Higher Convexity and Iterated Second Moment Estimates

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Abstract

We prove bounds for the number of solutions to the energy equation

$$a_1 + \cdots + a_k = a'_1 + \cdots + a'_k$$

over N-element sets of reals, which are sufficiently convex or near-convex. A near-convex set will be the image of a set with small additive doubling under a convex function with sufficiently many strictly monotone derivatives. We show, roughly, that every time the number of terms in the equation is doubled, an additional saving of 1 in the exponent of the trivial bound N^{2k-1} is made, starting from the trivial case $k = 1$. In the context of near-convex sets we also provide explicit dependencies on the additive doubling parameters.

Higher convexity is necessary for such bounds to hold, as evinced by sets of perfect powers of consecutive integers. We exploit these stronger assumptions using a different methodology and avoiding the use of the Szemerédi-Trotter theorem, which has not been adapted to embrace higher convexity.

As an application of our new estimates for $k > 2$ we improve the best known bounds for sumsets of convex sets under additional convexity assumptions.

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1 Introduction

Let $A := \{a_1 < \cdots < a_N\}$ be a finite subset of reals of cardinality $|A| = N$. A is called convex if the sequence of adjacent differences $a_{i+1} - a_i$ is strictly monotone for all $1 \leqslant i \leqslant N - 1$.

Convexity is thought to pose an obstruction to additive structure in A. A key measure of the presence of additive structure in A is the smallness of the cardinality of its sum or difference set

$$A \pm A := \{a_1 \pm a_2 : a_1, a_2 \in A\}.$$

The first nontrivial result in the opposite direction is due to Hegyvári [7] who showed, in response to an earlier question by Erdős, that if A is convex,

$$|A + A| \gg \frac{\log|A|}{\log\log|A|}|A|.$$

Current best known lower bounds for $|A \pm A|$ are considerably stronger: they are summarised in the Theorem 20 in the sequel. In Theorem 21 in this paper we slightly improve these bounds under mild additional convexity assumptions.

However, this improvement becomes only an application of the new upper bounds for "long" second moment quantities (which imply corresponding lower bounds for cardinalities via the Cauchy-Schwarz inequality) that this paper focuses on, defined next.

The additive energy of A is defined as

$$E(A) := |\{(a_1, a_2, a_1', a_2') \in A^4 : a_1 + a_2 = a_1' + a_2'\}|.$$

More generally, the aforesaid "longer" moments are as follows.

Definition 1. Let $A_1, \ldots, A_k$ be sets of reals. Define

$$T(A_1, \ldots, A_k) := \left|\left\{(a_1, \ldots, a_k, a_1', \ldots, a_k') \in \prod_{i=1}^k A_i \times \prod_{i=1}^k A_i : a_1 + \cdots + a_k = a_1' + \cdots + a_k'\right\}\right|. \tag{1}$$

Here and henceforth $\prod_{i=1}^k A_i$ means Cartesian product, which will also be denoted as A^k when all $A_i = A$. In the latter case we also write

$$T_k(A) := T(\underbrace{A, \ldots, A}_{k \text{ times}}).$$

For the special case when there are only two summands, we write $E(A) := T_2(A)$ and $E(A, B) := T(A, B)$.

Notation. We use Vinogradov's symbol extensively. We write $X \ll Y$ to mean that $X \leqslant CY$ for some absolute constant C, and $X \lesssim Y$ to mean that $X \leqslant C_1 Y(\log Y)^{C_2}$ for some absolute constants C_1, C_2. For several proofs herein, a suppressed constant $C := C(k)$ may depend on the dimension of the problem k. These dependencies can be easily calculated explicitly in the proofs, however we will not, since we canonically think of k as fixed and small compared to the other parameters in our results.

The quantities $T_k(A)$ are easily interpreted via L^{2k}-norms of trigonometric polynomials with frequencies in A, see e.g. [8].

Their analogues, for $k > 2$, have been studied quite extensively in the context of the Erdős-Szemerédi sum-product conjecture [4], see e.g. [1], [17], [12], [18], [2]. However, although the questions about (the lack of) additive structure in convex sets may be viewed as a generalisation of pivotal questions in sum-product literature (such as the "few products – many sums question), the methods developed in the sum-product literature rely heavily on algebraic ring properties of addition and multiplication that have no analogue in the convex setting.

Hence, for $k > 2$ the estimates in this paper have no precedents, and the methodology enabling us to induct in k is new, once we have adopted an idea from an over a 20 years old paper by Garaev [5], to be described shortly.

If $|A| = N$, then trivially $N^k \leqslant T_k(A) \leqslant N^{2k-1}$, and we are interested in proving non-trivial upper bounds for $T_k(A)$, under the heuristic that having "enough" convexity should push these upper bounds, ideally, close to the lower bound N^k. The fact that more and more convexity should be required for higher k is illustrated by setting A to be the set of consecutive perfect powers.

Many of today's state of the art results concerning additive properties of convex sets have been obtained via a particular version of the Szemerédi–Trotter theorem, which assumes that one can write a convex set A in the form $A = f([N])$ where the function f has strictly monotone derivative. From now on we just say "monotone", meaning "strictly monotone". Here $[N] = \{1, \ldots, N\}$, the integer interval. Indeed, spline interpolation can be used to produce such an f, so there is no loss of generality in assuming $A = f([N])$.

The Szemerédi–Trotter theorem was fetched to study the sums of convex sets by Elekes, Nathanson and Ruzsa [3]. In fact, the special Cartesian product set case of the theorem needed can be proved by an elementary *lucky pairs* method. The lucky pairs terminology has been adopted from J. Solymosi; the argument goes back, in particular, to [19].

In this paper, we also define a notion of lucky pairs, however a different one and without proceeding towards geometric incidence arguments. Instead, we develop the idea of Garaev from [5], which underlies his elementary proof of the following energy bound. This bound was previously established by Konyagin [9] via the Szemerédi-Trotter theorem.

Theorem 2 (Konyagin–Garaev). *Let A be a convex set of N elements. Then*

$$E(A) \ll N^{5/2}.$$

For the purposes of this paper, one can inductively define higher convexity for finite sets of reals, beginning by saying a 0-convex set is a set, written in monotone order. For $s \geqslant 1$, a set A with N elements is s-convex if the set of neighbouring differences $\{a_{i+1} - a_i, 1 \leqslant i \leqslant N - 1\}$ is $(s-1)$-convex. Under this definition, a 1-convex set is simply a convex set. A telescoping argument implies that if A is s-convex, then for any $1 \leqslant h < N$, the set

$$\Delta_h A := \{a_{i+h} - a_i, 1 \leqslant i \leqslant N - h\}$$

is $(s-1)$-convex. At times, we may implicitly assume that the h-difference set $\Delta_h A$ has N elements (rather than $N-h$), but this will not be of any consequence.

Higher convexity was recently investigated by Roche-Newton and the last two authors [6] aiming at proving ad infinitum with k, growth of k-fold sumsets of sufficiently convex sets. This paper develops a different, although not unrelated and also elementary approach to moments $T_k(A)$. Corollary 4 and estimate (3) in Corollary 7 imply, at least on the qualitative level, the main results in [6]. Hence in some sense, this paper is a stronger sequel to [6]. Garaev's method, explored ihere, has been brought to our attention by the work [13] by Olmezov.

Garaev's argument uses only the following weaker implication of the convexity of A, namely that for every $1 \leqslant h < N$, the collection of differences $\{a_{i+h} - a_i\}$ is indeed a set, rather than a multiset. In our forthcoming induction of Garaev's argument for s-convex sets, we will iterate this property, applying it to the difference sets $\{a_{i+h} - a_i\}$ as well as to A.

