Naruse hook formula
for linear extensions of mobile posets

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Abstract

Linear extensions of posets are important objects in enumerative and algebraic combinatorics that are difficult to count in general. Families of posets like Young diagrams of straight shapes and d-complete posets have hook-length product formulas to count linear extensions, whereas families like Young diagrams of skew shapes have determinant or positive sum formulas like the Naruse hook-length formula from 2014. In 2020, Garver et. al. gave determinant formulas to count linear extensions of a family of posets called mobile posets that refine d-complete posets and border strip skew shapes. We give a Naruse type hook-length formula to count linear extensions of such posets by proving a major index q-analogue. We also give an inversion index q-analogue of the Naruse formula for mobile tree posets.

Mathematics Subject Classifications: 05A05, 05A15

1 Introduction

1.1 Hook-length formulas for linear extensions

Linear extensions of posets are fundamental objects in combinatorics. In general, computing the number $e(P)$ linear extensions of any poset is a difficult problem, it is #P-complete [2]. For certain posets like Young diagrams, rooted trees, and more generally d-complete posets, there are product formulas that compute the number of linear extensions efficiently, such as the classical *hook-length formula* (HLF) for the number of standard Young tableaux (SYT) of shape λ, that we denote by $|\mathrm{SYT}(\lambda)|$.

Theorem 1 (Frame-Robinson-Thrall [4]). *Let λ be a partition of n. We have*

$$|\mathrm{SYT}(\lambda)| = n! \prod_{u \in [\lambda]} \frac{1}{h(u)} \tag{HLF}$$

where $h(u) = \lambda_i + \lambda'_j - i - j + 1$ is the hook-length of the square $u = (i, j)$.

For skew shapes λ/μ, there is no known product formula, however, Naruse introduced a generalization of the hook-length formula for the number of standard Young tableaux of skew shape as a positive sum over excited diagrams of products of hook-lengths. We call this the *Naruse hook-length formula* (NHLF).

Theorem 2 (Naruse [11]). *For a skew shape λ/μ of size n, we have*

$$|\mathrm{SYT}(\lambda/\mu)| = n! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)}, \tag{NHLF}$$

where $\mathcal{E}(\lambda/\mu)$ is the set of excited diagrams of λ/μ.

The number of SYT of shape λ/μ can also be interpreted as the number of linear extension of a poset induced by the Young diagram of λ/μ. In [13], Proctor defined the family of *d-complete* posets, that includes Young diagrams of shape λ and rooted trees and have a hook-length formula to count the number of linear extensions.

Theorem 3 (Peterson-Proctor [13]). *The number of linear extensions of a d-complete poset P with n elements is*

$$e(P) = \frac{n!}{\prod_{x \in P} h_P(x)},$$

where $h_P(x)$ is the hook-length of $x \in P$.

1.2 q-analogue of hook-length formulas

There are the following q-analogues of both the HLF and the NHLF for semistandard Young tableaux. We state these results in terms of $e_q^{\mathrm{maj}}(P, \omega) := \sum_\sigma q^{\mathrm{maj}(\sigma)}$ where (P, ω) is a labeled poset and the sum is over all linear extensions σ. This polynomial also encodes the generating functions of P-partitions [16].

Theorem 4 (Stanley [15]). *For a shape λ with associated poset Q_λ of size n with a Schur labeling ω, we have:*

$$\frac{e_q^{\mathrm{maj}}(Q_\lambda, \omega)}{\prod_{i=1}^n (1 - q^i)} = q^{b(\lambda)} \prod_{u \in [\lambda]} \frac{1}{1 - q^{h(u)}}, \tag{1}$$

where $b(\lambda) = \sum_i (i - 1)\lambda_i$.

Theorem 5 (Morales-Pak-Panova [10]). *For a skew shape λ/μ with associated poset $Q_{\lambda/\mu}$ of size n with a Schur labeling ω, we have:*

$$\frac{e_q^{\mathrm{maj}}(Q_{\lambda/\mu},\omega)}{\prod_{i=1}^{n}(1-q^i)} = \sum_{D\in\mathcal{E}(\lambda/\mu)} q^{w(D)} \prod_{u\in[\lambda]\setminus D}\frac{1}{1-q^{h(u)}}, \tag{2}$$

where $w(D) = \sum_{u\in Br(D)} h(u)$ is the sum of hook-lengths of the support of broken diagonals.

For d-complete posets, there is also the following q-analogue in terms of major index (see Section 2.5).

Theorem 6 (Peterson and Proctor [13]). *For a labeled d-complete poset (P,ω) of size n with ω any labeling, we have:*

$$e_q^{\mathrm{maj}}(P,\omega) = q^{\mathrm{maj}(P,\omega)}\frac{[n]_q!}{\prod_{x\in P}[h_P(x)]_q},$$

where $h_P(x)$ is the hook-length of $x\in P$ and $\mathrm{maj}(P,\omega) = \sum_{x\in\mathrm{Des}(P,\omega)} h_P(x)$.

1.3 Hook formulas for mobile posets

A *border strips* is a connected skew-shaped diagram with no 2×2 box. A *mobile poset* is a recent common refinement of border strips and d-complete posets introduced in [5] (see Figure 1). The authors found a determinantal formula for the number of linear extensions of these posets, similar to *Jacobi–Trudi* formula and asked whether there was a Naruse-type formula [5, Sec. 6.1] for this number. The first main result of this paper is a q-analogue Naruse hook-length formula for mobile posets for the major index, generalizing Theorem 5.

Theorem 7. *Let $P_{\lambda/\mu}(\mathbf{p})$ be a mobile poset of size n with with underlying border strip λ/μ and $\mathbf{p} = (p_{(r_1,s_1)},\ldots,p_{(r_k,s_k)})$ the d-complete posets hanging on (r,s). For a labeled mobile poset $(P_{\lambda/\mu}(\mathbf{p}),\omega)$ with ω reversed Schur labeling on $[\lambda/\mu]$ and natural labeling on the d-complete posets, we have:*

$$\frac{e_q^{\mathrm{maj}}(P_{\lambda/\mu},\omega)}{\prod_{i=1}^{n}(1-q^i)} = \prod_{v\in\mathbf{p}}\frac{1}{1-q^{h(v)}} \sum_{D\in\mathcal{E}(\lambda/\mu)} q^{w'(D)} \prod_{u\in[\lambda]\setminus D}\frac{1}{1-q^{h'(u)}},$$

where $h(v)$ is the hook-length of the element v in the d-complete posets in $\mathbf{p}$, $h'(i,j) = \lambda_i - i + \lambda'_j - j + 1 + \sum_{r\geqslant i,s\geqslant j}|p_{(r,s)}|$ and $w'(D) = \sum_{u\in Br(D)} h'(u)$ is the sum of hook-lengths of the supports of broken diagonals.

This result is a common refinement of Theorem 5 and Theorem 6. Note that Naruse-Okada [12] have a different q-analogue of $e_q^{\mathrm{maj}}(P,\omega)$ for a family called *skew d-complete* posets with *natural labelings*. See Section 7.4.

The proof of Theorem 7 is based on the method used in [8] to prove (NHLF) for border strips. This involves the *Pieri–Chevalley formula* (6) and a recurrence of linear extensions

for mobile posets. The proof of the latter is combinatorial and uses a specialization of Stanley's theory of (P, ω)-partitions [16] (see Theore 25).

By taking $q = 1$, we obtain a Naruse hook-length formula for mobile posets as a corollary.

Corollary 8 (NHLF for mobiles). *For a free-standing mobile poset $P_{\lambda/\mu}(\mathbf{p})$ of size n, we have:*

$$e(P_{\lambda/\mu}(\mathbf{p})) = \frac{n!}{H(\mathbf{p})} \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \gamma} \frac{1}{h'(i,j)}, \tag{3}$$

where $H(\mathbf{p})$ is the product of hook-lengths of all elements in the d-complete posets hanging from λ/μ.

As an application of this corollary, we give bounds to generalizations of Euler number defined in [5]. See Corollary 37 and Corollary 38.

Our second result is a q-analogue of NHLF for mobile tree posets (the d-complete posets are restricted to rooted trees) in terms of the inversion statistic, $e_q^{\mathrm{inv}}(P, \omega) := \sum_{\sigma} q^{\mathrm{inv}(\sigma)}$, where the sum is over all linear extensions $\sigma \in (P, \omega)$.

Theorem 9. *For a labeled mobile tree poset $(P_{\lambda/\mu}, \omega)$ with ω reversed Schur labeling on $[\lambda/\mu]$ and natural labeling on d-complete posets,*

$$\frac{e_q^{\mathrm{inv}}(P_{\lambda/\mu}, \omega)}{\prod_{i=1}^{n}(1-q^i)} = \prod_{v \in \mathbf{p}} \frac{1}{1-q^{h(v)}} \sum_{D \in \mathcal{E}(\lambda/\mu)} q^{w(D)+p_D} \prod_{u \in [\lambda] \setminus D} \frac{1}{1-q^{h'(u)}}, \tag{4}$$

where $w(D) = \sum_{u \in Br(D)} h(u)$ is the sum of original hook-lengths of the supports of broken diagonals and $p_D = \sum_{(i,j) \in [\mu] \setminus D} \sum_{s=j} p_{r,s}$, the sum of d-complete posets hanging on the same column as $(i,j) \in [\mu] \setminus D$.

1.4 Paper outline

In Section 2, we give definitions and background results required for the proof. In Section 3, we give results for P-partition with a fixed point. In Section 4, we give an example and the proof of Theorem 7. In Section 5, we show an application to the Corollary 8. In Section 6, we give examples and the proof of the inversion index of the case of q-analogue. Lastly, we end with final remarks in Section 7.

2 Background and Preliminaries

2.1 Posets and linear extensions

A *partially-ordered set* (poset) is a pair $(P, \leqslant_P)$ where P is a finite set and $\leqslant_P$ is a binary relation that is reflexive, anti-symmetric, and transitive. A *linear extension* of an n-element poset P is a bijection $f : P \to [n]$ that is order-preserving. We denote the set of linear extensions of P as $\mathcal{L}(P)$, and $e(P) = |\mathcal{L}(P)|$. A poset of shape λ/μ is a poset

obtained from a Young diagram of shape λ/μ, where the inner corners of the diagram are the maximal elements of the poset. the number of *linear extensions* of such poset is equal to $|SYT(\lambda/\mu)|$

2.2 Border strips and Mobile Posets

A *border strip* is a connected skew shape λ/μ containing no 2×2 box. *d-complete posets* are a large class of posets containing rooted tree posets and posets arising from Young diagrams. Given a border strip, we can convert it into a poset by letting the inner corners of the diagram be the maximal points of the corresponding poset. Now we can construct a mobile poset (See Figure 1 for an example).

