Distinguishing number of universal homogeneous Urysohn metric spaces

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Abstract

The distinguishing number of a structure is the smallest size of a partition of its elements so that only the trivial automorphism of the structure preserves each cell of the partition. We show that for any countable subset of the positive real numbers, the corresponding countable homogeneous Urysohn metric space, when it exists, has distinguishing number 2 or is infinite.

While it is known that a sufficiently large finite primitive structure has distinguishing number 2, unless its automorphism group is the full symmetric group or alternating group, the infinite case is open and these countable Urysohn metric spaces provide further confirmation toward the conjecture that all primitive homogeneous countably infinite structures have distinguishing number 2 or else the distinguishing number is infinite.

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1 Introduction

The *asymmetric colouring number* of a graph was introduced by Babai long ago in [2], and it resurfaced more recently as the *distinguishing number* in the work of Albertson

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and Collins in [1]. First call a group of permutations $\boldsymbol{G}$ on a set A *k-distinguishable* if there exists a partition of A into k cells such that only the identity permutation in $\boldsymbol{G}$ fixes setwise all of the cells of the partition. It is evident that $\boldsymbol{G}$ is always $|A|$-distinguishable. The least cardinal number k such that $\boldsymbol{G}$ is k-distinguishable is its *distinguishing number* $D(\boldsymbol{G})$. We call a graph or any structure $\boldsymbol{S}$ k-distinguishable if its automorphism group $\mathrm{Aut}(\boldsymbol{S})$ satisfies $D(\mathrm{Aut}(\boldsymbol{S})) \leqslant k$.

The distinguishing number is the amount of symmetry found within a structure, leading to interesting structural information that comes from the investigation of what is needed to break that symmetry. Of particular interest to us are countable homogeneous structures, carrying any two finite substructures of the same size and type into each other by an automorphism of the entire structure. Their automorphism group is thus highly symmetric and in particular transitive as a permutation group. The set of rational numbers with its linear order relation is such a countable homogeneous structure, and is easily seen to have infinite distinguishing number. On the other hand, the Rado graph (or infinite random graph) is also homogeneous, but Imrich, Klavzar and Trofimov showed in [10] that its distinguishing number is 2, which is the smallest it can be because the Rado graph is not rigid. The distinguishing number of various other finite and countable homogeneous structures was determined in [4, 5, 12], including all simple and directed homogeneous graphs and posets. In particular in all cases of infinite homogeneous simple and directed graphs, it was shown in [12] that their distinguishing number is either 2 or infinite, with only obvious exceptions having imprimitive automorphism groups. The following was thus conjectured.

Conjecture ([12]). The distinguishing number of all primitive homogeneous countably infinite structures is 2 or infinite.

The conjecture is very much in the spirit of the finite case, where Cameron, Neumann and Saxl proved (see [3]) that a sufficiently large, finite primitive permutation group has distinguishing number 2, unless it is the full symmetric group or alternating group. The 43 exceptions were determined by Seress, and one of these exceptions is the dihedral group D_{10}, which is primitive and the automorphism group of the homogeneous graph C_5; it has distinguishing number 3 and hence the necessity for the conjecture to address only infinite structures. A tool developed in [12] appears in the right direction to confirm the conjecture, namely, that of a fixing type for the action of a group G on a set A. If the action does have such a fixing type, then the distinguishing number of G acting on A is 2. It may be possible that a more general result for all primitive groups exists (homogeneous or not), but we have no insight in that direction.

A graph having distinguishing number 2 has an interesting translation to permutation group theoretic properties of its automorphism group, see Section 2.2 of [3]. Hence it is reasonable to hope that the 2-distinguishability of the rational Urysohn space, and the distinguishing numbers of other Urysohn spaces addressed below, have interesting translations into properties of their automorphism groups.

In this paper, we consider the case of homogeneous countable *Urysohn metric spaces* $\boldsymbol{U}_S$ for a given countable spectrum $S \subseteq \mathbb{R}_{\geqslant 0}$, constructed as the Fraïssé limits of all finite metric spaces whose spectrum is a subset of S. Note that not every subset S

can be the spectrum of such a Urysohn metric space, and a necessary and sufficient condition is known as the "4-values" condition, which is precisely when metric triangles amalgamate; when this is the case we call S a *universal spectrum*. Depending on S, we will see that the automorphism group of $\boldsymbol{U}_{\mathcal{S}}$ may or may not be primitive; we do not have a characterization for a spectrum to yield a primitive automorphism group of its corresponding Urysohn space.

These countable metric spaces are very much related to the well known (uncountable) Urysohn space, the complete separable metric space which is both homogeneous and universal; it is the completion of the countable homogeneous Urysohn spaces using the rationals as spectrum. See, for example, [6, 13].

The main result of the paper is as follows.

Theorem (Main Theorem). Let $\mathcal{S} \subseteq \mathbb{R}_{\geqslant 0}$ be a countable universal spectrum and $\boldsymbol{U}_{\mathcal{S}}$ the countable homogeneous structure with spectrum S. We then have that $D(\boldsymbol{U}_{\mathcal{S}}) = 2$ or ω, and the following items hold.

1. If $\mathcal{S}$ has a positive limit (not necessarily in $\mathcal{S}$), then $D(\boldsymbol{U}_{\mathcal{S}}) = 2$.

2. If $\mathcal{S}$ has no positive limits but has 0 as a limit, then $D(\boldsymbol{U}_{\mathcal{S}}) = 2$ if and only if S contains arbitrarily large elements of arbitrarily small distance.

3. If $\mathcal{S}$ does not have a limit, then $D(\boldsymbol{U}_{\mathcal{S}}) = 2$ if and only if S contains two distinct and positive elements of distance smaller than or equal to the minimum positive element of $\mathcal{S}$.

The proof is the result of analyzing the existence of various limit structures for the set $\mathcal{S}$ along the following lines, and the necessary lemmas will be proved in the remaining sections of the paper.

Proof. Let $\mathcal{S}$ be a universal spectrum. If $\mathcal{S}$ has a positive limit (not necessarily in $\mathcal{S}$), then $D(\boldsymbol{U}_{\mathcal{S}}) = 2$ by Lemma 23. If $\mathcal{S}$ has no positive limit but has 0 as a limit, then $D(\boldsymbol{U}_{\mathcal{S}}) = 2$ or ω by Lemma 27. Moreover $D(\boldsymbol{U}_{\mathcal{S}}) = 2$ if and only if $\mathrm{gap}(\mathcal{S}_{\geqslant s}) = 0$ for some positive number $s \in \mathcal{S}$; as $\mathcal{S}$ is assumed to have no positive limit, this latter condition is equivalent to $\mathcal{S}$ containing arbitrarily large elements of arbitrarily small distance.

Finally, if $\mathcal{S}$ does not have a limit, then by Lemma 25 $D(\boldsymbol{U}_{\mathcal{S}}) = 2$ if and only if there exist numbers $a < b \in \mathcal{S}$ with $b - a \leqslant \min(\mathcal{S}_{>0})$, that is contains two elements of distance smaller than the minimum positive element of $\mathcal{S}$. □

It follows that the well known *rational Urysohn space*, that is $\boldsymbol{U}_{\mathcal{S}}$ for $\mathcal{S}$ consisting of the non-negative rational numbers, has distinguishing number 2. Moreover we shall see following Theorem 9 that various examples exist showing that all those cases of the Main Theorem do occur.

It is worth noting that the rigid subspaces which are used in the proofs of those necessary lemmas are almost always rigid forests determined by the s-distance graph of a metric space for some s in the spectrum: there is an edge between $x, y \in \boldsymbol{U}_{\mathcal{S}}$ if and only if the distance between x and y is s.

We conclude the introduction by drawing a parallel with other work. Imrich et al. in [11] have shown that a countable permutation group acting on a countable set has distinguishing number 2, under the additional assumption that the group has *infinite motion*, meaning that every non-identity group element moves infinitely many elements. The automorphism groups of Urysohn spaces in this paper have infinite motion, but are uncountable. Moreover recall that, up to (topological group) isomorphism, the closed subgroups of the infinite symmetric S_∞ are exactly the automorphism groups of countable structures, and thus our work can be viewed in that setting. The work of Imrich et al. in [11] is also focused on closed subgroups of S_∞, where in particular they conjecture that any closed subgroup of S_∞ having infinite motion and where all orbits of its point stabilizers are finite has distinguishing number 2; this is the so-called *Infinite Motion Conjecture for Permutation Groups*. However the point stabilizers of automorphism groups of homogeneous Urysohn spaces all are infinite.

2 General Notions and Preliminaries

A relational structure $\boldsymbol{R}$ is *rigid* if its group of automorphisms $\mathrm{Aut}(\boldsymbol{R})$ consists only of the identity. The idea behind the distinguishing number is to find the smallest number of predicates $\langle P_i : i < d\rangle$ such that the expanded structure $(\boldsymbol{R}; P_i : i < d)$ becomes rigid.

For a metric space $\boldsymbol{M} = (M, d_{\boldsymbol{M}})$, let $\mathrm{spec}(\boldsymbol{M})$, the *spectrum* of $\boldsymbol{M}$, be the set of the distances between points of $\boldsymbol{M}$. A metric space $\boldsymbol{M}$ is *universal* if it embeds every finite metric space $\boldsymbol{N}$ with spectrum $\mathrm{spec}(\boldsymbol{N}) \subseteq \mathrm{spec}(\boldsymbol{M})$.