We extend Theorem 2 to s-convex sets as follows.

Theorem 3. *For $s \geqslant 0$, let $k = 2^s$ and let $A_1, \ldots, A_k$ be s-convex sets with $|A_i| \leqslant N$ for all $1 \leqslant i \leqslant k$. Then*
$$T(A_1, \ldots, A_k) \ll N^{2^{s+1}-1-s+\alpha_s},$$
where $\alpha_0 = 0$ and $\alpha_s = \sum_{j=1}^s j 2^{-j}$.

Loosely speaking, Theorem 3 says that provided our sets are convex enough, each time we double the number of terms k in the energy equation (1), we essentially get a "saving" of an additional factor of N off the trivial estimate N^{2k-1} for the quantity $T(A_1, \ldots, A_k)$.

We remark that in a recent preprint [11] Mudgal has shown, using an ingenious application of the Balog-Szemerédi-Gowers theorem, that cardinality bounds of [6] alone enable one to conclude that for an s-convex set A one has the estimate $T_k(A) \leqslant N^{2k-1-s+\alpha_k}$, with $\alpha_k \to 0$ as $k \to \infty$, although quite slowly, the characteristic scale being roughly $k \sim 2^{2^s}$, rather than $k = 2^s$ here.

Theorem 3 has a standard sumset implication after an application of the Cauchy–Schwarz inequality, which also illustrates the rough saving of N every time k doubles.

Corollary 4. *For $s \geqslant 1$, let $k = 2^s$ and let $A_1, \ldots, A_k$ be s-convex sets with $|A_i| = N$ for all $1 \leqslant i \leqslant k$. Then*
$$|A_1 \pm A_2 \pm \cdots \pm A_k| \gg N^{1+s-\sum_{j=1}^s j 2^{-j}}.$$

We remark that for $s \geqslant 2$, the proof of Theorem 3 can be refined to give the improved $\alpha_s = -\frac{2}{13} + \sum_{j=1}^s j 2^{-j}$ and slightly more for higher values of s. This is follows from us bounding $\frac{2^s-1}{2^s}$ by 1 in the induction proof of Theorem 3 plus the fact that the induction can start at $s = 1$, where we have Shkredov's [17] estimate $E(A) \lesssim |A|^{32/13}$, see Theorem 20 below. However, this is not the focus of the theorem and perturbs the exposition so we only comment on the modifications needed to admit this improvement.

However, since we will use the explicit bounds for $k = 4$ in the last section of this paper, we state the improved result, according to the remark above.

Theorem 5. *If A_1, A_2, A_3, A_4 are 2-convex sets all of size N, then*

$$T(A_1, A_2, A_3, A_4) \lesssim N^{4+24/13} \qquad and \qquad |A_1 \pm A_2 \pm A_3 \pm A_4| \gtrsim N^{2+2/13}.$$

Moreover, for A_1, A_2, A_3, A_4 being $s > 1$-convex and $1 \leqslant r \leqslant N^3$, one has

$$|\{x : r_{A_1 \pm A_2 \pm A_3 \pm A_4}(x) \geqslant r\}| \lesssim \frac{N^4}{r^{7/3}} E_{s-1}, \tag{2}$$

where

$$E_s = \sup_{B \ s\text{-convex}, \ |B| = N} E(B).$$

We proceed towards formulating our results concerning near-convex sets. It is easy to see (we show it explicitly in the forthcoming Lemma 11) that if $A = f([N])$ where f is a $C^s(\mathbb{R})$ function with monotone derivatives $f^{(0)}, f^{(1)}, \ldots, f^{(s)}$, then A is s-convex. A function f with this property is henceforth referred to as an s-convex function.

Using s-convex functions is not necessary to prove the above results about s-convex sets, but they provide a natural way to generalise to *near-convex sets*. We say that a set $A = f(B)$ is *near-convex* (more specifically $K-$ near-convex) if f is an s-convex function and the set B is such that

$$|B - B + B| \leqslant K|B|.$$

The parameter K will be referred to as the doubling constant associated with B.

Our main result, Theorem 6, reflects the maxim that "convex functions destroy additive structure", and that the more convex a function, the more it destroys additive structure. Its proof arises from generalising Garaev's method to longer sums and more convex sets. In light of our main theorem, and one can view Theorem 3 as its corollary by setting $B_1 = \cdots = B_k = [N]$. [1]

Theorem 6. *Let $B_1, \ldots B_k$ be any sets with $|B_i| = N$, $|B_i + B_i - B_i| = K_i N$ for all $1 \leqslant i \leqslant k$. With $s \geqslant 0$ and $k = 2^s$, let $A_i = f_i(B_i)$ for some s-convex functions $f_1, \ldots, f_k$. Then we have*

$$T(A_1, \ldots, A_k) \ll \left(\prod_{i=1}^{k} K_i^{2-(2+2s-2\alpha_s)2^{-s}} \right) \cdot N^{2^{s+1}-1-s+\alpha_s},$$

where $\alpha_0 = 0$ and $\alpha_s = \sum_{j=1}^{s} j 2^{-j}$.

Due to the generality of Theorem 6, it yields useful corollaries. The following follows by setting all the B_i and all the f_i to be the same.

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[1]For this to be exactly true, one needs to check that all s-convex sets are of the form $f([N])$ for some s-convex function f, for $s > 1$.

Corollary 7. *Let B be any set with $|B| = N$ and $|B + B - B| = KN$. If $A := f(B)$ where f is an s-convex function and $k = 2^s$, then*

$$T_k(A) \ll K^{2^{s+1}-2-2s+2\alpha_s} \cdot N^{2^{s+1}-1-s+\alpha_s},$$

where $\alpha_0 = 0$ and $\alpha_s = \sum_{j=1}^{s} j2^{-j}$.
  Thus

$$|\underbrace{A \pm A \pm \cdots \pm A}_{k\ times}| \gg K^{-2^{s+1}+2+2s-2\alpha_s} N^{1+s-\alpha_s}. \tag{3}$$

It should be mentioned that such bounds which depend on doubling constants can be used to obtain sum-product-type results, along the lines of [6, Corollary 1.5]. We will not however explicitly discuss sum-product phenomena further. Other sum-product type results in the context of convex sets can be seen in recent work of Stevens and Warren [20].

We also prove the following asymmetric energy bound:

Theorem 8. *Let B, C be sets with $|B| = N$, $|B + B - B| = KN$ and $|C| = L$. If $A := f(B)$ for some convex function f, then*

$$E(A, C) \ll K^{1/2} N L^{3/2}.$$

Our result improves the following result, which was previously the best-known and which follows from a straightforward extension of Konyagin's Szemerédi–Trotter proof of Theorem 2. The improvement is in the dependence on K.

Theorem 9. *Let B, C be sets with $|B| = N$, $|B - B| = KN$ and $|C| = L$. If $A := f(B)$ for some convex function f, then*

$$E(A, C) \ll K N L^{3/2}.$$

Indeed, the improved Theorem 8 is sharp when $|A| = |C|$. Let $A = C = f(B)$ where $B = \{x^2 : x \in [N]\}$ and $f(x) := \sqrt{x}$. Then we get $E(A, C) = K^{1/2}|A||C|^{3/2} = N^3$.

