Perfect Domination Ratios of Archimedean Lattices

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Abstract

An Archimedean lattice is an infinite graph constructed from a vertex-transitive tiling of the plane by regular polygons. A dominating set of vertices is a perfect dominating set if every vertex that is not in the set is dominated exactly once. The perfect domination ratio is the minimum proportion of vertices in a perfect dominating set. Seven of the eleven Archimedean lattices can be efficiently dominated, which easily determines their perfect domination ratios. The perfect domination ratios are determined for the four Archimedean lattices that can not be efficiently dominated.

Mathematics Subject Classifications: 05C88, 05C89

1 Introduction

In a simple graph $G = (V_G, E_G)$, a vertex x *dominates* a vertex y if either x is adjacent to y or $x = y$. A subset $D \subseteq V_G$ is a *dominating set* if every vertex in V_G is dominated by at least one vertex in D. More formally, define the *closed neighborhood* of a vertex $v \in V_G$ by $N[v] = \{u \in V_G : u = v \text{ or } u \text{ is adjacent to } v\}$. Vertices in $N[v] \setminus \{v\}$ are *neighbors* of v. A vertex v is said to dominate itself and all of its neighbors. A *dominating set* is a set $D \subseteq V_G$ such that every vertex in V_G is dominated by at least one vertex in D. An *efficient dominating set* is a set $D \subseteq V_G$ such that every vertex in V_G is dominated by exactly one vertex in D. A *perfect dominating set* is a set $D \subseteq V_G$ such that every vertex in $V_G \setminus D$ is dominated by exactly one vertex in D. For a finite graph G, the *domination number* $\gamma(G)$ is the minimum number of vertices in a dominating set in G.

There is an extensive literature on domination in finite graphs. The classic comprehensive reference is the two-volume series by Haynes, Hedetniemi, and Slater [15]. Many variants of domination motivated by different applications are defined and studied in, for example, [4, 5, 16, 17]. Perfect domination was introduced in [1] in the study of perfect codes, and has applications to facility location [21] and to efficient resource placement in a computer network [25]. A relatively recent survey of the main results in the literature on perfect domination is provided by Klostermeyer [24], which provides 57 references.

Recently there has been considerable and growing interest in subsets of vertices of infinite graphs with a variety of dominating or domination-like properties. Slater [27] considered fault-tolerant locating-dominating sets on the square lattice. Honkala [19] studied locating-dominating sets on the triangular lattice. Perfect domination, quasi-perfect domination, and rainbow perfect domination on square and triangular lattices are treated in [6, 7, 11]. Kincaid, Oldham, and Yu [22] found optimal open-locating-dominating sets in the triangular lattice. Efficient total domination on vertex-transitive graphs was considered by Hu, Li, and Liu [20]. Kinawi, Hussain, and Niepel [23] studied locating-paired domination on the triangular and king lattices. Bouznif *et al* [2] considered identifying codes, locating-dominating codes, and location-total-dominating codes on the square, triangular, and king lattices. Optimal (t, r) domination on the square and triangular lattices is treated in [3, 10, 14, 18]. The focus of all this research is on domination properties of a few infinite vertex-transitive graphs. In this article, we establish the existence or non-existence of efficient domination and find optimal perfect dominating sets for every graph in a larger class of infinite vertex-transitive graphs called Archimedean lattices.

Archimedean lattices are defined as follows: A *regular tiling* is a tiling of the plane by regular polygons. Considering the vertices and edges of the polygons in a regular tiling to be the vertices and edges of an infinite graph, an *Archimedean lattice* is a regular tiling which is vertex-transitive. Due to the restriction that the sum of the angles in polygons surrounding a vertex must equal 2π, there are only 21 possibilities for regular polygons to surround a vertex, and only eleven of these can be continued indefinitely to produce a vertex-transitive lattice. All eleven of the Archimedean lattices are illustrated in the figures in this article. There is a naming convention for the Archimedean lattices, in which the numbers of edges of the polygons incident to a vertex are listed in the order they appear around the vertex, with exponents indicating the number of successive polygons of a given size. In fact, the naming convention provides a prescription for constructing the lattice. The most commonly recognized Archimedean lattices are the square (4^4) lattice, the triangular (3^6) lattice, and the hexagonal (6^3) lattice. For a complete discussion, see the beautiful monograph by Grünbaum and Shephard [13, pp. 58–64].

The concept of periodic graph is defined in Section 2. Since the dominating set of an Archimedean lattice must be infinite, we define the *domination ratio* of an infinite periodic graph, which is the smallest proportion of vertices that constitute a dominating set. We also define the *perfect domination ratio* of an infinite periodic graph, which is the smallest proportion of vertices that constitute a perfect dominating set. Technicalities justifying these definitions and showing that they are independent of the periodic embedding of the lattice are provided in the Appendix. This article shows that exactly seven of the Archimedean lattices can be efficiently dominated, and determines the perfect domination ratios of all eleven Archimedean lattices.

The concept of efficient domination is central to this work. Let $|S|$ denote the cardinality of set S. A dominating set $D \subseteq V_G$ is an *efficient dominating set* if $|N[v] \cap D| = 1$ for all $v \in V_G$. Thus, an efficient dominating set must dominate every vertex in the graph exactly once. The existence of efficient dominating sets is studied in coding theory [1], since it is a variant of the classical problem of the existence and non-existence of perfect codes as a set in a vector space. Each Archimedean lattice is vertex-transitive, and thus k-regular, with $k \in \{3, 4, 5, 6\}$. If it is efficiently dominated, its domination ratio and perfect domination ratio are both equal to $\frac{1}{k+1}$. Section 3 shows that seven of the Archimdean lattices can be efficiently dominated, determining their domination ratios and perfect domination ratios, and proves that the remaining four cannot be efficiently dominated.

However, an efficient dominating set may not exist for a specific graph, as is proved for four of the Archimedean lattices. For those lattices, $\frac{1}{k+1}$ is a trivial lower bound, while the proportion of dominating vertices in any dominating set or perfect dominating set provides an upper bound for domination ratio and perfect domination ratio respectively. Section 4 exhibits perfect dominating sets to establish upper bounds on the perfect domination ratios for the (3, 6, 3, 6), (3, 4, 6, 4), and (3^2, 4, 3,4) lattices. A "local" approach uses elementary "forcing" arguments using triangles to prove that the upper bounds are correct, establishing that the perfect domination ratios are $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{4}$ respectively. For the (4, 6, 12) lattice, which does not contain any triangles, a local forcing proof was not discovered. Section 5 employs a "global" approach using a longer, more complicated linear programming argument to prove that the perfect domination ratio of the (4, 6, 12) lattice is equal to $\frac{5}{18}$. The reasoning had the fortunate consequence of proving that the domination ratio is also equal to $\frac{5}{18}$. Our results are summarized in Table 1.

In Section 6, ongoing research and open questions are briefly mentioned.

Archimedean Lattice	Efficient Domination	Perfect Domination Ratio γ_p
$(3, 12^2)$	Yes	1/4
$(4, 6, 12)$	No	5/18
$(4, 8^2)$	Yes	1/4
(6^3)	Yes	1/4
$(3, 4, 6, 4)$	No	1/4
$(3, 6, 3, 6)$	No	1/3
(4^4)	Yes	1/5
$(3^4, 6)$	Yes	1/6
$(3^2, 4, 3, 4)$	No	1/4
$(3^3, 4^2)$	Yes	1/6
(3^6)	Yes	1/7

Table 1: Results for the eleven Archimedean lattices. The middle column indicates whether or not there exists an efficient dominating set for each lattice. The rightmost column provides the exact value of the perfect domination ratio for each lattice.

2 Definition of Domination Ratio and Perfect Domination Ratio

2.1 Periodicity

A *periodic graph* G is a locally-finite connected simple graph with a countably-infinite vertex set, which can be embedded in $\mathbb{R}^d$ for some $d < \infty$ such that G is invariant under translation by each unit vector in a coordinate axis direction in $\mathbb{R}^d$ and each compact set of $\mathbb{R}^d$ intersects only finitely many edges and vertices of G. Note that it is actually the embedding which is periodic. For convenience, we will identify a graph with its periodic embedding, although the properties of a dominating set only depend on the adjacency structure of the graph. Each of the eleven Archimedean lattices is a periodic graph in $\mathbb{R}^2$. Figures showing periodic embeddings of the Archimedean lattices are provided in [28] and throughout this article.

2.2 Domination Ratio

For a periodic graph G, denote the subgraph of G induced by the vertices in the rectangle $[m_1, m_2) \times [n_1, n_2) \subset \mathbb{R}^2$ by $R_G(m_1, m_2; n_1, n_2)$, where $m_1 < m_2, n_1 < n_2$, and $m_1, m_2, n_1, n_2 \in \mathbb{Z}$. Note that all induced subgraphs $R_G(m_1, m_2; n_1, n_2)$ corresponding to translations of rectangles with the same edge lengths are isomorphic. Denote the minimum size of a dominating set for $R_G(0, m; 0, n)$, known as its *domination number*, by $\gamma_{m,n}(G)$, and the number of vertices in $R(0, m; 0, n)$ by $N_{m,n}(G)$. Denote $N_{1,1}(G) = k$. We define the *domination ratio* of G by

$$\lim_{m,n\to\infty} \frac{\gamma_{m,n}(G)}{N_{m,n}(G)} = \inf_{r,s} \frac{1}{rsk}\gamma_{r,s}(G).$$

A proof that the limit exists relies on *subadditivity.* Let G_1 and G_2 be vertex-disjoint induced subgraphs of G. Since the union of dominating sets for G_1 and G_2 is a dominating set for G, but there might be a smaller dominating set for G,

$$\gamma(G_1 \cup G_2) \leqslant \gamma(G_1) + \gamma(G_2),$$

while

$$N(G_1 \cup G_2) = N(G_1) + N(G_2).$$

Together, these imply that, for example, doubling the length or width of the rectangle cannot increase the domination ratio of the subgraph, and may decrease it. While our literature search did not find a proof of the existence of the limit for deterministic multiparameter subadditive functions, one may find a proof for the more difficult stochastic case in [12]. For completeness, but to maintain the focus on domination for now, a proof for the deterministic case is relegated to the Appendix.

2.3 Domination Proportion

To discuss upper bounds for the domination ratio, we need to consider dominating sets which may not be minimum dominating sets and may not be periodic. For a finite graph G

and a dominating set D, let the *domination proportion*, denoted $\gamma_D(G)$, be the number of vertices in D divided by total number of vertices in G. To extend the notion of domination proportion to infinite graphs, given a dominating set, suppose the vertex set of an infinite graph can be partitioned into finite subsets such that the subgraph induced by each subset is connected and all these finite induced subgraphs have the same domination proportion. The domination proportion of the dominating set is defined as the common value of the domination proportion of the finite induced subgraphs.

For the induced subgraphs, we require connectedness and the same domination proportion to avoid ambiguity arising from one-to-one or many-to-one correspondences between subgraphs, which can be used to obtain different domination proportions for all the subgraphs.

If the same domination proportion is not required for the induced subgraphs, we will have the following issue: For simplicity, assume the domination proportion of induced subgraphs are either γ_1 or γ_2, where $\gamma_1 \neq \gamma_2$. Since the graph is infinite, we can pair every induced subgraph having domination proportion γ_1 with two induced subgraphs having domination proportion γ_2 to obtain $\frac{\gamma_1+2\gamma_2}{3}$ as the domination proportion of the infinite periodic graph. Similarly, we can pair every induced subgraph having domination proportion γ_1 with three induced subgraphs having domination proportion γ_2 to obtain $\frac{\gamma_1+3\gamma_2}{4}$ as the domination proportion of the infinite periodic graph. Therefore, the domination proportion of an infinite periodic graph is not well defined if the same domination proportion is not required for the induced subgraphs.

If connectedness is not required for the induced subgraphs, we could have the following issue: For simplicity, assume every induced subgraph is the disjoint union of two connected components. The two connected components may have different domination proportions, γ_1 and γ_2 respectively. The same reasoning as in the previous paragraph can be applied to show that the domination proportion of an infinite periodic graph is not defined if connectedness is not required for the induced subgraphs.

2.4 Perfect Domination Ratio

For a periodic graph G, recall the definition in Section 2.2 of $R_G(m_1, m_2; n_1, n_2)$, where $m_1 < m_2$ and $n_1 < n_2$. Denote the minimum size of a perfect dominating set for $R_G(0, m; 0, n)$, known as its *perfect domination number*, by $\gamma_{p;m,n}(G)$, and the number of vertices in $R(0, m; 0, n)$ by $N_{m,n}(G)$. We define the *perfect domination ratio* of G by

$$\gamma_p(G) = \lim_{m,n\to\infty} \frac{\gamma_{p;m,n}(G)}{N_{m,n}(G)},$$

To prove that this limit exists, we consider a variant of the perfect domination ratio, for which we introduce some definitions. Given a graph G with a subgraph H, the *internal boundary* of H is the set of vertices $v \in H$ such that v is adjacent to a vertex that is not in H. Given a graph $G = (V, E)$, if a subset S of vertices is specified to be *dominated for free*, then a perfect dominating set D of V is only required to perfectly dominate the vertices of $V \setminus S$. (Note that D is allowed to contain vertices in S in order to dominate other vertices in $V \setminus S$.)

Denote the minimum size of a perfect dominating set of $R_G(0, m; 0, n)$ with the internal boundary perfectly dominated for free by $\gamma^B_{p;m,n}(G)$. Define the variant of the perfect domination ratio by

$$\gamma^B_p(G) = \lim_{m,n\to\infty} \frac{\gamma^B_{p;m,n}(G)}{N_{m,n}(G)}.$$

A proof that the limit exists relies on *superadditivity*. Let G_1 and G_2 denote vertex-disjoint induced subgraphs of G. Let D denote the minimum perfect dominating set of $G_1 \cup G_2$ with internal boundary dominated for free. Let $D_1 = D \cap V(G_1)$ and $D_2 = D \cap V(G_2)$.

We claim that D_1 is a perfect dominating set of G_1 with the internal boundary dominated for free. To see this: Vertices of G_1 which are not in the internal boundary must be dominated by vertices in D_1 (of which some are allowed to be in the internal boundary). Some vertices in the internal boundary of G_1 may be dominated by vertices in D_2, but are dominated for free.