For $\mathcal{S}$ a set of reals and $r \in \mathbb{R}$, let $\mathcal{S}_{>r} = \{s \in \mathcal{S} : s > r\}$, and similarly for $\mathcal{S}_{\geqslant r}$. If $\mathcal{S} \subseteq \mathbb{R}_{\geqslant 0}$ is countable, then let $\mathfrak{A}(\mathcal{S})$ denote the set of finite metric spaces whose spectrum is a subset of $\mathcal{S}$. Note that $\mathfrak{A}(\mathcal{S})$ is an age (meaning that it is closed under isomorphism and substructures, and up to isomorphism has only countably many members), and we will need conditions on $\mathcal{S}$ for which the age $\mathfrak{A}(\mathcal{S})$ has the amalgamation property.

Definition 1. We call a pair of metric spaces $\boldsymbol{A}$ and $\boldsymbol{B}$ an *amalgamation instance* if $d_{\boldsymbol{A}}(x, y) = d_{\boldsymbol{B}}(x, y)$ for all $x, y \in A \cap B$. If so, then we define:

$$\amalg(\boldsymbol{A}, \boldsymbol{B}) = \{\boldsymbol{C} = (A \cup B; d_{\boldsymbol{C}}) : \ \boldsymbol{C} \restriction A = \boldsymbol{A} \text{ and } \boldsymbol{C} \restriction B = \boldsymbol{B}\}.$$

For $\mathcal{S} \subseteq \mathbb{R}_{\geqslant 0}$ and $\{\boldsymbol{A}, \boldsymbol{B}\} \subseteq \mathfrak{A}(\mathcal{S})$ let:

$$\amalg_{\mathcal{S}}(\boldsymbol{A}, \boldsymbol{B}) = \{\boldsymbol{C} \in \amalg(\boldsymbol{A}, \boldsymbol{B}) : \mathrm{spec}(\boldsymbol{C}) \subseteq \mathcal{S}\}.$$

Finally, we say that the set $\mathcal{S} \subseteq \mathbb{R}_{\geqslant 0}$ has the *amalgamation property* if $\amalg_{\mathcal{S}}(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset$ for all amalgamation instances $\{\boldsymbol{A}, \boldsymbol{B}\} \subseteq \mathfrak{A}(\mathcal{S})$.

We now define the "4-values" condition, which is the description that triangles amalgamate.

Definition 2. A set $\mathcal{S} \subseteq \mathbb{R}_{\geqslant 0}$ satisfies the *4-values* condition if $\amalg_{\mathcal{S}}(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset$ for any two amalgamation instances of the form $\boldsymbol{A} = (\{x, y, z\}; d_{\boldsymbol{A}})$ and $\boldsymbol{B} = (\{x, y, w\}; d_{\boldsymbol{B}})$ in $\mathfrak{A}(\mathcal{S})$.

There are several equivalent definitions of the 4-values condition. The first version has been given in [7] together with Theorem 4 below. In [15] there is an equivalent version of the 4-values condition given and Definition 2 is stated as Lemma 3.3. Definition 2 here is the one most suitable for our purposes.

It is evident that the spectrum of any metric space satisfies the 4-values condition, and it provides the leading condition for a set $\mathcal{S}$ to yield a non-trivial Urysohn metric space.

Definition 3. A countable set $\mathcal{S} \subseteq \mathbb{R}_{\geqslant 0}$ is a *universal spectrum* if the following items hold.

1. The element 0 is in $\mathcal{S}$ and $\mathcal{S}$ contains at least one positive number;
2. The set $\mathcal{S}$ satisfies the 4-values condition.

It follows from Theorem 3.8 of [15] that if $\mathcal{S}$ is a universal spectrum, then $\mathfrak{A}(\mathcal{S})$ is an age with amalgamation. Hence, the next theorem follows from the general Fraïssé theory.

Theorem 4. *[9] If $\mathcal{S} \subseteq \mathbb{R}_{\geqslant 0}$ is a universal spectrum, then $\mathfrak{A}(\mathcal{S})$ is an age with amalgamation and there exists a countable homogeneous, universal, metric space $\boldsymbol{U}_{\mathcal{S}}$ whose spectrum is $\mathcal{S}$.*

In particular, is $\boldsymbol{M} \in \mathfrak{A}(\mathcal{S})$ and such that $\boldsymbol{M} \cap \boldsymbol{U}_{\mathcal{S}}$ is a subspace of $\boldsymbol{U}_{\mathcal{S}}$, then then there exists an embedding of $\boldsymbol{M}$ into $\boldsymbol{U}_{\mathcal{S}}$ which is the identity on $\boldsymbol{M} \cap \boldsymbol{U}_{\mathcal{S}}$.

Even though by definition an amalgamation instance can be amalgamated, we will in many cases want to do so controlling the new distances. We therefore have the following.

Lemma 5. *Let $\mathcal{S} \subseteq \mathbb{R}_{\geqslant 0}$ be a universal spectrum, and $\boldsymbol{A}$ and $\boldsymbol{B}$ an amalgamation instance in $\mathfrak{A}(\mathcal{S})$. If there exists a number $s \in \mathcal{S}$ so that*

$$s \leqslant d_{\boldsymbol{A}}(a, x) + d_{\boldsymbol{B}}(x, b)$$

for all $a \in A \setminus B$, $x \in A \cap B$, and $b \in B \setminus A$, then there exists a metric space $\boldsymbol{C} \in \amalg_{\mathcal{S}}(\boldsymbol{A}, \boldsymbol{B})$ so that $d_{\boldsymbol{C}}(a, b) \geqslant s$ for all $a \in A \setminus B$ and $b \in B \setminus A$.

Proof. Because $\mathcal{S} \subseteq \mathbb{R}_{\geqslant 0}$ satisfies the 4-values condition, there is a metric space $\boldsymbol{D} \in \amalg_{\mathcal{S}}(\boldsymbol{A}, \boldsymbol{B})$. Let $\boldsymbol{C}$ be the binary relational structure obtained from $\boldsymbol{D}$ by replacing the new distances as follows:

$$d_{\boldsymbol{C}}(a, b) = \max\{d_{\boldsymbol{D}}(a, b), s\}$$

for every $a \in A \setminus B$ and every $b \in B \setminus A$.

We claim that $\boldsymbol{C}$ is indeed a metric space in $\mathfrak{A}(\mathcal{S})$, and hence, all triangles of $\boldsymbol{C}$ not in $\boldsymbol{A}$ and not in $\boldsymbol{B}$ must be verified to be metric.

One type of such triangles is of the form $\{a, x, b\}$ with $a \in A \setminus B$ and $b \in B \setminus A$ and $x \in A \cap B$. As a triangle of $\boldsymbol{D}$ it is metric, and together with the assumption on s we derive that

$$d_{\boldsymbol{C}}(a, b) = \max\{d_{\boldsymbol{D}}(a, b), s\} \leqslant d_{\boldsymbol{A}}(a, x) + d_{\boldsymbol{B}}(x, b).$$

The other type of triangles we need to verify is of the form either $\{a, a', b\}$ with $\{a, a'\} \subseteq A \setminus B$ and $b \in B \setminus A$, or the other way around of the form $\{a, b, b'\}$ with $a \in A \setminus B$ and $\{b, b'\} \subseteq B \setminus A$; it suffices to consider the first case.

If both $d_{\boldsymbol{D}}(a, b) \leqslant s$ and $d_{\boldsymbol{D}}(a,' b) \leqslant s$, then $d_{\boldsymbol{C}}(a, b) = d_{\boldsymbol{C}}(a', b) = s$, and $d_{\boldsymbol{C}}(a, a') = d_{\boldsymbol{D}}(a, a') \leqslant d_{\boldsymbol{D}}(a, b) + d_{\boldsymbol{D}}(a', b) \leqslant d_{\boldsymbol{C}}(a, b) + d_{\boldsymbol{D}}(a', b)$ shows that $\{a, a', b\}$ is metric. If both $d_{\boldsymbol{D}}(a, b) \geqslant s$ and also $d_{\boldsymbol{D}}(a', b) \geqslant s$, then the side lengths have not changed from $\boldsymbol{D}$ to $\boldsymbol{C}$ and hence, again metric.

For the remaining case, say $d_{\boldsymbol{D}}(a, b) < s$ but $d_{\boldsymbol{D}}(a', b) \geqslant s$. Then $d_{\boldsymbol{C}}(a, b) = s \leqslant d_{\boldsymbol{D}}(a', b) = d_{\boldsymbol{C}}(a', b) \leqslant d_{\boldsymbol{C}}(a, a') + d_{\boldsymbol{C}}(a', b)$. The verification of the other two sides is immediate. □

The following general notions will useful to analyze various universal spectra.

Definition 6. Let $\mathcal{S}$ be a universal spectrum. An element $s \in \mathcal{S}$ is called:

1. an *initial number* of $\mathcal{S}$ if $[\frac{1}{2}s, s) \cap \mathcal{S} = \emptyset$,
2. a *jump number* of $\mathcal{S}$ if $(s, 2s] \cap \mathcal{S} = \emptyset$,
3. an *insular number* of $\mathcal{S}$ if it both an initial and a jump number of $\mathcal{S}$.

A subset $B \subseteq \mathcal{S}$ is a *block* of $\mathcal{S}$ if:

1. B is an interval of $\mathcal{S}$,
2. $\min B$ is a positive initial number of $\mathcal{S}$,
3. either B is unbounded, or $\max B$ is a jump number of $\mathcal{S}$, and
4. $\max B$ (if it exists) is the only jump number (of either B of S).

Hence, if $s \in \mathcal{S}$ is insular, then $B = \{s\}$ is a block consisting of only one element. Note also for future reference that if $s > 0$ is a jump number of $\mathcal{S}$, then the relation $\overset{s}{\sim}$ given by $x \overset{s}{\sim} y$ if $d(x, y) \leqslant s$ is an equivalence relation on $\boldsymbol{U}_{\mathcal{S}}$, this will soon play an important role.