Definition 10 (Garver-Grosser-Matherne-Morales [5])**.** A *mobile[1] (tree) poset* is a poset obtained from a border strip Q, by allowing every element $x \in Q$ to cover the maximal element of a nonnegative number of disjoint d-complete (rooted tree) posets. We use $P_{\lambda/\mu}(p_{a_1,b_1}, p_{a_1,b_2}, \dots) = P_{\lambda/\mu}(\mathbf{p})$ to denote a mobile poset with a border strip of shape λ/μ and a set of d-complete posets $\mathbf{p}$, where each d-complete poset p_{a_i,b_i} is covered by $(a_i, b_i) \in [\lambda/\mu]$.
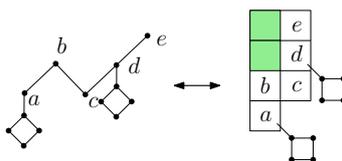

Figure 1: The conversion of a mobile poset to a young diagram with d-complete posets attached.

2.3 Excited diagrams and broken diagonals

Denote $[\lambda/\mu]$ the skew shape Young diagram of a shape λ/μ. An *excited diagram* of λ/μ, denoted by D, is a subset of $[\lambda]$ obtained from μ by applying a sequence of *excited moves* that we define next. Let $D \in \mathcal{E}(\lambda/\mu)$, then $(i, j) \in D$ is an *active cell* if $(i+1, j), (i, j+1)$, and $(i+1, j+1)$ are not in D. We obtain a new excited diagram by replacing an active cell by $(i+1, i+j)$ (see Figure 2, (b)). Note that for border strips, the excited diagrams can also be interpreted as the complement of its lattice paths γ from $(\lambda'_1, 1) \to (1, \lambda_1)$ that stay inside $[\lambda]$. (see[8, Sec. 3]). Let $\mathcal{E}(\lambda/\mu)$ be the set of all excited diagrams of λ/μ.

For each excited diagram $D \in \mathcal{E}(\lambda/\mu)$ we associate a set of *broken diagonals* $Br(D) \subset [\lambda] \setminus D$ as follows. Start with $D = [\mu]$, then $Br(D) = \{(i, j) \in \lambda/\mu \,|\, j - i = \mu_t - t\}$, where $1 < t < \ell(\lambda)$ and $\mu_t = 0$ if $\ell(\mu) < t < \ell(\lambda)$. For each active cell $u = (i, j)$ and its excited move $\alpha_u : D \to D'$, we have a corresponding move for the broken diagonal where $Br(D') = Br(D) \setminus \{(i+1, j+1)\} \cup \{(i+1, j)\}\}$. See Figure 2, (a) and (b).

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[1] What we call a mobile poset is called a free-standing mobile poset in [5].

Figure 2: (a) Excited move (b) Excited diagrams and the corresponding broken diagonals of $\lambda/\mu = (2,2,2,1)/(1,1)$

2.4 Multivariate function

For border strip λ/μ, let

$$F_{\lambda/\mu}(\mathbf{x},\mathbf{y}) = F_{\lambda/\mu}(x_1,\ldots,x_{\lambda_1'},y_1,\ldots,y_{\lambda_1}) := \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in[\lambda]\backslash D} \frac{1}{x_i-y_j}. \qquad (5)$$

Let λ/ν be the shape obtained by removing an inner corner of λ/μ. We denote this subtraction by $\mu \to \nu$. Note that $[\lambda/\nu]$ is a disconnected skew shape.

We need the following identity of $F_{\lambda/\mu}(\mathbf{x},\mathbf{y})$ from [8].

Lemma 11 (Pieri–Chevalley formula [8, Eq. (6.3)]).

$$F_{\lambda/\mu}(\mathbf{x},\mathbf{y}) = \frac{1}{x_1-y_1} \sum_{\mu\to\nu} F_{\lambda/\nu^1}(\mathbf{x},\mathbf{y})F_{\lambda/\nu^2}(\mathbf{x},\mathbf{y}), \qquad (6)$$

where λ/ν^1 and λ/ν^2 are the two connected border strips that form λ/ν.

2.5 q-analogues of linear extensions

A *labeled poset* (P,ω) is a poset P with a labeling $\omega : P \to [n]$. We call ω a *natural labeling* if for any $x,y \in P$ with $x <_P y$, we have $\omega(x) < \omega(y)$ [16]. We call ω a *reversed Schur labeling* if the labeling increases as it follows the path of the outer borderstrip of λ/μ from bottom right to top left. Such labeling is derived from the Schur labeling of λ/μ, which increases from bottom left to top right [16]. For Theorem 9 and Theorem 7, we use reversed Schur labeling on $[\lambda/\mu]$ and natural labeling on the d-complete posets. In the case of the inversion statistic, for each $\mu \to \nu$, we need $\omega(x_1) > \omega(x_2)$ for all $x_i \in \lambda/\nu_i$ to satisfy the condition for Proposition 21

Given a linear extension $f : P \to [n]$, the permutation $\omega \circ f^{-1} \in \mathfrak{S}_n$ is a *linear extension of the labeled poset, $\mathcal{L}(P,\omega)$*. For the major index, we label the poset using the *natural labeling* on the d-complete posets and the *reversed Schur labeling* on the borderstrip (see Figure 3 (b)). Such labeling is derived from the Schur labeling of λ/μ. We use the reversed Schur labeling instead of the Schur labeling because of the orientation of the conversion from a Young diagram to a poset we have chosen (see Figure 1).

The two common statistics for q-analogues of the number of linear extensions for a labeled poset (P,ω) are the major index and inversions. Let $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in \mathfrak{S}_n$ be a permutation. The *descent set* of σ is given by $\mathrm{Des}(\sigma) := \{i \in [n-1] \,|\, \sigma_i > \sigma_{i+1}\}$.

The *major index* of σ is $\mathrm{maj}(\sigma) = \sum_{i \in \mathrm{Des}(\sigma)} i$, and the *inversion index* of σ is $\mathrm{inv}(\sigma) = \#\{(i,j) \,|\, i < j$ and $\sigma_i > \sigma_j\}$.

There is a version of the major index for labeled d-complete posets and the inversion index for labeled posets. The *descent set* of labeled poset (P, ω) is given by $\mathrm{Des}(P, \omega) := \{x \in P \,|\, x <_P y$ and $\omega(x) > \omega(y)\}$.

Definition 12. The major index of a labeled poset (P, ω) is

$$\mathrm{maj}(P, \omega) = \sum_{x \in \mathrm{Des}(P, \omega)} h_P(x),$$

where $h_P(x)$ is the hook-length of $x \in P$.

Definition 13. The inversion index of a labeled poset (P, ω) is

$$\mathrm{inv}(P, \omega) = \#\{(x, y) \,|\, \omega(y) < \omega(x) \text{ and } x <_P y\}.$$

Let $\mathrm{stat} \in \{\mathrm{maj}, \mathrm{inv}\}$, the *major index (inversion) q-analogue* of the number of linear extensions of a labeled poset (P, ω) is

$$e_q^{\mathrm{stat}}(P, \omega) := \sum_{\sigma \in \mathcal{L}(P, \omega)} q^{\mathrm{stat}(\sigma)}.$$

Now we state the q-analogue of the hook-length formulas. We define the *q-integer* $[n]_q := 1 + q + \cdots + q^{n-1}$ and the *q-factorial* $[n]_q! := [n]_q \cdots [2]_q [1]_q$. Peterson and Proctor [13] gave a maj q-analogue formula for d-complete posets.

Theorem 14 (Peterson and Proctor [13]). *Let (P, ω) be a labeled d-complete poset of size n with any labeling. Then,*

$$e_q^{\mathrm{maj}}(P, \omega) = q^{\mathrm{maj}(P, \omega)} \frac{[n]_q!}{\prod_{x \in P} [h_P(x)]_q}.$$

Björner and Wachs [1] gave the inversion q-analogue formula for rooted tree posets.

Theorem 15 (Björner and Wachs [1]). *Let (P, ω) be a rooted tree poset with a natural labeling. Then*

$$e_q^{\mathrm{inv}}(P, \omega) = q^{\mathrm{inv}(P, \omega)} \frac{[n]_q!}{\prod_{x \in P} [h_P(x)]_q}.$$

2.6 Identities for e_q^{maj} from the theory of P-partitions

In this section, we state the definition of (P, ω)-partition and its connection to $e_q^{\mathrm{maj}}(P, \omega)$.

Definition 16. [16] A (P, ω)-partition is a map $f : P \to \mathbb{N}$ satisfying the conditions:

1. If $s \leqslant t$ in P, then $f(s) \geqslant \mathrm{f(t)}$.

2. If $s \leqslant t$ and $\omega(s) > \omega(t)$, then $f(s) > f(t)$.

If $\sum_{t \in P} f(t) = m$, then we say f is a (P, ω)-partition of m. We denote the set of all (P, ω)-partitions as $\mathcal{A}(P, \omega)$. First, we have the following definition.

Definition 17. Let $w = w_1 w_2 \ldots w_n \in \mathfrak{S}_n$. We say that the function $f' : [n] \to \mathbb{N}$ is w-compatible if the following two conditions hold.

1. $f'(w_1) \geqslant f'(w_2) \geqslant \cdots \geqslant f'(w_n)$.

2. $f'(w_i) > f'(w_{i+1})$ if $w_i > w_{i+1}$.

Define $f' : [n] \to \mathbb{N}$ by $f'(i) = f(\omega^{-1}(i))$, and let S_w be the set of all functions f such that f' is w-compatible. We have the following result called the fundamental lemma on (P, ω)-partitions.

Lemma 18 ([16], Lemma 3.15.3)**.** *A function $f : P \to \mathbb{N}$ is a (P, ω)-partition if and only if f' is w-compatible with some $w \in \mathcal{L}(P, \omega)$. In other words,*

$$\mathcal{A}(P, \omega) = \dot{\bigcup}_{w \in \mathcal{L}(P, \omega)} S_w,$$

where $\dot{\cup}$ denotes disjoint union.

Let $a_P(n)$ denote the number of (P, ω)-partitions of n and let $G_{P,\omega}(x) = \sum a_P(n) x^n$ be generating function of these partitions. Stanley gave the following specialization of the generating function associated with (P, ω)-partition:

Theorem 19 (Stanley [16])**.** *Let (P, ω) be a labeled poset of size p. Then we have,*

$$G_{P,\omega}(q) = \frac{e_q^{\mathrm{maj}}(P, \omega)}{\prod_{i=1}^{p}(1 - q^i)}.$$

For any disjoint union of labeled posets $(P, \omega_1) + (Q, \omega_2)$, by definition, we have

$$G_{P+Q,\omega_1+\omega_2}(q) = G_{P,\omega_1}(q) \cdot G_{Q,\omega_2}(q).$$

By applying Theorem 19 to the above equation, we have the following corollary.