Since D_1 may not be the minimum perfect dominating set of G_1 with internal boundary dominated for free, $|D_1| \geqslant \gamma^B_p(G_1)$. Similarly, $|D_2| \geqslant \gamma^B_p(G_2)$. Therefore,

$$\gamma^B_p(G_1 \cup G_2) \geqslant \gamma^B_p(G_1) + \gamma^B_p(G_2),$$

while

$$N(G_1 \cup G_2) = N(G_1) + N(G_2).$$

Together, these imply that, for example, doubling the length or width of the rectangle cannot decrease the variant of the perfect domination ratio of the subgraph, and may increase it. By Corollary 73 in the Appendix, $\gamma^B_{p;m,n}(G)$ has a limit as $m, n \to \infty$, and the limit equals $\sup_{r,s} \frac{1}{rsk}\gamma^B_{p;r,s}(G)$. Also, as $m, n \to \infty$, one may apply similar reasoning to show the proportion of vertices on the internal boundary approaches zero, which implies that the perfect domination ratio approaches a limit as $m, n \to \infty$, and the limit is

$$\lim_{m,n\to\infty} \frac{\gamma_{p;m,n}(G)}{N_{m,n}(G)} = \lim_{m,n\to\infty} \frac{\gamma^B_{p;m,n}(G)}{N_{m,n}(G)} = \sup_{r,s} \frac{1}{rsk}\gamma^B_{p;r,s}(G).$$

The Appendix also proves that the perfect domination ratio is independent of the choice of the periodic embedding.

3 Efficient Domination Results

3.1 Efficient Dominating Sets for Seven Archimedean Lattices

It is well-known that for finite graphs, efficient domination is optimal domination, and all efficient dominating sets have the same cardinality [15]. Since the definition of domination ratio for infinite periodic graphs is in terms of domination numbers for finite graphs, all efficient dominating sets are optimal and have the same domination ratio.

Existence of an efficient dominating set was previously proved for the three most common Archimedean lattices – the square (4^4) lattice [6, 9], the triangular (3^6) lattice [7], and the hexagonal (6^3) lattice [8]. For completeness, we illustrate the efficient dominating sets in these three lattices in Figures 1 – 3. In Figures 4 – 7, we illustrate efficient dominating sets for the $(3, 12^2)$, $(4, 8^2)$, $(3^4, 6)$, and $(3^3, 4^2)$ lattices, respectively. Each of the figures shows a subgraph of the lattice that is sufficiently large to demonstrate a periodic pattern that can be extended indefinitely to efficiently dominate the infinite lattice. In each of the figures, a star with bold edges is centered at each vertex in the dominating set, with the edges with arrows pointing to vertices that are dominated by the central vertex. Notice that every non-central vertex is the endpoint of exactly one arrow, so every vertex is dominated exactly once.

Since the Archimedean lattices are vertex-transitive, each is a regular graph. Each is k-regular for some $k \in \{3, 4, 5, 6\}$. For each k-regular Archimedean lattice which can be efficiently dominated, the domination ratio is $1/(k+1)$, since each vertex in the dominating set dominates itself and precisely k neighbors, and no vertex is dominated more than once.

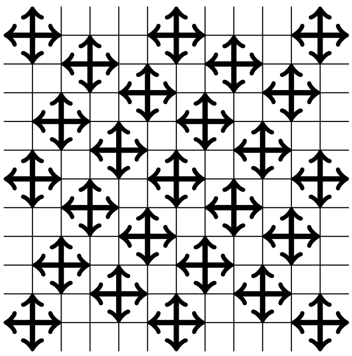


Figure 1: An efficient dominating set in the square lattice.

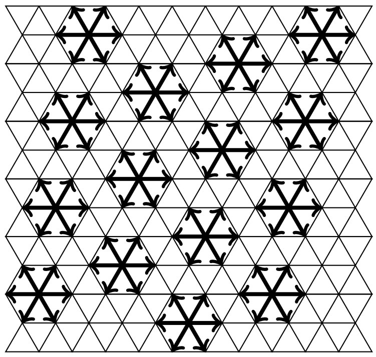


Figure 2: An efficient dominating set in the triangular lattice.

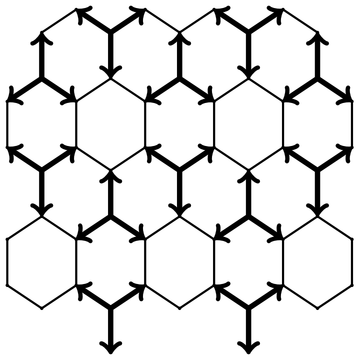


Figure 3: An efficient dominating set of the hexagonal lattice.

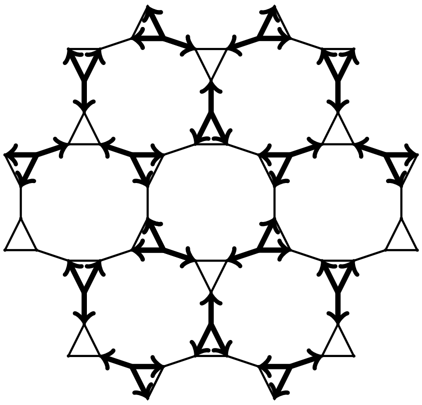


Figure 4: An efficient dominating set in the $(3, 12^2)$ lattice.

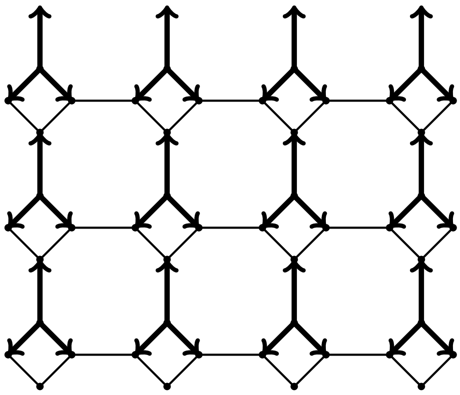


Figure 5: An efficient dominating set in the $(4, 8^2)$ lattice.

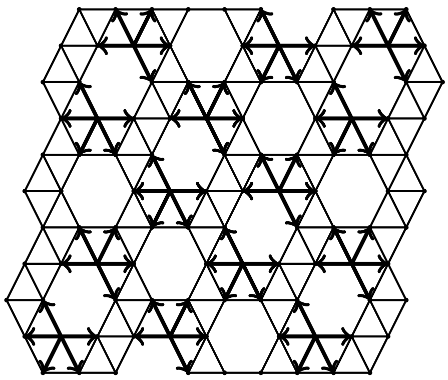


Figure 6: An efficient dominating set in the $(3^4, 6)$ lattice.

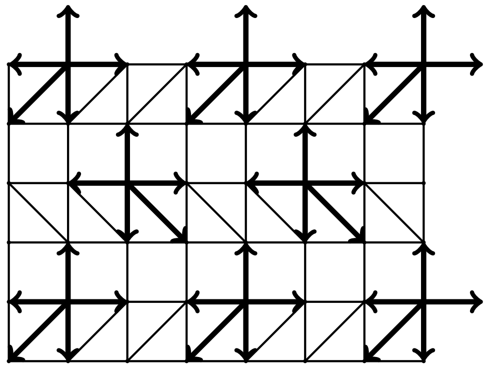


Figure 7: An efficient dominating set in the $(3^3, 4^2)$ lattice.

Note that, for convenience, the $(3^3, 4^2)$ lattice is drawn in a periodic rectangular structure, rather than using regular polygons.

3.2 Non-existence of Efficient Domination in Four Archimedean Lattices

Efficient dominating sets do not exist in the $(3, 4, 6, 4)$, $(3^2, 4, 3, 4)$, $(4, 6, 12)$, and $(3, 6, 3, 6)$ lattices, as is shown in the following four lemmas. Since the proofs are similar, the first lemma is proved in detail, while the later proofs are more abbreviated.

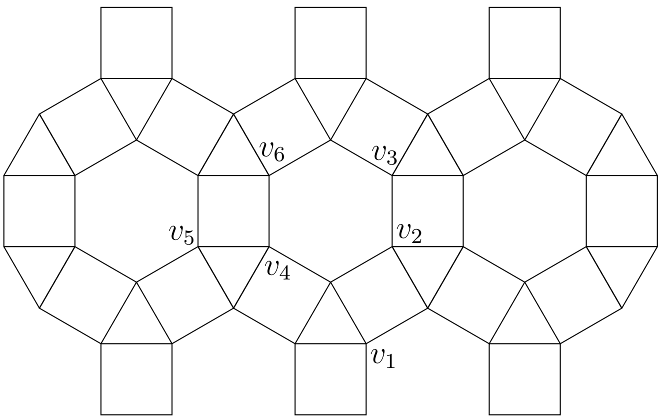


Figure 8: An illustration of the proof of non-existence of an efficient dominating set in the $(3, 4, 6, 4)$ lattice.

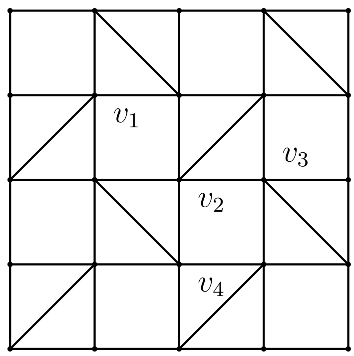


Figure 9: An illustration of the proof of non-existence of an efficient dominating set in the $(3^2, 4, 3, 4)$ lattice.

Lemma 1. *There does not exist an efficient dominating set in the* $(3, 4, 6, 4)$ *lattice.*

Proof. The proof is by contradiction. Assume that there exists an efficient dominating set D. Since $D \neq \emptyset$, there exists a vertex $v_1 \in D$. Figure 8 illustrates the following reasoning. By vertex-transitivity, any vertex may be chosen to represent v_1.

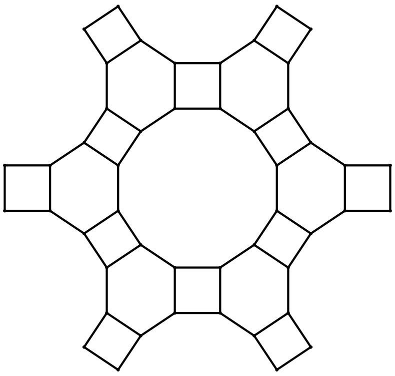
Vertex v_2 is adjacent to a vertex in $N[V_1]$, so $v_2 \notin D$ or the adjacent vertex would be dominated by both v_1 and v_2. Therefore, v_2 must be dominated by one of its neighbors. The only neighbor v for which $N[v] \cap N[v_1] = \emptyset$ is v_3, so $v_3 \in D$ if D is to be an efficient dominating set.

Similarly, $v_4 \notin D$ and must be dominated by $v_5 \in D$.

Continuing, $N[v_6] \cap N[v_5] \neq \emptyset$ and $N[v_6] \cap N[v_3] \neq \emptyset$, so $v_6 \notin D$. However, every neighbor v of v_6 satisfies either $N[v] \cap N[v_5] \neq \emptyset$ or $N[v] \cap N[v_3] \neq \emptyset$, so there does not exist any vertex $v \in D$ such that $v \in N[v_6]$. Since there is no $v \in D$ which dominates v_6, D is not a dominating set, and thus not an efficient dominating set, contradicting our original assumption. □

Lemma 2. *There does not exist an efficient dominating set in the* $(3^2, 4, 3, 4)$ *lattice.*

Proof. The proof, by contradiction, is similar to that for the $(3, 4, 6, 4)$ lattice, and is illustrated in Figure 9. We provide an abbreviated description. Any vertex may represent a vertex $v_1 \in D$. Then $v_2 \notin D$, and must be dominated by $v_3 \in D$. However, any vertex that would dominate v_4 would also dominate a vertex that is already dominated by v_1 or v_3. Thus, efficient domination is not possible. □



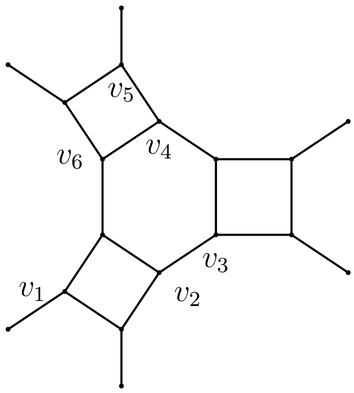


Figure 10: The left figure is a subgraph of the $(4, 6, 12)$ lattice. The right figure is an illustration of the proof of non-existence of an efficient dominating set in the $(4, 6, 12)$ lattice.

Lemma 3. *There does not exist an efficient dominating set in the* $(4, 6, 12)$ *lattice.*

Proof. The proof follows the model of the previous two proofs, and is illustrated in Figure 10. Any vertex may represent $v_1 \in D$. Vertex $v_2 \notin D$ and must be dominated by $v_3 \in D$. Then vertex $v_4 \notin D$ and must be dominated by $v_5 \in D$. However, v_6 cannot be dominated if D is an efficient dominating set. □

Lemma 4. *There does not exist an efficient dominating set in the* $(3, 6, 3, 6)$ *lattice.*

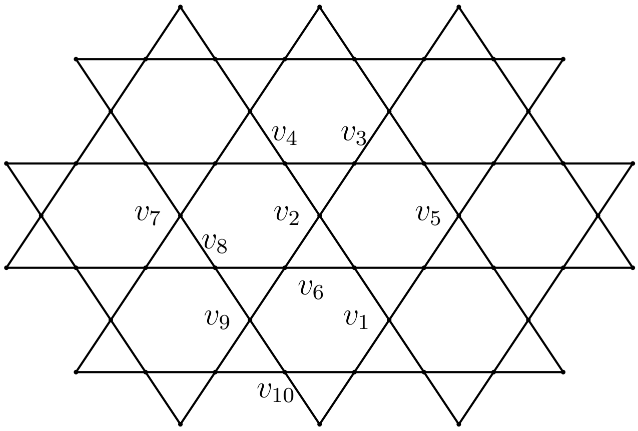


Figure 11: An illustration of the proof of non-existence of an efficient dominating set in the $(3, 6, 3, 6)$ lattice.

Proof. The proof is similar to those above, but is longer and has a more intricate sequence of steps. It is illustrated in Figure 11.

Any vertex may represent $v_1 \in D$. Then $v_2 \notin D$ and must be dominated by either v_3 or v_4, but not both.

If $v_3 \in D$, then $v_5 \notin D$ but also cannot be dominated by D.

If $v_4 \in D$, then $v_6 \notin D$ and must be dominated by either v_8 or v_9. If $v_8 \in D$, then v_{10} cannot be dominated. If $v_9 \in D$, then v_7 cannot be dominated.

Thus, every case leads to the contradiction that D cannot be a dominating set. □

4 Proof of Perfect Domination Ratio by Forcing Arguments

We now provide some definitions, terminology, and lemmas that apply to perfect domination on all the Archimedean lattices.

If G is a graph with vertex set V_G and edge set E_G, for simplicity we will write $v \in G$ rather than $v \in V_G$ and write $e \in G$ rather than $e \in E_G$.

In the remainder of this article, we will abbreviate perfect dominating set as "PDS." As in any graph, given a PDS D in a graph G, the subgraph of G induced by vertices in D is a disjoint union of connected components. Our proofs use certain features of the structure of the the boundary of the components, described in the remainder of this section. For brevity, proofs of some Lemmas and Facts are omitted.