3 Universal spectra without positive limits

In this section, we develop tools to handle the case where the spectrum does not have a positive limit. In particular, no positive element r of $\mathbb{R}$ is a limit of $\mathcal{S}$, whether $r \in \mathcal{S}$ or not.

To handle this case, we first call a set $\mathcal{S} \subseteq \mathbb{R}_{\geqslant 0}$ *inversely well ordered* if every non-empty bounded above subset of $\mathcal{S}$ has a maximum. We recall the $\oplus$ operation from [16], defined there for a closed set of reals, but which for our purpose can also be defined for inversely well ordered sets.

Definition 7. Let $\mathcal{S}$ be an inversely well ordered, and for $r, t \in \mathcal{S}$ define

$$r \oplus t = \max\{s \in \mathcal{S} : s \leqslant r + t\}.$$

Observe that if $\{r, s, t\} \subseteq \mathcal{S}$, then $r \oplus t \geqslant s$ if and only if $r + t \geqslant s$; and hence, we note the following obvious observation which will be used without warning.

Lemma 8. *A triangle is metric if and only if the $\oplus$ sum of any two of the three side lengths is larger than or equal to the third side length.*

The $\oplus$ operation on $\mathcal{S}$ can easily be verified to be commutative and monotone. Further it was shown [16] that the associativity of the $\oplus$ operation is equivalent to the 4-value condition for a closed set. That assumption, however, was only used to justify the operation and the same argument can be used for the following.

Theorem 9. *If $\mathcal{S}$ is an inversely well ordered set, then it satisfies the 4-value condition if and only if the $\oplus$ operation on $\mathcal{S}$ is associative.*

This result can be used to easily verify that the following inversely well ordered sets all satisfy the 4-values condition:

$$\begin{aligned}
\mathcal{S}_1 &= 0 \cup \{1 + 1/n : n \in \omega\}, \\
\mathcal{S}_{2a} &= 0 \cup \{1/2^{2n} : n \in \omega\}, \\
\mathcal{S}_{2b} &= 0 \cup \{1/2^{2n} : n \in \omega\} \cup \{2^n, 2^n + 1/n : n \in \omega\}, \\
\mathcal{S}_{3a} &= \{0, 1\}, \\
\mathcal{S}_{3b} &= \{0, 1, 2\}.
\end{aligned}$$

The set $\mathcal{S}_1$ has 1 as a limit. Both sets $\mathcal{S}_{2a}$ and $\mathcal{S}_{2b}$ have no positive limits but do have 0 as a limit; $\mathcal{S}_{2b}$ has arbitrarily large elements if arbitrarily small distance, while $\mathcal{S}_{2a}$ does not. Both sets $\mathcal{S}_{3a}$ and $\mathcal{S}_{3b}$ do not have any limit; $\mathcal{S}_{3b}$ has two distinct and positive elements of distance smaller than or equal to the minimum positive element, while $\mathcal{S}_{3a}$ does not have such elements. Hence this shows that all types universal spectrum of the Main Theorem do occur.

We note that the homogenous Urysohn space $\boldsymbol{U}_{\mathcal{S}_{3b}}$ is nothing else but the Rado graph, hence providing another (albeit lengthy) proof that its distinguishing number is 2.

Our arguments below rely on an analysis of inversely well ordered set, and moreover we use the $\oplus$ operation to construct a specific and controlled amalgamation.

Lemma 10. *Let $\mathcal{S}$ be an inversely well ordered universal spectrum, and $\boldsymbol{A}$ and $\boldsymbol{B}$ in $\mathfrak{A}(\mathcal{S})$ an amalgamation instance. Then there exists a unique metric space $\boldsymbol{C} \in \amalg_{\mathcal{S}}(\boldsymbol{A}, \boldsymbol{B})$, which we denote by $\amalg^{\oplus}_{\mathcal{S}}(\boldsymbol{A}, \boldsymbol{B})$, such that:*

$$d_{\boldsymbol{C}}(a, b) = \min\{d_{\boldsymbol{A}}(a, x) \oplus d_{\boldsymbol{B}}(x, b) : \ x \in A \cap B\}$$

for all $a \in A \setminus B$ and $b \in B \setminus A$.

Proof. Let $\boldsymbol{C}$ be defined as above, we must show that every triangle of $\boldsymbol{C}$ is metric.

Let $\boldsymbol{D} \in \amalg(\boldsymbol{A}, \boldsymbol{B})$. Note first that $d_{\boldsymbol{D}}(a, b) \leqslant d_{\boldsymbol{C}}(a, b)$ for all $a \in A \setminus B$ and $b \in B \setminus A$; this is because $d_{\boldsymbol{D}}(a, b) \leqslant d_{\boldsymbol{A}}(a, x) + d_{\boldsymbol{B}}(x, b)$ for any $x \in A \cap B$, and therefore, $d_{\boldsymbol{D}}(a, b) \leqslant d_{\boldsymbol{A}}(a, x) \oplus d_{\boldsymbol{B}}(x, b)$ by definition of $\oplus$.

The first kind of triangles to consider are of the form $\{a, x, b\}$ for $a \in A \setminus B$, $x \in A \cap B$ and $b \in B \setminus A$. Since $\boldsymbol{D}$ is metric and $d_{\boldsymbol{D}}(a, b) \leqslant d_{\boldsymbol{C}}(a, b)$, all we need to verify is the inequality $d_{\boldsymbol{C}}(a, b) \leqslant d_{\boldsymbol{A}}(a, x) + d_{\boldsymbol{B}}(x, b)$. Let $x' = \mu(a, b)$. By definition, we have that $d_{\boldsymbol{C}}(a, b) = d_{\boldsymbol{A}}(a, x') \oplus d_{\boldsymbol{B}}(b, x') \leqslant d_{\boldsymbol{A}}(a, x') + d_{\boldsymbol{B}}(b, x') \leqslant d_{\boldsymbol{A}}(a, x) + d_{\boldsymbol{B}}(b, x)$.

Now, by symmetry, the only other case is a triangle $\{a, a', b\}$ for $\{a, a'\} \subseteq A \setminus B$ and $b \in B \setminus A$. Because $d_{\boldsymbol{D}}(a, b) \leqslant d_{\boldsymbol{C}}(a, b)$ and $d_{\boldsymbol{D}}(a', b) \leqslant d_{\boldsymbol{C}}(a', b)$, we then have that

$$\begin{aligned} d_{\boldsymbol{C}}(a, a') &= d_{\boldsymbol{D}}(a, a') \\ &\leqslant d_{\boldsymbol{D}}(a, b) + d_{\boldsymbol{D}}(b, a') \\ &\leqslant d_{\boldsymbol{D}}(a, b) + d_{\boldsymbol{C}}(b, a'). \end{aligned}$$

For the other sides, it remains to show without loss of generality that $d_{\boldsymbol{C}}(a', b) \leqslant d_{\boldsymbol{C}}(a', a) + d_{\boldsymbol{C}}(a, b)$, or equivalently that $d_{\boldsymbol{C}}(a', b) \leqslant d_{\boldsymbol{C}}(a', a) \oplus d_{\boldsymbol{C}}(a, b)$.

For this let $z = \mu(a, b) \in A \cap B$ be such that

$$d_{\boldsymbol{C}}(a, b) = d_{\boldsymbol{A}}(a, z) \oplus d_{\boldsymbol{B}}(z, b)$$

and such that $d_{\boldsymbol{A}}(a, z) + d_{\boldsymbol{B}}(z, b)$ is as small as possible.

But $d_{\boldsymbol{A}}(a', z) \leqslant d_{\boldsymbol{A}}(a', a) + d_{\boldsymbol{A}}(a, z)$, equivalently

$$d_{\boldsymbol{C}}(a', z) \leqslant d_{\boldsymbol{C}}(a', a) \oplus d_{\boldsymbol{C}}(a, z).$$

Hence,

$$\begin{aligned} d_{\boldsymbol{C}}(a', b) &\leqslant d_{\boldsymbol{C}}(a', z) \oplus d_{\boldsymbol{C}}(z, b) \\ &\leqslant (d_{\boldsymbol{C}}(a', a) \oplus d_{\boldsymbol{C}}(a, z)) \oplus d_{\boldsymbol{C}}(z, b) \\ &= d_{\boldsymbol{C}}(a', a) \oplus (d_{\boldsymbol{C}}(a, z) \oplus d_{\boldsymbol{C}}(z, b)) \\ &= d_{\boldsymbol{C}}(a', a) \oplus d_{\boldsymbol{C}}(a, b). \end{aligned}$$

This completes the proof. □

We are now ready to further analyze the structure of the universal spectrum without positive limits, but first some useful terminology.

Definition 11. Let $\mathcal{S}$ a universal spectrum without positive limits.

1. For $s \in \mathcal{S}$, let s^- be the largest number in $\mathcal{S}$ smaller than s if $s > 0$, and $0^- = 0$.
2. If $s \neq \max \mathcal{S}$, then let s^+ be the smallest number in $\mathcal{S}$ larger than s, and let $s^+ = s$ if $s = \max \mathcal{S}$.
3. Two numbers $s < t \in \mathcal{S}$ are said to be *consecutive* if $s^+ = t$ (or if $t^- = s$).
4. The *cover* of $\{r, t\}$, is the number (in $\mathcal{S}$):

$$\min\{s \in \mathcal{S} : \ |r - t| \leqslant s\}.$$

5. The *gap* at $s \in \mathcal{S}$, denoted by $\mathrm{gap}(s)$, is the number (in $\mathbb{R}$):

$$\min\{|s-t| : \ t \in \mathcal{S} \setminus \{s\}\}.$$

6. If $T \subseteq \mathcal{S}_{>0}$, then $\mathrm{gap}(T) = \min\{\mathrm{gap}(t) : t \in T\}$.

Note that if $\{s,t\} \in \mathcal{S}$ with $s \neq t$, then $\mathrm{gap}(s) \leqslant |s-t|$. Hence, we immediately have the following general fact.