Corollary 20. *[16, Exercise 3.162(a)] Let $(P + Q, \omega)$ be a labeled disjoint sum of posets with $|P + Q| = n$ and $|P| = p$. For any labeling ω, we have*

$$e_q^{\mathrm{maj}}(P + Q, \omega) = \begin{bmatrix} n \\ p \end{bmatrix}_q e_q^{\mathrm{maj}}(P, \omega_1) \cdot e_q^{\mathrm{maj}}(Q, \omega_2),$$

where ω_1 and ω_2 are the labelings obtained by restricting ω to P and Q respectively.

2.7 Identities for e_q^{inv} for disjoint union of posets

We also have the following identity for the disjoint union of posets in the case of the inversion index.

Proposition 21 (Björner-Wachs,[1])**.** *Let $(P + Q, \omega)$ be a labeled disjoint sum of posets with $|P + Q| = n$ and $|P| = p$. Suppose that ω has the property that the label of every element of P is smaller than the label of every element of Q. We have*

$$e_q^{\text{inv}}(P + Q, \omega) = \begin{bmatrix} n \\ p \end{bmatrix}_q e_q^{\text{inv}}(P, \omega_1) \cdot e_q^{\text{inv}}(Q, \omega_2),$$

where ω_1 and ω_2 are the labeling obtained by restricting ω to P and Q respectively.

Remark 22. Note that the disjoint poset identity for the inversion statistic has a more specific condition on the poset labeling than the major index does. When applying the Theorem 9, we need to label the mobile posets so that it satisfies the condition of Proposition 21. More detail about the labeling of the poset for the case of inversion index is stated in Section 6.

3 *P*-partition with a fixed point

In this section, we discuss a variation of Theorem 19, where we fix the position of an element in P. As a result, we have Corollary 27, which is used to prove the recurrence of e_q^{maj}.

Denote $\mathcal{L}(P, \omega; \{s\})$ the set of linear extensions of (P, ω) that end with $\omega(s)$ for a fixed $s \in P$ and $e_q^{\text{maj}}(P, \omega; \{s\}) = \sum_{\sigma \in \mathcal{L}(P, \omega; \{s\})} q^{\text{maj}(\sigma)}$. We omit ω and assume that the poset is labeled (P, ω) unless specified. There is analogue of Corollary 20 for $e_q^{\text{maj}}(P; \{s\})$.

We can restrict the conditions on $\mathcal{A}(P, \omega)$ so that it only considers the linear extensions that ends with a fixed element $s \in P$. We define the following restricted (P, ω)-partition:

Definition 23. A $(P, \omega; \{s\})$-partition is a map $f : P \to \mathbb{N}$ satisfying the following conditions:

1. f is a (P, ω)-partition

2. $f(s) \leqslant f(t)$ for all $t \in P$

3. $f(s) = f(t)$ then $\omega(s) > \omega(t)$

Let $\mathcal{A}(P, \omega; \{s\})$ denote the set of such partitions.

Lemma 24. *A function $f : P \to \mathbb{N}$ is a $(P, \omega; \{s\})$-partition if and only if f' is w-compatible with some $w \in \mathcal{L}(P, \omega)$. In other words,*

$$\mathcal{A}(P, \omega; \{s\}) = \dot{\bigcup}_{w \in \mathcal{L}(P; \{s\})} S_w.$$

Proof. By the definition of a $(P, \omega; \{s\})$-partition, the set $\mathcal{A}(P, \omega; \{s\})$ consists of the (P, ω)-partitions such that f' is w-compatible with some $w \in \mathcal{L}(P; \{s\})$. Then we can restrict the Lemma 18 to $\mathcal{L}(P; \{s\})$. □

Let $a_P^s(n)$ be the number of $(P, \omega; \{s\})$-partition of size n, and let $G_{P;\{s\}}(x) = \sum a_P^s(n) x^n$ be the generating function. Then we can restrict Theorem 19 to $\mathcal{L}(P, \omega; \{s\})$. Denote $e_q^{\mathrm{maj}}(P; \{s\}) := \sum_{\sigma \in \mathcal{L}(P_{\lambda/\mu}; \{s\})} q^{\mathrm{maj}(\sigma)}$.

Theorem 25. *Let (P, ω) be a labeled poset of size p and $s \in P$. Then we have,*

$$G_{P;\{s\}}(q) = \frac{e_q^{\mathrm{maj}}(P; \{s\})}{\prod_{i=1}^{p}(1 - q^i)}.$$

Proof. This is a consequence of Lemma 24. □

We have the following lemma for the disjoint union of posets with the fixed point.

Lemma 26. *Let $(P + Q, \omega)$ be a labeled disjoint union of posets and fix $s \in P$ such that $\omega(s) > \omega(t)$ for all $t \in Q$. Let $p = |P|$ and $n = |P + Q|$. Then we have*

$$(1 - q^n) \cdot G_{P+Q;\{s\}}(q) = (1 - q^p) \cdot G_{P;\{s\}}(q) \cdot G_Q(q).$$

Proof. To prove the lemma, we first give a combinatorial interpretation of $H_{P;\{s\}}(q) := (1 - q^p) \cdot G_{P;\{s\}}(q)$. Consider the difference $G_{P;\{s\}} - q^p G_{P;\{s\}}$. We build an injection from the $(P; \{s\})$-partition of size $m - p$ to the $(P; \{s\})$-partition of size m. The coefficients of q^m from the generating function $q^p \cdot G_{P;\{s\}}(q)$ counts the number of $(P; \{s\})$-partitions of size $m - p$. Note that for any $(P; \{s\})$-partition f of size $m - p$, we can obtain a P-partition of size m by adding an element for each part of f. Likewise, given any $(P; \{s\})$-partition g of size m such that $g(i) > 0$ for all i, we can obtain a $(P; \{s\})$-partition of size $m - p$ by subtracting one from each part of g. Then the coefficients of q^m of $H_{P;\{s\}}(q)$ counts the number of $(P; \{s\})$-partitions g of size m with $g(s)$ is a minimum value, $\omega(s) > \omega(t)$ for all $t \in P$ where $g(s) = g(t)$, and with at least one zero value. Thus it is necessary and sufficient for $g(s) = 0$. Similarly, $H_{P+Q;\{s\}}(q)$ is a generating function for $(P + Q; \{s\})$-partition f of size n such that (i) $f(s) = 0$ and (ii) $f(s) > f(t)$ for all $t \in P + Q$ such that $f(t) = 0$.

Then consider the right hand side of the equation. The coefficients of q^m counts size of a disjoint union of $(P; \{s\})$-partition f_1 of size k counted in $H_{P;\{s\}}(q)$ and Q-partition f_2 of size $n - k$ for some $k = 0, \ldots, n$. Such disjoint union is equivalent to the $(P + Q; \{s\})$-partitions counted in $H_{+Q;\{s\}}(q)$. To see this note that such $(P + Q; \{s\})$-partition f satisfies condition (i), namely $f(s) = f_1(s) = 0$. Next, we verify that f satisfies the condition (ii). If $t \in P$ with $f(t) = f_1(t) = 0$ then $\omega(s) > \omega(t)$ since σ_1 satisfies condition (ii). If $t \in Q$ with $f_2(t) = 0$, then by assumption $\omega(s) > \omega(t)$. Thus, we have that $H_{P+Q;\{s\}}(q) = H_{P;\{s\}}(q) \cdot G_Q(q)$ as desired. □

Then by applying Theorem 25 to Lemma 26, we have the following corollary which we will use to prove our main result in the next section.

Corollary 27. *Let $(P + Q, \omega)$ be a labeled disjoint sum of posets and fix $s \in P$ such that $\omega(s) > \omega(t)$ for all $t \in Q$. Let $|P + Q| = n$ and $|P| = p$. Then we have,*

$$e_q^{\mathrm{maj}}(P + Q; \{s\}) = \begin{bmatrix} n - 1 \\ p - 1 \end{bmatrix}_q e_q^{\mathrm{maj}}(P; \{s\}) \cdot e_q^{\mathrm{maj}}(Q).$$

4 A Major index q-analogue

In this section we give the proof of Theorem 7. The proof follows the proof of the NHLF for border strips in [8]. We need to first define the hook-lengths of mobile posets. Given a mobile poset $P_{\lambda/\mu}(\mathbf{p})$, define the hook-length of $(i, j) \in [\lambda]$ as follows:

$$h'(i, j) = \lambda_i - i + \lambda'_j - j + 1 + \sum_{a \geqslant i, b \geqslant j} p_{a,b}. \tag{7}$$

In other words, it is the usual hook-length of the cell in λ plus the size of the d-complete posets that are attached on the segment of the border strip inside of the hook of (i, j) (see Figure 3 (a)). We provide an example of the of the theorem below.
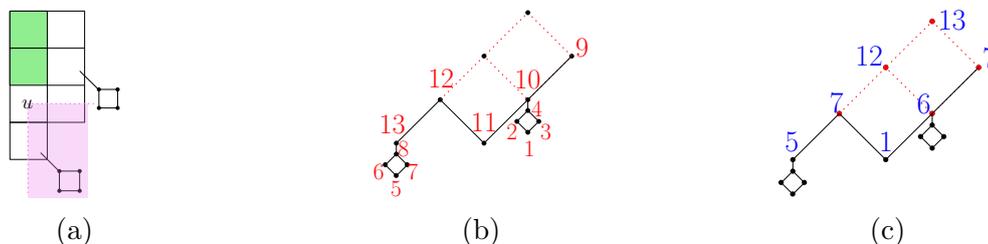

(a)   (b)   (c)

Figure 3: (a): $h'(u)$ is the usual hook-length plus the size of d-complete posets in the shaded area. (b): mobile poset with reversed Schur labeling. (c): mobile poset with hook-lengths.

Example 28. Consider the poset $(P_{2221/11}, \omega)$ from Figure 3 (b). By Theorem 7, one can check that

$$\begin{aligned}
e_q^{\mathrm{maj}}(P) &= \frac{[13]!}{[1]^2[2]^4[3]^2} \left(\frac{q^{12}}{[5][7][1][6][7]} + \frac{q^{18}}{[5][7][12][6][7]} + \frac{q^{24}}{[5][7][12][13][7]} \right) \\
&= q^{61} + 2q^{60} + 6q^{59} + 11q^{58} + \cdots + 6q^{14} + 2q^{13} + q^{12}.
\end{aligned}$$

We first introduce two lemmas required for the proof. We provide the proof of each lemma in Section 4.1 and Section 4.2.