Definition 5. Given a PDS D, let D_n denote *a connected component of size n in the subgraph induced by vertices in D.*

Note 6. For a fixed positive integer n, there may exist components D_n which are not isomorphic.

Definition 7. For two vertices v and u in a graph G, let $d_G(v, u)$ denote *the number of edges in a shortest path between v and u.* For a vertex v and a subgraph S of G, define

$d_G(v, S) = \min_{u\in S}\{d_G(v, u)\}$. For brevity, when the graph G is clear from the context, we omit the subscript G.

Definition 8. Given a subgraph S in a graph G, define the *external boundary of S* as *the set of vertices v such that $d_G(v, S) = 1$.*

Definition 9. Given a subgraph S in a graph G, define the *double external boundary of S* as *the set of vertices v such that $d_G(v, S) \in \{1, 2\}$.*

Lemma 10. *Given a component D_n in a PDS D, no vertex in the double external boundary of D_n is in D.*

Proof. Let v be in the double external boundary of D_n.

If $d(v, D_n) = 1$, then v is adjacent to a vertex in D_n, so if $v \in D$ then v is in the component D_n, contradicting $d(v, D_n) = 1$. Therefore, no vertex in the external boundary is in D.

If $d(v, D_n) = 2$, there exists a path of length two with vertices v, w, and x, where $w \notin D_n$ and $x \in D_n$. If $v \in D$, then vertex w is dominated by both v and x. Thus, $w \in D$ and thus also in D_n. This implies that $v \in D_n$ also, contradicting that v is in the double external boundary of D_n. $\square$

Lemma 11. *Given a PDS D, if $v \notin D$, u is a neighbor of v, and every other neighbor of v is not in D, then $u \in D$.*

Definition 12. Let *v pulls in u* indicate that *for a PDS D and a vertex $v \notin D$, u is a neighbor of v and every other neighbor of v is not in D, requiring that $u \in D$ by Lemma 11.*

By the definition of PDS, a vertex that is not in the PDS must be dominated exactly once. Thus, given a PDS D, if a vertex v has two neighbors u and w in D, then $v \in D$.

Definition 13. Let u and w *double force in v* indicate that *for a PDS D, if a vertex v has two neighbors u and w in D, then $v \in D$.*

Lemma 14. *Given a PDS D, if a vertex $v \notin D$ has a neighbor $u \in D$, then no other neighbor of v is in D.*

Definition 15. Let *v and u force out w* indicate that *vertex $v \notin D$ has a neighbor $u \in D$, so for any other neighbor w of v, $w \notin D$.*

Note 16. In each of subsections 4.1, 4.2, and 4.3, we consider a specific Archimedean lattice. In each section, the notations such as PDS, γ_p, and D_n refer to only that specific lattice.

4.1 The Kagome or $(3, 6, 3, 6)$ lattice

Lemma 17. $\gamma_p(3, 6, 3, 6) \leqslant \frac{1}{3}$.

Proof. A periodic PDS D with $\gamma_p(D) = \frac{1}{3}$ is shown in Figure 12, establishing $\frac{1}{3}$ as an upper bound. □

Definition 18. *Let* "a row of D_1s" *denote a sequence (possibly doubly-infinite) of at least two consecutive D_1s such that every two consecutive D_1s in the sequence are distance three apart in a 6-cycle.*

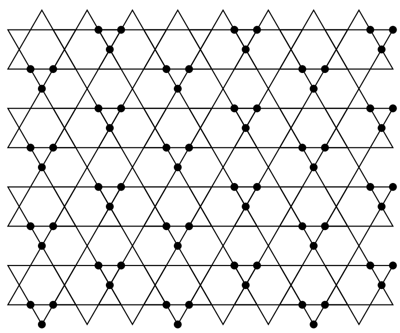


Figure 12: A PDS D of the $(3, 6, 3, 6)$ lattice with $\gamma_p(D) = \frac{1}{3}$.

Lemma 19. *A D_1 must appear in a doubly-infinite row of D_1s.*

Proof. Suppose $v_1 \in D$ is a D_1. Figure 13 (left) illustrates the following reasoning. By vertex-transitivity, any vertex may be chosen to represent v_1. Notice that v_2 and v_5 are not in D since they are in the double external boundary of v_1. Thus, v_2 pulls in either v_3 or v_4, and v_5 pulls in either v_4 or v_6.

If $v_3 \in D$, then $v_4 \notin D$, so v_6 is pulled in to dominate v_5. However, then v_4 is dominated twice, by v_3 and v_6, providing the contradiction that $v_4 \in D$. Hence $v_3 \notin D$. The same reasoning applies if $v_6 \in D$. Thus, $v_4 \in D$ and is a D_1.

The same reasoning regarding v_1 can be applied to v_4 to show v_9 is a D_1. Thus, one can show by induction that any vertex v on the line (extending infinitely in both directions) going through v_1 and v_4 must be a D_1. □

Lemma 20. *Two rows of D_1s must be parallel.*

Proof. To deduce a contradiction, suppose there exist two rows of D_1s that are not parallel. By Lemma 19, the two rows of D_1s must extend infinitely and therefore must intersect. There are only three possible directions for a row of D_1s, so these two rows of D_1s must form an angle of $\frac{\pi}{3}$. Figure 13 (right) illustrates the reasoning. Notice that v_1 and v_2 are in a row of D_1s, and v_3 and v_4 are in another row of D_1s. Thus, v_2 and v_4 are in the same D_n. Then, v_2 is in a D_2 or larger D_n, contradicting that v_2 is a D_1. □

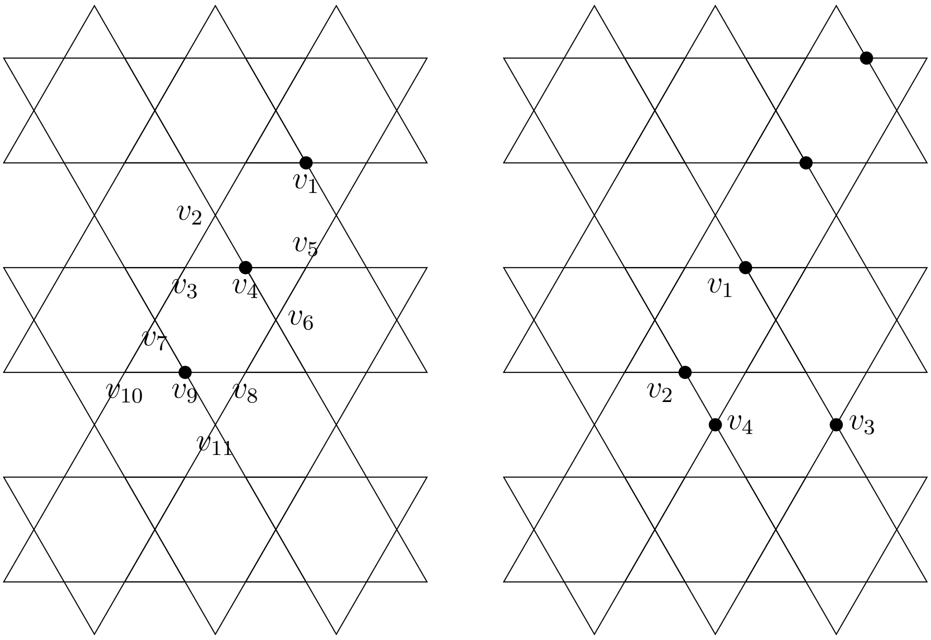


Figure 13: The figure on the left illustrates the proof of Lemma 19. The figure on the right illustrates the proof of Lemma 20.

Lemma 21. *A D_2 cannot exist.*

Proof. To deduce a contradiction, suppose there exists a PDS D that contains a D_2. Let u and v be vertices in this D_2. Since any edge in the Kagome lattice is in a 3-cycle, u and v are in a 3-cycle $\{u, v, w\}$. Then $w \notin D$ is dominated by both u and v, contradicting that D is a PDS. $\square$

Lemma 22. *If a PDS D of an induced subgraph of the Kagome lattice does not contain a D_1, then the perfect domination proportion of D is at least $\frac{1}{3}$.*

Proof. Suppose there exists a PDS D that does not contain a D_1. By Lemma 21, any vertex $v \in D$ must be in a D_3 or larger D_n. Observe that a vertex v in a D_3 or larger D_n has at least two neighbors in D. Thus, v dominates at most two vertices not in D, which implies that the perfect domination proportion of D is greater than or equal to $\frac{1}{3}$.

The same reasoning can be applied to any induced subgraph to show that if D is a PDS that does not contain a D_1, then any vertex $v \in D$ dominates at most two vertices not in D. Thus, the domination proportion of the induced subgraph is at least $\frac{1}{3}$. $\square$

Lemma 23. *A PDS D with perfect domination proportion strictly less than $\frac{1}{3}$ must include infinitely many rows of D_1s.*

Proof. Suppose there exists a PDS D that includes only finitely many rows of D_1s. Let W denote the set of vertices that are neither D_1s nor dominated by D_1s. Consider the subgraph H induced by W, which by Lemma 22 has a perfect domination proportion at least $\frac{1}{3}$. Since the effect of finitely many rows of D_1s is negligible, the perfect domination proportion of D is at least $\frac{1}{3}$. Thus, any PDS D with a perfect domination proportion strictly less than $\frac{1}{3}$ must include infinitely many rows of D_1s. $\square$

Lemma 24. $\gamma_p(3, 6, 3, 6) \geqslant \frac{1}{3}$.

Proof. The proof is by contradiction. Let V be the vertex set of the Kagome lattice. Assume there exists a PDS D with perfect domination proportion strictly less than $\frac{1}{3}$. By Lemma 23, D must contain infinitely many rows of D_1s. By Lemma 20, the rows of D_1s in D must be parallel. Let W be a row of D_1s. Figure 14 illustrates the reasoning.

Let v_1 be a D_1 in W. Notice that $v_2 \notin D$ since it is in the double external boundary of v_1. Thus, v_2 pulls in either v_3 or v_4. The two cases are equivalent by symmetry. Without loss of generality, let $v_3 \in D$ and $v_4 \notin D$. Notice that v_3 is not a D_1, since otherwise by Lemma 19, v_3 and v_5 form a row of D_1s that intersects W, which contradicts Lemma 20. By Lemma 21, v_3 is in a D_3 or larger D_n. Thus, v_6 and v_7 are in D.

Notice that $v_8 \notin D$ since it is in the double external boundary of v_1. Thus $v_8 \notin D$ and $v_6 \in D$ force out v_9. Then $v_9 \notin D$ and $v_6 \in D$ force out v_{10} and v_{11}. A similar argument as for v_2 can be applied to v_{12} to show that $v_{12} \notin D$ and v_{13}, v_{14}, and v_{15} are in D. Then v_{10} pulls in either v_{16} or v_{17}. The two cases are equivalent by symmetry. Without loss of generality, let $v_{16} \in D$, and $v_{17} \notin D$. Then v_7 and v_{16} double force in v_{18}. Thus, v_7 and v_{18} double force in v_{19}, and v_{16} and v_{18} double force in v_{20}.

Next, $v_4 \notin D$ and $v_3 \in D$ force out v_{21} and v_{22}. Then $v_{23} \notin D$, since otherwise v_{19} and v_{23} double force in v_{24} and consequently v_{23} and v_{24} double force in v_{21}, contradicting our previous argument that $v_{21} \notin D$. Thus, v_{21} pulls in v_{24}. Finally, v_{24} and v_{19} in D double force in v_{25}.

The same reasoning can be applied to show that v_{26}, v_{27}, v_{28}, v_{29}, v_{30}, v_{31}, v_{32}, v_{33}, and v_{34} are in D.

Next, we calculate a lower bound for the perfect domination proportion of such a PDS D, given the reasoning above. Let W denote the line of D_1s containing v_1, and let H_1 denote $W \cup N(W)$. As illustrated by Figure 14, let H_2 denote the set of alternating D_3s and subgraphs of D_ns with $n \geqslant 9$, together with the vertices that they dominate within the region indicated, just above $W \cup N(W)$. Let H_3 denote the isomorphic subgraph obtained by reflecting H_2 through the line corresponding to W. Within $H_1 \cup H_2 \cup H_3$ we can form connected subgraphs consisting of four D_1s, one D_3 on each side, and one D_7 on each side, and the vertices that they dominate.

Number of Components	Subgraph	Vertices in D	Vertices Dominated
4	D_1	1	5
2	D_3	3	9
2	$H_2 \cap D_n, n \geqslant 9$	7	15

Table 2: Data for calculation of the perfect domination proportion.

Denoting the vertex sets of H_1, H_2, H_3 by $V_{H_1}, V_{H_2}, V_{H_3}$ respectively, we have

$$\frac{|D \cap (V_{H_1} \cup V_{H_2} \cup V_{H_3})|}{|V_{H_1} \cup V_{H_2} \cup V_{H_3}|} = \frac{4 \times 1 + 2 \times 3 + 2 \times 7}{4 \times 5 + 2 \times 9 + 2 \times 15} = \frac{6}{17} > \frac{1}{3}.$$

Let G denote the union of all rows of D_1s and their corresponding H_1, H_2, H_3. We have shown above that the perfect domination proportion of G is strictly larger than $\frac{1}{3}$.

By definition of G, the subgraph induced by $V \setminus V_G$ does not contain any D_1s. However, it does contain neighbors of H_2 or H_3 which are in D because they are part of a $D_n, n \geqslant 9$, such as v_{25}, In the subgraph induced by $V \setminus V_G$, let V_B denote the set of such vertices and the two additional vertices dominated by each of them. The perfect domination proportion of the subgraph induced by V_B is exactly $\frac{1}{3}$.