Fact 12. *Let $\mathcal{S}$ be a universal spectrum without positive limits, and $\boldsymbol{M}$ be a metric space with $\mathrm{spec}(\boldsymbol{M}) \subseteq \mathcal{S}$. However, if $d(x,y) < \mathrm{gap}(d(x,z))$ for any three distinct points $\{x,y,z\} \subseteq M$, then $d(x,z) = d(y,z)$.*

Lemma 13. *If $\mathcal{S}$ is a universal spectrum without positive limits, then the cover c of two consecutive numbers $r < t \in \mathcal{S}$ with $r + r \geqslant t$ is an initial number of $\mathcal{S}$.*

Proof. Note that $c \leqslant r$ because $r+r \geqslant t$ and c is the smallest number in $\mathcal{S}$ with $r+c \geqslant t$. Assume that c is not initial. Then there exists a number $p < c$ with $p + p \geqslant c$; this implies that the two triangles, T_0 with side lengths p, p, c, and T_1 of side lengths $\{r, t, c\}$, are metric. These two triangles form an amalgamation instance via the common side c. Hence, because $\mathcal{S}$ satisfies the 4-values condition there exists a number $s \in \mathcal{S}$ and a metric space $\boldsymbol{M} \in \amalg(\mathrm{T}_0, \mathrm{T}_1)$ with amalgamation distance s.

This is not possible. Indeed first note that $r + p < t$ because $p < c$ and again c is the smallest number in $\mathcal{S}$ with $r + c \geqslant t$. Now, if $s \leqslant r$, then $s + p \leqslant r + p < t$ and the triangle $\{t, p, s\}$ is not metric. If on the other hand $s > r$, then $s \geqslant t$ because $r < t$ are consecutive; but now $r + p < t \leqslant s$ and the triangle $\{r, p, s\}$ is not metric. ☐

This gives the following.

Lemma 14. *Let $\mathcal{S}$ be a universal spectrum without positive limits. If 0 is a limit of $\mathcal{S}$, then 0 is also a limit of the set of initial numbers of $\mathcal{S}$, and also a limit of the set of jump numbers of $\mathcal{S}$.*

Proof. Let $r < t < \ell \in \mathcal{S}$ be two consecutive numbers, and let c be their cover. If $r+r \geqslant t$, then $c \leqslant r$ and it follows from Lemma 13 that c is initial. However, if $r + r < t$, then t itself is initial by definition. Thus, $\mathcal{S}$ contains arbitrarily small initial numbers.

Moreover, is s is an initial number, then s^- is a jump number. We therefore have that $\mathcal{S}$ contains arbitrarily small jump numbers as well. ☐

3.1 The case where $\mathcal{S}$ has no positive limits

In this subsection we continue with $\mathcal{S}$ a universal spectrum without positive limits, but we will focus on $\boldsymbol{U}_{\mathcal{S}}$ the homogeneous structure with spectrum $\mathcal{S}$ and construct some well chosen automorphisms.

Recall that if $s > 0$ is a jump number of $\mathcal{S}$, then the relation $\stackrel{s}{\sim}$ given by $x \stackrel{s}{\sim} y$ if $d(x,y) \leqslant s$ is an equivalence relation on $\boldsymbol{U}_{\mathcal{S}}$. If E is an $\stackrel{s}{\sim}$ equivalence class and $\boldsymbol{M}$ the metric space induced by E, then $\operatorname{spec}(\boldsymbol{M}) = \{r \in \mathcal{S} : r \leqslant s\} = \mathcal{S}_{\leqslant s}$. On the other hand, it is evident that $\operatorname{spec}(\boldsymbol{M})$ satisfies the 4-values condition, and hence, it is a universal spectrum. It follows that $\boldsymbol{M}$ is isomorphic to the universal homogeneous metric space $\boldsymbol{U}_{\mathcal{S}_{\leqslant s}}$.

We now define the notion of dense subset of $\boldsymbol{U}_{\mathcal{S}}$, and show that similar to the rationals, if $\boldsymbol{U}_{\mathcal{S}}$ is partitioned into finitely many dense sets, then there is a non-trivial automorphism preserving the partition.

Definition 15. Let $\mathcal{S}$ be a universal spectrum. A subset A of $\boldsymbol{U}_{\mathcal{S}}$ is *dense* if $A \cap E \neq \emptyset$ for every jump number $s \in \mathcal{S}_{>0}$ and for every equivalence class E of the relation $\stackrel{s}{\sim}$.

We will in fact build a non-trivial automorphism which is an involution.

Lemma 16. *Let $\mathcal{S}$ be a universal spectrum without positive limits but with 0 as a limit, and let $\{A_i : i \in n\}$ form a finite partition of $\boldsymbol{U}_{\mathcal{S}}$ into dense sets. It then follows that there exists, for every number $s \in \mathcal{S}_{>0}$, an automorphism f of $\boldsymbol{U}_{\mathcal{S}}$ such that:*

1. *f preserves the partition, that is $f[A_i] = A_i$ for every $i \in n$, and*
2. *$f(f(x)) = x$ and $d(x, f(x)) = s$ for all $x \in \boldsymbol{U}_{\mathcal{S}}$.*

Proof. The proof is an inductive construction on the countable domain of $\boldsymbol{U}_{\mathcal{S}}$, and is a consequence of the following claim handling the inductive step.

Claim. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two disjoint and finite subspaces of $\boldsymbol{U}_{\mathcal{S}}$ for which there exists an automorphism g of the subspace induced by $A \cup B$ so that:

1. g preserves the partition restricted to $A \cup B$, and
2. $g(x) \in B$, $g(g(x)) = x$, and $d(x, g(x)) = s$ for all $x \in A$.

Let $u \in \boldsymbol{U}_{\mathcal{S}} \setminus (A \cup B)$. Then there exists a point $v \in \boldsymbol{U}_{\mathcal{S}}$ and an automorphism g' of the subspace induced by $A \cup B \cup \{u, v\}$ so that:

1. g' extends g, that is $g'(x) = g(x)$ for all $x \in A \cup B$,
2. $d(u, v) = s$,
3. u and v are in the same member of the partition, and
4. $g'(u) = v$ and $g'(v) = u$.

To prove the claim, we first show that there exists a metric space $\boldsymbol{M}$ with $M = A \cup B \cup \{u, v\}$ so that the following hold.

1. $\boldsymbol{M}$ restricted to $A \cup B \cup \{u\}$ is equal to $\boldsymbol{U}_{\mathcal{S}}$ restricted to $A \cup B \cup \{u\}$.
2. $d_{\boldsymbol{M}}(u, v) = s$.
3. $d_{\boldsymbol{M}}(v, x) = d(u, g(x))$ (and so $d_{\boldsymbol{M}}(v, g(x)) = d(u, x)$) for all $x \in A \cup B$.

To verify that $\boldsymbol{M}$ will indeed be a metric space under these conditions, it suffices that every triangle of $\boldsymbol{M}$ is metric. Let $\{x, y, z\}$ be a triangle of $\boldsymbol{M}$. If $v \notin \{x, y, z\}$, then the triangle $\{x, y, z\}$ is metric because every triangle of $\boldsymbol{U}_{\mathcal{S}}$ is metric. Now let $\{x, y, v\}$ be a triangle of $\boldsymbol{M}$ with $u \notin \{x, y\}$; but the triangle $\{g(x), g(y), u\}$ is metric and has the same side lengths as the triangle $\{x, y, v\}$, hence, the latter is metric. Let $\{x, u, v\}$ be a triangle of $\boldsymbol{M}$. The sides have lengths $d(x, u)$, $d_{\boldsymbol{M}}(x, v) = d(g(x), u)$, and s; but the triangle $\{x, g(x), u\}$ is metric and has the same side lengths.

Now the bijection $\tilde{g}$ of $A \cup B \cup \{u, v\}$ extending g and interchanging u and v is an automorphism of $\boldsymbol{M}$ because $d_{\boldsymbol{M}}(u, x) = d(u, x) = d_{\boldsymbol{M}}(v, g(x))$ for all $x \in A \cup B$.

Because $\boldsymbol{U}_{\mathcal{S}}$ is homogeneous there exists an embedding h of $\boldsymbol{M}$ into $\boldsymbol{U}_{\mathcal{S}}$ with $h(x) = x$ for all $x \in A \cup B \cup \{u\}$ (see Theorem 4). By Lemma 14, let $0 < r \in \mathcal{S}$ be a jump number with $0 < r < \operatorname{gap}(\{d(h(v), x) : x \in A \cup B \cup \{u\}\}$, and let E be the $\stackrel{r}{\sim}$ equivalence class containing the point $h(v)$. If $i \in n$ is such that $h(v) \in A_i$, then choose $w \in A_i \cap E$; this is possible because A_i is assumed to be dense. It follows from the choice of r and from Fact 12 that $d(w, x) = d(h(v), x) = d_{\boldsymbol{M}}(v, x)$ for all $x \in A \cup B \cup \{u\}$.

The required automorphism g' is simply the map corresponding to $\tilde{g}$ interchanging u and w. □

4 Distinguishing Number of Homogeneous Urysohn Metric Spaces

In this section, we show that the distinguishing number $\boldsymbol{U}_{\mathcal{S}}$ is either 2 or infinite for any countable universal spectrum $\mathcal{S}$. When it is 2, we will show this is so by decomposing $\boldsymbol{U}_{\mathcal{S}}$ into a rigid subspace particularly constructed so that all automorphisms fixing this subspace also fix its complement.