We have the following recurrence lemma for the the q-analogue of linear extensions for the major index.

Lemma 29. *For a labeled mobile poset $(P_{\lambda/\mu}(\mathbf{p}), \omega)$, where ω is a reversed Schur labeling,*

$$e_q^{\mathrm{maj}}(P_{\lambda/\mu}, \omega) = \sum_{\mu \to \nu} q^{|P_{\lambda/\nu_1}|} e_q^{\mathrm{maj}}(P_{\lambda/\nu}, \omega_\nu) \tag{8}$$

where P_{λ/ν_1} is the left disconnected poset of $P_{\lambda/\nu}$, and ω_ν is the restricted labeling of ω onto $P_{\lambda/\nu}$.

Next, we have the following Pieri–Chevalley formula. Denote the RHS of (7) as $H_{\lambda/\mu}(q)$.

$$H_{\lambda/\mu}(q) := \prod_{v \in \mathbf{p}} \frac{1}{1 - q^{h(v)}} \sum_{D \in \mathcal{E}(\lambda/\mu)} q^{w'(D)} \prod_{u \in [\lambda] \setminus D} \frac{1}{1 - q^{h'(u)}}$$

Lemma 30.

$$(1 - q^n) \cdot H_{\lambda/\mu}(q) = \sum_{\mu \to \nu} \frac{q^{|P_{\lambda/\nu^1}|}}{\prod_{v \in T_\nu}(1 - q^{h(v)})} \cdot H_{\lambda/\nu^1}(q) \cdot H_{\lambda/\nu^2}(q), \tag{9}$$

where T_ν is the union of the d-complete posets that were hanging on the removed inner corner u.

We are now ready to give the proof of Theorem 7.

Proof of Theorem 7. We first evaluate the multivariate formula $F_{\lambda/\mu}$ at $x = q^{\lambda_i - i + 1 - \sum_{a < i} p_{a,b}}$ and $y_j = q^{j - \lambda'_j - \sum_{b \geqslant j} p_{a,b}}$. We denote $e_q^{\mathrm{maj}}(P, \omega)$ as $e_q(P)$ unless specified. We show that $e_q^{\mathrm{maj}}(P_{\lambda/\mu}(\mathbf{p})) = \prod_{i=1}^n (1 - q^i) H_{\lambda/\mu}(q)$ by induction on $|\lambda/\mu|$ using Lemma 29. Note that λ/ν is disconnected and

$$P_{\lambda/\nu} = P_{\lambda/\nu^1} + P_{\lambda/\nu^2} + T_\nu, \tag{10}$$

where T_ν is the union of the d-complete posets that were hanging on the removed inner corner u. Denote $|P_{\lambda/\nu^j}|$ as p_j. By induction, we have for $j = 1, 2$

$$\frac{e_q(P_{\lambda/\nu^j})}{[p_j]_q!} = \prod_{i=1}^{p_j} (1 - q^i) \cdot H_{\lambda/\nu^j} \cdot \frac{(1 - q)^{p_j}}{\prod_{i=1}^{p_j}(1 - q^i)} = (1 - q)^{p_j} \cdot H_{\lambda/\nu^j},$$

and by Theorem 14, for each $T_i \subset T_\nu$ we have,

$$\frac{e_q(T_i)}{[t_i]_q!} = \frac{q^{\mathrm{maj}(T_i)}}{\prod_{v \in T_i}[h(v)]_q} = \frac{(1 - q)^{t_i}}{\prod_{v \in T_i}(1 - q^{h(v)})}.$$

Note that T_i are natural labeling, so $\mathrm{maj}(T_i) = 0$.
Using Corollary 20 and the equations above, we have

$$e_q(P_{\lambda/\nu}) = \frac{\prod_{i=1}^{n-1}(1 - q^i)}{\prod_{v \in T_\nu}(1 - q^{h(v)})} H_{\lambda/\nu^1}(q) \cdot H_{\lambda/\nu^2}(q).$$

We now apply the equation to (8):

$$e_q(P_{\lambda/\mu}) = \prod_{i=1}^{n-1}(1 - q^i) \cdot \sum_{\mu \to \nu} \frac{q^{|\lambda/\nu_1|}}{\prod_{v \in T_\nu}(1 - q^{h(v)})} \cdot H_{\lambda/\nu^1}(q) \cdot H_{\lambda/\nu^2}(q) \tag{11}$$

When ω is a reversed Schur labeling, we have $\lambda'_1 - 1 + c(u) + p_1 = |\lambda/\nu_1|$, where $c(u) = j - i$ is the content of $u = (i, j)$. By (9), we can show the sum on the RHS of (11) equals $(1 - q^n) \cdot H_{\lambda/\mu}(q)$, completing the proof. $\qquad\square$

Remark 31. We can generalize the labeling of the mobile poset by allowing non-natural labeling on the d-complete posets. In such case, we would have non-trivial values for $q^{\mathrm{maj}(T_i)}$ in our final formula.

4.1 Proof of Lemma 29

To prove Lemma 29, we need the following lemmas.

Let $(P_{\lambda/\mu}, \omega; U) := \{\sigma \in \mathcal{L}(P_{\lambda/\mu}, \omega) \mid \sigma \text{ ends with } u \in U\}$ and

$$e_q^{\mathrm{maj}}(P_{\lambda/\mu}, \omega; U) := \sum_{\sigma \in (P_{\lambda/\mu}, \omega; U)} q^{\mathrm{maj}(\sigma)}.$$

Unless specified otherwise, we denote this as $e_q^{\mathrm{maj}}(P_{\lambda/\mu}; U)$.

Lemma 32. *For a labeled mobile poset $(P_{\lambda/\mu}(\mathbf{p}), \omega)$, where ω is a reversed Schur labeling, let $\mu \to \nu$ be the removal of an inner corner u, and P_{λ/ν_1} and P_{λ/ν_2} be the two disconnected parts of λ/ν. Then we have*

$$e_q^{\mathrm{maj}}(P_{\lambda/\mu}; \{u\}) = q^{n-1} e_q^{\mathrm{maj}}(P_{\lambda/\nu}; [\lambda/\nu_1]) + e_q^{\mathrm{maj}}(P_{\lambda/\nu}; [\lambda/\nu_2]). \tag{12}$$

Proof. The linear extension on the left is $\sigma \in \mathcal{L}(P_{\lambda/\mu})$, such that $\sigma = \sigma_1 \ldots \sigma_{n-1} \sigma_n$, where $\sigma_1 \ldots \sigma_{n-1} \in \mathcal{L}(P_{\lambda/\nu})$ and $\sigma_n = \omega(u)$. Then σ_{n-1} is either an element in Q_{λ/ν_1} or Q_{λ/ν_2}. If $\sigma_{n-1} \in Q_{\lambda/\nu_1}$, since ω is a reversed Schur labeling, $\omega(\sigma_{n-1}) > \omega(\sigma_n)$, so $\mathrm{maj}(\sigma) = \mathrm{maj}(\sigma_1 \ldots \sigma_{n-1}) + n - 1$. If $\sigma_{n-1} \in Q_{\lambda/\nu_2}$, then $\omega(\sigma_{n-1}) < \omega(\sigma_n)$, so $\mathrm{maj}(\sigma) = \mathrm{maj}(\sigma_1 \ldots \sigma_{n-1})$. $\square$

Now we are ready to prove Lemma 29.

Proof of Lemma 29. By Corollary 20 and a standard recurrence for q-binomial coefficients we have

$$e_q^{\mathrm{maj}}(P_{\lambda/\nu}) = \left(\begin{bmatrix} n-2 \\ |\lambda/\nu_1| \end{bmatrix}_q + q^{|\lambda/\nu_1|} \begin{bmatrix} n-2 \\ |\lambda/\nu_2| - 1 \end{bmatrix}_q \right) e_q^{\mathrm{maj}}(P_{\lambda/\nu_1}) e_q^{\mathrm{maj}}(P_{\lambda/\nu_2})$$

$$= \begin{bmatrix} n-2 \\ |\lambda/\nu_2| \end{bmatrix}_q e_q^{\mathrm{maj}}(P_{\lambda/\nu_1}) e_q^{\mathrm{maj}}(P_{\lambda/\nu_2}) + q^{|\lambda/\nu_1|} \begin{bmatrix} n-2 \\ |\lambda/\nu_2| - 1 \end{bmatrix}_q e_q^{\mathrm{maj}}(P_{\lambda/\nu_1}) e_q^{\mathrm{maj}}(P_{\lambda/\nu_2}). \tag{13}$$

Then the two parts of the sum can be interpreted in the following ways:

Proposition 33.

$$e_q^{\mathrm{maj}}(P_{\lambda/\nu}; [\lambda/\nu_1]) = \begin{bmatrix} n-2 \\ |\lambda/\nu_1| - 1, |\lambda/\nu_2| \end{bmatrix}_q e_q^{\mathrm{maj}}(P_{\lambda/\nu_1}) e_q^{\mathrm{maj}}(P_{\lambda/\nu_2}), \tag{14}$$

and

$$e_q^{\mathrm{maj}}(P_{\lambda/\nu}; [\lambda/\nu_2]) = q^{|\lambda/\nu_1|} \begin{bmatrix} n-2 \\ |\lambda/\nu_2| - 1 \end{bmatrix}_q e_q^{\mathrm{maj}}(P_{\lambda/\nu_1}) e_q^{\mathrm{maj}}(P_{\lambda/\nu_2}). \tag{15}$$

Proof. Let $\{x_1, \ldots x_k\}$ be the maximal elements of P_{λ/ν_1}. Note that

$$e_q^{\mathrm{maj}}(P_{\lambda/\nu}; [\lambda/\nu_1]) = \sum_{i=1}^{k} e_q^{\mathrm{maj}}(P_{\lambda/\nu}; x_i), \tag{16}$$

and

$$e_q^{\mathrm{maj}}(P_{\lambda/\nu_1}) = \sum_{i=1}^{k} e_q^{\mathrm{maj}}(P_{\lambda/\nu_1}; x_i). \tag{17}$$

We know that P_{λ/ν_1} and P_{λ/ν_2} satisfy the condition in Lemma 26. Then by Corollary 27, we have

$$e_q^{\mathrm{maj}}(P_{\lambda/\nu}; x_i) = \begin{bmatrix} n-2 \\ |\lambda/\nu_1| - 1, |\lambda/\nu_2| \end{bmatrix}_q e_q^{\mathrm{maj}}(P_{\lambda/\nu_1}; x_i) \cdot e_q^{\mathrm{maj}}(P_{\lambda/\nu_2}). \tag{18}$$

Applying (18) and (17) to (16), we have the desired result.

