A simple proof of the Gan-Loh-Sudakov conjecture

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Abstract

We give a new unified proof that any simple graph on n vertices with maximum degree at most Δ has no more than $a\binom{\Delta+1}{t} + \binom{b}{t}$ cliques of size t ($t \geqslant 3$), where $n = a(\Delta+1) + b$ ($0 \leqslant b \leqslant \Delta$).

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1 Introduction

For a positive integer $t \geqslant 3$, let $k_t(G)$ be the number of cliques of size t in a simple graph $G = G(V, E)$. In [3], Gan, Loh, and Sudakov asked how large $k_t(G)$ can be for graphs with maximum degree at most Δ. They made a conjecture, which we henceforth refer to as the *GLS Conjecture*, that $k_t(G)$ is maximized by a disjoint union of a cliques of size $\Delta + 1$ and one clique of size b, where $|V| = a(\Delta + 1) + b$ for $0 \leqslant b \leqslant \Delta$. Moreover, they proved in [3] that

$$\text{the GLS Conjecture holds for } t = 3 \quad \Longrightarrow \quad \text{the GLS Conjecture holds for } t \geqslant 4.$$

The proof is an application of the Lovász version of the famed Kruskal–Katona theorem (see [2]).

Later on, Chase proved that the GLS Conjecture holds for $t = 3$ in [1], and hence resolved the GLS Conjecture completely. In this short note we present a new proof of the GLS conjecture that works for all $t \geqslant 3$ uniformly without using the Kruskal–Katona theorem. The proof can be viewed as a simplification and a generalization of Chase's proof in [1]. We prove the following statement:

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Theorem 1. *Let G be a simple graph on n vertices with maximum degree at most Δ. For any integer $t \geqslant 3$, if $n = a(\Delta+1)+b$ where $a, b \in \mathbb{Z}$ and $0 \leqslant b \leqslant \Delta$, then $k_t(G) \leqslant a\binom{\Delta+1}{t} + \binom{b}{t}$.*

For every simple graph $G = G(V, E)$, write $u \sim v$ if uv is an edge, and $u \nsim v$ if uv is a nonedge. We denote by $\overline{N(v)} \stackrel{\text{def}}{=} \{v\} \cup \{u \in V : u \sim v\}$ the closed neighborhood of v. Let T_v be the set of all t-cliques intersecting $\overline{N(v)}$. The proof of Theorem 1 relies on the following lemma:

Lemma 2. *For any integer $t \geqslant 3$, if $G = G(V, E)$ is a simple graph, then*

$$\sum_{v \in V} |T_v| \leqslant \sum_{v \in V} \binom{\deg(v)+1}{t}.$$

This note is organized as follows: We first show that Theorem 1 follows from Lemma 2, and then prove Lemma 2 in a separate section.

Proof of Theorem 1 assuming Lemma 2. Fix $t \geqslant 3$ and $\Delta \in \mathbb{N}_+$, and let G be an n-vertex graph. Then there exists $v \in V$ such that $|T_v| \leqslant \binom{\deg(v)+1}{t}$, by Lemma 2.

We induct on n. The base case is obvious, as Theorem 1 is trivially true for $n = 0, 1, \ldots, \Delta+1$. Suppose Theorem 1 is true for $n-1, n-2, \ldots, n-\Delta-1$. Then we have that

$$k_t(G) \leqslant \begin{cases} \binom{\deg(v)+1}{t} + a\binom{\Delta+1}{t} + \binom{b-\deg(v)-1}{t}, & \text{when } b \geqslant \deg(v)+1, \\ \binom{\deg(v)+1}{t} + (a-1)\binom{\Delta+1}{t} + \binom{b+\Delta-\deg(v)}{t}, & \text{when } b < \deg(v)+1 \leqslant b+\Delta+1. \end{cases}$$

Since the sequence $\left\{\binom{n}{t}\right\}_{n \geqslant 0}$ is convex, we have that $\binom{\deg(v)+1}{t} + \binom{b-\deg(v)-1}{t} \leqslant \binom{b}{t}$ when $b \geqslant \deg(v)+1$, and $\binom{\deg(v)+1}{t} + \binom{b+\Delta-\deg(v)}{t} \leqslant \binom{\Delta+1}{t} + \binom{b}{t}$ otherwise. We conclude that $k_t(G) \leqslant a\binom{\Delta+1}{t} + \binom{b}{t}$. $\square$

2 Proof of Lemma 2

Define the set

$$\Phi \stackrel{\text{def}}{=} \{(u, x_1, \ldots, x_t) \in V^{t+1} : x_1, \ldots, x_t \text{ form a } t\text{-clique in } G, \text{ and } u \sim x_i \text{ for some } i \in [t]\}.$$

Observe that each $(v, x_1, \ldots, x_t) \in \Phi$ consists of a vertex $v \in V$ and a t-clique $x_1 \cdots x_t \in T_v$. Since for every t-clique in G, there are $t!$ ways to label its t vertices as $x_1, \ldots, x_t$, we have that

$$|\Phi| = t! \sum_{v \in V} |T_v|. \tag{1}$$

For each tuple $(u, x_1, \ldots, x_t) \in \Phi$, the vertices $u, x_1, \ldots, x_t$ are not necessarily distinct. However, there are at least t distinct vertices among $u, x_1, \ldots, x_t$, because $x_1, \ldots, x_t$ form a t-clique. For every tuple $(u, x_1, \dots, x_t) \in V^{t+1}$, we call it *good* if $u, x_1, \dots, x_t$ are distinct, and *bad* otherwise. Let

$$\Phi_{\text{good}} \stackrel{\text{def}}{=} \{(u, x_1, \dots, x_t) \in \Phi : (u, x_1, \dots, x_t) \text{ is good}\},$$
$$\Phi_{\text{bad}} \stackrel{\text{def}}{=} \{(u, x_1, \dots, x_t) \in \Phi : (u, x_1, \dots, x_t) \text{ is bad}\}.$$

Then Φ_{good} and Φ_{bad} partition Φ.