Since $V \setminus (V_G \cup V_B)$ does not contain any D_1, by Lemma 22, the perfect domination proportion of the rest of the lattice is greater than or equal to $\frac{1}{3}$. Combining these, we conclude that the perfect domination proportion of the lattice is at least $\frac{1}{3}$, contradicting our original assumption. □

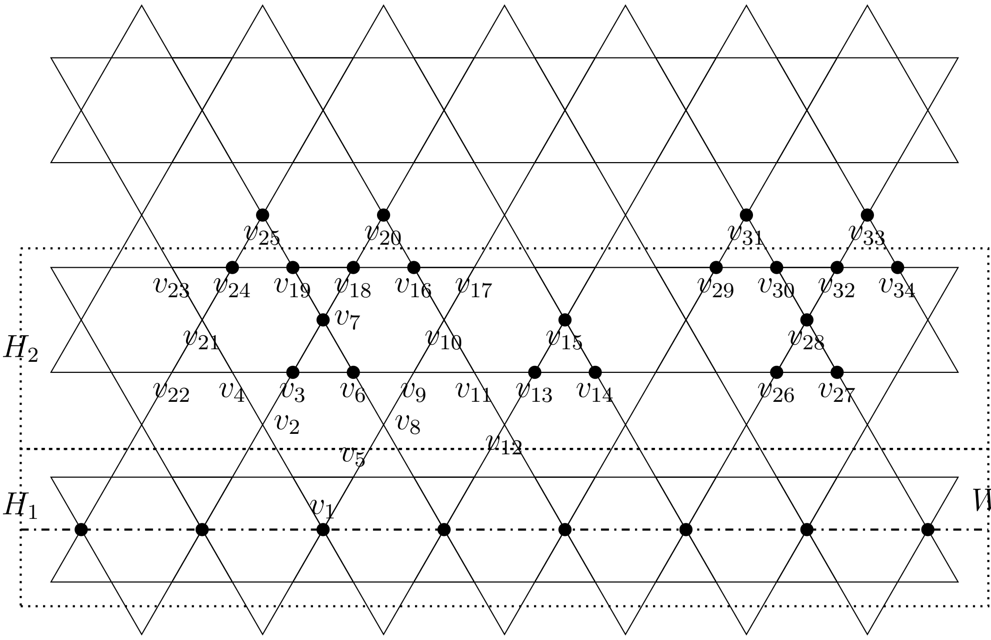


Figure 14: An illustration of the proof of Lemma 24.

Theorem 25. $\gamma_p(3, 6, 3, 6) = \frac{1}{3}$.

Proof. The result is immediate from Lemma 17 and Lemma 24. □

4.2 The $(3, 4, 6, 4)$ lattice

Lemma 26. $\gamma_p(3, 4, 6, 4) \leqslant \frac{1}{4}$.

Proof. A periodic PDS D with perfect domination proportion $\frac{1}{4}$ is shown in Figure 15, establishing $\frac{1}{4}$ as an upper bound. $\square$

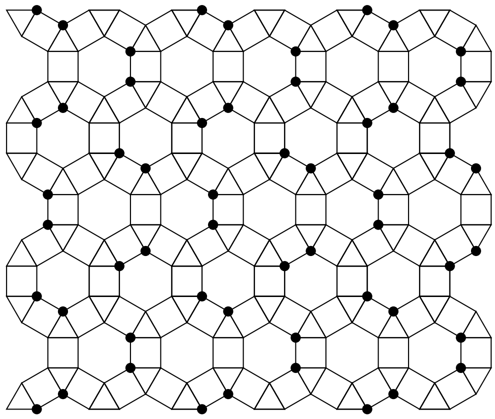


Figure 15: A PDS D on the $(3, 4, 6, 4)$ lattice with $\gamma_p(D) = \frac{1}{4}$.

The following definitions, facts, and lemmas apply for any PDS D.

Definition 27. *A D-adequate subgraph is a connected subgraph of the $(3, 4, 6, 4)$ lattice with at most 12 vertices that has at least $\frac{1}{4}$ of its vertices in D.*

Definition 28. *A triangle is a 3-cycle. A k-triangle is a triangle that has k vertices in D.*

Fact 29. *A triangle cannot have exactly 2 vertices in D, since otherwise the third vertex is dominated twice. Thus, a k-triangle only exists for $k \in \{0, 1, 3\}$.*

Definition 30. Let *two neighboring triangles* denote *two 3-cycles such that an edge in the (3,4,6,4) lattice has an endpoint in each 3-cycle.*

Fact 31. *Given two neighboring triangles, there exist two edges with an endpoint in each triangle. Each triangle has exactly three neighbors.*

Lemma 32. *Any 3-triangle, together with its three neighboring triangles, form a D-adequate subgraph.*

Proof. The subgraph has 12 vertices, at least 3 of which are in D, because it contains a 3-triangle. Thus, the subgraph is D-adequate. $\square$

Definition 33. Let R_3 denote *the union of all 3-triangles and neighbors of 3-triangles in the* $(3, 4, 6, 4)$ *lattice.*

Lemma 34. *R_3 is a disjoint union of D-adequate subgraphs.*

Proof. We prove the claim by constructing a disjoint set of D-adequate subgraphs whose union is R_3.

The set of 3-cycles in the $(3,4,6,4)$ lattice is countably infinite. Order them in a sequence $\{T_k\}_{k=1}^{\infty}$. For each k, denote the union of T_k and its neighboring triangles that are not 3-triangles by G_k. If T_k is a 3-triangle, we define $H_k = G_k$, and otherwise $H_k = \emptyset$. Let $H_0 = \emptyset$. Define a sequence of disjoint subgraphs by

$$J_k = H_k \setminus \left\{ \bigcup_{i=0}^{k-1} H_i \right\}.$$

Note that each nonempty J_k contains at least the three vertices of T_k in D, and no more than 12 vertices, so each J_k is D-adequate.

Then R_3 is the union of the nonempty J_k, $k \in \{1,2,3,\ldots\}$, which are disjoint and D-adequate. □

Fact 35. *Each triangle in R_3^c is a 0-triangle or a 1-triangle.*

Lemma 36. *A 0-triangle in R_3^c has no 0-triangle neighbor.*

Proof. Figure 16 illustrates the reasoning. To deduce a contradiction, suppose T_1 is a 0-triangle in R_3^c and T_2 ia a neighboring 0-triangle. Notice that v_1 pulls in v_4, and v_2 pulls in v_7. Since T_1 is in R_3^c, N_1 and N_2 are not 3-triangles. Thus, $v_5, v_6 \notin D$. Consequently, v_3 cannot be dominated, contradicting that D is a dominating set. □

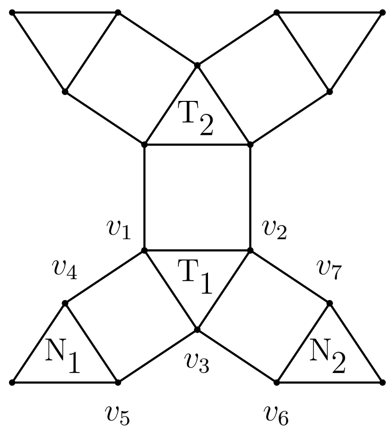


Figure 16: An illustration of the proof of Lemma 36.

Lemma 37. *A 0-triangle in R_3^c cannot have a neighbor in R_3.*

Proof. Figure 17 illustrates the reasoning. To deduce a contradiction, suppose a 0-triangle T in R_3^c has a neighbor N_1 in R_3. Since T is in R_3^c, N_1 is not a 3-triangle. Since N_1 is in R_3, N_1 must have a neighbor M that is a 3-triangle. Without loss of generality, assume T, N_1, and M are positioned as in Figure 17.

By Lemma 36, N_1 is not a 0-triangle. Since M is a 3-triangle, and N_1 has a vertex in D, all vertices in N_1 are forced in D, contradicting the assumption that N_1 is not a 3-triangle. □

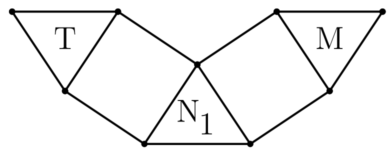


Figure 17: An illustration of the proof of Lemma 37.

Lemma 38. *Two 0-triangles in R_3^c cannot have a common neighbor in R_3^c.*

Proof. Figure 18 illustrates the reasoning. To deduce a contradiction, suppose T_1 and T_2 are two 0-triangles in R_3^c and they have a common neighbor C in R_3^c. Since C is in R_3^c, it is not a 3-triangle. By Lemma 36, C is not a 0-triangle. Thus, C is a 1-triangle. We will consider different cases based on the location of the vertex of C which is in D.

Case 1 ($\boldsymbol{v_1 \in D}$): Suppose that $v_1 \in D$, and thus v_2 and v_3 are not in D. Since T_2 is a 0-triangle, v_6 pulls in v_8. Since T_2 is in R_3^c, N_4 is not a 3-triangle, and thus $v_7 \notin D$. Consequently, v_5 pulls in v_{11}. Since T_2 is in R_3^c, N_2 is not a 3-triangle. Consequently, $v_{11} \in D$ and $v_9 \notin D$ force out v_{13}, and $v_{11} \in D$ and $v_{10} \notin D$ force out v_{14}. By symmetry, the same reasoning applies to the left side of the figure to show that v_{12} and v_{15} cannot be in D either. We have achieved a contradiction, since v_{14} cannot be dominated by any vertex in D.

Case 2 ($\boldsymbol{v_2 \in D}$): Suppose that $v_2 \in D$, and thus v_1 and v_3 are not in D. (By symmetry, the following reasoning also applies to the case when $v_3 \in D$.)

Since T_2 is a 0-triangle, v_6 pulls in v_8 and v_4 pulls in v_{10}. Since T_2 is in R_3^c, N_2 and N_4 are not 3-triangles. Consequently, $v_7 \notin D$ and $v_{11} \notin D$. We have achieved a contradiction, since v_5 cannot be dominated by any vertex in D. □

Lemma 39. *A 0-triangle in R_3^c has three 1-triangles as neighbors which do not have a different 0-triangle in R_3^c as neighbor.*

Proof. Suppose that T is a 0-triangle in R_3^c. By Lemma 37, T does not have a neighbor in R_3. Therefore, each neighboring triangle is either a 0-triangle or a 1-triangle which is in R_3^c.

By Lemma 36, T cannot have a neighbor which is a 0-triangle.

By Lemma 38, none of T's neighbors is a neighbor of a different 0-triangle in R_3^c.

Therefore, each neighbor of T is a 1-triangle which is in R_3^c which is not a neighbor of a different 0-triangle in R_3^c. □

Theorem 40. $\gamma_p(3, 4, 6, 4) = \frac{1}{4}$.

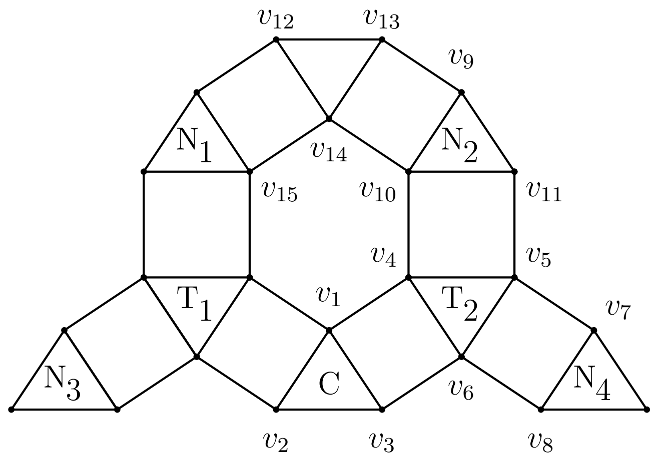


Figure 18: An illustration of the proof of Lemma 38.

Proof. By Lemma 39, each 0-triangle in R_3^c has three 1-triangles as neighbors which do not have a different 0-triangle in R_3^c as neighbor.

Notice a 0-triangle in R_3^c together with its three neighboring 1-triangles contains 12 vertices with exactly 3 vertices in D, and thus it is a D-adequate subgraph. Furthermore, by Lemma 39, these D-adequate subgraphs are disjoint. Denote the union of all such subgraphs by R_0.

Each triangle in $R_3^c \cap R_0^c$ is a 1-triangle, which has $\frac{1}{3}$ of its vertices in D and is a D-adequate subgraph. Denote the union of all triangles in $R_3^c \cap R_0^c$ by R_1.

Since $R_3 \cup R_0 \cup R_1$ contains every vertex of the $(3,4,6,4)$ lattice, the lattice can be decomposed into disjoint connected induced subgraphs which are all D-adequate. Thus, we have $\gamma_p(3,4,6,4) \geqslant \frac{1}{4}$.

By Lemma 26, $\gamma_p(3,4,6,4) \leqslant \frac{1}{4}$. We conclude that $\gamma_p(3,4,6,4) = \frac{1}{4}$. $\square$

4.3 The $(3^2, 4, 3, 4)$ lattice

We first provide a PDS that establishes an upper bound, then prove this PDS is actually a minimal PDS, to conclude that $\gamma_p(3^2,4,3,4) = \frac{1}{4}$.

Lemma 41. $\gamma_p(3^2,4,3,4) \leqslant \frac{1}{4}$.

Proof. Figure 19 shows a periodic PDS D on the $(3^2,4,3,4)$ lattice. To calculate the domination ratio of this PDS, note that there are pairs of D_1s which are distance three apart. In the figure, there are D_4s above and below each such pair of D_1s. These four components of D and their external boundaries induce a subgraph with 40 vertices which are dominated by 10 vertices, giving a domination proportion of $\frac{1}{4}$. The lattice may be decomposed into disjoint isomorphic connected subgraphs, so $\gamma_p(D) = \frac{1}{4}$. Thus, $\frac{1}{4}$ is an upper bound for $\gamma_p(3^2,4,3,4)$. $\square$

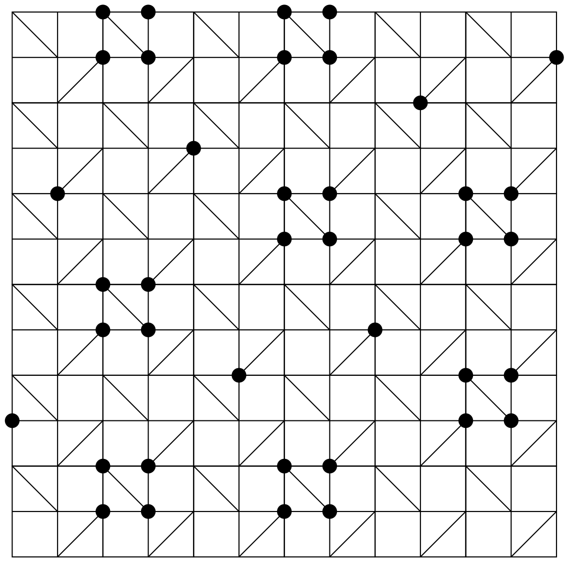


Figure 19: A PDS D in the $(3^2, 4, 3, 4)$ lattice with $\gamma_p(D) = \frac{1}{4}$.

Lemma 42. *A PDS of the* $(3^2, 4, 3, 4)$ *lattice cannot contain a* D_2*.*

Proof. To deduce a contradiction, suppose there exists a PDS D that contains a D_2. Let x and y denote the vertices in this D_2. Since every edge is in a 3-cycle, there exists vertex $z \notin D$ that is a common neighbor of x and y. Then z is dominated by both x and y, contradicting the assumption that D is a perfect dominating set. □

Lemma 43. *A PDS of the* $(3^2, 4, 3, 4)$ *lattice cannot contain a* D_3*.*

Proof. To deduce a contradiction, suppose there exists a PDS D that contains a D_3. Let x, y and z denote vertices in this D_3. There are 2 possible types of D_3s: a 3-path and a 3-cycle.