For the second equation, we know that $e_q^{\mathrm{maj}}(P_{\lambda/\nu}) = e_q^{\mathrm{maj}}(P_{\lambda/\nu}; [\lambda/\nu_1]) + e_q^{\mathrm{maj}}(P_{\lambda/\nu}; [\lambda/\nu_2])$, so subtracting (14) from $e_q^{\mathrm{maj}}(P_{\lambda/\nu})$, we have the desired result as well. $\qquad\square$

We now apply (14) and (15) to (12) to get the following equation. Let $p_1 = |P_{\lambda/\nu_1}|$ and $p_2 = |P_{\lambda/\nu_2}|$.

$$\begin{aligned}
&e_q^{\mathrm{maj}}(P_{\lambda/\mu}; \{u\}) \\
&= q^{n-1} \begin{bmatrix} n-2 \\ p_2 \end{bmatrix}_q e_q^{\mathrm{maj}}(P_{\lambda/\nu_1}) \cdot e_q^{\mathrm{maj}}(P_{\lambda/\nu_2}) + q^{p_1} \begin{bmatrix} n-2 \\ p_2 - 1 \end{bmatrix}_q e_q^{\mathrm{maj}}(P_{\lambda/\nu_1}) \cdot e_q^{\mathrm{maj}}(P_{\lambda/\nu_2})
\end{aligned}$$

After simplifying everything, we get

$$e_q^{\mathrm{maj}}(P_{\lambda/\mu}; \{u\}) = q^{p_1} \cdot e_q^{\mathrm{maj}}(P_{\lambda/\nu}).$$

Such equation is true for all inner corners u of $\mu \to \nu$, which completes the proof of Lemma 29. $\qquad\square$

4.2   Proof of Lemma 30

We first evaluate the Pieri–Chevalley formula (6) for $x_i = q^{\lambda_i - i + 1 - \sum_{a<i} p_{a,b}}$ and $y_j = q^{j - \lambda'_j - \sum_{b \geqslant j} p_{a,b}}$. The LHS of this formula becomes

$$F_{\lambda/\mu}(\mathbf{x}, \mathbf{y})\big|_{\substack{x_i = q^{\lambda_i - i + 1 - \sum_{a<i} p_{a,b}}, \\ y_j = q^{j - \lambda'_j - \sum_{b \geqslant j} p_{a,b}}}} = (-1)^n \cdot \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in [\lambda] \setminus D} \frac{q^{\lambda'_j - j + \sum_{b \geqslant j} p_{a,b}}}{1 - q^{h'(i,j)}} \tag{19}$$

By [10, Proposition 4.7] we have

$$\sum_{(i,j) \in [\lambda] \setminus D} \left((\lambda'_j - j) + \sum_{b \geqslant j} p_{a,b} \right) = \sum_{(i,j) \in [\lambda] \setminus D} \left((\lambda'_j - i) + \sum_{b \geqslant j} p_{a,b} \right) - \sum_{(i,j) \in [\lambda] \setminus [\mu]} c(i,j),$$

where in the last equality, we use [10, Proposition 7.16] to obtain,

$$\sum_{(i,j)\in[\lambda]\setminus D}\left((\lambda'_j-j)+\sum_{b\geqslant j}p_{a,b}\right)=w'(\mathrm{Br}(D))+\left(\sum_{(i,j)\in[\lambda]\setminus D}\sum_{b\geqslant j}p_{a,b}-\sum_{\substack{a\geqslant i,b\geqslant j\\(i,j)\in\mathrm{Br}(D)}}p_{a,b}\right)$$
$$-\sum_{(i,j)\in[\lambda]\setminus[\mu]}c(i,j) \tag{20}$$

where $w'(\mathrm{Br}(D))=\sum_{(i,j)\in\mathrm{Br}(D)}h'(i,j)$, and the subtraction in the second sum is the d-complete posets that are included in the new hook-lengths of the broken diagonals. We denote the quantity in parenthesis on the RHS as $\widehat{p}_D$.

$$\widehat{p}_D:=\sum_{(i,j)\in[\lambda]\setminus D}\sum_{b\geqslant j}p_{a,b}-\sum_{\substack{a\geqslant i,b\geqslant j\\(i,j)\in\mathrm{Br}(D)}}p_{a,b}$$

We claim that $\widehat{p}_D$ is invariant under the changes of D's.

Lemma 34. *The quantity $\widehat{p}_D$ is invariant among all $D\in\mathcal{E}(\lambda/\mu)$.*

Proof. We prove by induction on excited moves $\beta:D\to D'$. Denote $a_i=(s_i,t_i)$ as the active cell and $b=(s+1,t+1),b'=(s+1,t)$ be the old and new broken diagonal of an excited move $\beta:D\to D'$, Then for each excited move β, $[\lambda]\setminus D'=([\lambda]\setminus D)\setminus\{b\}\cup\{a\}$. Also, $\mathrm{Br}(D')=\mathrm{Br}(D)\setminus\{b\}\cup\{b'\}$. Thus

$$\widehat{p}_{D'}=\left(\sum_{(i,j)\in[\lambda]\setminus D}\sum_{b\geqslant j}p_{a,b}-\sum_{b\geqslant t_i+1}p_{a,b}+\sum_{b\geqslant t_i}p_{a,b}\right)-\left(\sum_{\substack{a\geqslant i,b\geqslant j\\(i,j)\in\mathrm{Br}(D)}}p_{a,b}-\sum_{\substack{a\geqslant s_i+1,\\b\geqslant t_i+1}}p_{a,b}+\sum_{\substack{a\geqslant s_i+1,\\b\geqslant t_i}}p_{a,b}\right)$$
$$=\left(\sum_{(i,j)\in[\lambda]\setminus D}\sum_{b\geqslant j}p_{a,b}+\sum_{b=t_i}p_{a,b}\right)-\left(\sum_{\substack{a\geqslant i,b\geqslant j\\(i,j)\in\mathrm{Br}(D)}}p_{a,b}+\sum_{\substack{a\geqslant s_i+1,\\b=t_i}}p_{a,b}\right)$$
$$=\widehat{p}_D+\sum_{b=t_i}p_{a,b}-\sum_{\substack{a\geqslant s_i+1,\\b=t_i}}p_{a,b}$$

For λ/μ border strip, there cannot be any d-complete posets hanging above an active cell in the same column, so the difference on the RHS of the equation above is zero and so $\widehat{P}_{D'}=\widehat{P}_D$. $\square$

We then let $\widehat{p}_D=\widehat{p}$ for all $D\in\mathcal{E}(\lambda/\mu)$. Putting $\widehat{p}$ and $c(i,j)$ outside of the sum, we can rewrite (19) as:

$$F_{\lambda/\mu}(\mathbf{x}|\mathbf{y})\Big|_{\substack{x_i=q^{\lambda_i-i+1-\sum_{a<i}p_{a,b}},\\y_j=q^{j-\lambda'_j-\sum_{b\geqslant j}p_{a,b}}}}=(-1)^n\cdot q^{\widehat{p}_\mu-\mathrm{con}(\lambda/\mu)}\prod_{v\in\mathbf{p}}(1-q^{h(v)})\cdot H_{\lambda/\mu}, \tag{21}$$

where $\mathrm{con}(\lambda/\mu) := \sum_{(i,j)\in[\lambda/\mu]} c(i,j)$. Next we see what happens to the RHS of Pieri–Chevalley formula (6) when we evaluate at $x_i = q^{\lambda_i - i + 1 - \sum_{a<i} p_{a,b}}$ and $y_j = q^{j - \lambda'_j - \sum_{b\geqslant j} p_{a,b}}$, where the $p_{a,b}$ appearing in the sums are the sizes of the d-complete posets in $P_{\lambda/\mu}(\mathbf{p})$. The linear factor on the RHS of Pieri–Chevalley formula becomes

$$\frac{1}{x_1 - y_1}\bigg|_{\substack{x_i=q^{\lambda_i-i+1-\sum_{a<i} p_{a,b}}, \\ y_j=q^{j-\lambda'_j-\sum_{b\geqslant j} p_{a,b}}}} = \frac{(-1)^{n-1}}{q^{\lambda_1-\sum_{a<1} p_{a,b}} - q^{1-\lambda'_1-\sum_{b\geqslant 1} p_{a,b}}} = \frac{(-1)^n q^{\lambda'_1-1+\sum_{(a,b)\in[\lambda/\mu]} p_{a,b}}}{(1-q^n)}.$$

(22)

Given an inner corner removed $\mu \to \nu$, denote $\mathbf{p}_1$ and $\mathbf{p}_2$ be the set of d-complete poset hanging on P_{λ/ν^1} and P_{λ/ν^2} respectively. Then for the shapes λ/ν^k where $k = 1, 2$ we have

$$F_{\lambda/\nu^k}(\mathbf{x}|\mathbf{y})\big|_{\substack{x_i=q^{\lambda_i-i+1-\sum_{a<i} p_{a,b}}, \\ y_j=q^{j-\lambda'_j-\sum_{b\geqslant j} p_{a,b}}}} = (-1)^{|\lambda/\nu^k|} \cdot q^{\widehat{p}_{\nu^k}-\mathrm{con}(\lambda/\nu^k)} \prod_{v\in\mathbf{p}_k}(1-q^{h(v)}) \cdot H_{\lambda/\nu^k}, \quad (23)$$

$$\frac{\prod_{v\in\mathbf{p}_1}(1-q^{h(v)}) \cdot \prod_{v\in\mathbf{p}_2}(1-q^{h(v)})}{\prod_{v\in\mathbf{p}}(1-q^{h(v)})} = \frac{1}{\prod_{v\in T_\nu}(1-q^{h(v)})}$$

Thus, by (21), (22), and (23) the Pieri–Chevalley formula (6) evaluated at such x_i and y_j becomes,

$$q^{\widehat{p}_\mu-\mathrm{con}(\lambda/\mu)} H_{\lambda/\mu} = \frac{q^{\lambda'_1-1+\sum_{(a,b)\in[\lambda/\mu]} p_{a,b}}}{(1-q^n)} \sum_{\mu\to\nu} \frac{q^{\widehat{p}_{\nu_1}+\widehat{p}_{\nu_2}-\mathrm{con}(\lambda/\nu_1)-\mathrm{con}(\lambda/\nu_2)}}{\prod_{v\in T_\nu}(1-q^{h(v)})} H_{\lambda/\nu_1} H_{\lambda/\nu_2}. \quad (24)$$

Note that for each inner corner $u : \mu \to \nu$, we have $\mathrm{con}(\lambda/\mu) - \mathrm{con}(\lambda/\nu_1) - \mathrm{con}(\lambda/\nu_2) = c(u)$. Thus the previous equation becomes

$$(1-q^n) H_{\lambda/\mu} = \sum_{\mu\to\nu} \frac{q^{\lambda'_1-1+c(u)+\sum_{(a,b)\in[\lambda/\mu]} p_{a,b}+\widehat{p}_{\nu_1}+\widehat{p}_{\nu_2}-\widehat{p}_\mu}}{\prod_{v\in T_\nu}(1-q^{h(v)})} H_{\lambda/\nu_1} H_{\lambda/\nu_2}.$$

Lemma 35. *For the equation defined above,*

$$\sum_{(a,b)\in[\lambda/\mu]} p_{a,b} + \widehat{p}_{\nu_1} + \widehat{p}_{\nu_2} - \widehat{p}_\mu = |\mathbf{p}_1|.$$

Proof. Consider $\sum_{b\geqslant 1} p_{a,b} + \widehat{p}_{\nu_1} + \widehat{p}_{\nu_2} - \widehat{p}_\mu$. We know that $\widehat{p}$ is invariant among the excited diagrams, so without loss of generality, assume $D = [\mu]$.

