \max\left\{\binom{2k}{2}, \frac{k(n+d-k)}{2}\right\}, \quad \text{if } d \geq 2k-1, n \leq d + k;
\]
otherwise.
Proof. Let us divide into two cases to complete the proof.

**Case 1.** \( d \geq 2k - 1 \) and \( n \leq d + k \).

If \( d \leq 5k - n - 2 \), then

\[
\max \left\{ \left( \frac{2k}{2} \right), \left[ \frac{k(n - k + d)}{2} \right] \right\} = \left( \frac{2k}{2} \right).
\]

We can check that \( K_{2k} \cup K_{n-2k} \in \mathcal{F}(n, k, d) \) and its size is \( \left( \frac{2k}{2} \right) \). Thus \( f(n, k, d) \geq \left( \frac{2k}{2} \right) \).

If \( d \geq 5k - n - 2 \), then

\[
\max \left\{ \left( \frac{2k}{2} \right), \left[ \frac{k(n - k + d)}{2} \right] \right\} = \left[ \frac{k(n - k + d)}{2} \right].
\]

Consider the parity of \( k(n - k + d) \). When \( k(n - k + d) \) is even, we take a \((k - n + d)\)-regular graph \( G_0 \) with \( k \) vertices. When \( k(n - k + d) \) is odd, we take a graph \( G_0 \) with \( k - 1 \) vertices of degree \( k - n + d \) and a vertex of degree \( k - n - d - 1 \). Add \( n - k \) independent vertices to \( G_0 \) such that they are adjacent to each vertex of \( G_0 \), which contains a matching covering each vertex of \( G_0 \) between \( G_0 \) and these independent vertices, and denote the resulting graph by \( G \). Clearly, \( \nu_f(G) = k \) and \( \Delta(G) = d \), and then \( G \in \mathcal{F}(n, k, d) \). By the construction of \( G \), we have

\[
|E(G)| = \left[ \frac{k(n - k + d)}{2} \right] + k(n - k) = \left[ \frac{k(n - k + d)}{2} \right].
\]

Thus \( f(n, k, d) \geq \left[ \frac{k(n - k + d)}{2} \right]. \)

**Case 2.** \( d \leq 2k - 1 \) or \( n \geq d + k \).

If \( d \leq 2k - 1 \) and \( n \leq d + k \), we take a \( d \)-regular graph \( G_0 \) with \( 2k \) vertices. Proposition 10 ensures that \( G_0 \) has a fractional perfect matching, and then \( \nu_f(G_0) = k \). Clearly, \( G = G_0 \cup K_{n-2k} \in \mathcal{F}(n, k, d) \) and \( |E(G)| = dk \). Thus \( f(n, k, d) \geq dk \).

If \( n \geq d + k \), let \( X = \{x_1, x_2, \ldots, x_k\} \) and \( Y = \{y_1, y_2, \ldots, y_h\} \) be two sets of vertices, where \( h = n - k \geq k \). For \( i = 1, 2, \ldots, k \), joining the vertex \( x_i \) with every vertex \( y_j \) by an edge, where \( j = i + t \) \((\text{mod } h)\) for \( t = 0, 1, \ldots, d - 1 \), we call the resulting graph \( G \). Clearly, \( \nu_f(G) = k \), \( \Delta(G) = d \) and \( |E(G)| = dk \). Thus \( f(n, k, d) \geq dk \).

Next we give the lower bounds of \( f(n, k, d) \) when \( 2k \) is odd.