Fix $v \in V$. If $(v, x_1, \dots, x_t) \in \Phi_{\text{bad}}$, then $v, x_1, \dots, x_t$ are vertices of a t-clique in G, where exactly one x_i happens to be v. There are t choices for this x_i, and at most $\binom{\deg(v)}{t-1}$ choices for the rest of the vertices $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_t$, and $(t-1)!$ choices for their possible permutations. Hence,

$$|\Phi_{\text{bad}}| \leqslant \sum_{v \in V} t \cdot \binom{\deg(v)}{t-1} \cdot (t-1)! = t! \sum_{v \in V} \binom{\deg(v)}{t-1}. \tag{2}$$

To upper bound $|\Phi_{\text{good}}|$, we need to introduce the auxiliary set

$$\Omega_{\text{good}} \stackrel{\text{def}}{=} \{(w, y_1, \dots, y_t) \in V^{t+1} : (w, y_1, \dots, y_t) \text{ is good, } w \sim y_i \text{ for all } i \in [t], \text{ and } y_1, \dots, y_t \text{ contain a } (t-1)\text{-clique in } G\}.$$

For any fixed $v \in V$, if $(v, y_1, \dots, y_t) \in \Omega_{\text{good}}$, then $y_1, \dots, y_t$ are distinct neighbors of v, and so

$$|\Omega_{\text{good}}| \leqslant t! \sum_{v \in V} \binom{\deg(v)}{t}. \tag{3}$$

We claim that

$$|\Phi_{\text{good}}| \leqslant |\Omega_{\text{good}}|. \tag{4}$$

Assume that (4) is established. From the combination of (1), (2), (3), and (4), we obtain

$$\begin{aligned} t! \sum_{v \in V} |T_v| = |\Phi| = |\Phi_{\text{bad}}| + |\Phi_{\text{good}}| &\leqslant |\Phi_{\text{bad}}| + |\Omega_{\text{good}}| \\ &\leqslant t! \sum_{v \in V} \left(\binom{\deg(v)}{t-1} + \binom{\deg(v)}{t} \right) \\ &= t! \sum_{v \in V} \binom{\deg(v)+1}{t}, \end{aligned}$$

which concludes the proof of Lemma 2. $\square$

Proof of estimate (4). When $\boldsymbol{u} \stackrel{\text{def}}{=} (u, x_1, \dots, x_t) \in \Phi_{\text{good}}$ or $\boldsymbol{w} \stackrel{\text{def}}{=} (w, y_1, \dots, y_t) \in \Omega_{\text{good}}$, the induced subgraph $G[\boldsymbol{u}]$ or $G[\boldsymbol{w}]$ is connected and contains a t-clique. Consider any induced $(t+1)$-vertex subgraph H of G that is connected and contains a t-clique. Let $z_1, \dots, z_t$ be the vertices of the t-clique (choose arbitrary ones if there are several). Let z^* be the remaining vertex of H. Assume without loss of generality that $z^* \sim z_1, \dots, z^* \sim z_k$, and $z^* \nsim z_{k+1}, \dots, z^* \nsim z_t$. Note that $t \geqslant 3$, we count for different values of k the contribution of H to $|\Phi_{\text{good}}|$ and $|\Omega_{\text{good}}|$, respectively:

- $1 \leqslant k \leqslant t-2$. If $(u, x_1, \ldots, x_t) \in \Phi_{\text{good}}$, then $u = z^*$ since the degree of z^* in H is less than $t-1$, and hence $\{x_1, \ldots, x_t\} = \{z_1, \ldots, z_t\}$. If $(w, y_1, \ldots, y_t) \in \Omega_{\text{good}}$, then $w \in \{z_1, \ldots, z_k\}$, and hence $\{y_1, \ldots, y_t\} = \{z^*, z_1, \ldots, z_t\} \setminus \{w\}$. Such an H contributes $t!$ and $k \cdot t!$ elements to Φ_{good} and Ω_{good}, respectively.

- $k = t-1$. If $(u, x_1, \ldots, x_t) \in \Phi_{\text{good}}$, then $\{x_1, \ldots, x_t\} \supset \{z_1, \ldots, z_{t-1}\}$, and hence $u \in \{z_t, z^*\}$. If $(w, y_1, \ldots, y_t) \in \Omega_{\text{good}}$, then $w \in \{z_1, \ldots, z_{t-1}\}$, and hence $\{y_1, \ldots, y_t\} = \{z^*, z_1, \ldots, z_{t-1}\} \setminus \{w\}$. Such an H contributes $2 \cdot t!$ and $(t-1) \cdot t!$ elements to Φ_{good} and Ω_{good}, respectively.

- $k = t$. Then $H = K_{t+1}$. Such an H contributes $(t+1)!$ elements to both Φ_{good} and Ω_{good}.

The claimed estimate (4) follows from the cases above. $\square$

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