If the subgraph induced by $\{x, y, z\}$ is a 3-cycle, then the adjacent 3-cycle (i.e., sharing an edge with $\{x, y, z\}$) must be in D, and therefore $\{x, y, z\}$ must be in a D_4 or a larger D_n.

If the subgraph induced by $\{x, y, z\}$ is a 3-path, then the subgraph induced by $\{x, y, z\}$ includes an edge of a 3-cycle, and the 3-cycle must be in D. Thus, $\{x, y, z\}$ must be in a D_4 or a larger D_n.

In either case, we reach the contradiction that $\{x, y, z\}$ is not a D_3. □

Lemma 44. *If a PDS D contains a D_1, the PDS must be a union of D_1s and D_4s.*

Proof. Figure 21 illustrates the reasoning, which is rather long and intricate.

Suppose there exists a PDS D that contains a D_1. Let v_1 denote this D_1. The vertices in the double external boundary of v_1 are shown in Figure 21 as open circles. Therefore, v_2 pulls in v_3.

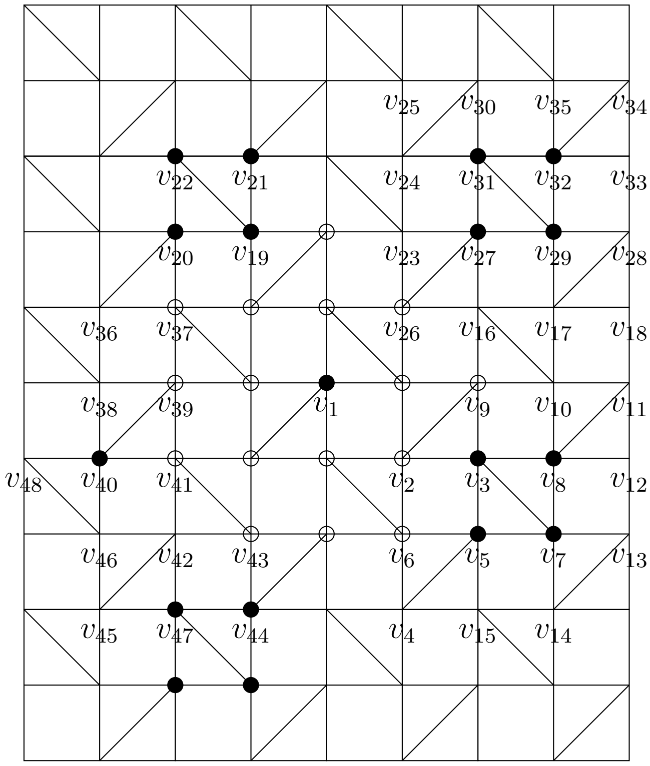


Figure 20: An illustration of the proof of Lemma 44.

We show that $v_4 \notin D$ by contradiction: If $v_4 \in D$, then v_3 and v_4 double force $v_5 \in D$, and consequently v_4 and v_5 double force $v_6 \in D$. This contradicts the fact that $v_6 \notin D$ because it is in double external boundary of v_1.

Since v_4 is not in D, v_6 pulls in v_5. Then v_3 and v_5 double force $v_7 \in D$, and consequently v_3 and v_7 double force $v_8 \in D$. Since $v_9 \notin D$ and $v_3 \in D$, the vertex v_{10} cannot double-dominate v_9, so v_{10} is forced out. Similarly, $v_{10} \notin D$ and $v_8 \in D$ forces out v_{11}, and by repeating this reasoning v_{12}, v_{13}, v_{14}, and v_{15} are forced out. Thus, v_3, v_5, v_7, and v_8 form a D_4. Furthermore, the double external boundary of this D_4 contains v_{16}, v_{17}, and v_{18}, so they are not in D.

By reflection through the 45^o line through v_1, the same reasoning applies to show that v_{19}, v_{20}, v_{21}, and v_{22} are a D_4, and, being in its double external boundary, v_{23}, v_{24}, and v_{25} are not in D.

Next, v_{26} pulls in v_{27}, and we show that $v_{28} \notin D$ by contradiction: Otherwise v_{27} and v_{28} would double force $v_{29} \in D$, and consequently v_{28} and v_{29} would double force $v_{17} \in D$, contradicting our previous conclusion that $v_{17} \notin D$ since it is in the double external boundary of a D_4.

Thus, v_{17} pulls in v_{29}. Vertices v_{27} and v_{29} then force in v_{31} which helps double force $v_{32} \in D$. Since $v_{24} \notin D$, it forces out v_{30}. Similarly, in sequence, the vertices v_{35}, v_{34}, and v_{33} are forced out. We conclude that v_{27}, v_{29}, v_{31}, and v_{32} are a D_4.

Next we consider vertices in the lower left part with respect to v_1 of the figure, where the reasoning proceeds somewhat differently. The double external boundary of the D_4 formed by v_{19}, v_{20}, v_{21} and v_{22} contains v_{37} and v_{38}, and therefore v_{37} and v_{38} are not in D. Therefore, v_{39} pulls in v_{40}.

Reason by contradiction that $v_{42} \notin D$: Otherwise v_{42} and v_{40} double force $v_{41} \in D$, contradicting the fact that $v_{41} \notin D$ because it is in double external boundary of v_1.

With no alternative, v_{43} pulls in v_{44}. By contradiction $v_{45} \notin D$: Otherwise v_{40} and v_{45} double force $v_{46} \in D$, and consequently v_{45} and v_{46} double force $v_{42} \in D$, contradicting our previous conclusion that $v_{42} \notin D$.

Since $v_{42} \notin D$, it pulls in either v_{46} or v_{47}. The two cases are equivalent by symmetry. Without loss of generality, let $v_{47} \in D$ and $v_{46} \notin D$. Then $v_{40} \in D$ and $v_{46} \notin D$ force out v_{48}, and we conclude that v_{40} is a D_1. On the other hand, v_{47} and v_{44} double force in two neighbors to form a possible D_4, and reasoning as in the previous cases forces out the boundary to confirm that it must be a D_4. (Note that if we had chosen $v_{46} \in D$ and $v_{47} \notin D$, the resulting PDS would be isomorphic, but rotated by 90^o.)

In the remainder of the proof, we show that the reasoning above can be extended to the entire $(3^2, 4, 3, 4)$ lattice. This second part the proof is illustrated in Figures 21, 22, and 23. First, the entire argument so far can be repeated starting from v_{40} instead of v_1, to show that there are four D_4s around v_{40}, as is shown in the Figure 21. In particular, note that v_{59} is in one of the four D_4s around v_{40}.

Next, notice that the double external boundary of the D_4 formed by v_{19}, v_{20}, v_{21}, and v_{22} contains v_{49} and v_{50}, and therefore $v_{49}, v_{50} \notin D$. Consequently, v_{25} pulls in v_{51}. Similarly, the double external boundary of the D_4 formed by v_{27}, v_{29}, v_{31}, and v_{32} contains v_{52}, and therefore $v_{52} \notin D$. Thus, $v_{51} \in D$ and $v_{52} \notin D$ force out v_{53}, and then $v_{51} \in D$ and $v_{53} \notin D$ force out v_{54}. We conclude that v_{51} is a D_1.

The same reasoning as starting from v_1 can be applied to v_{51} to show that exactly one of v_{55} and v_{56} is a D_1 and the other is in a D_4.

Similarly, v_{57} can be shown to be a D_1. The same reasoning as starting from v_1 can be applied to v_{57} to show that exactly one of v_{58} and v_{59} is a D_1 and the other is in a D_4. Since we have showed that v_{59} is in a D_4, we conclude that v_{58} is a D_1.

Similarly, both v_{60} and v_{61} can be shown to be D_1s. Thus, such an arrangement of D_1s and D_4s must extend periodically in both directions, as illustrated by the dash dot line in Figure 21.

Finally, we show that D is a union of only D_1s and D_4s. We have showed that exactly one of v_{55} and v_{66} is a D_1 and the other is in a D_4.

If v_{55} is a D_1 and v_{56} is in a D_4, then the same reasoning as starting from v_1 can be applied to v_{51} to show that v_{51} is in an arrangement of D_1s and D_4s extending periodically in both directions, as illustrated by Figure 22.

If v_{56} is a D_1 and v_{55} is in a D_4, then the same reasoning as starting from v_1 can be applied to v_{51} to show that v_{51} is in an arrangement of D_1s and D_4s extending periodically in both directions, as illustrated by Figure 23.

Similarly, v_{62} can be shown to be a D_1. The same reasoning as starting from v_1 can be applied to v_{62} to show that v_{62} is in an arrangement of D_1s and D_4s extending periodically in both directions.

We conclude that such an arrangement of D_1s and D_4s must extend periodically in all directions, so the PDS D is a union of only D_1s and D_4s. $\square$

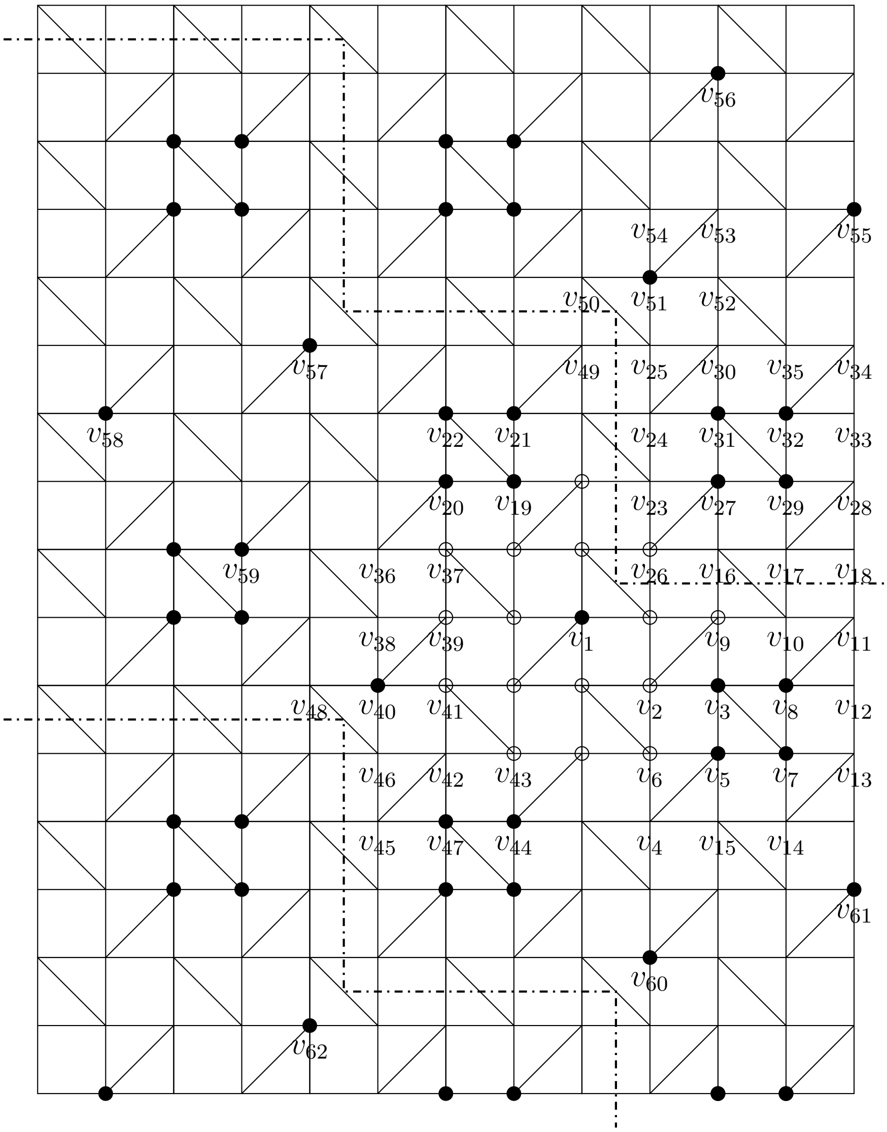


Figure 21: An illustration of the proof of Lemma 44.

Theorem 45. $\gamma_p(3^2, 4, 3, 4) = \frac{1}{4}$.

Proof. We show that any PDS that consists of only D_4s and larger D_ns are less efficient than a union of D_1s and D_4s. Any vertex x in a $D_n, n \geqslant 4$ must have a neighbor y in this D_n. Since every edge is in a 3-cycle and $\{x, y\}$ is an edge, there exists a vertex z dominated by both x and y. Thus, z is also in this D_n. Since every vertex has degree 5 and x has two neighbors y and z in the same D_n, vertex x dominates at most 3 vertices

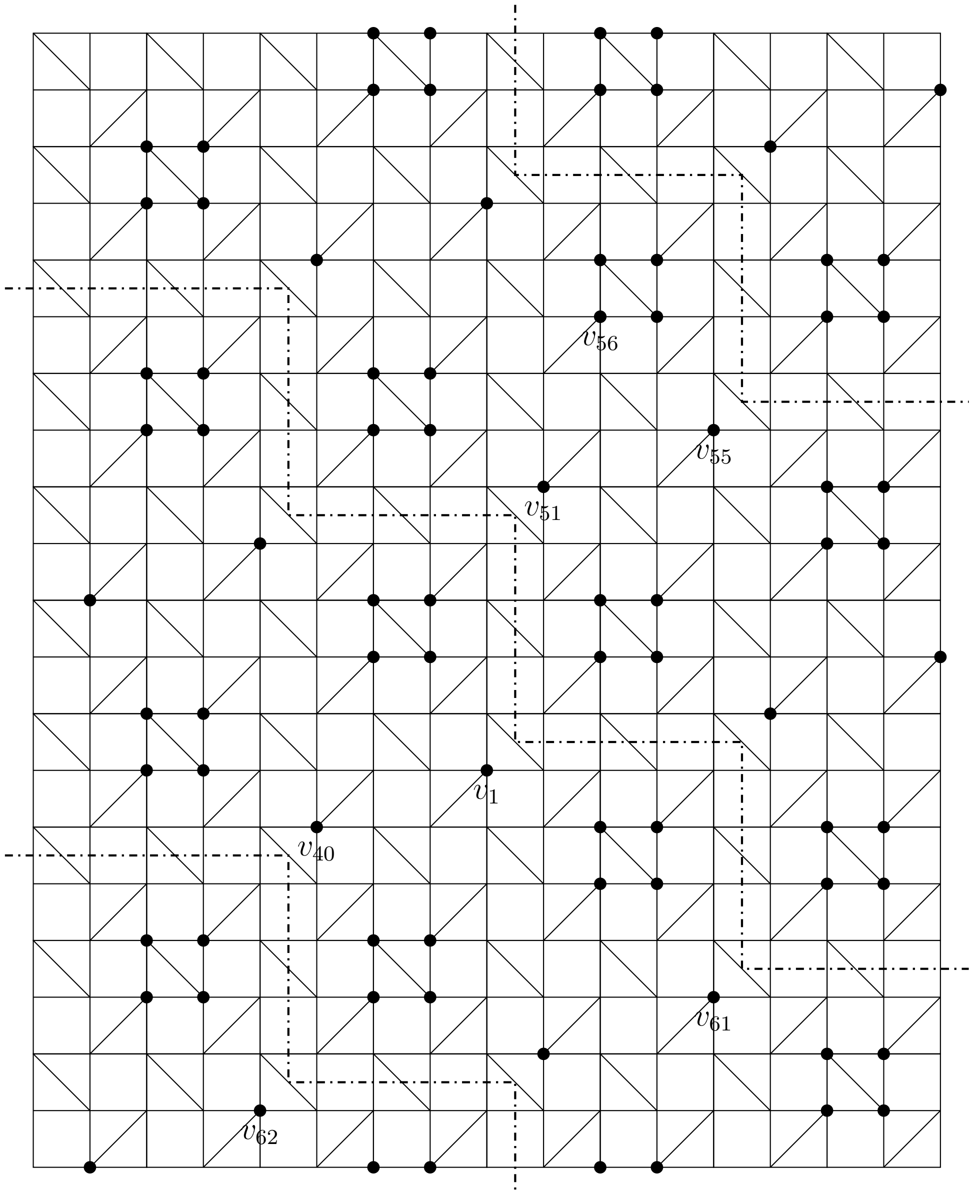


Figure 22: An illustration of the proof of Lemma 44.