The utility of incorporating higher convexity into the main results is as follows. For 1-convex sets, the best known estimates for $T_k(A)$, have been derived in [8] using induction and Szemerédi-Trotter bounds, namely

$$T_k(A) \ll N^{2k-2+2^{-(k-1)}}. \tag{4}$$

This in particular implies that $|kA| \gg N^{2-2^{-(k-1)}}$. This estimate cannot be improved beyond N^2, as evinced by the first N squares, which form a 1-convex (but not 2-convex) set. Consequently, the energy bound (4) is almost best-possible. One would naturally expect better estimates for more convex sets, but the choice techniques based on the Szemerédi–Trotter theorem have not been adapted to allow for nontrivial estimates. A different elementary technique has been developed in [6] to account for the following growth of particular sumsets:

$$|2^s A - (2^s - 1)A| \gg |A|^{s+1}.$$

In this paper we show that Garaev's idea underlying his proof of Theorem 2, allows for upper bounds for the quantities T_{2^s}, which are in line with the sumset bounds in [6].

Note that bounds for $T_k(A)$ can be immediately recycled into L^{2k} bounds for exponential sums over A (see [8] and the references contained therein).

In the last section of this paper we use our estimates for the quantities $T_3(A)$ and $T_4(A)$ to get small improvements on the best known sumset and energy estimates for sets which are s-convex, for $s \geqslant 2$. Although our quantitative improvements are quite modest (and most likely not best possible within the technology we present) they do break the ice in some sense, for previously used methods based on the Szemerédi-Trotter theorem did not enable one to benefit by using higher convexity.

1.1 Organisation of this paper

Section 2 contains the notation used herein which relates to energy and difference sets. It also briefly discusses the relevant properties of convex sets and convex functions. Section 3 introduces the Garaev argument and a generalised definition of "lucky pairs". The key result is Proposition 15, which forms the framework for proving the main results of the paper. Sections 4 and 5 respectively contain the proofs of the main results in the convex and near-convex cases. Section 6 is devoted to proving new sumset results for s-convex functions where $s \geqslant 2$.

2 Preliminaries and Notation

Throughout this paper, if $x \in A_1 + \cdots + A_k$, then

$$r_{A_1 + \cdots + A_k}(x) := |\{(a_1, \ldots, a_k) \in \prod_{i=1}^{k} A_i : x = a_1 + \cdots + a_k\}|.$$

Also for $r \geqslant 1$, we define

$$X_r := \{x \in A_1 + \cdots + A_k : r \leqslant r_{A_1 + \cdots + A_k}(x) < 2r\},$$

the set of r-rich sums in a given sumset. Whenever we use (extensively) the notation X_r, context will make it clear what sumset it is contained in.

In this language we can express the energy in either of the following ways:

$$T(A_1, \ldots, A_k) = \sum_{x \in A_1 + \cdots + A_k} r_{A_1 + \cdots + A_k}^2(x),$$

or

$$T(A_1, \ldots, A_k) \approx \sum_{r \text{ dyadic}} r^2 |X_r|.$$

For a convex set A with N elements, and $1 \leqslant h < N$ its h-difference set is

$$\Delta_h A := \{a_{i+h} - a_i : i \in [N - h]\}.$$

We will not explicitly deal with the fact that $|\Delta_h A| < |A| = N$, but rather simply add h extra elements to $\Delta_h A$ ad hoc.

We also define "discrete derivatives". Given a function f, let its h-derivative be

$$\Delta_h f(x) := f(x+h) - f(x).$$

Lemma 10. *If f is an s-convex function, then for any h, $\Delta_h f$ is an $(s-1)$-convex function.*

Proof. We use induction on s. Suppose f is 1-convex. We have

$$\Delta_h f(x) := f(x+h) - f(x) = \int_x^{x+h} f'(y) dy.$$

Since f' is monotone, it follows that $\Delta_h f$ is also monotone, and hence 0-convex.

Next assume the statement holds for $(s-1)$-convex functions. Let f be an s-convex function. By definition, this implies that f' is an $(s-1)$-convex function. The induction hypothesis implies that $\Delta_h(f')$ is an $(s-2)$-convex function. But since $\Delta_h(f') = (\Delta_h f)'$, it follows that $\Delta_h f$ is $(s-1)$-convex, completing the induction. $\square$

A one-way relationship between s-convex functions and s-convex sets is intuitive and is summarised in the following lemma.

Lemma 11. *If f is an s-convex function, then $f([N])$ is an s-convex set.*

Proof. By induction on s. If f is a 0-convex function, then $f([N])$ is clearly ordered as a 0-convex set.

Assume the statement holds for $(s-1)$-convex functions. Let f be an s-convex function. Then $(\Delta_1 f)(x) := f(x+1) - f(x)$ is an $(s-1)$-convex function. By the induction hypothesis, $(\Delta_1 f)([N])$ is an $(s-1)$-convex set, which proves that $f([N])$ is an s-convex set, completing the induction. $\square$

We expect the converse to the statement of Lemma 11 to be true, as it is for $s = 1$. However, proving this may require interpolation techniques beyond the scope of this paper.

3 Methods

We begin by presenting a version of Garaev's proof of Theorem 2. This proof is essentially synthesised from its exposition by Olmezov [13], with an additional observation that convexity can be used more sparingly, which enables one to extend the estimate for $E(A)$ to $E(A, B)$, where A is a convex set and B is any set. This is based on replicating estimate (8) below, known earlier via the Szemerédi-Trotter theorem.

In the forthcoming argument, we only need the following property of a *convex* set

$$A = \{a_1, a_2, \ldots a_N\},$$

written in increasing order: For each $h < N$, the differences $a_{i+h} - a_i$, $i = 1, \ldots, N - h$ are all distinct.

Proof of Theorem 2. We are estimating the number of solutions to

$$a_{i_1} + a_{j_1} = a_{i_2} + a_{j_2} \; : (a_{i_1}, a_{j_1}, a_{i_2}, a_{j_2}) \in A^4 \,. \tag{5}$$

Consider some $x \in X_r$. Recall that this means $r \leqslant r_{A+A}(x) < 2r$ where $r_{A+A}(x)$ is the number of realisations of x as a sum of two elements in A. Write

$$x = a_{i_1} + a_{j_1} = \cdots = a_{i_r} + a_{j_r},$$

with $i_1 < i_2 < \ldots < i_r$. Since $a_{i_u} + a_{j_u} = x$ for all u, we also have $j_1 > j_2 > \ldots > j_r$. We may also assume that $j_u \geqslant i_u$ for all u, affecting only the multiplicative constant implied in the $\ll$ notation of the final estimate. It follows that

$$\sum_{u=1}^{r-1} (i_{u+1} - i_u) \leqslant N \qquad \text{and} \qquad \sum_{u=1}^{r-1} (j_u - j_{u+1}) \leqslant N.$$

By the pigeonhole principle, at least $3r/4$ of the summands in each sum cannot exceed $4N/r$. This implies that there is a set of indices $U \subset [r-1]$ with $|U| \geqslant r/2$ such that for every $u \in U$, $i_{u+1} - i_u \leqslant 4N/r$ and $j_u - j_{u+1} \leqslant 4N/r$. For $u \in U$, we say the pair $(a_{i_u}, a_{j_u}), (a_{i_{u+1}}, a_{j_{u+1}})$ is a *lucky pair*, so there are at least $r/2$ lucky pairs.

Since each lucky pair gives rise to a solution to the energy equation (5), there are least $r/2$ distinct solutions of the equation

$$a_{i_1+h_1} - a_{i_1} = a_{i_2+h_2} - a_{i_2} \,,$$

where $i_1, i_2 \in [N]$ and $1 \leqslant h_1, h_2 \leqslant 4N/r$.