**Lemma 15.** Let \( n, 2k, d \) be three positive integers with \( n \geq 2k \). If \( 2k \) is odd, then

\[
f(n, k, d) \geq \begin{cases} 
\max \left\{ \left( \frac{2k}{2} \right), d(k - \frac{3}{2}) + 3 \right\}, & \text{if } d \geq 2k - 1, n \leq d + k - \frac{3}{2}; \\
\max \left\{ \left( \frac{2k}{2} \right), \left[ \frac{(k-\frac{1}{2})(n+d-k+\frac{1}{2})}{2} \right] + 3 \right\}, & \text{if } d \geq 2k - 1, n \leq d + k - \frac{3}{2}; \\
dk, & \text{if } \frac{k}{2} \leq 2k - 1.
\end{cases}
\]

Proof. The fact that \( 2k \) is odd implies \( d \geq 2 \); otherwise \( d = 1 \), which implies fractional matching number \( k \) must be an integer, a contradiction. If \( k = \frac{3}{2} \), then we have \( G = \)
Let us discuss three cases.

**Case 1.** \( d \geq 2k - 1 \) and \( n \geq d + k - \frac{3}{2} \).

If \( d \leq \frac{k(2k-1)-3}{k-\frac{3}{2}} \), then
\[
\max \left\{ \left( \frac{2k}{2} \right), d(k-\frac{3}{2})+3 \right\} = \left( \frac{2k}{2} \right).
\]

Clearly, \( K_{2k} \cup K_{n-2k} \in F(n,k,d) \). Thus \( f(n,k,d) \geq \left( \frac{2k}{2} \right) \).

If \( d \geq \frac{k(2k-1)-3}{k-\frac{3}{2}} \), then
\[
\max \left\{ \left( \frac{2k}{2} \right), d(k-\frac{3}{2})+3 \right\} = d(k-\frac{3}{2})+3.
\]

Let \( X = \{x_1, x_2, \ldots, x_l\} \) and \( Y = \{y_1, y_2, \ldots, y_h\} \) be two sets of vertices, where \( l = k - \frac{3}{2} \) and \( h = n - k + \frac{3}{2} \). Note that \( h - l \geq 3 \) as \( n \geq 2k \). For \( i = 1, 2, \ldots, l \), joining the vertex \( x_i \) with every vertex \( y_j \) by an edge, where \( j = i + t \) (mod \( h \)) for \( t = 0, 1, \ldots, d - 1 \), we call the resulting graph \( G_0 \). Clearly, \( M = \{x_iy_i \in E(G_0) : i = 1, 2, \ldots, l\} \) forms a matching covering \( X \), and there exist at least three independent vertices \( y_{t+1}, y_{t+2} \) and \( y_{t+3} \) in \( Y \) which are not covered by \( M \) as \( h - l \geq 3 \). Add three edges in \( \{y_{t+1}, y_{t+2}, y_{t+3}\} \) such that they form a \( K_3 \) to \( G_0 \), and call the resulting graph \( G \). Clearly, \( \nu_f(G) = l + \frac{3}{2} = k \) and \( \Delta(G) = d \), and hence \( G \in F(n,k,d) \). Thus \( f(n,k,d) \geq d(k-\frac{3}{2})+3 \).

**Case 2.** \( d \geq 2k - 1 \) and \( n \leq d + k - \frac{3}{2} \).

If \( n \geq \frac{4k^2-2k-6}{k-\frac{3}{2}} + k - d - \frac{3}{2} \), then
\[
\max \left\{ \left( \frac{2k}{2} \right), \left\lfloor \frac{(k-\frac{3}{2})(n-k+d+\frac{3}{2})}{2} \right\rfloor +3 \right\} = \left( \frac{2k}{2} \right).
\]

Clearly, \( K_{2k} \cup K_{n-2k} \in F(n,k,d) \). Thus \( f(n,k,d) \geq \left( \frac{2k}{2} \right) \).