We have $\mathrm{Br}([\nu_1]) \cup \mathrm{Br}([\nu_2]) \cup \{u_0\} = \mathrm{Br}([\lambda/\mu])$, where $u_0 = (u_1 + 1, u_2)$ is a broken diagonal of $[\mu]$ below $u = (u_1, u_2)$. Denote p_i as the size of the d-complete posets hanging on λ/ν_i. Then,
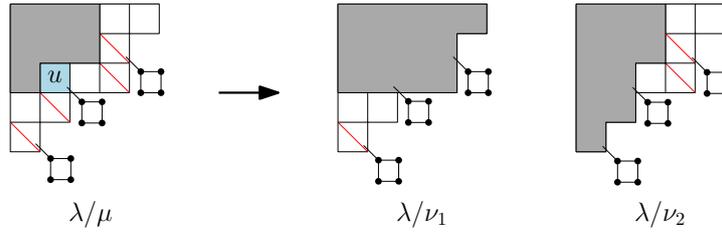
Figure 4: $[\lambda/\mu] = [\lambda/\nu_1] + [\lambda/\nu_2] + u$. Note that in λ/ν_1, we are missing a broken diagonal underneath u.

$$\sum_{b\geqslant 1} p_{a,b} + \widehat{p}_{\nu_1} + \widehat{p}_{\nu_2} - \widehat{p}_{\mu} = \sum_{b\geqslant 1} p_{a,b} + \left(\sum_{(i,j)\in[\lambda/\nu_1]}\sum_{b\geqslant j} p_{a,b} + \sum_{(i,j)\in[\lambda/\nu_2]}\sum_{b\geqslant j} p_{a,b} - \sum_{(i,j)\in[\lambda/\mu]}\sum_{b\geqslant j} p_{a,b} \right)$$

$$- \left(\sum_{\substack{a\geqslant i,b\geqslant j \\ (i,j)\in \mathrm{Br}([\nu_1])}} p_{a,b} + \sum_{\substack{a\geqslant i,b\geqslant j \\ (i,j)\in \mathrm{Br}([\nu_2])}} p_{a,b} - \sum_{\substack{a\geqslant i,b\geqslant j \\ (i,j)\in \mathrm{Br}([\mu])}} p_{a,b} \right) \quad (25)$$

Let S_1 and S_2 be the sums in parenthesis on the RHS above. Since λ/ν_1 and λ/ν_2 do not contain the inner corner u (see Figure 4), S_1 and S_2 simplify to

$$S_1 = \sum_{b\geqslant u_2} p_{a,b}, \qquad S_2 = \sum_{a\geqslant u_1+1, b\geqslant u_2} p_{a,b}. \quad (26)$$

Thus Equation (25) becomes

$$\sum_{b\geqslant 1} p_{a,b} + \widehat{p}_{\nu_1} + \widehat{p}_{\nu_2} - \widehat{p}_{\mu} = \sum_{b\geqslant 1} p_{a,b} - \left(\sum_{b\geqslant u_2} p_{a,b} - \sum_{a\geqslant u_1+1, b\geqslant u_2} p_{a,b} \right).$$

Note that the equation in the parenthesis counts the number of d-complete posets hanging on and to the the right of u. These are exactly the d-complete posets on P_{λ/ν^1}. $\square$

Finally, note that $\lambda_1' - 1 + c(u) + |\mathbf{p}_1| = |P_{\lambda/\nu_1}|$. Then we can simplify (4.2) to obtain the desired formula.

5 Application: bounds for the number of linear extensions

In this section, we provide a short proof of Corollary 8 along with an example. We also discuss an application of the formula to the bounds of generalized Euler numbers.

5.1 The case of $q = 1$

proof of Corollary 8. Evaluating $q = 1$ in Theorem 7 gives the desired identity (3).

One can also prove Corollary 8 directly by evaluating the multivariate formula $F_{\lambda/\mu}$ from (6) at $x_i = \lambda_i - i + 1 - \sum_{a<i} p_{a,b}$ and $y_j = j - \lambda_j' - \sum_{b\geqslant j} p_{a,b}$ $\square$

Figure 5: (a) the ω_{inv} labeled mobile tree poset, (b) illustration of the posets $\mathcal{C}_p(k)$ (top) and $\mathcal{A}_p(k)$ (bottom).

We give an example of the theorem below.

Example 36. Consider the mobile poset $P_{2221/11}$ from Figure 3 (b). Then by Corollary 8 we have

$$e(P) = \frac{13!}{2^4 \cdot 3^2} \left(\frac{1}{5 \cdot 6 \cdot 7^2} + \frac{1}{5 \cdot 6 \cdot 7^2 \cdot 12} + \frac{1}{5 \cdot 7^2 \cdot 12 \cdot 13} \right) = 33000. \tag{27}$$

5.2 Bounds to generalizations of Euler numbers

As an application to Corollary 8 gives bounds to $e(P_{\lambda/\mu}(\mathbf{p}))$ just as in [9].

Corollary 37. *For any mobile poset $e(P_{\lambda/\mu}(\mathbf{p}))$ of size n,*

$$\frac{n!}{H(\mathbf{p}) \prod_{u \in [\lambda/\mu]} h'(u)} \leqslant e(P_{\lambda/\mu}(\mathbf{p})) \leqslant |\mathcal{E}(\lambda/\mu)| \cdot \frac{n!}{H(\mathbf{p}) \prod_{u \in [\lambda/\mu]} h'(u)}$$

where $[\lambda/\mu]$ is the border strip of the mobile poset.

Proof. For any skew shape λ/μ, we have $[\mu] \in \mathcal{E}(\lambda/\mu)$, so the lower bound is given by Corollary 8. For the upper bound, note that under the excited move, the product $\prod_{v \in [\lambda/\mu]} h'(u)$ increases. Then this product is minimal when the excited diagram is $[\mu]$ and the upper bound follows from Corollary 8. $\square$

For more detail about asymptotic of linear extensions of skew shaped tableaux, see [9].

One application of the formula is that it provides bounds to generalizations of *Euler numbers* defined in [5]. The authors give two generalizations of Euler number using two different families of posets, up-down posets with $k - 1$ downs and chains (or anti-chains) of size p hanging on every minimal element, denoted as $\mathcal{C}_p(k)$ and $\mathcal{A}_p(k)$ (see Figure 3 (b)). See [14, A332471] and [14, A332568] for examples of these sequences.

Corollary 38. *For a mobile poset $P_{\lambda/\mu}(\mathbf{p})$ that is either $\mathcal{C}_p(k)$ or $\mathcal{A}_p(k)$ for k and p non-negative integers, we have*

$$\frac{(2k+kp)!}{(p+1)!^k(2p+3)^{k-1}(p+2)} \leqslant e(\mathcal{C}_p(k)) \leqslant \mathrm{Cat}(k) \cdot \frac{(2k+kp)!}{(p+1)!^k(2p+3)^{k-1}(p+2)}$$

$$\frac{(2k+kp)!}{(p+1)^k(2p+3)^{k-1}(p+2)} \leqslant e(\mathcal{A}_p(k)) \leqslant \mathrm{Cat}(k) \cdot \frac{(2k+kp)!}{(p+1)^k(2p+3)^{k-1}(p+2)}$$

where $\mathcal{Z}$ is the up-down border strip with $k-1$ many down steps and $\mathrm{Cat}(k) = \frac{1}{k+1}\binom{2k}{k}$ is the kth Catalan number.

Proof. The result follows from Corollary 37, a routine calculations of hooks, and the fact that the excited diagrams of up-down posets are given by the Catalan numbers [8, Corollary 8.1]

$\square$

6   An Inversion index q-analogue

In this section we give an example and the proof of Theorem 9. Unless specified otherwise, $(P_{\lambda/\mu}(\mathbf{p}), \omega)$ is a labeled mobile tree poset.

6.1   Labeling of the poset for the case of inversion index

The mobile tree poset must satisfy a very specific labeling for the case of the inversion statistic. One of the reasons why is because of the condition stated in Proposition 21. Another reason is that the labeling needs to satisfy Lemma 41. To satisfy both conditions, we must label the poset in the following way: let $\{u_1, \ldots, u_k\}$ be the list of inner corners of $P_{\lambda/\mu}$ from $(1, \lambda_1)$ to $(\lambda_1', 1)$, and $u_0 = (1, \lambda_1)$ and $u_{k+1} = (\lambda_1', 1)$. Partition the mobile posets into $P_1, \ldots, P_{k+1}$ such that for each P_i, it contains all elements $(s, t) \in P_{\lambda/\mu}$ for $u_i < t \leqslant u_{i-1}$ and all the elements of rooted trees hanging on such (s, t). If $u_{k+1} = (\lambda_1', 1)$, then $P_{k+1} = u_{k+1}$. Starting from P_1, we label each P_i so that all the hanging rooted trees are naturally labeled and the elements in the border strip have reversed Schur labeling. See Figure 6 for an example. We denote such labeling as ω_{inv}.