By considering all $x \in X_r$, it follows that

$$r|X_r| \ll (N/r)^2 \max_{1 \leqslant h_1, h_2 \ll N/r} \left| \{ (i_1, i_2) \in [N]^2 : a_{i_1+h_1} - a_{i_1} = a_{i_2+h_2} - a_{i_2} \} \right|.$$

Now comes the only part of the argument where we use the convexity of A: given h_1, all differences $a_{i_1+h_1} - a_{i_1}$ are distinct, hence for any fixed h_1 and h_2, we have trivially that

$$\left| \{ (i_1, i_2) \in [N]^2 : a_{i_1+h_1} - a_{i_1} = a_{i_2+h_2} - a_{i_2} \} \right| \leqslant N \,. \tag{6}$$

It follows that

$$|X_r| \ll N^3/r^3. \tag{7}$$

We write $E(A) = \sum_{r \text{ dyadic}} r^2 |X_r|$ and, for some parameter r_* to be chosen, use the trivial bound $|X_r| \ll N^2/r$ for $r \leqslant r_*$ and estimate (7) for $r > r_*$. Choosing the optimal $r_* = N^{1/2}$ yields the desired

$$E(A) \ll N^{5/2}. \qquad \square$$

We remark that since the lucky pairs argument itself involves solely the pigeonhole principle and no assumptions on the set A, the above proof generalises immediately to the case of $E(A, B)$, where A is convex and B any set. Bound (7) becomes

$$|X_r| \ll |A||B|^2/r^3 \,, \tag{8}$$

with X_r now being the set of r-rich sums in $A + B$. Indeed, the only necessary changes to the proof are that now h_2 pertain to the set

$$B = \{b_1, b_2, \ldots, b_{|B|}\},$$

so that $1 \leqslant h_2 \ll |B|/r$, and the trivial bound (6) is replaced by $|B|$. This takes into account that given h_2, the quantities $b_{i_2+h_2} - b_{i_2}$ are not necessarily all distinct. What matters is that $a_{i_1+h_1} - a_{i_1}$ are all distinct.

Hence, Garaev's method enables one to obtain the standard corollary of estimate (8), which is usually proved using the Szemerédi-Trotter theorem.

Corollary 12. *If A is a convex set, then for any B,*

$$
\begin{aligned}
E_3(A, B) &:= \sum_x r_{A \pm B}^3(x) &\ll& \quad |A||B|^2 \log |A|, \\
E_{1+p}(A, B) &:= \sum_x r_{A \pm B}^{1+p}(x) &\ll& \quad |A||B|^{1+p/2}, \quad \text{for } 1 < p < 2.
\end{aligned}
$$

Earlier expositions of Garaev's method appear to overlook the fact that it generalises easily to embrace two different sets A and B, owing to an overreliance on convexity in the proof.

In order to generalise Theorem 2 to the quantity $T_{2^k}(A)$, we need to generalise the concept of lucky pairs from above. In order not to repeat ourselves, we do it in the general setting, suitable for all the results in this paper. In the convex set setting, $B_1, \ldots, B_k$ below are all just the interval $[N]$. In the near-convex setting, the full generality of Definition 13 and Proposition 15 will be needed.

Definition 13 (Lucky Pairs). For $1 \leqslant i \leqslant k$, suppose B_i is a finite set of real numbers, g_i is a monotone function and $A_i = g_i(B_i)$. Given any r, where $r^{1/(k-1)} \ll |B_i + B_i - B_i|$ for all $1 \leqslant i \leqslant k$, let

$$X_r = \{x \in A_1 + \cdots + A_k : r \leqslant r_{A_1 + \cdots + A_k}(x) < 2r\}$$

be the r-rich sums in $A_1 + \cdots + A_k$. Suppose $P := (b_1, \ldots, b_k)$ and $P' := (b'_1, \ldots, b'_k)$ are *distinct* points, each belonging to $\prod_{i=1}^k B_i$. For $x \in X_r$, we say (P, P') forms a *lucky pair* associated with x if the following two conditions hold.

1. The pair (P, P') gives rise to a solution to the energy equation for the sum x. That is,
$$g_1(b_1) + \cdots + g_k(b_k) = x = g_1(b'_1) + \cdots + g_k(b'_k).$$

2. In all coordinates, there are not many elements of $B_i + B_i - B_i$ between P and P'. That is, if $n_{B_i}(b, b')$ is the number of elements of $B_i + B_i - B_i$ lying in $(b, b']$ (or $(b', b]$ if $b > b'$), then
$$n_{B_i}(b_i, b'_i) \ll |B_i + B_i - B_i|/r^{1/(k-1)}, \tag{9}$$
for all $1 \leqslant i \leqslant k$.

Remark 14. In (9), we will always be treating the upper bound on $n_{B_i}(b_i, b_i')$ as an integer. This is why we insist that $r^{1/(k-1)} \ll |B_i + B_i - B_i|$ in Definition 13. For all the results in this paper, this condition will hold trivially, so it will not be discussed further.

Proposition 15. *Let $r, k \geqslant 1$ and for $1 \leqslant i \leqslant k$, suppose B_i is a finite set of real numbers, g_i is a monotone function and $A_i = g_i(B_i)$. Let*

$$X_r = \{x \in A_1 + \cdots + A_k : r \leqslant r_{A_1 + \cdots + A_k}(x) < 2r\}$$

be the r-rich sums in $A_1 + \cdots + A_k$. Then for each $x \in X_r$, there are $\gg r$ lucky pairs associated with x.

We will need the following lemma in the proof:

Lemma 16. *Suppose we have a k-dimensional box in $\mathbb{R}^k$ (a Cartesian product of k orthogonal intervals) which is composed of r^k smaller (nonidentical) boxes (or cells) in an $r \times \cdots \times r$ grid. Then any generic hyperplane H (not parallel to any one-dimensional edge of the box) can pass through at most kr^{k-1} cells.*

Proof. By translation and scaling, we may assume that the origin is one of the corners of the box, the facets of the box are all parallel to coordinate hyperplanes and that the hyperplane H is of the form $X_1 + \cdots + X_k = C$ for some constant C.

We can index each cell by a k-tuple $(e_1, \ldots, e_k)$ which denotes its position among the cells on each axis, starting from the origin. Now for each cell in which at least one of the e_i is 1, we define its associated *diagonal* as the set of cells with indices $(e_1 + a, \ldots, e_k + a)$ for $0 \leqslant a \leqslant r - \max_i e_i$.

There are at most kr^{k-1} such diagonals which cover all the cells, and H intersects each diagonal in at most one cell, completing the proof. $\square$

Proof of Proposition 15. For each i, partition $B_i + B_i - B_i$ into $r^{1/(k-1)}/4$ intervals, each containing $4|B_i + B_i - B_i|/r^{1/(k-1)}$ elements. Since $B_i \subset B_i + B_i - B_i$, this also partitions the elements of B_i. Doing this for each i partitions $\prod_i B_i$ into boxes and hence, since the g_i are all monotone functions, also partition $\prod_i A_i$ into boxes (or cells).