If \( n \leq \frac{4k^2-2k-6}{k-\frac{3}{2}} + k - d - \frac{3}{2} \), then
\[
\max \left\{ \left( \frac{2k}{2} \right), \left\lfloor \frac{(k-\frac{3}{2})(n-k+d+\frac{3}{2})}{2} \right\rfloor +3 \right\} = \left\lfloor \frac{(k-\frac{3}{2})(n-k+d+\frac{3}{2})}{2} \right\rfloor +3.
\]

Consider the parity of \( (k-\frac{3}{2})(n-k+d+\frac{3}{2}) \). When \( (k-\frac{3}{2})(n-k+d+\frac{3}{2}) \) is even, we take a graph \( G_0 \) with \( k-\frac{3}{2} \) vertices of degree \( k-n+d-\frac{3}{2} \). When \( (k-\frac{3}{2})(n-k+d+\frac{3}{2}) \) is odd, we take a graph \( G_0 \) with \( k-\frac{5}{2} \) vertices of degree \( k-n+d-\frac{3}{2} \) and a vertex of degree \( k-n+d-\frac{5}{2} \). Add \( n-k-\frac{3}{2} \) independent vertices to \( G_0 \) such that they are adjacent to each vertex of \( G_0 \), which contains a matching covering each vertex of \( G_0 \) between \( G_0 \) and these independent vertices, and denote the resulting graph by \( G_1 \). Therefore \( \nu_f(G_1) = k-\frac{3}{2} \). Furthermore, add a \( K_3 \) formed by remainder three vertices to \( G_1 \) such that each vertex of \( K_3 \) are adjacent to each vertex of \( G_0 \), and denote the resulting graph by \( G \). Then we
have \( \nu_f(G) = \nu_f(G_1) + \frac{3}{2} = k \) and \( \Delta(G) = d \). Then \( G \in \mathcal{F}(n, k, d) \). By the construction of \( G \), we have

\[
|E(G)| = \left\lfloor \frac{(k - \frac{3}{2})(n - k + 3 - \frac{3}{2})}{2} \right\rfloor + (k - \frac{3}{2})(n - k + \frac{3}{2}) + 3
\]

Thus \( f(n, k, d) \geq \left\lfloor \frac{(k - \frac{3}{2})(n - k + d + \frac{3}{2})}{2} \right\rfloor + 3 \).

**Case 3.** \( d \leq 2k - 1 \).

When \( d \) is even, we can take a \( d \)-regular graph \( G_0 \) with \( 2k \) vertices. By Proposition 10, \( G_0 \) has a fractional perfect matching. When \( d \) is odd, Corollary 13 ensures that we can find a graph \( G_0 \) with \( 2k - 1 \) vertices of degree \( d \) and one vertex of degree \( d - 1 \) such that \( G_0 \) contains a fractional perfect matching. Therefore, \( \nu_f(G_0) = k \). Clearly, \( G = G_0 \cup \overline{K}_{n-2k} \in \mathcal{F}(n, k, d) \) and \( |E(G)| = [dk] \). Thus \( f(n, k, d) \geq [dk] \). □

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**References**


**Appendix. Proof of Lemma 8**

Let us recall that $n \geq 2k$, $d \leq n - 1$ and the function

$$F(x) = \min \left\{ dx, \frac{x(n + d - x)}{2} \right\} + \min \left\{ d(k - x), \frac{(2k - 2x)}{2} \right\},$$

where $x \geq 0$.

It is easy to see that $dx = \frac{x(n + d - x)}{2}$ iff $x = n - d$, and $d(k - x) = \frac{(2k - 2x)}{2}$ iff $x = k - \frac{d+1}{2}$.

If $k - \frac{d+1}{2} \geq 0$, then $k - \frac{d+1}{2} < k - \frac{d}{2} \leq 2k - d \leq n - d$; otherwise $k - \frac{d+1}{2} \leq 0 < n - d$. So $k - \frac{d+1}{2} < n - d$ always holds. We can rewrite the function

$$F(x) = \begin{cases} dk, & \text{if } x \leq k - \frac{d+1}{2}; \\
\frac{dx + (2k - 2x)}{x(n - d - x)} + \frac{(2k - 2x)}{2}, & \text{if } k - \frac{d+1}{2} \leq x \leq n - d; \\
\frac{(2k - 2x)}{2}, & \text{if } x \geq n - d. 
\end{cases}$$

Let $F_1(x) = dk$, $F_2(x) = dx + \frac{(2k - 2x)}{2}$ and $F_3(x) = \frac{x(n + d - x)}{2} + \frac{(2k - 2x)}{2}$. Note that $F_i(x)$ is convex for $i = 1, 2, 3$. To obtain the maximum value of $F(x)$, it suffices to discuss maximum values of $F_i(x)$’s in corresponding intervals.