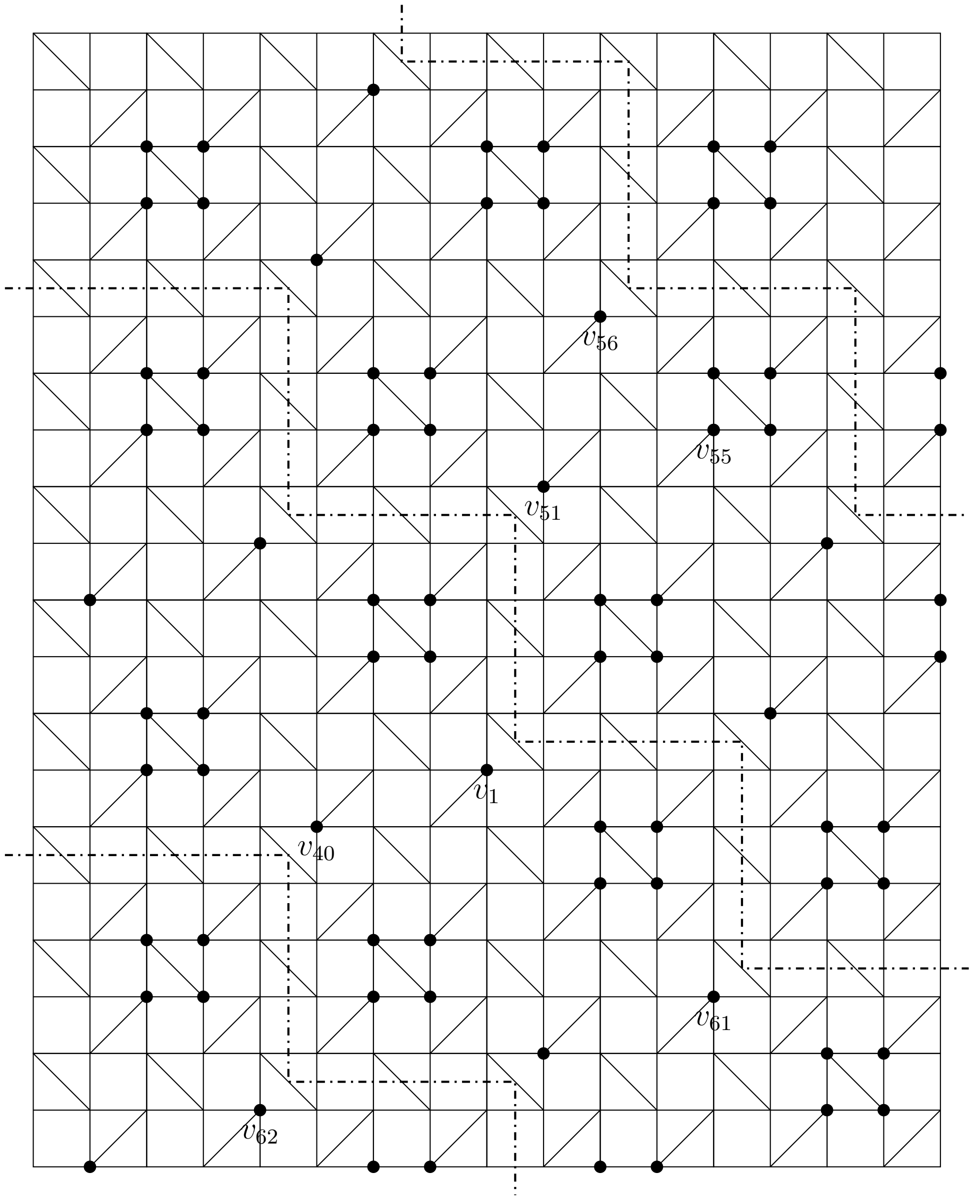


Figure 23: An illustration of the proof of Lemma 44.

outside this D_n. Thus, a PDS that consists of only D_4s and larger D_ns has domination proportion at least $\frac{1}{4}$.

Lemma 44 shows that any PDS that contains a D_1 must be a union of D_1s and D_4s. By Lemma 42 and Lemma 43, D_2s and D_3s do not exist. Thus, the PDS given in Lemma 41 is the minimal PDS, and $\gamma_p(3^2, 4, 3, 4) = \frac{1}{4}$. □

5 Proof of the Perfect Domination Ratio of the $(4, 6, 12)$ Lattice by Linear Programming

For domination number problems, the generic integer programming method requires an integral variable for every vertex of the graph. The vertex set of an infinite periodic graph is infinite. Therefore, the generic integer program will have infinitely many variables and contraints.

To solve the minimum dominating set problem on the $(4, 6, 12)$ lattice, we introduce a linear programming relaxation on an infinite periodic graph. The relaxation is a minimization problem on a particular polytope. (A polyhedron is the solution set of a finite system of linear inequalities. A polytope is a polyhedron that contains no infinite half-line. An inequality $w^T x \leqslant t$ is valid for a polyhedron P if $P \subseteq \{x : w^T x \leqslant t\}$.) Furthermore, the relaxation has finitely many constraints and the number of constraints does not depend on the number of vertices. Therefore, the relaxation can be solved in polynomial time by any linear programming solver. Formulating the relaxation requires choosing a subgraph of the infinite periodic graph and examining the properties of the subgraph.

One can use the relaxation to compute a lower bound for the domination ratio of an infinite periodic graph. One can also use the relaxation to compute a lower bound for the domination number of a finite subgraph of an infinite periodic graph.

Using the relaxation, we computed a lower bound for the domination ratio of the $(4, 6, 12)$ lattice. The lower bound equals an upper bound we obtained from a dominating set. Therefore, we obtain the exact value of the domination ratio and the perfect domination ratio of the $(4, 6, 12)$ lattice.

Lemma 46. $\gamma(4, 6, 12) \leqslant \gamma_p(4, 6, 12) \leqslant \frac{5}{18}$.

Proof. A periodic PDS D with $\gamma_p(D) = \frac{5}{18}$ is shown in Figure 24, establishing $\frac{5}{18}$ as an upper bound. The vertex set of the $(4, 6, 12)$ lattice can be partitioned into subsets of size 36 such that the subgraph induced by vertices in every subset is isomorphic to G' as shown in Figure 24.

To calculate the domination proportion, notice that every subgraph isomorphic to G' has 10 vertices in D. Thus,

$$\gamma_p(D) = \frac{10}{36} = \frac{5}{18}.$$

Since any PDS is a dominating set, we have $\gamma(4, 6, 12) \leqslant \gamma_p(4, 6, 12) \leqslant \frac{5}{18}$. □

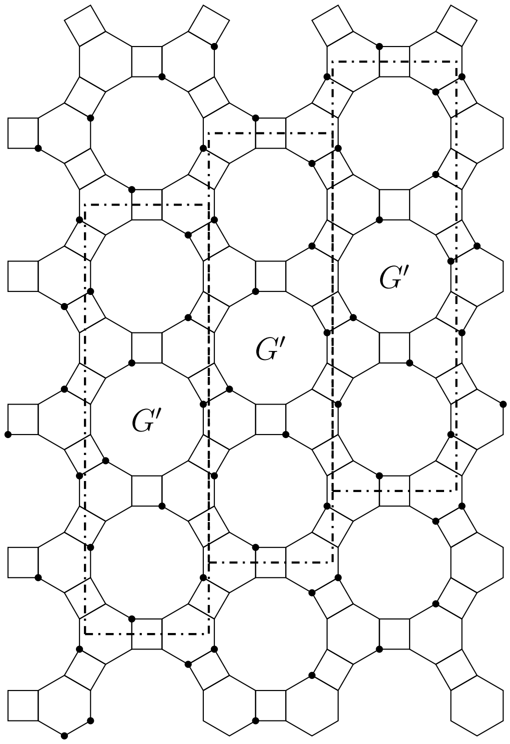


Figure 24: A PDS D of (4,6,12) lattice with $\gamma_p(D) = \frac{5}{18}$

Note 47. The vertex set of the (4,6,12) lattice can be partitioned into disjoint subsets such that the subgraph induced by vertices in every subset is isomorphic to H shown in Figure 25 with sub-divided edges on its boundary.

Note 48. The internal boundary of H is illustrated by $\{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$. Throughout Section 5, we do not consider the sub-dividing vertices at the ends of half-edges to be vertices of H.

Definition 49. An H_n is *a pair (G, D), where G is a graph isomorphic to H, and D is an n-vertex dominating set of G assuming boundary vertices of G are dominated for free.*

Recall that the definition of dominated for free is provided in Section 2.4.

Definition 50. Let $H^{(1)} = (G^{(1)}, D^{(1)})$ and $H^{(2)} = (G^{(2)}, D^{(2)})$ be two H_ns. Insert a self-loop (i.e. an edge that connects a vertex to itself) edge in $G^{(1)}$ at every vertex in $D^{(1)}$ and a self-loop in $G^{(2)}$ at every vertex in $D^{(2)}$. If the resulting $G^{(1)*}$ and $G^{(2)*}$ are isomorphic, then $H^{(1)}$ and $H^{(2)}$ are *isomorphic*. (Note that $G^{(1)*}$ and $G^{(2)*}$ are not isomorphic when (but not only when) $|D^{(1)}| \neq |D^{(2)}|$.)

Definition 51. For a given n, let $\mathcal{H}_n$ denote *the set of all non-isomorphic H_n.*

Figure 25 (right) illustrates the following definitions. Let (G, D) be an H_n, where G is an induced subgraph of the $(4, 6, 12)$ lattice. We have the following notation and definitions: Let V_G denote the set of vertices in G.

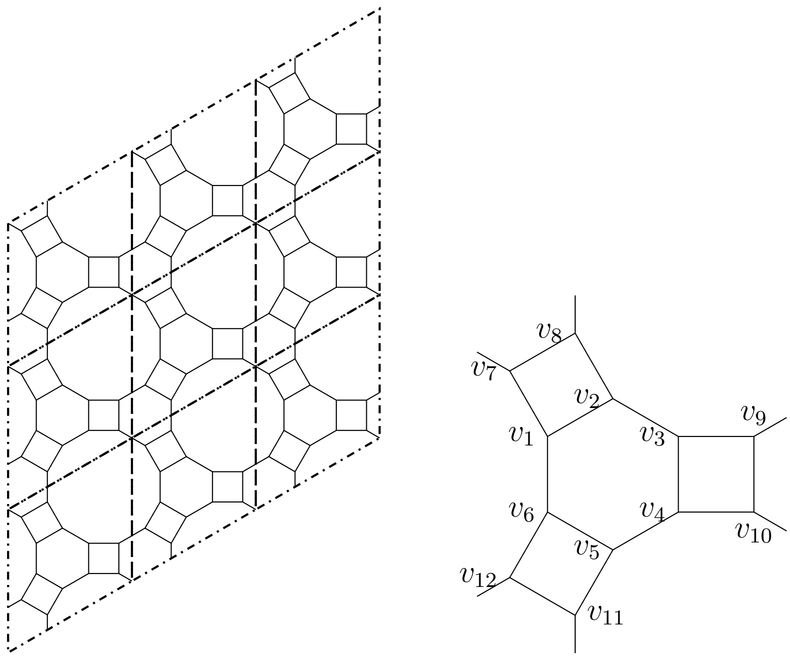


Figure 25: Left: A subgraph of the $(4, 6, 12)$ lattice; right: H

Definition 52. The graph G contains a unique 6-cycle, illustrated by $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Let C_G denote the set of vertices in the unique 6-cycle in G.

Definition 53. The graph G has six vertices on its internal boundary, illustrated by $\{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$. Let B_G denote the set of vertices on the internal boundary of G.

Definition 54. Let $lend(G, D)$ denote *the number of vertices in* $(4, 6, 12) \setminus G$ *dominated by a vertex in* D.

Definition 55. Let $borrow(G, D)$ denote *the number of vertices in* G *not dominated by a vertex in* D.

Note 56. If a vertex $v \in V_G$ is not dominated by vertices in D, then we must have $v \in B_G$ for D to be a dominating set of G assuming boundary vertices of G are dominated for free.

Definition 57. For a fixed n,

$$netlend(\mathcal{H}_n) = \max_{(G,D)\in\mathcal{H}_n} \Big(lend(G, D) - borrow(G, D)\Big).$$

Lemma 58. *If* (G, D) *is an* H_n*, then* $lend(G, D) = |D \cap B_G|$.