Now consider some $x \in X_r$. Each solution to

$$x = g_1(b_1) + \cdots + g_k(b_k)$$

corresponds to a point $(g_1(b_1), \ldots, g_k(b_k))$ on the hyperplane

$$x = X_1 + \cdots + X_k.$$

By Lemma 16 this hyperplane can pass through at most $(k/4^{k-1}) \cdot r \leqslant r/2$ cells and the hyperplane has r points on it. By the pigeonhole principle, there must be $\gg r$ pairs of points which lie together in the same cell. By construction, these are lucky pairs, which completes the proof. $\square$

4 Proof of Theorem 3

Despite the fact that Theorem 3 is essentially a less general version of Theorem 6, we present its proof separately to illustrate exactly how much convexity is needed.

Proof of Theorem 3. Let us denote the desired universal bound for $T_k(A_1, \ldots, A_k)$ as

$$\mathcal{T}_k := \mathcal{T}_k(N) = \sup T_k(A_1, \ldots, A_k)$$

where the supremum is taken over all k-tuples of s-convex sets of size N. Let $A_i := \{a_1^{(i)} < \ldots < a_N^{(i)}\}$ for each $1 \leqslant i \leqslant k$. We are counting solutions to the equation

$$a_{e_1}^{(1)} + \cdots + a_{e_k}^{(k)} = a_{e_1'}^{(1)} + \cdots + a_{e_k'}^{(k)}, \tag{10}$$

for some indices $e_1, \ldots, e_k, e_1', \ldots, e_k' \in [N]$.

The proof is by induction on s where $k = 2^s$, the base case $s = 0$ being trivial: the number of solutions of

$$a = a' : \quad a, a' \in A_1$$

is at most (in fact precisely) N.

Let us assume that in the equation (10) no two terms $a_{e_i}^{(i)}$ and $a_{e_i'}^{(i)}$ are the same for $i = 1, \ldots, k$. More precisely, suppose that such non-degenerate solutions to equation (10) constitute at least half of the quantity $T_k(A_1, \ldots, A_k)$. Indeed, the number of degenerate solutions is at most

$$\sum_{j=1}^{k} T_{k-1}(A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_k)$$

and by freezing all but $k/2$ of the variables on each side and applying Cauchy-Schwarz, we get

$$T_k(A_1, \ldots, A_k) \ll_k N^{k-1} \mathcal{T}_{k/2} \ll \mathcal{T}_k.$$

Thus if the degenerate solutions constituted more than half of the upper bound, the proof would be complete.

Recall that X_r is the set of r-rich sums. For each $x \in X_r$, we apply Proposition 15 with $B_1 = \ldots = B_k = [N]$ and $g_i(y) := a_y^{(i)}$, for all $1 \leqslant i \leqslant k$. By considering all lucky pairs arising from any $x \in X_r$, one obtains

$$r|X_r| \ll \# \text{ solutions to (10)},$$

where $|e_i - e_i'| \ll N/r^{1/(k-1)}$ for all $1 \leqslant i \leqslant k$. We now choose the $h_i := e_i - e_i'$ for $1 \leqslant i \leqslant k$ which maximise the number of solutions to (10).

Notice that for each h_i, $\Delta_{h_i} A_i := \{a_{e_i'+h_i}^{(i)} - a_{e_i'}^{(i)}\}$ is an $(s-1)$-convex set (not a multiset) and has $\leqslant N$ elements. Here we have used that h_i is non-zero, which is a consequence of the non-degeneracy assumption. We can subtract all the elements on the right-hand side

of (10), and since there are $N^k/r^{k/(k-1)}$ ways altogether of choosing $e_1 - e_1', \ldots, e_k - e_k'$, it follows that

$$r|X_r| \ll \frac{N^k}{r^{k/(k-1)}} \cdot \# \text{ of solutions to } a_1 + \ldots + a_k = 0$$

where $a_i \in \Delta_{h_i} A_i$ for $1 \leqslant i \leqslant k$. Rearranging the terms of the above equation so there are $k/2$ terms on each side of the equation and using Cauchy-Schwarz, one can then apply the induction hypothesis to obtain

$$|X_r| \ll \frac{N^k}{r^{(2k-1)/(k-1)}} \cdot \mathcal{T}_{k/2} \,. \tag{11}$$

Using $T_k(A_1, \ldots, A_k) = \sum_{r \text{ dyadic}} r^2 |X_r|$, we optimise in r by taking, for some r_* to be determined, the trivial bound $r_* N^k$ for $r \leqslant r_*$, and the dyadic sum with (11) over the values of $r \geqslant r_*$. Thus

$$T_k(A_1, \ldots, A_k) \ll r_* N^k + \frac{N^k}{r_*^{1/(k-1)}} \mathcal{T}_{k/2} \,.$$

Taking the optimal choice of

$$r_* = \mathcal{T}_{k/2}^{1-1/k},$$

we get

$$T_k(A_1, \ldots, A_k) \ll N^k \mathcal{T}_{k/2}^{1-1/k} = N^{2^s} \cdot N^{(2^s-s+\alpha_{s-1})(1-2^{-s})} \ll N^{2^{s+1}-1-s+\alpha_s}.$$

This closes the induction and completes the proof. □

Remark 17. The step where we apply Cauchy–Schwarz is a variation on the well-known procedure to prove that
$$E(A, B) \leqslant E(A)^{1/2} E(B)^{1/2}.$$

As alluded to in the introduction, we can refine this approach to obtain a slightly better bound, specifically a smaller value of α_s. If we assume that $s \geqslant 2$ then $s = 2$ becomes the base case of the induction. Using the bound (11), $T(A_1, A_2, A_3, A_4)$ can be bounded in terms of $E(A)$ where A is a 1-convex set of size N. Estimating $E(A)$ using Theorem 2 produces the improvement $\alpha_s = -\frac{1}{8} + \sum_{j=1}^{s} j2^{-j}$. Using instead Shkredov's stronger bound [17]
$$E(A) \ll N^{32/13},$$
gives the further improvement $\alpha_s = -\frac{2}{13} + \sum_{j=1}^{s} j2^{-j}$.

In the above proof, s-convexity is only used in one place. Since all the A_i are s-convex, the sets $\Delta_{h_i} A_i$ are $(s-1)$-convex. In particular, this implies that $\Delta_{h_i} A_i$ will always be a set rather than a multiset which is essential when iterating the argument.

5 Proofs of Theorems 8 and 6

In this section, we focus on the results pertaining to sets with small additive doubling. The following lemma is included to clarify the key step for generalising our earlier results.

Lemma 18. *Let $D := \{d_1 < d_2 < \cdots < d_{|D|}\}$ be the positive differences in $B - B$. If $b, b' \in B$ with $b > b'$ and $n_B(b, b') \leqslant Z$, then $b - b' \leqslant d_Z$. In other words, if $n_B(b, b') \leqslant Z$ then there are at most Z possible values that $b - b'$ can take.*

Proof. If not then $b - b' = d_Y$ where $Y > Z$. But then

$$b' < b' + d_i \leqslant b,$$

for $i = 1, \ldots, Y$. Thus there are at least $Y > Z$ elements of $B + B - B$ in $(b', b]$, contradicting that $n_B(b', b) \leqslant Z$. □

Proof of Theorem 8. Let $C := \{c_1 < \cdots < c_L\}$. For each $x \in X_r$, we apply Proposition 15 with $k = 2, B_1 = B, B_2 = [L]$ and functions $g_1(t) := f(t), g_2(t) := c_t$. By considering all $x \in X_r$, this implies that the total number of lucky pairs from r-rich sums is $\gg r|X_r|$. Since lucky pairs give rise to solutions to the energy equation, it follows that

$$r|X_r| \ll \# \text{ solutions to } f(b_1) - f(b_2) = c_{e_2} - c_{e_1}, \tag{12}$$

where $n_B(b_1, b_2) \ll KN/r$ and $|e_2 - e_1| \ll L/r$. It follows from Lemma 18 that there are at most KN/r possible values for $|b_2 - b_1|$.