In all cases but one, the rigid subspace is made from a rigid forest. First we show how to use the graph structure of a metric space.

Definition 17. Consider a metric space $\boldsymbol{M}$ with distances in $\mathcal{S} \subseteq \mathbb{R}$. For $s \in \mathcal{S}$, the *s-distance graph* of $\boldsymbol{M}$ is the (simple) graph on the elements of $\boldsymbol{M}$ (as vertices), and two vertices are adjacent if and only if their distance is s.

This following observation will play a crucial role in building rigid subspaces.

Observation. If the s-distance graph of a metric space is rigid, then the metric space is rigid.

4.1 Basic Construction

Here is the first such construction.

Lemma 18. *If $\mathcal{S}$ be a universal spectrum, then the distinguishing number of $\boldsymbol{U}_{\mathcal{S}}$ is 2 if there exists a positive number $s \in \mathcal{S}$ for which the following hold.*

1. *The element s is not a jump number; that is, there exists a number $r \in \mathcal{S}$ with $s < r \leqslant s + s$.*

2. *For every positive $t \in \mathcal{S}$, there exist numbers $\{h_t, k_t\} \subseteq \mathcal{S}$ so that:*

   (a) $s < h_t < k_t$.

   (b) $h_t + k_t \geqslant t \geqslant k_t - h_t$.

Proof. We say that a set P of pairs of points in $\boldsymbol{U}_\mathcal{S} \setminus \boldsymbol{M}$ is *stabilized by the subspace $\boldsymbol{M}$ of $\boldsymbol{U}_\mathcal{S}$* if:

1. The s-distance graph of $\boldsymbol{M}$ is a rigid forest.

2. For all $(x, y) \in P$, there exists a point $z \in M$ with $d(x, z) \neq d(y, z)$.

The proof of the lemma is an inductive construction on the countable domain of $\boldsymbol{U}_\mathcal{S}$, building such an $\boldsymbol{M}$ as an increasing union of finite spaces such that any pair $(x, y) \in \boldsymbol{U}_\mathcal{S} \setminus \boldsymbol{M}$ is stabilized by the subspace $\boldsymbol{M}$. We then have that the partition of $\boldsymbol{M}$ and its complement shows that the distinguishing number of $\boldsymbol{U}_\mathcal{S}$ is 2.

The construction of $\boldsymbol{M}$ is a consequence of the following claim handling the inductive finite stages.

Claim. Let P be a set of pairs of points in $\boldsymbol{U}_\mathcal{S}$ which is stabilized by the finite subspace $\boldsymbol{M}$ of $\boldsymbol{U}_\mathcal{S}$, and let $\{x, y\}$ be two points in $\boldsymbol{U}_\mathcal{S} \setminus \boldsymbol{M}$. We then have that there exists a finite subspace $\boldsymbol{N}$ of $\boldsymbol{U}_\mathcal{S}$ containing $\boldsymbol{M}$ and stabilizing $P' = P \cup \{(x, y)\}$.

To prove the claim, if there already exists a point $z \in M$ with $d(x, z) \neq d(y, z)$, then we can let $\boldsymbol{N} = \boldsymbol{M}$. Hence, we assume that $d(x, z) = d(y, z)$ for all $z \in M$.

Let $r \in \mathcal{S}$ with $s < r \leqslant s + s$. Let $\boldsymbol{C}$ be a metric space with spectrum $\{s, r\}$ whose s-distance graph is a rigid tree G which is not isomorphic to one of the trees of the s-distance graph of the space $\boldsymbol{M}$. Let e be an endpoint of the tree G. Let $\boldsymbol{T}$ be the metric space with $T = \{x, y, e\}$ so that $d_\mathrm{T}(x, y) = t$ and $d_T(x, e) = h_t$ and $d_\mathrm{T}(y, e) = k_t$ as per the hypothesis. Then $\boldsymbol{T}$ is indeed a metric space because $h_t + k_t \geqslant t \geqslant k_t - h_t$ and $h_t < k_t$. The pair of metric spaces $(\boldsymbol{T}, \boldsymbol{C})$ forms an amalgamation instance. Now because $s < r$ and $s < h_t$, then $s < r' = \min\{r, h_t\}$. It follows from Lemma 5 that there exists a metric space $\boldsymbol{C}' \in \amalg_\mathcal{S}(\mathrm{T}, \boldsymbol{C})$ with $d_{\boldsymbol{C}'}(v, x) \geqslant r'$ and with $d_{\boldsymbol{C}'}(v, y) \geqslant r'$ for all $v \in C$ and so that $d_{\boldsymbol{C}'}(v, x) \leqslant d_{\boldsymbol{C}'}(v, y)$ for every $v \in C$.

Now let $\boldsymbol{M}'$ be the subspace of $\boldsymbol{U}_\mathcal{S}$ induced by the set $M \cup \{x, y\}$ of points. The metric spaces $\boldsymbol{M}'$ and $\boldsymbol{C}'$ form an amalgamation instance. It follows from Lemma 5 again that there exists a metric space $\boldsymbol{M}'' \in \amalg_\mathcal{S}(\boldsymbol{M}', \boldsymbol{C}')$ so that $d_{\boldsymbol{M}''}(v, z) \geqslant r' > s$ for every $v \in C$ and every $z \in M$.

Because $\boldsymbol{U}_\mathcal{S}$ is homogeneous there exists an embedding f of $\boldsymbol{M}''$ into $\boldsymbol{U}_\mathcal{S}$ with $f(z) = z$ for all $z \in M'$. Then the image $\boldsymbol{N}$ of $\boldsymbol{M}''$ under the embedding f, removing x and y, is as required. ☐

This yields the following case.

Corollary 19. *Let $\mathcal{S}$ be a universal spectrum. If $\mathcal{S}$ contains a positive number s which is not a jump number, and has a limit r (not necessarily in $\mathcal{S}$) with $s < r$, then the distinguishing number of $\boldsymbol{U}_\mathcal{S}$ is 2.*

Proof. By Lemma 18, if suffices to show that for every positive $t \in \mathcal{S}$ there exist numbers $\{h_t, k_t\} \subseteq \mathcal{S}$ so that $s < h_t < k_t$, and $h_t + k_t \geqslant t \geqslant k_t - h_t$.

If $t \leqslant r$, then because r is a limit point of $\mathcal{S}$ one can find the required numbers $s < h_t < k_t \in \mathcal{S}$ (close enough to r). If $t > r$, then choose $h_t \in \mathcal{S}$ close enough to r so that $s < h_t < t$, and let $k_t = t$; again this is possible because r is a limit point of $\mathcal{S}$. $\square$

The case of $\mathcal{S}$ having a positive limit in $\mathcal{S}$ can be handle in a similar manner, but the more general case of the positive limit not necessarily in $\mathcal{S}$ is more delicate. In that case the rigid forest will be replaced by a "crab nest".

4.2 Crab Nest

In the more general situation of $\mathcal{S}$ having a limit point not in $\mathcal{S}$ and all points below are jump numbers, we may not be able to retain the connected components in the intended rigid s-graph, and therefore, we need a new structure.

First call a finite graph S a *spider* if it is a tree which contains exactly one vertex, the *centre* of S, of degree larger than 2 and then all of the other vertices have degree 2 or are endpoints having degree one.

Two cliques $\boldsymbol{C}$ and $\boldsymbol{C}'$ of a graph G are called *adjacent* if they are vertex disjoint, the order of one of them (say $\boldsymbol{C}'$) is less than or equal to the order of the other ($\boldsymbol{C}$), and there exists an injection f of $V(\boldsymbol{C}')$ to $V(\boldsymbol{C})$ so that a vertex $x \in V(\boldsymbol{C}')$ is adjacent to a vertex $y \in V(\boldsymbol{C})$ if and only if $f(x) = y$.

A finite graph $G = G(S, n, \mathcal{C})$ is a *crab* if S is a rigid spider, $n \geqslant 5$ is an integer, called the *heft* of G denoted by $\mathrm{heft}(G)$, $\mathcal{C}$ is a set of maximal cliques of the graph G:

1. Every vertex of G is a vertex of exactly one of the cliques in $\mathcal{C}$.
2. Exactly one of the cliques in $\mathcal{C}$, the *centre clique of* G, has order $n+1$ and all other cliques in $\mathcal{C}$ have order n.
3. Cliques in $\mathcal{C}$ are either adjacent, or have no edges between them.
4. If $\boldsymbol{C}$ is the centre clique, then for every vertex $x \in V(\boldsymbol{C})$ there exists exactly one clique $\boldsymbol{C}' \in \mathcal{C}$ which is adjacent to $\boldsymbol{C}$ and for which no vertex in $V(\boldsymbol{C}')$ is adjacent to x.
5. There exists a bijection π of $\mathcal{C}$ to $V(S)$ which maps the centre clique of G to the centre of S, and such that $\boldsymbol{C}'$ is adjacent to $\boldsymbol{C}''$ (in $\mathcal{C}$) if and only if $\pi(\boldsymbol{C}')$ is adjacent to $\pi(\boldsymbol{C}'')$ (in S).

A clique $\boldsymbol{C} \in \mathcal{C}$ is an *end clique* of the crab G if $\pi(\boldsymbol{C})$ is an endpoint of the spider S. Note that the degree of the centre of the spider S is equal to $n+1$.

Lemma 20. *Every crab is rigid.*

Proof. Let $G = G(S, n, \mathcal{C})$ be a crab with associated spider S, $\text{heft}(G) = n$, and $\mathcal{C}$ its set of maximal cliques. Note that the structure of such a crab implies that any triangle in G must be contained in one of the cliques $\boldsymbol{C} \in \mathcal{C}$; indeed any edge (x, y) across two different cliques implies the cliques are adjacent.