Example 39. Consider the labeled mobile poset $(P_{2221/11}, \omega_{\mathrm{inv}})$ from Figure 5 (a). Then by Theorem 9 we have

$$e_q^{\mathrm{inv}}(P) = \frac{[11]!}{[1]^4[3]^2} \left(\frac{q^4}{[4][6][1][5][6]} + \frac{q^9}{[4][6][10][5][6]} + \frac{q^{14}}{[4][6][10][11][6]} \right)$$
$$= q^{38} + 4q^{37} + 9q^{36} + 17q^{35} + \cdots + 9q^6 + 4q^5 + q^4.$$

Remark 40. Note that Theorem 9 is only for mobile trees, where the d-complete posets are restricted to rooted trees. This is because there is no known hook-length formula for $e_q^{\mathrm{inv}}(P)$ when P is a general d-complete poset.
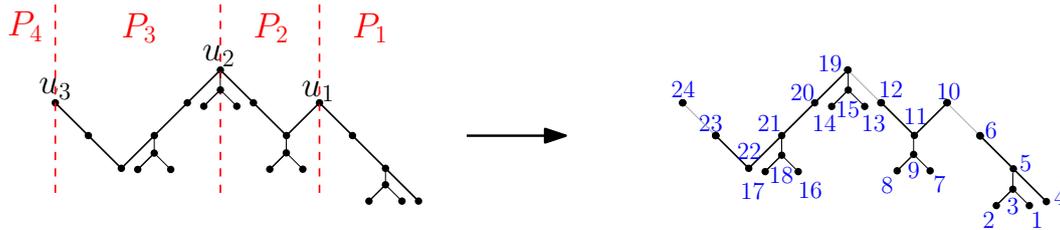
Figure 6: A mobile tree poset with labeling ω_{inv}. On the left we show the partitions of the poset into $P_1, \ldots, P_4$. We label each P_i so that the d-complete posets are naturally labeled and the elements in the border-strip have reversed Schur labeling.

We need the following recursion for the inversion index q-analogue.

Lemma 41.

$$e_q^{\text{inv}}(P_{\lambda/\mu}, \omega) = \sum_{\mu \to \nu} q^{n - \omega(u)} e_q^{\text{inv}}(P_{\lambda/\nu}, \omega_\nu), \tag{28}$$

where ω is a reversed Schur labeling and $\omega(u)$ is the label of the inner corner u from $\mu \to \nu$.

We also need the following Pieri–Chevalley formula for the inversion index. Denote the RHS of (4) as $\widetilde{H}_{\lambda/\mu}(q)$:

$$\widetilde{H}_{\lambda/\mu}(q) := \prod_{v \in \mathbf{p}} \frac{1}{1 - q^{h(v)}} \sum_{D \in \mathcal{E}(\lambda/\mu)} q^{w(D) + p_D} \prod_{u \in [\lambda] \setminus D} \frac{1}{1 - q^{h'(u)}},$$

where $p_D = \sum_{(i,j) \in [\mu] \setminus D} \sum_{s=j} p_{r,s}$.

Lemma 42.

$$(1 - q^n) \cdot \widetilde{H}_{\lambda/\mu} = \sum_{\mu \to \nu} \frac{q^{q^{n - \omega_{\text{inv}}(u)}}}{\prod_{v \in T_\nu}(1 - q^{h(v)})} \cdot \widetilde{H}_{\lambda/\nu^1} \cdot \widetilde{H}_{\lambda/\nu^2}, \tag{29}$$

where $\omega_{\text{inv}}(u)$ is the label of u, the inner corner from $\mu \to \nu$

We provide the proof of the lemmas in Section 6.2.

We are now ready to give the proof of Theorem 9.

Proof of Theorem 9. Similarly as in the case of major index, we show that $e_q^{\text{inv}}(P_{\lambda/\mu}(\mathbf{p})) = \prod_{i=1}^{n}(1 - q^i) \cdot \widetilde{H}_{\lambda/\mu}(q)$ by induction on $|\lambda/\mu|$ using Lemma 41. Recall $P_{\lambda/\nu}$ can be expressed as (10). By induction and Theorem 14, we have

$$\frac{e_q(P_{\lambda/\nu^j})}{[p_j]_q!} = \prod_{i=1}^{p_j}(1 - q^i) \cdot \widetilde{H}_{\lambda/\nu^j} \cdot \frac{(1 - q)^{p_j}}{\prod_{i=1}^{p_j}(1 - q^i)} = (1 - q)^{p_j} \cdot \widetilde{H}_{\lambda/\nu^j},$$

where $p_j = |P_{\lambda/\nu^j}|$. Also, for each $T_i \subset T_\nu$,

$$\frac{e_q^{\text{inv}}(T_i)}{[t_i]_q!} = \frac{q^{\text{inv}(T_i)}}{\prod_{v \in T_i}[h(v)]_q} = \frac{(1 - q)^{t_i}}{\prod_{v \in T_i}(1 - q^{h(v)})}. \tag{30}$$

Note that T_i are naturally labeled, so $\mathrm{inv}(T_i) = 0$. Using Proposition 21 and (30), we have

$$e_q(P_{\lambda/\nu}) = \frac{\prod_{i=1}^{n-1}(1-q^i)}{\prod_{v \in T_\nu}(1-q^{h(v)})} \widetilde{H}_{\lambda/\nu^1} \cdot \widetilde{H}_{\lambda/\nu^2}.$$

By this equation and Lemma 41.

$$e_q(P_{\lambda/\mu}) = \prod_{i=1}^{n-1}(1-q^i) \sum_{\mu \to \nu} \frac{q^{n-\omega_{\mathrm{inv}}(u)}}{\prod_{v \in T_\nu}(1-q^{h(v)})} \cdot \widetilde{H}_{\lambda/\nu^1} \cdot \widetilde{H}_{\lambda/\nu^2}. \tag{31}$$

By (29), we can show the sum on the RHS of (31) equals $(1-q^n) \cdot \widetilde{H}_{\lambda/\mu}$, completing the proof. $\qquad\square$

6.2 Proof of Lemma 41 and Lemma 42

Proof of Lemma 41. Recall that for a fixed tree mobile $P_{\lambda/\mu}$, a linear extension of $\sigma \in \mathcal{L}(P_{\lambda/\mu})$ consists of an inner corner of λ/μ followed by a linear extension of the remaining poset of shape λ/ν, where $\mu \to \nu$. Conversely, given a linear extension $\sigma' \in \mathcal{L}(P_{\lambda/\nu})$, by inserting the new element in the beginning we obtain a linear extension of $P_{\lambda/\mu}$. Note that $\mathrm{inv}(\sigma) = \mathrm{inv}(\sigma') + n - \omega$, where $n - \omega$ is the number of inversion caused by the inner corner. The result follows from this correspondence $\qquad\square$

To prove Lemma 42, we first evaluate $F_{\lambda/\mu}(\mathbf{x}, \mathbf{y})$ at $x_i = q^{\lambda_i - i + 1 - \sum_{a<i} p_{a,b}}$ and $y_j = q^{j - \lambda'_j - \sum_{b \geqslant j} p_{a,b}}$.

$$F_{\lambda/\mu}(\mathbf{x}, \mathbf{y}) \Big|_{\substack{x_i = q^{\lambda_i - i + 1 - \sum_{a<i} p_{a,b}}, \\ y_j = q^{j - \lambda'_j - \sum_{b \geqslant j} p_{a,b}}}} = (-1)^n \sum_{\substack{\gamma: A \to B, \\ \gamma \subset \lambda}} \prod_{(i,j) \in \gamma} \frac{q^{\lambda'_j - j + \sum_{b \geqslant j} p_{a,b}}}{1 - q^{h'(i,j)}} \tag{32}$$

By [10, Prop 4.7] and [10, Lemma 7.17], we have

$$\sum_{(i,j) \in \lambda \backslash D} \left((\lambda'_j - j) + \sum_{b \geqslant j} p_{a,b} \right) = \sum_{(i,j) \in [\lambda] \backslash D} \left((\lambda'_j - i) + \sum_{b \geqslant j} p_{a,b} \right) - \sum_{(i,j) \in [\lambda] \backslash [\mu]} c(i,j)$$

$$= w(\mathrm{Br}(D)) + \sum_{(i,j) \in [\lambda] \backslash D} \sum_{b \geqslant j} p_{a,b} - \sum_{(i,j) \in [\lambda] \backslash [\mu]} c(i,j), \tag{33}$$

where $w(\mathrm{Br}(D)) = \sum_{(i,j) \in Br(D)} h(i,j)$. Note that unlike the case of major index, we do not include the size of the rooted trees in $w(\mathrm{Br}(D))$ (see (20)).

Denote $\tilde{p}_{\lambda/\mu} := \sum_{(i,j) \in [\lambda]/[\mu]} \sum_{b \geqslant j} p_{a,b}$. For each $D \in \mathcal{E}(\lambda/\mu)$, we have

$$\sum_{(i,j) \in [\lambda] \backslash D} \sum_{b \geqslant j} p_{a,b} - \tilde{p}_{\lambda/\mu} = \sum_{(i,j) \in [\mu]/D} \sum_{b = j} p_{a,b} = p_D.$$

Then $\tilde{p}_{\lambda/\mu}$ and $c(i,j)$ from (33) do not depend on D, so we can take them outside of the sum to rewrite (32) as:

$$F_{\lambda/\mu}(\mathbf{x}|\mathbf{y}) \big|_{\substack{x_i = q^{\lambda'_1 - i + 1 - \sum_{a<i} p_{a,b}}, \\ y_j = q^{j - \lambda'_j - \sum_{b \geqslant j} p_{a,b}}}} = (-1)^n \cdot q^{\tilde{p}_{\lambda/\mu} - \operatorname{con}(\lambda/\mu)} \prod_{v \in \mathbf{p}} (1 - q^{h(v)}) \cdot \widetilde{H}_{\lambda/\mu} \qquad (34)$$

Now as done for the case of major index in Section 4, we evaluate the Pieri–Chevalley formula at such x_i and y_j. Then applying (34) to (6), and simplifying everything as we did in the case of major index, we have

$$(1 - q^n)\widetilde{H}_{\lambda/\mu} = \sum_{\mu \to \nu} \frac{q^{\lambda_1 - 1 + c(u) + \sum_{b \geqslant 1} p_{a,b} + \tilde{p}_{\lambda/\nu_1} + \tilde{p}_{\lambda/\nu_2} - \tilde{p}_{\lambda/\mu}}}{\prod_{v \in T_\nu} (1 - q^{h(v)})} \widetilde{H}_{\lambda/\nu_1} \widetilde{H}_{\lambda/\nu_2}. \qquad (35)$$

Note that from (26), the exponent of q is equivalent to

$$\sum_{b \geqslant 1} p_{a,b} + \tilde{p}_{\lambda/\nu_1} + \tilde{p}_{\lambda/\nu_2} - \tilde{p}_{\lambda/\mu} = \sum_{b \geqslant 1} p_{a,b} - \sum_{b \geqslant u_2} p_{a,b},$$

where u_2 is the column of the inner corner $u = (u_1, u_2)$. It is left to show the following lemma to complete the proof.