After fixing $b_1 - b_2, e_2 - e_1$ and c_{e_1} in (12), which can be done in KNL^2/r^2 ways, the energy equation admits at most one solution since $f_d(x) := f(x + d) - f(x)$ is a monotone function.

It follows that $|X_r| \ll KNL^2/r^3$. Since

$$E(A, B) \ll \sum_{r \text{ dyadic}} r^2|X_r| \ll r_* \sum_{\substack{r \text{ dyadic} \\ r \leqslant r_*}} r|X_r| + \sum_{\substack{r \text{ dyadic} \\ r > r_*}} \frac{KNL^2}{r} \ll r_* NL + \frac{KNL^2}{r_*}$$

we get that, upon choosing $r_* = (KL)^{1/2}$,

$$E(A, B) \ll K^{1/2} NL^{3/2}. \qquad □$$

Proof of Theorem 6. As in the proof of Theorem 3, denote the desired universal bound for $T_k(A_1, \ldots, A_k)$ as

$$\mathcal{T}_k(N; K_1, \ldots, K_k).$$

We are counting solutions to the equation

$$f_1(b_1) + \cdots + f_k(b_k) = f_1(b_1') + \cdots + f_k(b_k'), \tag{13}$$

where $b_i, b_i' \in B_i$ for all i.

The proof is by induction on s where again the base case $s = 0$ is trivial: the number of solutions of
$$f_1(b) = f_1(b') : \quad b, b' \in B_1$$
is at most N.

Let us assume that for each solution to (13) no two terms $f_i(b_i)$ and $f_i(b'_i)$ are equal for any $i = 1, \ldots, k$. More precisely, suppose that such non-degenerate solutions to equation (13) constitute at least half of the quantity $T_k(A_1, \ldots, A_k)$. For otherwise, as in the proof of Theorem 3, using a trivial upper bound and the induction hypothesis, we would have
$$T_k(A_1, \ldots, A_k) \ll N^{k-1} \mathcal{T}_{k/2}(N; K_{\iota_1}, \ldots, K_{\iota_{k/2}}) \ll \mathcal{T}_k(N; K_1, \ldots, K_k),$$
where $K_{\iota_1}, \ldots, K_{\iota_{k/2}}$ are the largest $k/2$ terms among all the K_i. This would complete the proof immediately.

As previously seen, X_r contains the sums $x \in A_1 + \cdots + A_k$ with $r \leqslant r_{A_1+\cdots+A_k}(x) < 2r$. For each $x \in X_r$, we now apply Proposition 15 with $g_i(b) := f_i(b)$ for $1 \leqslant i \leqslant k$. This obtains
$$r|X_r| \ll \# \text{ solutions to (13)},$$
where $n_{B_i}(b_i, b'_i) \ll K_i N / r^{1/(k-1)}$ for all $1 \leqslant i \leqslant k$.

We now choose the $h_i := b_i - b'_i$ for $1 \leqslant i \leqslant k$ which maximise the number of solutions to (13), and then rearrange to obtain
$$(\Delta_{h_1} f_1)(b'_1) + \cdots + (\Delta_{h_{k/2}} f_{k/2})(b'_{k/2}) = (\Delta_{h_{k/2+1}} f_{k/2+1})(b_{k/2+1}) + \cdots + (\Delta_{h_k} f_k)(b_k). \quad (14)$$

By Lemma 18, there are at most $\prod_{i=1}^{k}(K_i N)/r^{k/(k-1)}$ ways altogether of choosing the $b_i - b'_i$, so we have
$$r|X_r| \ll \frac{\prod_{i=1}^{k}(K_i N)}{r^{k/(k-1)}} \cdot \# \text{ solutions to (14)}.$$

Applying Cauchy–Schwarz proves that the number of solutions to (14) is bounded above by
$$T((\Delta_{h_1} f_1)(B_1), \ldots, (\Delta_{h_{k/2}} f_{k/2})(B_{k/2}))^{1/2} T((\Delta_{h_{k/2+1}} f_{k/2+1})(B_{k/2+1}), \ldots, \Delta_{h_k}(f_k)(B_k))^{1/2}. \quad (15)$$

Since all the functions $\Delta_{h_i} f_i$ are $(s-1)$-convex, the induction hypothesis upper bounds (15) by
$$\mathcal{T}_{k/2}(N; K_1, \ldots, K_{k/2})^{1/2} \mathcal{T}_{k/2}(N; K_{k/2+1}, \ldots, K_k)^{1/2},$$
whence
$$|X_r| \ll \frac{\prod_{i=1}^{k}(K_i N)}{r^{(2k-1)/(k-1)}} \cdot \mathcal{T}_{k/2}(N; K_1, \ldots, K_{k/2})^{1/2} \mathcal{T}_{k/2}(N; K_{k/2+1}, \ldots, K_k)^{1/2}. \quad (16)$$

Using $T_k(A_1, \ldots, A_k) = \sum_{r \text{ dyadic}} r^2 |X_r|$, we optimise in r by taking, for some r_* to be determined, the trivial bound $r_* N^k$ for $r \leqslant r_*$, and the dyadic sum with (16) over the values of $r \geqslant r_*$. Thus
$$T_k(A_1, \ldots, A_k) \ll r_* N^k + \frac{\prod_{i=1}^{k}(K_i N)}{r_*^{1/(k-1)}} \mathcal{T}_{k/2}(N; K_1, \ldots, K_{k/2})^{1/2} \mathcal{T}_{k/2}(N; K_{k/2+1}, \ldots, K_k)^{1/2}.$$

Taking the optimal choice of

$$r_* = \left(\prod_{i=1}^{k} K_i^{1-\frac{1}{k}} \right) \cdot \mathcal{T}_{k/2}(N; K_1, \ldots, K_{k/2})^{(\frac{1}{2}-\frac{1}{2k})} \mathcal{T}_{k/2}(N; K_{k/2+1}, \ldots, K_k)^{(\frac{1}{2}-\frac{1}{2k})},$$

it is elementary to check that

$$T_k(A_1, \ldots, A_k) \ll r_* N^k \ll \left(\prod_{i=1}^{k} K_i^{2-(2+2s-2\alpha_s)2^{-s}} \right) \cdot N^{2^{s+1}-1-s+\alpha_s}.$$

This closes the induction and completes the proof. $\qquad\square$

Similar to Theorem 3, we can refine this approach to obtain a bound with a slightly smaller (at least by $1/8$) value of α_s for $s \geqslant 2$.

The above proofs apply to sums of length $k = 2^s$, where we start the induction with the trivial estimate for $s = 0$. One can also easily develop similar inductions that start with the quantity $T(A_1, A_2, A_3)$ and formulate analogues of Theorems 3 and 6 for $k = 3 \cdot 2^s$. We leave this to the interested reader, concluding this section by stating the base case $k = 3$, since it will be once used in the next section.

Theorem 19. *If A_1, A_2, A_3 are 2-convex sets with N elements. Let X_r be the set of r-rich sums from $A_1 + A_2 + A_3$. Then*

$$|X_r| \ll \frac{N^{14/3}}{r^{5/2}}. \tag{17}$$

In particular

$$T_3(A_1, A_2, A_3) \ll N^{4+\frac{1}{9}}.$$

Proof. By the familiar lucky pairs argument

$$r|X_r| \ll N^3/r^{3/2} \cdot S_{A+B=C},$$

where $S_{A+B=C}$ is the maximum number of solutions to

$$a + b = c: \quad a \in A, \, b \in B, \, c \in C.$$

for some 1-convex sets A, B, C with $|A| = |B| = |C| = N$. It remains to show that $S_{A+B=C} \ll N^{5/3}$.