This implies that any automorphism g of G induces a permutation of of $\mathcal{C}$, and fixes the centre clique $\boldsymbol{C}$ because it is the only one of size $n+1$. Hence, g induces an automorphism of its associated spider S and thus, because the latter is assumed to be rigid, $\mathcal{C}$ as a set is actually fixed by g.

We claim that g is the identity map on G. First let $x \in V(\boldsymbol{C})$ the centre clique, and assume that $x \neq g(x)$. There exists a (unique) clique $\boldsymbol{C}'$ which is adjacent to the clique $\boldsymbol{C}$ and which does not contain a vertex which is adjacent to $g(x)$. We then have that $V(\boldsymbol{C}')$ contains a vertex x' which is adjacent to x, implying that $g(x') \notin V(\boldsymbol{C}')$, a contradiction. Hence, $g(x) = x$ for all $x \in V(\boldsymbol{C})$. But this implies inductively on the distance from the centre that $g(x) = x$ for all $x \in V(G)$. □

The rigid subspaces we are looking for will be built of crabs into what we call crab nests.

Definition 21. A graph $G = G(H = \{H_i, i \in I\}, R)$ (where $I = \omega$ or $I = n \in \omega$) is a *crab nest* if $V(G)$ is the disjoint union of the $V(H_i)$'s, $R \subseteq V(G)$ (called the *distinguished endpoint set* of G), so that for every $i \in I$:

1. H_i as an induced subgraph of G is a crab (with its specified spider, heft and decomposition into maximal cliques).
2. $\text{heft}(H_i) + 2 < \text{heft}(H_{i+1})$.
3. R contains exactly one vertex $r_i \in V(H_i)$, belonging to an end clique of H_i.
4. If $(x, y) \in E(G) \setminus E(H)$, then the following hold:
   (a) If $x \in V(H_j)$ and $y \in V(H_i)$ for some $j < i$, then $y = r_i$.
   (b) If $(z, r_i) \in E(G) \setminus E(H)$ and $z \in H_k$ with $k < i$, then $z = x$.

Thus 4a implies that only $r_i \in H_i$ may be connected to a previous H_j, and if so only once by 4b.

Lemma 22. *Every crab nest is rigid.*

Proof. Let $G = G(H = \{H_i, i \in I\}, R)$ be a crab nest, and let g be an automorphism of G. We show that g must be the identity.

Assume that there exists a vertex $x \in V(H_i)$ with $i > 0$ such that $g(x) \in V(H_0)$. Let $\boldsymbol{C}$ be the maximal clique (thus, of size at least three) of H_i containing x, and set:

$$j = \max\{k \in I : \text{ for some } y \in V(\boldsymbol{C}), g(y) \in V(H_j)\}.$$

First assume that $j > 0$. Note that if $y \in V(\boldsymbol{C})$ is such that $g(y) \in V(H_j)$, then $(g(y), g(x)) \in E(G) \setminus E(H)$, and thus, by item (4a) we must have $f(y) = r_j$. But now if z is a third element of $V(\boldsymbol{C})$ and $g(z) \in V(H_k)$, then $k \leqslant j$ by definition. But $k < j$ since $g(y)$ is the only vertex of $V(H_j)$ connected to $g(x)$, and $k \geqslant j$ since $g(x)$ is the only vertex of any $V(H_\ell)$ for $\ell < j$ connected to $g(y)$.

It follows that $j = 0$. That is, the automorphism g maps every element of $V(\boldsymbol{C})$ into $V(H_0)$. But every triangle of H_0 is in one of the cliques of H_0, implying that g maps the maximal clique $\boldsymbol{C}$ into a maximal clique of H_0. But this is not possible because $\text{heft}(H_i) > \text{heft}(H_0) + 2$. It follows that g does not map any vertex of $V(G) \setminus V(H_0)$ into $V(H_0)$. Implying that if $x \in V(H_0)$, then $g(x) \in V(H_0)$.

This in turn implies, because the crab H_0 is rigid, that $g(x) = x$ for all $x \in V(H_0)$. Then via induction on the index set I it follows that $g(x) = x$ for all $x \in V(G)$. □

We are now ready to handle the general case of $\mathcal{S}$ having a positive limit not necessarily in $\mathcal{S}$.

Lemma 23. *Let $\mathcal{S}$ be a universal spectrum. If $\mathcal{S}$ has a positive limit (not necessarily in $\mathcal{S}$), then the distinguishing number of $\boldsymbol{U}_\mathcal{S}$ is 2.*

Proof. If $\mathcal{S}$ has a positive limit r (not necessarily in $\mathcal{S}$) and one can find a non-jump number $s < r$, then Corollary 19 applies. This is the case if $\mathcal{S}$ has two positive limits $r' < r$, in which case one can find such a non-jump number $s \in \mathcal{S}$ close to r'. Similarly, this is the case if r is a limit of the elements of $\mathcal{S}$ below r.

Hence, we may assume that $\mathcal{S}$ has only one positive limit r, every number in $\mathcal{S}$ less than r is insular, and the elements of $\mathcal{S}$ above r form a (possibly two way) sequence converging to r. In particular this means that every non-empty and bounded above subset of $\mathcal{S}$ has a maximum, that is $\mathcal{S}$ is inversely well ordered. Thus the operation $\oplus$ is defined for $\mathcal{S}$ and we will be using this fact.

We are now ready to undertake the construction of the stabilizing subspace using a rigid s-graph, where $s \in \mathcal{S}$ is chosen close enough to r so that $r < s < s' < s'' < r + r$ for some $s', s'' \in \mathcal{S}$; this is possible due to the elements of $\mathcal{S}$ above r converging to r. That space and its complement will show that the distinguishing number of $\boldsymbol{U}_\mathcal{S}$ is 2.

For the sake of this proof, we say that a set P of pairs of points in $\boldsymbol{U}_\mathcal{S}$ is *stabilized* by the subspace $\boldsymbol{M}$ if the following properties hold.

1. The s-distance graph of $\boldsymbol{M}$ is a crab nest $G = G(H = \{H_i : i \in n\}, R)$.
2. If $u \neq v$ are two points of $\boldsymbol{M}$, then $d(u, v) \geqslant r$.
3. For every pair $(x, y) \in P$, there exists a crab H_i and point $r_i \in R \cap H_i$ such that $d(x, r_i) \neq d(y, r_i)$.

The proof of Lemma 23 is an inductive construction on the countable domain of $\boldsymbol{U}_\mathcal{S}$, building such an $\boldsymbol{M}$ as an increasing union of finite spaces such that any pair $(x, y) \in \boldsymbol{U}_\mathcal{S} \setminus \boldsymbol{M}$ is stabilized by the subspace $\boldsymbol{M}$. We then have that the partition of $\boldsymbol{M}$ and its complement shows that the distinguishing number of $\boldsymbol{U}_\mathcal{S}$ is 2.

The construction is a consequence of the following claim handling the inductive step.

Claim. Let P be a set of pairs of points in $\boldsymbol{U_\mathcal{S}}$ which is stabilized by the finite subspace $\boldsymbol{M}$ of $\boldsymbol{U_\mathcal{S}}$. Let $\{x, y\}$ be two points in $\boldsymbol{U_\mathcal{S}} \setminus \boldsymbol{M}$ such that $d(x, y) = t > 0$, and $d(x, z) = d(y, z)$ for all $z \in M$. Then there exists a finite subspace $\boldsymbol{N}$ of $\boldsymbol{U_\mathcal{S}}$ containing $\boldsymbol{M}$ and stabilizing $P' = P \cup \{(x, y)\}$.

To prove the claim, let $G = G(H = \{H_i : i \in n\}, R)$ be the s-distance graph of $\boldsymbol{M}$. If $t < r$, then let $r < h_t < k_t$ be two numbers in $\mathcal{S}$ (close to r) with $k_t - h_t \leqslant t$. If $r \leqslant t < r + r$, then let $h_t = s'$ and $k_t = s''$. Finally if $r + r \leqslant t$, then let $h_t = s'$ and $k_t = t$. Note that $h_t < k_t$, $k_t - h_t \leqslant t$, and $t \leqslant h_t + k_t$ in all cases, and thus the triangle with distances $\{t, h_t, k_t\}$ is metric in all cases.

Let $\boldsymbol{C}$ be a metric space with spectrum $\{r, s\}$ whose s-distance graph is a crab H_n disjoint from each H_i for $i \in n$ and for which $\mathrm{heft}(H_{n-1}) + 2 < \mathrm{heft}(H_n)$. Let r_n be a vertex of an end clique of the crab H_n and let $R' = R \cup \{r_n\}$. Let $\boldsymbol{T}$ be the metric space with $T = \{x, y, r_n\}$ so that $d_\mathrm{T}(x, y) = t$ and $d_T(x, r_n) = h_t$ and $d_\mathrm{T}(y, r_n) = k_t$. Note that $\boldsymbol{T}$ is in all cases indeed a metric space in $\mathfrak{A}(\mathcal{S})$. The pair of metric spaces $(\boldsymbol{T}, \boldsymbol{C})$ forms an amalgamation instance. It follows from Lemma 5 and from $s' \leqslant r+r \leqslant r+h_t < r+k_t$ that there exists a metric space $\boldsymbol{C'} \in \amalg_\mathcal{S}(\boldsymbol{T}, \boldsymbol{C})$ with

$$d_{\boldsymbol{C'}}(v, x) \geqslant s' \text{ and } d_{\boldsymbol{C'}}(v, y) \geqslant s' \text{ for all } r_n \neq v \in C \tag{i.}$$

and so that $d_{\boldsymbol{C'}}(v, x) \leqslant d_{\boldsymbol{C'}}(v, y)$ for every $v \in C$.