Lemma 43. *Let* $(P_{\lambda/\mu}(\mathbf{p}), \omega_{\mathrm{inv}})$ *be a mobile tree poset of size n with a labeling ω_{inv} and u be the inner corner for $\mu \to \nu$. Then we have,*

$$n - \omega_{\mathrm{inv}}(u) = \lambda'_1 - 1 + c(u) + \sum_{b \geqslant 1} p_{a,b} - \sum_{b \geqslant u_2} p_{a,b}.$$

Proof. First, we show that for a border strip $(P_{\lambda/\mu}, \omega)$ of size n_0 with a reversed Schur labeling we have $n_0 - \omega(x) = \lambda'_1 - 1 + c(x)$ for all $x \in P_{\lambda/\mu}$. Note that in a border strip, there is only one element per content. Also, for Schur labeling of a border strip, we have $\omega(\lambda'_1, 1) = 1$ and $\omega(1, \lambda_1) = n_0$. The element $(\lambda'_1, 1)$ satisfies the equation. Then as you follow the border strip, the content decreases by one while the label increases by one, so the rest of the elements satisfy the equation $n_0 - \omega(x) = \lambda'_1 - 1 + c(x)$.

Now for a labeled mobile tree poset $(P_{\lambda/\mu}(\mathbf{p}), \omega_{\mathrm{inv}})$ of size n, for any $x \in P_{\lambda/\mu}(\mathbf{p})$, $\omega_{\mathrm{inv}}(x)$ gets shifted by $\sum_{b \geqslant x_2} p_{a,b}$. Then $\omega_{\mathrm{inv}}(x) = \omega(x) + \sum_{b \geqslant x_2} p_{a,b}$. Also, we have that $n = n_0 + \sum_{b \geqslant 1} p_{a,b}$. Then applying such shifts to the equation obtained from a border strip, we have the desired equation. $\qquad \square$

Then we can simplify (35), completing the proof of Lemma 42.

7 Final remarks

7.1 Theorem 8 for border-strips

In [10] Morales, Pak, and Panova gave a proof of Theorem 5 using factorial Schur functions. In [8] the same authors gave another proof of Theorem 5 reducing it to the case of border

strips. The latter proof included an analogue of Lemma 30 for border strips, but there was no explicit analogue of Lemma 29. Instead they relied on an identity [8, Lemma 7.2] proved using factorial Schur functions. Our Lemma 29 can be reduced to the case of border strips as follows.

Corollary 44. *For a labeled border-strip poset* $(Q_{\lambda/\mu}, \omega)$*, where* ω *is a reversed Schur labeling,*

$$e_q^{\mathrm{maj}}(Q_{\lambda/\mu}, \omega) = \sum_{\mu \to \nu} q^{|Q_{\lambda/\nu_1}|} e_q^{\mathrm{maj}}(Q_{\lambda/\nu}, \omega_\nu),$$

where Q_{λ/ν_1} *is the left disconnected poset of* $Q_{\lambda/\nu}$*, and* ω_ν *is the restricted labeling of* ω *onto* $Q_{\lambda/\nu}$*.*

Then using the Pieri–Chevalley formula ((6) proved in [8]) and Corollary 44, we obtain a proof of Theorem 5 for border strips without using factorial Schur functions.

7.2 Bijective proof between maj and inv index for border strips

Lemma 41 is the inversion statistic analogue of Lemma 29. In the case of border strips $Q_{\lambda/\mu}$, since $n - \omega(u) = |Q_{\lambda/\nu_1}|$, we obtain the *same* recurrence as $e_q^{\mathrm{maj}}(Q_{\lambda/\mu})$ as in Corollary 44. Thus, we obtain the following equation of the q-analogues for border strips.

Corollary 45. *For a border strip* $Q_{\lambda/\mu}$*,*

$$e_q^{\mathrm{inv}}(Q_{\lambda/\mu}, \omega) = e_q^{\mathrm{maj}}(Q_{\lambda/\mu}, \omega),$$

where ω *is a reversed Schur labeling*

This identity can also be proved bijectively using Foata's classical bijection on permutations, denoted by φ, defined as follows (see [16, Sec. 1.4]). Let $w = w_1 \cdots w_n \in \mathfrak{S}_n$, and we define $\gamma_1, \ldots \gamma_n$, where γ_k is a permutation of $\{w_1, \ldots, w_k\}$. Let $\gamma_1 = w_1$. For each $k \geqslant 1$, if the last letter of γ_k is greater (respectively smaller) than w_{k+1}, then split γ_k after each letter greater (respectively smaller) than w_{k+1}. To obtain γ_{k+1}, cyclically shift each compartment of γ_k to the right, then place w_{k+1} at the end. We set $\varphi(w) = \gamma_n$. We have the following theorem.

Theorem 46 (Foata [3])**.** *Let* φ *be the Foata bijection. For all* $w \in \mathfrak{S}_n$*,*

$$\mathrm{Des}(w^{-1}) = \mathrm{Des}(\varphi(w)^{-1}).$$

Because Foata's bijection preserves descent sets, we have the following bijection between the major and inversion index.

Lemma 47. *Given a* $\sigma \in \mathcal{L}(P, \omega)$*, where* ω *is a (reversed) Schur labeling,* $\varphi(\sigma)$ *is also in* $\mathcal{L}(P, \omega)$*, and* $\mathrm{maj}(\sigma) = \mathrm{inv}(\varphi(\sigma))$

More detailed information about the equidistribution of major and inversion index in trees can be found in [1]. The situation for mobile posets is more subtle since the q-analogues do not agree (see Example 28 and Example 39).

7.3 Mobiles of general skew shapes

The formula (NHLF) holds true for all posets coming from skew shapes, but the combinatorial proof of the formula is restricted to the case of border strips. Recall that a mobile is obtained by attaching d-complete posets from a border-strip. It would be interesting to see if Theorem 7 holds for posets where the border strip is replaced by general skew shape. Calculations suggest that the Naruse formula (7) would need some adjustments.

On the other hand, we use a version of Pieri–Chevalley formula and the recurrence for our proof. There is a version of the Pieri–Chevalley formula for general skew shapes, shown algebraically by Ikeda and Naruse [6], and combinatorially by Konvalinka [7].

7.4 Relation with Naruse-Okada hook-length formula

Naruse-Okada [12] have a different q-analogue of $e_q^{\mathrm{maj}}(P, \omega)$ for a family called *skew d-complete* posets, which intersects with the family of mobile posets [5, Section 6.1].

Definition 48. [12] A *skew d-complete poset* is a d-complete poset P with an order filter I removed. We denote such a poset by $P \setminus I$.

The Naruse-Okada formula for counting linear extensions of skew d-complete posets uses the hook-length of *excited peaks* (see [10, Section 6] and [12]) instead of broken diagonals.

Theorem 49 (Naruse-Okada [12]). *Let $P \setminus I$ be a skew d-complete poset with n elements. Then*

$$e_q^{\mathrm{maj}}(P \setminus I) = \prod_{i=1}^{n}(1 - q^i) \sum_{D \in \mathcal{E}(P \setminus I)} \frac{\prod_{v \in B(D)} q^{h(v)}}{\prod_{v \in P \setminus D}(1 - q^{h(v)})},$$

where $h(v)$ is the hook length of element v in $P \setminus I$ and $B(D)$ is a set of excited peaks of D.

For posets that are both mobiles and skew d-complete, the notion of hook-lengths are the same (see Figure 7). Then for such posets Theorem 49 at $q = 1$ and Corollary 8 agree.

However, the q-analogues in Theorem 8 and Theorem 49 are different (see Example 50). This is because the NHLF formula for skew d-complete posets uses the natural labeling of the poset as opposed to the reversed Schur labeling. For the case of skew shapes, their q-analogue agrees with the reverse plane partition q-analogue of the Naruse formula in (see [10] Corollary 6.17) instead of SSYT q-analogue, Theorem 5, which uses the Schur labeling (see Figure 7).

Example 50. Consider the poset $Q = P \setminus I$ in Figure 7 that is both a mobile poset and a skew d-complete poset [5, Ex. 6.3]. If we label it using the reversed Schur labeling on the border strip and natural labeling on the d-complete posets, then by Theorem 7, we have

$$e_q^{\mathrm{maj}}(Q, \omega') = q^{11} + 2q^{10} + 3q^9 + 3q^8 + 3q^7 + 2q^6 + 1q^5 + q^4$$

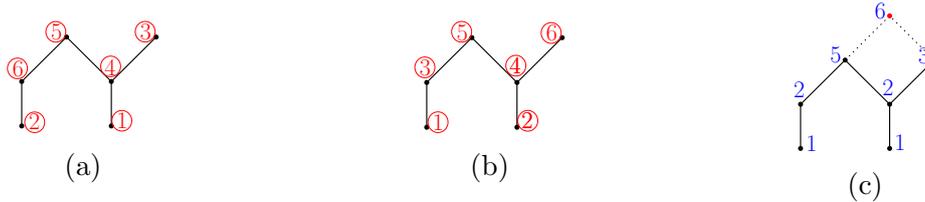$$= [6]! \left(\frac{q^4}{[1][1][2][2][3][5]} + \frac{q^7}{[1][1][2][3][5][6]} \right).$$

Figure 7: A skew d-complete poset $P \setminus I$ that is also a mobile labeled using (a) the Schur labeling ω' and (b) the natural labeling ω. (c) The hook lengths of the elements in the poset and the excited peak colored in blue.

Now, label the same skew d-complete poset $Q = P \setminus I$ using the natural labeling. Then, by the Naruse–Okada formula (Theorem 49), we have

$$e_q^{\text{maj}}(Q, \omega) = q^9 + q^8 + 2q^7 + 2q^6 + 2q^5 + 2q^4 + 2q^3 + 2q^3 + 2q^2 + q + 1$$

$$= [6]! \left(\frac{q^0}{[1][1][2][2][3][5]} + \frac{q^6}{[1][1][2][3][5][6]} \right).$$

It would be interesting to see if one can give a proof of Theorem 49 using the technique from [8]. This would involve proving a variation of Lemma 29 where ω is a natural labeling instead of reversed Schur labeling.

Acknowledgement

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