Consider the r_0-rich sums $a + b \in A + B$ and recall the corresponding bound (8). Combining with a trivial bound, we get

$$S_{A+B=C} \ll |A||B|^2/r_0^2 + |C|r_0 = N^3 r_0^{-2} + N r_0,$$

and optimising in r_0 obtains

$$S_{A+B=C} \ll N^{5/3}. \qquad\square$$

6 Applications to convex sumsets

We use our estimates to improve the state of the art sumset bounds for convex sets. The best known bounds to date are respectively due to Schoen and Shkredov [16], Rudnev and Stevens [15] and Shkredov [17], and are summarised below.

Theorem 20. *If A is convex, then*

$$|A - A| \gtrsim |A|^{8/5 \, = \, 1.6}$$
$$|A + A| \gtrsim |A|^{30/19 \, \approx \, 1.579},$$
$$E(A) \lesssim |A|^{32/13 \, \approx \, 2.4615}.$$

We can get small improvements of all these bounds for s-convex sets, with $s \geqslant 2$. These estimates rely on using our new bounds apropos of the quantity $T_4(A)$ in Theorem 5, as well as $T_3(A)$ in (17). We will incorporate them into existing methods developed by Shkredov and collaborators (see, e.g. [16], [17], [12]), which rely extensively on the use of the third moment

$$E_3(A) := \sum_x r_{A-A}^3(x) \ll N^3 \log N ,$$

by (7) (the same bound would normally be attributed to the use of Szemerédi-Trotter theorem).

We have attempted to make exposition in this section prerequisite-free. Hence, observe that after resummation the quantity $E_3(A)$ (not to be confused with $T_3(A)$) has the following meaning. If $[a, b, c]$ denotes an equivalence class of triples $(a, b, c) \in A^3$ by translation, with $r([a, b, c])$ triples therein, and $[A^3]$ denotes the set of these equivalence classes, then

$$E_3(A) = \sum_{x \in [A^3]} r^2(x). \tag{18}$$

Let A be an s-convex set. When $s = 2$, the improvement comes from fetching the equation, accounting for the moment $T_4(A)$, for which we have estimate (2), where for the quantity E_{s-1} one can substitute the energy bound from Theorem 20 for 1-convex sets. Furthermore, for $s > 2$ this process can then be iterated to obtain incrementally better energy bounds for more convex sets, the iterations rapidly converging. We note that even the simpler energy estimate $E(A) \ll N^{5/2}$ for 1-convex sets would already improve the estimates of Theorem 20 for 2-convex sets. We present only estimates for 2-convex sets in the next theorem; the small improvements for more convex sets can be found in the forthcoming proof.

Theorem 21. *If A is a 2-convex set with sufficiently large[2] size N, then*

$$|A - A| \gtrsim N^{1 + 151/234 \, \approx \, 1.645},$$
$$|A + A| \gtrsim N^{1 + 229/309 \, \approx \, 1.587},$$
$$E(A) \leqslant N^{2.4554}.$$

---

[2]If N is sufficiently large, then decimal approximation of the exponents enable one to replace the $\lesssim, \gtrsim$ symbols by, respectively, $\leqslant, \geqslant$.

Proof. We begin with the technically least demanding bound for the set $A - A$, where A is s-convex for $s \geqslant 2$. Consider a tautology on triples $(a, b, c) \in A^3$:

$$(a - c) = (a - b) + (b - c).$$

If $|A - A| = KN$ and D be the set of popular differences with $\gg N/K$ realisations, by the pigeonhole principle the above tautology is valid for $\gg N^3$ triples

$$(a, b, c) \in A^3 : b - c, a - b \in D.$$

On the other hand, considering equivalence classes of triples (a, b, c) by translation and using (18) in combination with Cauchy-Schwarz, one has

$$
\begin{aligned}
N^6 &\ll E_3(A) \left|\{d_1 + d_2 \in A - A : (d_1, d_2) \in D^2\}\right| \\
&\lesssim N^3 (K/N)^2 \left|\{a_1 - a_2 + a_3 - a_4 = d : (a_1, \ldots, a_4) \in A^4, d \in A - A\}\right|.
\end{aligned}
\tag{19}
$$

Using the Hölder inequality and (2) with dyadic summation yields

$$\left|\{a_1 - a_2 + a_3 - a_4 = d : (a_1, \ldots, a_4) \in A^4, d \in A - A\}\right| \lesssim (KN)^{4/7} (N^4 E_{s-1})^{3/7}, \quad (20)$$

with E_{s-1} as in (2). Thus

$$N^{19/7} \lesssim K^{18/7} E_{s-1}^{3/7}.$$

Using Shkredov's bound for E_{s-1} yields, for 2-convex A:

$$K \gtrsim N^{\frac{151}{234}}.$$

If A is more than 2-convex, one can asymptotically use the forthcoming bound (24) for E_{s-1}, which improves the exponent for $|A - A|$ just by slightly over .001. Namely, if A is sufficiently convex and N is large enough, it follows that

$$K \geqslant N^{.646},$$

the decimal approximation having accounted for replacing $\lesssim$ by $\leqslant$ for sufficiently large N and s.

Furthermore, to bound $|A + A|$ we use a slightly more involved pigeonholing technique which is exposed in more detail in [15]. Suppose, $|A + A| = KN$, define P as a set of sums with $\gtrsim N/K$ realisations, and D as a popular set of differences by energy (we will use energy to connect the difference set with the sum set). Namely D is defined as follows.

By the dyadic pigeonhole principle, there exists $D \subseteq A - A$, and a real number $1 \leqslant \Delta < |A|$, such that for every $d \in D$, $\Delta \leqslant r_{A-A}(d) < 2\Delta$, and on top of this

$$E(A) \lesssim |D|\Delta^2. \tag{21}$$

Moreover, by (7) one has $|D|\Delta^3 \ll |A|^3$ so

$$\Delta \lesssim |A|^3/E(A). \tag{22}$$

Then (after possibly passing to a large subset of A, see [15, Proof of Theorem 5]) the analogue of the argument underlying the above estimate (19) becomes the tautology

$$(a+b) - (b+c) = a - c \in D, \ a+b, \ b+c \in P,$$

with the number of triples $(a, b, c) \in A^3$ realising this being $\gtrsim |A||D|\Delta$.

Hence, using (18) and Cauchy-Schwarz as in (19) yields

$$
\begin{aligned}
(|A||D|\Delta)^2 &\ll E_3(A) \left| \{ s_1 - s_2 = d : (s_1, s_2, d) \in P^2 \times D \} \right| \\
&\lesssim N^3 (K/N)^2 \left| \{ a_1 + a_2 - a_3 - a_4 = d : (a_1, \dots, a_4) \in A^4, \ d \in D^3 \} \right| \\
&\lesssim K^2 N^{19/7} E_{s-1}^{3/7} |D|^{4/7},
\end{aligned}
$$

after using the Hölder inequality and (2), as in (20).