Now let $\boldsymbol{M'}$ be the subspace of $\boldsymbol{U_\mathcal{S}}$ induced by the set $M \cup \{x, y\}$. The metric spaces $\boldsymbol{M'}$ and $\boldsymbol{C'}$ form an amalgamation instance, and we let $\boldsymbol{M''} = \amalg^\oplus_\mathcal{S}(\boldsymbol{M'}, \boldsymbol{C'})$ provided by Lemma 10. Then by that construction we have for $v \in C$ and $z \in M$:

$$d_{\boldsymbol{M''}}(v, z) \geqslant d_{\boldsymbol{C'}}(v, x) \oplus d(x, z) \geqslant d_{\boldsymbol{C'}}(v, x). \tag{ii.}$$

Thus $d_{\boldsymbol{M''}}(v, z) \geqslant s'$ for any $r_n \neq v \in C$ and $z \in M$. Moreover, for $v = r_n$ and any $z \in M$, we have:
if $t \geqslant r$, then $h_t = s'$, so

$$d_{\boldsymbol{M''}}(r_n, z) \geqslant s' > s, and \tag{iii.}$$

if $t < r$, then by construction

$$d_{\boldsymbol{M''}}(r_n, z) = h_t \oplus d(x, z). \tag{iv.}$$

We claim that $\boldsymbol{M'''} = \boldsymbol{M''} \setminus \{x, y\}$ is the desired subspace.

First we must verify that the s-distance graph of $\boldsymbol{M'''}$ is a crab nest $G''' = G'''(H''' = \{H_i, i \in n+1\}, R')$, and only Item (4) of Definition 21 remains to be verified. To do so, let $(z, v) \in E(G''') \setminus E(H''')$, and hence $d_{\boldsymbol{M'''}}(z, v) = s$.
If $\{z, v\} \subseteq M$, then the requirements of Item (4) in Definition 21 for the pair (z, v) will be satisfied by the assumptions on $\boldsymbol{M}$. If $\{z, v\} \subseteq C$, then $\{z, v\} \in E(H_n)$ because the spectrum of $\boldsymbol{C}$ is $\{s, r\}$ and the s-distance graph of $\boldsymbol{C}$ is the crab H_n. Hence, we may assume that $v \in C$ and $z \in M$.
If $t \geqslant r$ or if $v \neq r_n$, it follows from ii. and iii. above that $d_{\boldsymbol{M'''}}(v, z) \geqslant s' > s$,

contradicting $d_{\boldsymbol{M}'''}(z,v) = s$.
Thus, $t < r$ and $v = r_n$. If $r \leqslant d(x,z)$ ($=d(y,z)$ by assumption), then by iv. $d_{\boldsymbol{M}''}(r_n, z) = h_t \oplus d(x,z) \geqslant r \oplus r \geqslant s' > s$, which is again a contradiction. If on the other hand $p = d(x,z) < r$, then it is possible that $d_{\boldsymbol{M}'''}(r_n, z) = h_t \oplus d(x,z) = h_t \oplus p = s$, and we must show that z is the only such vertex. So assume that $q = d(r_n, w) < r$ for a point $w \in M$. We then have that p and q are insular numbers of $\mathcal{S}$ and both smaller than r. If $z \neq w$, then $d(z,w) \geqslant r$ by assumption, which is a contradiction because as insular points $p \oplus q = \max\{p,q\} < r$. Hence, $z = w$ verifying Item (4) of Definition 21 and G''' is a crab nest.

Finally by construction (see ii., iii., and iv.), $d_{\boldsymbol{M}'''}(u,v) \geqslant r$ for any $u \neq v$ in $\boldsymbol{M}'''$, and $d_{\boldsymbol{M}'''}(x, r_n) = h_t \neq k_t = d_{\boldsymbol{M}'''}(y, r_n)$.

The space $\boldsymbol{M}'''$ is not immediately a subspace of $\boldsymbol{U}_{\mathcal{S}}$ as constructed, but since $\boldsymbol{U}_{\mathcal{S}}$ is homogeneous, there exists an embedding f of $\boldsymbol{M}'''$ into $\boldsymbol{U}_{\mathcal{S}}$ with $f(a) = a$ for all $a \in M$. Then the image of $\boldsymbol{M}'''$ under the embedding f is the required subspace stabilizing $P \cup (x,y)$ and the proof of the lemma is complete. $\square$

4.3 The Remaining Cases

There are a few remaining cases to handle, made possible from previous results and techniques.

Lemma 24. *Let $\mathcal{S}$ be a universal spectrum. If S contains two distinct and positive elements of distance smaller than or equal to the minimum positive element of $\mathcal{S}$, then the distinguishing number of $\mathcal{S}$ is 2.*

Proof. Let $p = \inf \mathcal{S}_{>0} \geqslant b - a$, where $0 < a < b \in \mathcal{S}$. If $\mathcal{S}$ has a positive limit, then the distinguishing number of $\boldsymbol{U}_{\mathcal{S}}$ is 2 according to Lemma 23. Thus, we assume that $\mathcal{S}$ does not have a positive limit. In particular, $p \in \mathcal{S}$.

Moreover, the set $\mathcal{S}$ must contain an initial non-jump number. Assume to the contrary that there is no such number. Note that $a < b \leqslant p + a \leqslant a + a = 2a$ and thus, a is a non-jump number and hence must be non-initial. Thus there is $a_1 \in \mathcal{S} \cap [a/2, a]$, a non-jump number, and hence a_1 must be non-initial. Continuing in this manner yields a positive limit in $\mathcal{S}$, a contradiction.

Now, let s be the smallest positive initial non-jump number of $\mathcal{S}$, and let r be the smallest number in $\mathcal{S}$ larger than s. We cannot have $a < s$ because being a non-jump number and smaller that s would make a again a non-initial number; and similar to above would yield $\mathcal{S}$ with a positive limit point. If $s < a$, then we claim that Lemma 18 applies. Consider an arbitrary positive element $t \in \mathcal{S}$: if $t > r$, let $h_t = a$ and $k_t = b$, and if $t \leqslant r$, let with $h_t = r$ and $k_t = t$. Thus the conditions of Lemma 18 are indeed satisfied and hence, $\boldsymbol{U}_{\mathcal{S}}$ has then distinguishing number 2.

Hence, we may assume that $s = a$ and thus r can be used in the role of b, meaning that that $r - s \leqslant p$, and of course $q \geqslant p$ for all $q \in \mathcal{S}$. We then proceed analogously to the proof of Lemma 18, and construct a subspace together with its complement showing that the distinguishing number of $\boldsymbol{U}_{\mathcal{S}}$ is 2.

For the sake of this proof, we say that a set of pairs P in $\boldsymbol{U}_{\mathcal{S}}$ is *stabilized* by the subspace $\boldsymbol{M}$ of $\boldsymbol{U}_{\mathcal{S}}$ if for all $(x, y) \in P$:

1. There exists a point $z \in M$ with $d(x, z) \neq d(y, z)$.

2. The s-distance graph of $\boldsymbol{M}$ is a rigid forest.

The proof of the Lemma is an inductive construction on the countable domain of $\boldsymbol{U}_{\mathcal{S}}$, and is a consequence of the following claim handling the inductive step.

Claim. Let P be a set of pairs of points in $\boldsymbol{U}_{\mathcal{S}}$ which is stabilized by the finite subspace $\boldsymbol{M}$ of $\boldsymbol{U}_{\mathcal{S}}$. Let $\{x, y\}$ be two points in $\boldsymbol{U}_{\mathcal{S}} \setminus \boldsymbol{M}$ with $t = d(x, y) > 0$. Then there exists a finite subspace $\boldsymbol{N}$ of $\boldsymbol{U}_{\mathcal{S}}$ containing $\boldsymbol{M}$ and stabilizing $P' = P \cup \{(x, y)\}$.

To prove the claim, if there exists a point $z \in M$ with $d(x, z) \neq d(y, z)$ let $\boldsymbol{N} = \boldsymbol{M}$. Hence, we may assume that $d(x, z) = d(y, z)$ for all $z \in M$.

Let $\boldsymbol{C}$ be a metric space with spectrum $\{s, r\}$ whose s-distance graph is a rigid tree S which is not isomorphic to one of the trees of the s-distance graph of the space $\boldsymbol{M}$. Let e be an endpoint of the tree S. If $t \leqslant r$ let $h_t = s$ and $k_t = r$; and if $t > r$ let $h_t = r$ and $k_t = t$. Let $\boldsymbol{T}$ be the metric space with $T = \{x, y, e\}$ so that $d_{\boldsymbol{T}}(x, y) = t$, $d_{\boldsymbol{T}}(x, e) = h_t$ and $d_{\boldsymbol{T}}(y, e) = k_t$. Note that $\boldsymbol{T}$ is indeed a metric space in all cases. The pair of metric spaces $(\boldsymbol{T}, \boldsymbol{C})$ forms an amalgamation instance, and it follows from Lemma 5 and from $r \leqslant s + h_t < s + k_t$ that there exists a metric space $\boldsymbol{C}' \in \amalg_{\mathcal{S}}(\boldsymbol{T}, \boldsymbol{C})$ with $d_{\boldsymbol{C}'}(v, x) \geqslant r$ and with $d_{\boldsymbol{C}'}(v, y) \geqslant r$ for all $e \neq v \in C$ and so that $d_{\boldsymbol{C}'}(v, x) \leqslant d_{\boldsymbol{C}'}(v, y)$ for every $v \in C$. Hence, $d_{\boldsymbol{C}'}(v, y) \geqslant d_{\boldsymbol{C}'}(v, x) \geqslant s$ for all $v \in C$

Let $\boldsymbol{M}'$ be the subspace of $\boldsymbol{U}_{\mathcal{S}}$ induced by the set $M \cup \{x, y\}$ of points. The metric spaces $\boldsymbol{M}'$ and $\boldsymbol{C}'$ form an amalgamation instance. Let $\boldsymbol{M}'' = \amalg^{\oplus}_{\mathcal{S}}(\boldsymbol{M}', C')$. It follows from the definition of $\oplus$ that:

$$d_{\boldsymbol{M}''}(v, z) \geqslant d_{\boldsymbol{C}'}(v, x) \oplus d(x, z) \geqslant s \oplus d(x, z) \geqslant r.$$

Thus the s-distance graph of $\boldsymbol{M}'' \setminus \{x, y\}$ is a rigid forest, and there exists a point $z \in M$ with $d(x, z) \neq d(y, z)$ (namely $z = r_n$). Because $\boldsymbol{U}_{\mathcal{S}}$ is homogeneous there exists an embedding f of $\boldsymbol{M}''$ into $\boldsymbol{U}_{\mathcal{S}}$ with $f(a) = a$ for all $a \in M'$ (see Theorem 4), and the image $\boldsymbol{N}$ of $\boldsymbol{M}'' \setminus \{x, y\}$ under the embedding f is as required for the claim. ☐

We arrive at the first case where $\boldsymbol{U}_{\mathcal{S}}$ has infinite distinguishing number.