Proof. No vertex in C_G could dominate any vertex in $(4, 6, 12) \setminus G$. Every vertex in B_G could dominate one vertex in $(4, 6, 12) \setminus G$. Thus, $lend(G, D) = |D \cap B_G|$. ☐

Lemma 59. *If (G, D) is an H_n, then $borrow(G, D) \geqslant 6 - |D \cap C_G| - 2 \times |D \cap B_G|$.*

Proof. Each vertex in C_G can dominate only one vertex in B_G. Every vertex in B_G could dominate two vertices in B_G. Since some vertices in B_G may be dominated twice, at most $|D \cap C_G| + 2 \times |D \cap B_G|$ vertices in B_G are dominated by vertices in $V_G \cap D$. Thus, at least $6 - |D \cap C_G| - 2 \times |D \cap B_G|$ vertices in B_G are not dominated by vertices in V_G. Thus, $borrow(G, D) \geqslant 6 - |D \cap C_G| - 2 \times |D \cap B_G|$. Note that all vertices of C_G will be dominated, by the definition of D. □

Fact 60. *If (G, D) is an H_n, then $|D \cap B_G| = n - |D \cap C_G|$.*

Lemma 61. *If (G, D) is an H_n, then $|D \cap C_G| \geqslant \lceil \frac{6-n}{2} \rceil$.*

Proof. Every vertex in C_G could dominate three vertices in C_G. Every vertex in B_G could dominate one vertex in C_G. To dominate all six vertices in C_G, we must have

$$3 \times |D \cap C_G| + |D \cap B_G| \geqslant 6.$$

By Fact 60, $|D \cap B_G| = n - |D \cap C_G|$. Thus,

$$3 \times |D \cap C_G| + (n - |D \cap C_G|) \geqslant 6,$$

so

$$|D \cap C_G| \geqslant \frac{6-n}{2}.$$

However, $|D \cap C_G|$ must be an integer, so $|D \cap C_G| \geqslant \lceil \frac{6-n}{2} \rceil$. □

Lemma 62. *$netlend(\mathcal{H}_2)$ = -4.*

Proof. Assume (G, D) is an H_2. By Lemma 61, $|D \cap C_G| \geqslant \lceil \frac{6-2}{2} \rceil = 2$. Since (G, D) is an H_2, by Fact 60, $|D \cap B_G| = 2 - |D \cap C_G| = 0$. By Lemma 58, $lend(G, D) = |D \cap B_G| = 0$. By Lemma 59,

$$borrow(G, D) \geqslant 6 - |D \cap C_G| - 2 \times |D \cap B_G| = 6 - 2 - 0 = 4.$$

Thus,

$$netlend(\mathcal{H}_2) = \max_{(G,D) \in \mathcal{H}_2} \Big(lend(G, D) - borrow(G, D) \Big) \leqslant 0 - 4 = -4.$$

Figure 26 demonstrates a pair (G', D') that is an H_2 where $lend(G', D') - borrow(G', D') = -4$. Thus, $netlend(\mathcal{H}_2) = -4$. □

Lemma 63. *$netlend(\mathcal{H}_3) = -1$.*

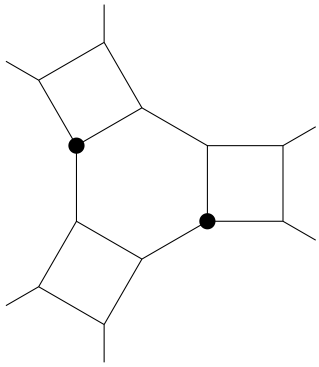


Figure 26: An illustration of the proof of Lemma 62.

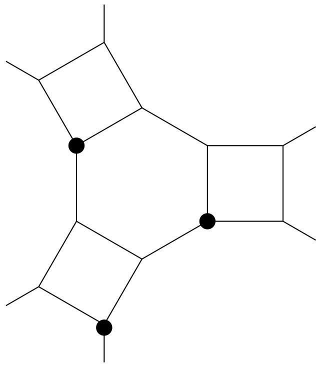


Figure 27: An illustration of the proof of Lemma 63.

Proof. Assume (G, D) is an H_3. By Lemma 61, $|D \cap C_G| \geqslant \lceil \frac{6-3}{2} \rceil = 2$. We consider two cases, depending on the number of vertices of D in C_G.

Case 1: $|D \cap C_G| = 3$. Since (G, D) is an H_3, by Fact 60, $|D \cap B_G| = 3 - |D \cap C_G| = 0$. By Lemma 58, $lend(G, D) = |D \cap B_G| = 0$. By Lemma 59,

$$borrow(G, D) \geqslant 6 - |D \cap C_G| - 2 \times |D \cap B_G| \geqslant 6 - 3 = 3.$$

Thus,

$$lend(G, D) - borrow(G, D) \leqslant 0 - 3 = -3.$$

Case 2: $|D \cap C_G| = 2$. Since (G, D) is an H_3, by Fact 60, $|D \cap B_G| = 3 - |D \cap C_G| = 1$. By Lemma 58, $lend(G, D) = |D \cap B_G| = 1$. By Lemma 59,

$$borrow(G, D) \geqslant 6 - |D \cap C_G| - 2 \times |D \cap B_G| \geqslant 6 - 2 - 2 = 2,$$

so

$$lend(G, D) - borrow(G, D) \leqslant 1 - 2 = -1.$$

Thus,

$$netlend(\mathcal{H}_3) = \max_{(G,D) \in \mathcal{H}_3} \Big(lend(G, D) - borrow(G, D) \Big) \leqslant -1.$$

Figure 27 demonstrates a pair (G', D') that is an H_3 where $lend(G', D') - borrow(G', D') = -1$. Thus, $netlend(\mathcal{H}_3) = -1$. $\square$

Lemma 64. *$netlend(\mathcal{H}_4) = 2$.*

Proof. Assume (G, D) is an H_4. By Lemma 61, $|D \cap C_G| \geqslant \lceil \frac{6-4}{2} \rceil = 1$. We consider four cases, depending on the number of vertices of D in C_G.

Case 1: $|D \cap C_G| = 1$. Since (G, D) is an H_4, by Fact 60, $|D \cap B_G| = 4 - |D \cap C_G| = 3$. Figure 28 represents the reasoning. Since $|D \cap C_G| = 1$ and choices of vertex in $|D \cap C_G|$ are equivalent by symmetry, let $v_1 \in D \cap C_G$. To dominate v_3, v_4, v_5, we must have

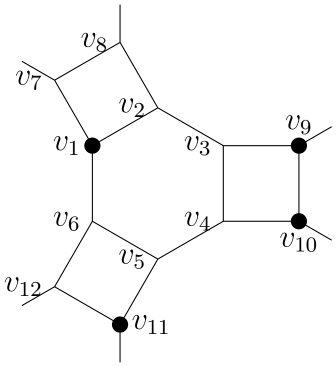


Figure 28: An illustration of the proof of Lemma 64.

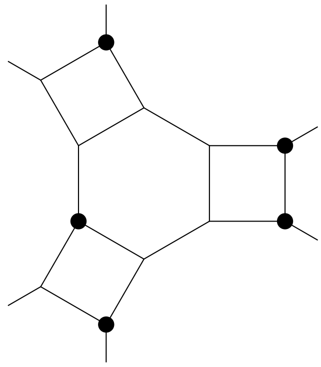


Figure 29: An illustration of the proof of Lemma 65.

$v_9, v_{10}, v_{11} \in D$. Since v_8 is not dominated by a vertex in $V_G \cap D$, $borrow(G, D) = 1$. By Lemma 58, $lend(G, D) = |D \cap B_G| = 3$. Thus,

$$lend(G, D) - borrow(G, D) \leqslant 3 - 1 = 2.$$

Case 2: $|D \cap C_G| = 2$. Since (G, D) is an H_4, by Fact 60, $|D \cap B_G| = 4 - |D \cap C_G| = 2$. By Lemma 58, $lend(G, D) = |D \cap B_G| = 2$. By Lemma 59,

$$borrow(G, D) \geqslant 6 - |D \cap C_G| - 2 \times |D \cap B_G| \geqslant 6 - 2 - 4 = 0.$$

Thus,

$$lend(G, D) - borrow(G, D) \leqslant 2 - 0 = 2.$$

Case 3: $|D \cap C_G| = 3$. Since (G, D) is an H_4, by Fact 60, $|D \cap B_G| = 4 - |D \cap C_G| = 1$. By Lemma 58, $lend(G, D) = |D \cap B_G| = 1$. By Lemma 59,

$$borrow(G, D) \geqslant 6 - |D \cap C_G| - 2 \times |D \cap B_G| \geqslant 6 - 3 - 2 = 1.$$

Thus,

$$lend(G, D) - borrow(G, D) \leqslant 1 - 1 = 0.$$

Case 4: $|D \cap C_G| = 4$. Since (G, D) is an H_4, by Fact 60, $|D \cap B_G| = 4 - |D \cap C_G| = 0$. By Lemma 58, $lend(G, D) = |D \cap B_G| = 0$. By Lemma 59,

$$borrow(G, D) \geqslant 6 - |D \cap C_G| - 2 \times |D \cap B_G| \geqslant 6 - 4 - 0 = 2.$$

Thus,

$$lend(G, D) - borrow(G, D) \leqslant 0 - 2 = -2.$$

In every case,

$$netlend(\mathcal{H}_4) = \max_{(G,D) \in \mathcal{H}_4} \Big(lend(G, D) - borrow(G, D) \Big) \leqslant 2.$$

Figure 28 demonstrates a pair (G', D') that is an H_4 where $lend(G', D') - borrow(G', D') = 2$. Thus, $netlend(\mathcal{H}_4) = 2$. $\square$

Lemma 65. *$netlend(\mathcal{H}_5) = 4$.*

Proof. Assume (G, D) is an H_5. By Lemma 61, $|D \cap C_G| \geqslant \lceil \frac{6-5}{2} \rceil = 1$. Since (G, D) is an H_5, by Fact 60, $|D \cap B_G| = 5 - |D \cap C_G| \leqslant 4$. By Lemma 58, $lend(G, D) = |D \cap B_G| \leqslant 4$. Notice $borrow(G, D) \geqslant 0$. Thus,

$$netlend(\mathcal{H}_5) = \max_{(G,D) \in \mathcal{H}_5} \Big(lend(G, D) - borrow(G, D) \Big) \leqslant 4 - 0 = 4.$$

Figure 29 demonstrates a pair (G', D') that is an H_5 where $lend(G', D') - borrow(G', D') = 4$. Thus, $netlend(\mathcal{H}_5) = 4$. $\square$

Lemma 66. *For $n \geqslant 6$, $netlend(\mathcal{H}_n) = 6$.*

Proof. Assume (G, D) is an H_n, where $n \geqslant 6$. Notice $lend(G, D) \leqslant 6$ and $borrow(G, D) \geqslant 0$. Thus,

$$netlend(\mathcal{H}_n) = \max_{(G,D) \in \mathcal{H}_n} \Big(lend(G, D) - borrow(G, D) \Big) \leqslant 6.$$

Since $n \geqslant 6$, we can choose all vertices in B_G to be in D such that D is a dominating set of G. In this case, $lend(G, D) = 6$ and $borrow(G, D) = 0$. Consequently, $lend(G, D) - borrow(G, D) = 6$. Thus, $netlend(\mathcal{H}_n) = 6$. $\square$

Definition 67. Let D be a dominating set of the $(4, 6, 12)$ lattice. Let G be a subgraph of the $(4, 6, 12)$ lattice whose vertex set can be partitioned into disjoint subsets $S_1, S_2, \ldots, S_m$ such that for every subset S_i, the pair $(G_i, D \cap S_i)$ is an H_n for some n, where G_i is the subgraph induced by vertices in S_i.

For $n \in \{2, 3, 4, \ldots, 12\}$, let $p_n(G)$ denote *the proportion of H_n in the vertex disjoint subgraphs of G.*

Note 68. We can embed the $(4, 6, 12)$ lattice in the plane such that the subgraph induced by vertices in every unit square with integer coordinates is isomorphic to H as shown in Figure 25. In Lemma 69 and Theorem 70, we consider such embedding.

Lemma 69. *Let $R_{l,m}$ denote a rectangular region $R_G(0, l; 0, m)$, where $l, m > 0$. We have*

$$\sum_{k=2}^{12} p_k \times netlend(\mathcal{H}_k) \geqslant -\epsilon_{l,m},$$

where $\epsilon_{l,m} \to 0^+$ as $l, m \to \infty$.

Proof. Let D be any dominating set of the $(4, 6, 12)$ lattice. The vertex set of $R_{l,m}$ can be partitioned into disjoint subsets $S_1, S_2, \ldots, S_{lm}$ such that for every subset S_i, the pair $(G_i, D \cap S_i)$ is an H_n, where G_i is the subgraph induced by vertices in S_i. For any $i \in \{1, \ldots, lm\}$, let $D_i = D \cap S_i$. Let $D^{(l,m)} = \bigcup_{i=1}^{lm} D_i$.

Let $N_k(R_{l,m})$ denote the number of H_k in $(G_1, D_1), (G_2, D_2), \ldots, (G_{lm}, D_{lm})$. Let a, b, c, d denote the number of vertices in the upper, lower, left and right internal boundary of $R_{l,m}$ respectively.

Notice that

$$p_k(R_{l,m}) = \frac{N_k(R_{l,m})}{lm}.$$

Therefore,

$$\sum_{k=2}^{12} p_k(R_{l,m}) \times netlend(\mathcal{H}_k) = \sum_{k=2}^{12} \frac{N_k(R_{l,m})}{lm} netlend(\mathcal{H}_k).$$

Since $netlend(\mathcal{H}_k) = \max_{(G,D)\in\mathcal{H}_k}\Big(lend(G,D) - borrow(G,D)\Big)$, for $i \in \{1,\ldots,m\}$, if (G_i, D_i) is an H_k, then $netlend(\mathcal{H}_k) \geqslant lend(G_i, D_i) - borrow(G_i, D_i)$. Therefore,

$$\sum_{k=2}^{12} p_k(R_{l,m}) \times netlend(\mathcal{H}_k) \geqslant \sum_{i=1}^{lm} \frac{lend(G_i, D_i) - borrow(G_i, D_i)}{lm}.$$

For D to be a dominating set of the $(4, 6, 12)$ lattice, every vertex $v \in B_{G_i}$ not dominated by a vertex in D_i must be dominated by a vertex in $D \setminus D_i$. In addition, a vertex $v \in B_{G_i}$ may be dominated both by a vertex in D_i and by a vertex in $D \setminus D_i$. Therefore,

$$\sum_{i=1}^{lm} \Big(lend(G_i, D_i) - borrow(G_i, D_i)\Big) \geqslant lend(R_{l,m}, D^{(l,m)}) - borrow(R_{l,m}, D^{(l,m)}).$$

Since $lend(R_{l,m}, D^{(l,m)}) \geqslant 0$ and $borrow(R_{l,m}, D^{(l,m)}) \leqslant a+b+c+d$, we have

$$lend(R_{l,m}, D^{(l,m)}) - borrow(R_{l,m}, D^{(l,m)}) \geqslant 0 - (a+b+c+d).$$

Consequently,

$$\sum_{i=1}^{lm} \Big(lend(G_i, D_i) - borrow(G_i, D_i)\Big) \geqslant -(a+b+c+d).$$

Since $l, m > 0$, we divide both sides by lm and obtain

$$\sum_{i=1}^{lm} \frac{lend(G_i, D_i) - borrow(G_i, D_i)}{lm} \geqslant -\frac{a+b+c+d}{lm},$$

so

$$\sum_{k=2}^{12} p_k(R_{l,m}) \times netlend(\mathcal{H}_k) \geqslant -\frac{a+b+c+d}{lm}.$$

Letting $\epsilon_{l,m} = \frac{a+b+c+d}{lm}$, we have

$$\sum_{k=2}^{12} p_k(R_{l,m}) \times netlend(\mathcal{H}_k) \geqslant -\epsilon_{l,m}.$$

Since $a + b = O(l)$ and $c + d = O(m)$, as $m, n \to \infty$, we have

$$\epsilon_{l,m} = \frac{a+b+c+d}{lm} \to 0^+.$$

$\square$

Theorem 70. $\gamma(4,6,12) = \gamma_p(4,6,12) = \frac{5}{18}$.