Multiplying both sides by $\Delta^{6/7} \lesssim \left(\frac{N^3}{E(A)} \right)^{6/7}$ (see (22)) to balance the powers of $|D|$ and Δ, so one can use $|D|\Delta^2 \gtrsim E(A)$ (see (21)), yields

$$E(A)^{16/7} \lesssim K^2 E_{s-1}^{3/7} N^{23/7}. \tag{23}$$

Substituting Shkredov's bound for E_{s-1} and using the standard Cauchy-Schwarz bound

$$K \geqslant \frac{N^3}{E(A)}$$

yields

$$K \gtrsim N^{\frac{229}{390}}.$$

Once again, if A is more than 2-convex, one can asymptotically use the forthcoming bound (24) for E_{s-1}, in which case (optimising between the Cauchy-Schwarz bound and (23))

$$K \gtrsim N^{\frac{16}{27}} \approx 0.592,$$

the decimal approximation having accounted for replacing $\lesssim$ by $\leqslant$ for sufficiently large N and s.

We now turn to the energy $E(A)$ estimate, where the analysis ends up being somewhat more involved. For the reader's convenience we briefly recall the key steps of Shkredov's spectral (alias operator) method we use, following, e.g. [17] (for an overview of the method see [14]). The operator method is used to replace the lower bounds from easy tautologies that enabled estimate (19) and its analogue in the bounds for $|A \pm A|$ derived above.

Once again, let D be the set of popular differences by energy, satisfying (21), (22).

Identifying D with its characteristic function, consider the quantity

$$S := \sum_{a, b, c \in A} D(a - b) D(b - c) r_{A-A}(a - c).$$

The quantity S takes triples $(a, b, c) \in A^3$ for which $a - b, b - c$ are in the "popular" set D, and counts each one the number of times the difference $a - c$ repeats itself. The spectral

method enables one to get a lower bound on S, to be compared with the upper bound we will again obtain by (18) and Cauchy-Schwarz.

Let us view D as a $|A| \times |A|$ symmetric boolean matrix with 1 at the position (a, b) if $a - b \in D$ and 0 otherwise. Similarly r_{A-A} can be seen as a square symmetric matrix R (where $R_{ij} := r_{A-A}(i - j)$), which in addition is non-negative definite (checking this is tantamount to rearrangement of the energy equation, see e.g. [17]). Thus $S = \operatorname{tr} DDR$.

Let μ_1 be a positive eigenvalue of D with the largest modulus and normalised eigenvector $\boldsymbol{v} \geqslant 0$ with all non-negative entries; this is possible by the Perron-Frobenius theorem. Since D is symmetric, one can estimate

$$\mu_1 = \boldsymbol{v} \cdot D\boldsymbol{v} \geqslant \frac{|D|\Delta}{|A|}\,,$$

replacing $\boldsymbol{v}$ by the vector $\frac{1}{\sqrt{|A|}}\mathbf{1}$.

Since D is symmetric, one can write $D = Q\tilde{D}Q^{\top}$ so that $\tilde{D}$ is diagonal with μ_1 in the top left corner, and $\boldsymbol{v}$ is the first column of orthogonal matrix Q. The basis-invariance of trace gives

$$S = \operatorname{tr}(\tilde{D}^2 Q^{\top} R Q).$$

Noting that R is nonnegative definite, the trace can be bounded from below by the $(1, 1)$-entry, whence

$$S \geqslant \mu_1^2\, \boldsymbol{v} \cdot R\boldsymbol{v}.$$

Since $\boldsymbol{v} \geqslant 0$, this can be estimated from below by making the matrix R entry-wise smaller, namely replacing it with ΔD. But for the latter matrix, once again, we can replace $\boldsymbol{v}$ with $\frac{1}{\sqrt{|A|}}\mathbf{1}$ to get a lower bound $\boldsymbol{v} \cdot R\boldsymbol{v} \gtrsim E(A)/|A|$, and hence

$$S \gtrsim \frac{(|D|\Delta)^2 E(A)}{|A|^3}\,.$$

On the other hand the quantity S, tautologically, is the number of solutions of the equation

$$(a - b) + (b - c) = a' - c' \,:\, a - b,\, b - c \in D,\, a', c' \in A\,.$$

It follows from Cauchy-Schwarz that

$$S^2 \leqslant E_3(A) \sum_{d_1, d_2 \in D} r_{A-A}^2(d_1 + d_2)\,,$$

and since each of d_1, d_2 has at least Δ representations in $A - A$, this means

$$S^2 \lesssim |A|^3 \Delta^{-2} \sum_x r_{A-A+A-A}(x) r_{A-A}^2(x).$$

We partition $A - A$ into "rich and poor" sets D_1 and D_2, so that for some τ to be determined, $r_{A-A}(x) \leqslant \tau$, for every $x \in D_1$.

We firstly consider the poor differences D_1. By the Hölder inequality

$$\sum_{x \in D_1} r_{A-A+A-A}(x) r_{A-A}^2(x) \leqslant \left(\sum_{x \in D_1} r_{A-A+A-A}(x)^{7/3} \right)^{3/7} \left(\sum_{x \in D_1} r_{A-A}(x)^{7/2} \right)^{4/7} .$$

From (2) we have, once again,

$$\sum_x r_{A-A+A-A}(x)^{7/3} \lesssim N^4 E_{s-1} ,$$

and from the definition of D_1,

$$\sum_{x \in D_1} r_{A-A}(x)^{7/2} \lesssim N^3 \tau^{1/2} .$$

As for the set D_2, we have, from (7),

$$|D_2| \leqslant N^3 / \tau^3 .$$

Without changing the notation, let us replace D_2 by its subset $\{x : \tau \leqslant r_{A-A}(x) < 2\tau\}$: this will not have consequences, after a subsequent dyadic summation. Then the quantity to be estimated is

$$\sum_{x \in D_2} r_{A-A+A-A}(x) r_{A-A}^2(x) \leqslant \tau^2 |\{d = a_1 + a_2 - a_3 - a_4 : d \in D_2; a_1, \ldots, a_4 \in A\}| .$$

By the Hölder inequality, this is bounded by

$$\tau^2 \left(\sum_x r_{A+A-A}(x)^{5/2} \right)^{2/5} \left(\sum_x r_{A+D}(x)^{5/3} \right)^{3/5} .$$

The first bracketed term is estimated directly using (17). Moreover, by the second bound of Corollary 12,

$$\sum_x r_{A+D}(x)^{5/3} \ll N |D|^{4/3} .$$

Putting everything together,

$$|D|^4 \Delta^6 E^2(A) \lesssim |A|^9 \left(N^{24/7} E_{s-1}^{3/7} \tau^{2/7} + N^{73/15} \tau^{-2/5} \right) .$$

Optimising in τ yields

$$\tau = N^{151/72} E_{s-1}^{-5/8} .$$

Multiplying both sides by Δ^2, using $E(A) \lesssim |D| \Delta^2$ (see (21)) on the left and $\Delta \lesssim |A|^3 / E(A)$ (see (22)) on the right yields

$$E^8(A) \lesssim N^{15+24/7+151/252} E_{s-1}^{1/4} .$$

It remains to substitute an estimate for E_{s-1}. If A is 2-convex we can use Shkredov's bound $E_{s-1} \lesssim N^{32/13}$, and we arrive at

$$E(A) \lesssim N^{2+1705/3744} \leqslant N^{2+.4554},$$

for sufficiently large N.

One can iterate this bound for higher convexity (namely using it as E_{s-1} if $s = 3$, etc.) and it is easily seen that the iterates converge rapidly. In the limit when $E(A) = E_{s-1}$ in the above calculation one gets

$$E(A) \lesssim N^{2+127/279} \leqslant N^{2+.4552}. \tag{24}$$

$\square$

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