**Case 1.** $2k$ is even and $0 < x < k$.

(i) If $d \geq 2k - 1$ and $n \geq d + k$, then $k - \frac{d+1}{2} \leq 0 < k - d \leq k$. Clearly, $F(x) = F_2(x)$ for any $x$ with $0 \leq x < k$. By the convexity of $F_2(x)$, we have

$$F(x) \leq \max \{F_2(0), F_2(k)\} = \max \left\{ \frac{(2k)}{2}, dk \right\} = dk.$$

(ii) If $d \geq 2k - 1$ and $n \leq d + k$, then $k - \frac{d+1}{2} \leq 0 < n - d \leq k$. Clearly, $F(x) = F_2(x)$ when $0 \leq x \leq n - d$, and $F(x) = F_3(x)$ when $n - d \leq x \leq k$. By the convexity of $F_3(x)$, we have $F_3(n - d) \leq \max \{F_3(0), F_3(k)\}$. Since $F_2(0) = F_3(0)$, it follows that

$$F(x) \leq \max \{F_2(0), F_3(n - d), F_3(k)\} = \max \left\{ \frac{(2k)}{2}, \frac{k(n + d - k)}{2} \right\}.$$

(iii) If $d \leq 2k - 1$ and $n \geq d + k$, then $0 \leq x \leq k - \frac{d+1}{2} < k \leq n - d$. Clearly, $F(x) = F_1(x)$ when $0 \leq x \leq k - \frac{d+1}{2}$, and $F(x) = F_2(x)$ when $k - \frac{d+1}{2} \leq x \leq k$. Then we obtain

$$F(s) \leq \max \left\{ F_1(0), F_1(k - \frac{d+1}{2}), F_2(k) \right\} = dk.$$

(iv) If $d \leq 2k - 1$ and $n \leq d + k$, then $0 \leq k - \frac{d+1}{2} < n - d \leq k$. Clearly, $F(x) = F_1(x)$ when $0 \leq x \leq k - \frac{d+1}{2}$, and $F(x) = F_2(x)$ when $k - \frac{d+1}{2} \leq x \leq n - d$.
and \( F(x) = F_3(x) \) when \( n - d \leq x \leq k \). By the convexity of \( F_2(x) \), we have \( F_2(n - d) \leq \max\{F_2(k - \frac{d + 1}{2}), F_2(k)\} = dk \). This implies that

\[
F(x) \leq \max \left\{ F_1(0), F_2(k - \frac{d + 1}{2}), F_2(n - d), F_3(k) \right\}
= \max \left\{ dk, \frac{k(n + d - k)}{2} \right\}
= dk.
\]

Consequently, if \( 2k \) is even and \( 0 \leq x \leq k \), then

\[
F(x) \leq \begin{cases} 
\max \left\{ \frac{(2k)}{2}, \frac{k(n + d - k)}{2} \right\}, & \text{if } d \geq 2k - 1, n \leq d + k; \\
dk, & \text{otherwise}.
\end{cases}
\]

**Case 2.** \( 2k \) is odd and \( 0 \leq x \leq k - \frac{3}{2} \).

Note that \( d \geq 2 \). Otherwise \( d = 1 \) implies that \( k \) is an integer, which contradicts that \( 2k \) is odd.