Lemma 25. *Let $\mathcal{S}$ be a universal spectrum which does not have a limit, then the distinguishing number of $\boldsymbol{U}_{\mathcal{S}}$ is 2 or infinite. If $p = \min(\mathcal{S}_{>0})$, then the distinguishing number of $\boldsymbol{U}_{\mathcal{S}}$ is 2 if and only if there exist positive numbers $a < b \in \mathcal{S}$ with $b - a \leqslant p$.*

Proof. If there exist positive numbers $a < b \in \mathcal{S}$ with $b - a \leqslant p$, then the distinguishing number of $\boldsymbol{U}_{\mathcal{S}}$ is 2 again according to Lemma 24. This is the case if p is not a jump number, because if $p < q \leqslant p + p$, then $q - p \leqslant p$. Thus, we may assume that p is a jump number, and hence, an insular number because being the smallest positive number of $\mathcal{S}$ it is an initial number.

Thus, we assume that p is insular and $p < b-a$ for all $a < b \in \mathcal{S}$, and we show that the distinguishing number of $\boldsymbol{U}_\mathcal{S}$ is not finite. Let γ be a colouring function of $\boldsymbol{U}_\mathcal{S}$ with $n \in \omega$ colours, that is $\gamma[\boldsymbol{U}_\mathcal{S}] \subseteq n$. Note that because p is a jump number and $p = \min(\mathcal{S}_{>0})$, the relation $\sim$ on $\boldsymbol{U}_\mathcal{S}$ given by $x \sim y$ if and only if $d(x, y) = p$ is an equivalence relation. Let E be a $\sim$ equivalence class of $\boldsymbol{U}_\mathcal{S}$. By our assumption, $d(x, z) = d(y, z)$ for all points z of $\boldsymbol{U}_\mathcal{S} \setminus E$ and all points $x, y \in E$ (see Fact 12). The set E is infinite because every finite metric space $\boldsymbol{M}$ with $\mathrm{spec}(\boldsymbol{M}) = \{p\}$ is an element of the age $\mathfrak{A}(S)$. Hence, there are two points $x \neq y$ in E with $\gamma(x) = \gamma(y)$. The function f with $f(z) = z$ for all points z of $\boldsymbol{U}_\mathcal{S} \setminus \{x, y\}$, and $f(x) = y$ and $f(y) = x$, is then a colour preserving non-trivial automorphism of $\boldsymbol{U}_\mathcal{S}$. □

The following is another instance where we can show that the distinguishing number is infinite.

Lemma 26. *Let $\mathcal{S}$ be a universal spectrum with no positive limits but with 0 as a limit. If $\mathrm{gap}(\mathcal{S}_{\geqslant s}) > 0$ for every positive number $s \in \mathcal{S}$, then there exists for every $n \in \omega$ and every partition $\boldsymbol{P} = \{A_i : i \in n\}$ of $\boldsymbol{U}_\mathcal{S}$ a non-trivial automorphism f of $\boldsymbol{U}_\mathcal{S}$ preserving $\boldsymbol{P}$.*

Proof. Under these assumptions, 0 is also a limit of the set of initial numbers of $\mathcal{S}$ (see Lemma 14). Thus we can find a jump number s of $\mathcal{S}$, a non-empty subset $I \subseteq n$ and an $\overset{s}{\sim}$ equivalence class E so that for all $i \in I$ the set A_i is dense for the homogeneous metric space E (isomorphic to $\boldsymbol{U}_{\mathcal{S}_{\leqslant s}}$), and so that $A_i \cap E = \emptyset$ for all $i \notin I$.

Let $r \in \mathcal{S}$ be a jump number with $0 < r < \mathrm{gap}(\mathcal{S}_{\geqslant s})$. According to Lemma 16 (applied to E and r), there exists an automorphism f' of E with $d(x, f'(x)) = r$ for all $x \in E$ which preserves the partition of E induced by the partition $\boldsymbol{P}$ of $\boldsymbol{U}_\mathcal{S}$. It follows from Fact 12 that $d(y, x) = d(y, f'(x))$ for all points y in $\boldsymbol{U}_\mathcal{S} \setminus E$ and all points $x \in E$. It follows that the function $f : \boldsymbol{U}_\mathcal{S} \to \boldsymbol{U}_\mathcal{S}$ with $f(x) = f'(x)$ if $x \in E$ and with $f(y) = y$ if $y \notin E$ is a non-trivial automorphism of $\boldsymbol{U}_\mathcal{S}$ preserving $\boldsymbol{P}$. □

We can then characterize the case of a universal spectrum with no positive limits but with 0 as a limit.

Lemma 27. *Let $\mathcal{S}$ be a universal spectrum with no positive limits but with 0 as a limit, then the distinguishing number of $\boldsymbol{U}_\mathcal{S}$ is 2 or infinite. The distinguishing number of $\boldsymbol{U}_\mathcal{S}$ is infinite if and only if $\mathrm{gap}(\mathcal{S}_{\geqslant s}) > 0$ for every positive number $s \in \mathcal{S}$.*

Proof. On account of Lemma 26 it remains to prove that if there exists a positive number $s \in \mathcal{S}$ for which $\mathrm{gap}(\mathcal{S}_{\geqslant s}) = 0$, then the distinguishing number of $\boldsymbol{U}_\mathcal{S}$ is equal to 2.

But if every number in $\mathrm{gap}(\mathcal{S}_{\geqslant s})$ is insular, then $\mathrm{gap}(\mathcal{S}_{\geqslant s}) \geqslant s > 0$. Let r be the smallest non-insular initial number larger than or equal to s. Then $\mathrm{gap}(\mathcal{S}_{\geqslant r}) = 0$ and in turn then $\mathrm{gap}(\mathcal{S}_{\geqslant r^+}) = 0$. By Lemma 18, the distinguishing number of $\boldsymbol{U}_\mathcal{S}$ is 2. □

This completes all the results required to prove the Main Theorem characterizing the distinguishing number of universal homogeneous Urysohn metric spaces.

5 Conclusion

We have shown that the distinguishing number of every universal homogeneous Urysohn metric spaces is either 2 or infinite, and moreover characterized when each case occurs by structural properties of the corresponding universal spectrum. It is interesting that this is the case even though the permutation group of these Urysohn metric spaces is often imprimitive. This is for example the case when the spectrum $\mathcal{S}$ contains a jump number $s > 0$, then the relation $\overset{s}{\sim}$ given by $x \overset{s}{\sim} y$ if $d(x, y) \leqslant s$ is an equivalence relation on $\boldsymbol{U}_{\mathcal{S}}$. Hence if this is a non-trivial relation, for example if $\mathcal{S}$ contains an element larger than s, then the automorphism group of $\boldsymbol{U}_{\mathcal{S}}$ is imprimitive. We thus propose an open problem suggested by the referee.

Open Problem. Characterize universal spectra $\mathcal{S}$ such that the corresponding universal homogeneous Urysohn metric space $\boldsymbol{U}_{\mathcal{S}}$ has a primitive automorphism group.

In these cases of imprimitivity, it is due to the homogeneity and universality that the distinguishing number passes directly from 2 to infinity. However, one cannot expect the distinguishing number of every metric space to always be either 2 or infinite, even for homogeneous metric spaces. This is the case of the pentagon C_5 equipped with the graph distance, making it into an homogeneous metric space with primitive automorphism group D_{10} and distinguishing number 3. One can also produce an infinite homogeneous metric space of distinguishing number 3, but with imprimitive automorphism group. Indeed consider the Rado (homogeneous) graph, first turn it into a metric space $\boldsymbol{R}$ with spectrum $\{0, 3, 5\}$ by assigning distance 5 to every edge, distance 3 to non-edges, and then consider the wreath product $\boldsymbol{R}[\boldsymbol{M}]$ for $\boldsymbol{M}$ the metric space consisting of two points of distance 1. This creates an homogeneous metric space $\boldsymbol{R}[\boldsymbol{M}]$ with spectrum $\{0, 1, 3, 5\}$. Since the Rado graph has distinguishing number 2, we must use 3 colours to obtain two different sets of two different colours to assign to elements of $\boldsymbol{M}$. Finally, the automorphism group of $\boldsymbol{R}[\boldsymbol{M}]$ is imprimitive because points of distance 1 form a non-trivial equivalence relation.

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