(i) If \( d \geq 2k - 1 \) and \( n \geq d + k - \frac{3}{2} \), then \( k - \frac{d + 1}{2} \leq 0 \leq k - \frac{3}{2} \leq n - d \). Clearly, \( F(x) = F_2(x) \) for any \( x \) with \( 0 \leq x \leq k - \frac{3}{2} \). By the convexity of \( F_2(x) \), we have

\[
F(x) \leq \max \left\{ F_2(0), F_2(k - \frac{3}{2}) \right\}
= \max \left\{ \left( \frac{2k}{2} \right), d(k - \frac{3}{2}) + 3 \right\}.
\]

(ii) If \( d \geq 2k - 1 \) and \( n \leq d + k - \frac{3}{2} \), then \( k - \frac{d + 1}{2} \leq 0 < n - d \leq k - \frac{3}{2} \). Clearly, \( F(x) = F_2(x) \) when \( 0 \leq x \leq n - d \), and \( F(x) = F_3(x) \) when \( n - d \leq x \leq k - \frac{3}{2} \). By the convexity of \( F_3(x) \), we have \( F_3(n - d) \leq \max\{F_3(0), F_3(k - \frac{3}{2})\} \). Since \( F_2(0) = F_3(0) \), it follows that

\[
F(x) \leq \max \left\{ F_2(0), F_3(n - d), F_3(k - \frac{3}{2}) \right\}
= \max \left\{ \left( \frac{2k}{2} \right), \frac{(k - \frac{3}{2})(n + d - k + \frac{3}{2})}{2} + 3 \right\}.
\]

(iii) If \( d \leq 2k - 1 \) and \( n \geq d + k - \frac{3}{2} \), then \( 0 \leq k - \frac{d + 1}{2} \leq k - \frac{3}{2} \leq n - d \). Clearly, \( F(x) = F_1(x) \) when \( 0 \leq x \leq k - \frac{d + 1}{2} \), and \( F(x) = F_2(x) \) when \( k - \frac{d + 1}{2} \leq x \leq k - \frac{3}{2} \). Then we obtain

\[
F(x) \leq \max \left\{ F_1(0), F_1(k - \frac{d + 1}{2}), F_2(k - \frac{3}{2}) \right\}
= dk.
\]
(iv) If $d \leq 2k-1$ and $n \leq d + k - \frac{3}{2}$, then $0 \leq k - \frac{d+1}{2} < n - d \leq k - \frac{3}{2}$. Clearly, $F(x) = F_1(x)$ when $0 \leq x \leq k - \frac{d+1}{2}$, and $F(x) = F_2(x)$ when $k - \frac{d+1}{2} \leq x \leq n - d$, and $F(x) = F_3(x)$ when $n - d \leq x \leq k - \frac{3}{2}$. By the convexity of $F_2(x)$, we have $F_2(n - d) \leq \max\{F_2(k - \frac{d+1}{2}), F_2(k - \frac{3}{2})\} = dk$. This implies that

$$F(x) \leq \max \left\{ F_1(0), F_2(k - \frac{d+1}{2}), F_2(n - d), F_3(k - \frac{3}{2}) \right\} = \max \left\{ dk, \frac{(k - \frac{3}{2})(n + d - k + \frac{3}{2})}{2} \right\} = dk.$$

Consequently, if $2k$ is odd and $0 \leq x \leq k - \frac{3}{2}$, then

$$F(x) \leq \begin{cases} \max \left\{ \binom{2k}{2}, d(k - \frac{3}{2}) + 3 \right\}, & \text{if } d \geq 2k - 1, n \geq d + k - \frac{3}{2}; \\ \max \left\{ \binom{2k}{2}, \frac{(k-\frac{3}{2})(n+d-k+\frac{3}{2})}{2} + 3 \right\}, & \text{if } d \geq 2k - 1, n \leq d + k - \frac{3}{2}; \\ dk, & \text{if } d \leq 2k - 1. \end{cases}$$

We complete the proof of Lemma 8.