Proof. We prove that both the domination ratio and the perfect domination ratio of the $(4,6,12)$ lattice are equal to $\frac{5}{18}$.

Consider a rectangular region $R_{l,m}$ as above. We formulate the domination ratio problem in $R_{l,m}$ as a linear program. The set of all feasible solutions is described by a polytope. Lemma 69 provides a valid inequality for the polytope, which is a constraint for the linear program. We describe the constraints, objective function, linear program, dual program in parts 1,2,3, and 4 of the proof respectively. The optimal solution to the linear program provides a lower bound for the domination ratio of $R_{l,m}$, as described in part 3.

We show that $\frac{5}{18}$ is a lower bound for the domination ratio. Combined with Lemma 46, we conclude that $\gamma(4,6,12) = \gamma_p(4,6,12) = \frac{5}{18}$.

1. Constraints

Let $x = [p_2, p_3, p_4, p_5, p_{other}]^T$, where $p_{other} = \sum_{k \geqslant 6} p_k$.

By Lemma 69, we have

$$\sum_{k=2}^{12} p_k \times netlend(\mathcal{H}_k) \geqslant -\epsilon_{l,m},$$

where $\epsilon_{l,m} \to 0^+$ as $l, m \to \infty$.

By Lemma 66, for $n \geqslant 6$, $netlend(\mathcal{H}_n) = 6$, so

$$\left(\sum_{n=2}^{5} p_n \times netlend(\mathcal{H}_n) \right) + 6\, p_{other} \geqslant -\epsilon_{l,m}.$$

For $n \in \{2,3,4,5\}$, $netlend(\mathcal{H}_n)$ was calculated in Lemmas 62, 63, 64, and 65. Thus,

$$[-4,-1,2,4,6] \cdot x \geqslant [-4,-1,2,4,6] \cdot [p_2, p_3, p_4, p_5, p_{other}]^T \geqslant -\epsilon_{l,m}.$$

where $\epsilon_{l,m} \to 0^+$ as $l, m \to \infty$.

Notice that we also have constraints $\sum_k p_k = 1$ and $0 \leqslant p_k \leqslant 1$ for any p_k.

2. Objective function

Let $c = \frac{1}{12}[2,3,4,5,6]^T$. Notice c is multiplied by $\frac{1}{12}$ because H_n has 12 vertices. For any dominating set D of $R_{l,m}$,

$$\gamma(D) = \sum_{k=2}^{12} \frac{k}{12} p_k \geqslant \frac{1}{12}[2,3,4,5,6][p_2, p_3, p_4, p_5, p_{other}]^T = c^T x.$$

3. Linear program (LP)

The linear program below provides a lower bound for the domination ratio of $R_{l,m}$.

$$\min c^T x \text{ subject to}$$

$$[-4,-1,2,4,6]x \geqslant -\epsilon_{l,m}$$

$$\sum_i x_i = 1 \text{ and for any } i,\ 0 \leqslant x_i \leqslant 1.$$

The linear program provides a lower bound for the domination ratio of $R_{l,m}$ because a minimum dominating set D with associated vector x^* satisfies the constraints above and $\gamma(D) \geqslant c^T x^*$.

Writing the LP explicitly in matrix form:

$$\min c^T x = \frac{1}{12}[2,3,4,5,6]x \text{ subject to } x \geqslant \vec{0} \text{ and}$$

$$Ax = \begin{bmatrix} -4 & -1 & 2 & 4 & 6 \\ 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} x \geqslant \begin{bmatrix} -\epsilon_{l.m} \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = b.$$

4. **Dual program (DP)**
The dual program is

$$\max b^T y = [-\epsilon_{l.m}, 1, -1, -1, -1, -1, -1, -1]y \text{ subject to } y \geqslant \vec{0} \text{ and}$$

$$A^T y = \begin{bmatrix} -4 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 4 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 6 & 1 & -1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} y \leqslant \frac{1}{12}\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} = c.$$

5. **Optimal solution**
For the linear program, we obtain $x^* = [1/3+\epsilon_{l,m}/6, 0, 2/3-\epsilon_{l,m}/6, 0, 0]^T$ as an optimal solution with optimal objective function value $\frac{5}{18} - \epsilon_{l,m}/36$.

For the dual program, we obtain $y^* = [5/180, 5/18, 0, 0, 0, 0, 0, 0]^T$ as an optimal solution with optimal objective function value $\frac{5}{18} - \epsilon_{l,m}/36$.

To check that x^* is the optimal solution, one can verify that x^* is primal feasible and y^* is dual feasible. One can also verify that the primal objective function value at x^* and dual objective function value at y^* are both equal to $\frac{5}{18} - \epsilon_{l,m}/36$. By the Strong Duality Theorem, x^* and y^* are optimal solutions of primal and dual respectively.

Suppose to the contrary that $\gamma(4,6,12) \leqslant \frac{5}{18} - \delta$ for some $\delta > 0$. Since $\epsilon_{l,m} \to 0^+$ by Lemma 68, we can choose $\epsilon_{l,m} = 18 \cdot \delta$ for some l and m. The linear program provides the lower bound $\gamma(4,6,12) \geqslant \frac{5}{18} - \frac{1}{2}\delta$, contradicting $\gamma(4,6,12) \leqslant \frac{5}{18} - \delta$.

Therefore, $\frac{5}{18} \leqslant \gamma(4,6,12)$. By Lemma 46, $\gamma(4,6,12) \leqslant \gamma_p(4,6,12) \leqslant \frac{5}{18}$. Combining the two inequalities, we get

$$\frac{5}{18} \leqslant \gamma(4,6,12) \leqslant \gamma_p(4,6,12) \leqslant \frac{5}{18}.$$

Therefore, $\gamma(4,6,12) = \gamma_p(4,6,12) = \frac{5}{18}$. □

6 Open Questions and Future Research

We have determined exact perfect domination ratios for all of the eleven Archimedean lattices. We have determined exact domination ratios for eight of the eleven Archimedean lattices. Solving for the exact domination ratios of the $(3,6,3,6)$, $(3,4,6,4)$, and $(3^2,4,3,4)$ lattices remain open problems. Domination ratios and perfect domination ratios of other classes of infinite lattices such as 2-uniform lattices, or three dimensional lattices such as the simple cubic, face-centered cubic, and body-centered cubic, might also be investigated.

Further study might also consider possible perfect domination proportions and nonisomorphic perfect dominating sets that achieve the same perfect domination proportion. As examples, [30] shows that the number of possible perfect domination proportion values is infinite for the Kagome lattice and is only two for the $(3^2,4,3,4)$ lattice. It would be interesting to consider nonisomorphic perfect dominating sets of and possible perfect domination proportions for all Archimedean lattices, specifically, to determine whether the number of possible perfect domination proportion values is finite or infinite.

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Appendix

A Proof that the Domination Ratio Exists

Theorem 71. *For a periodic graph G, the limit*

$$\gamma(G) = \lim_{m,n\to\infty} \frac{\gamma_{m,n}(G)}{N_{m,n}(G)} = \inf_{m,n} \frac{\gamma_{m,n}(G)}{N_{m,n}(G)}$$

exists.

Proof. Let $k = N_{1,1}(G)$. Fix positive integers r and s. Any integers m and n sufficiently large may be expressed as

$$m = \alpha r + \beta, \text{ where } \alpha = \left\lfloor \frac{m}{r} \right\rfloor \text{ and } 0 \leqslant \beta < r,$$

$$n = \rho s + \sigma, \text{ where } \rho = \left\lfloor \frac{n}{s} \right\rfloor \text{ and } 0 \leqslant \sigma < s.$$

The vertex set of the rectangular region $R_G(0, m; 0, n)$ is the disjoint union of vertex sets of rectangular regions listed below [12].

$$R_{ij} = R_G((i-1)r, ir; (j-1)s, js), \text{ where } 1 \leqslant i \leqslant \alpha, 1 \leqslant j \leqslant \rho$$

$$S_i = R_G((i-1)r, ir; \rho s, \rho s + \sigma), \text{ where } 1 \leqslant i \leqslant \alpha$$
$$T_j = R_G(\alpha r, \alpha r + \beta; (j-1)s, js), \text{ where } 1 \leqslant j \leqslant \rho$$
$$U = R_G(\alpha r, \alpha r + \beta; \rho s, \rho s + \sigma).$$

The rectangular regions are labeled in Figure 30. For simplicity, we do not label all of R_{ij} in Figure 30.

Using subadditivity and the fact that the domination number of a subgraph is no greater than the number of vertices in the subgraph, we deduce that

$$\gamma_{m,n}(G) \leqslant \sum_{i=1}^{\alpha} \sum_{j=1}^{\rho} \gamma(R_{ij}) + (\alpha\sigma r + \rho\beta s + \beta\sigma)k.$$

Notice $\gamma(R_{ij})$ is the same for all R_{ij} from periodicity and the embedding of the graph. Furthermore, $\gamma(R_{ij}) = \gamma_{r,s}(G)$.

$$\gamma_{m,n}(G) \leqslant \alpha\rho\gamma_{r,s}(G) + (\alpha\sigma r + \rho\beta s + \beta\sigma)k.$$

$$\frac{\gamma_{m,n}(G)}{N_{m,n}(G)} \leqslant \frac{\alpha\rho\gamma_{r,s}(G)}{N_{m,n}(G)} + \frac{(\alpha\sigma r + \rho\beta s + \beta\sigma)k}{N_{m,n}(G)}.$$

Since $N_{m,n}(G) = mnk$, we have

$$\frac{\gamma_{m,n}(G)}{N_{m,n}(G)} \leqslant \frac{\alpha\rho\gamma_{r,s}(G)}{mnk} + \frac{\alpha\sigma r + \rho\beta s + \beta\sigma}{mn},$$

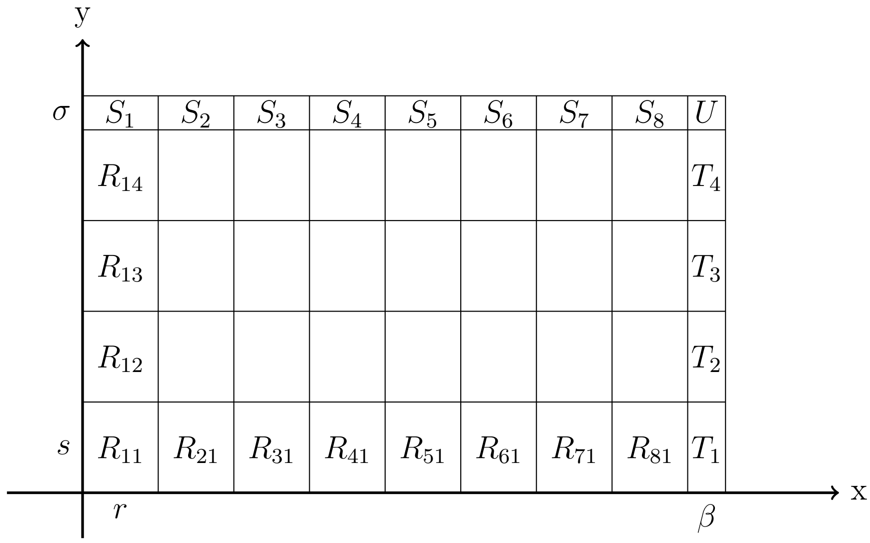

Figure 30: An illustration of the proof that the domination ratio exists.

so

$$\liminf_{m,n\to\infty} \frac{\gamma_{m,n}(G)}{N_{m,n}(G)} \leqslant \liminf_{m,n\to\infty} \frac{\alpha\rho\gamma_{r,s}(G)}{mnk} + \lim_{m,n\to\infty} \frac{\alpha\sigma r + \rho\beta s + \beta\sigma}{mn},$$

since $\alpha \leqslant m$ and $\rho \leqslant n$, while σ, r, β, and s are fixed, so the rightmost term tends to zero as $m, n \to \infty$. Therefore we have

$$\liminf_{m,n\to\infty} \frac{\gamma_{m,n}(G)}{N_{m,n}(G)} \leqslant \liminf_{m,n\to\infty} \frac{\alpha\rho}{mnk}\gamma_{r,s}(G).$$

Since $\alpha r \leqslant m$ and $\rho s \leqslant n$, we have $\alpha r\rho s \leqslant mn$, so

$$\frac{\alpha\rho}{mnk}\gamma_{r,s}(G) \leqslant \frac{1}{rsk}\gamma_{r,s}(G).$$

Since $\frac{1}{rsk}\gamma_{r,s}(G)$ does not depend on m or n, we have

$$\liminf_{m,n\to\infty} \frac{\alpha\rho}{mnk}\gamma_{r,s}(G) \leqslant \frac{1}{rsk}\gamma_{r,s}(G).$$

Since the inequality above holds for any r and s, we have

$$\liminf_{m,n\to\infty} \frac{\alpha\rho}{mnk}\gamma_{r,s}(G) \leqslant \inf_{r,s} \frac{1}{rsk}\gamma_{r,s}(G),$$

so

$$\liminf_{m,n\to\infty} \frac{\gamma_{m,n}(G)}{N_{m,n}(G)} \leqslant \inf_{r,s} \frac{1}{rsk}\gamma_{r,s}(G).$$

Since $\frac{\gamma_{m,n}(G)}{N_{m,n}(G)} = \frac{1}{rsk}\gamma_{r,s}(G)$ when $r = m, s = n$, we have

$$\frac{\gamma_{m,n}(G)}{N_{m,n}(G)} \geqslant \inf_{r,s} \frac{1}{rsk}\gamma_{r,s}(G),$$

and therefore

$$\liminf_{m,n\to\infty} \frac{\gamma_{m,n}(G)}{N_{m,n}(G)} \geqslant \inf_{r,s} \frac{1}{rsk}\gamma_{r,s}(G).$$

Thus, we conclude the limit exists and

$$\lim_{m,n\to\infty} \frac{\gamma_{m,n}(G)}{N_{m,n}(G)} = \inf_{r,s} \frac{1}{rsk}\gamma_{r,s}(G).$$

□

Remark 72. If a bounded function $f(m,n)$ is subadditive, where m,n are length and width of a rectangular region in an infinite periodic graph, then $f(m,n)$ has a limit as $m,n \to \infty$, and the limit equals $\inf_{r,s} \frac{1}{rsk} f(r,s)$.

Proof. Let $f(m,n)$ be a bounded subadditive function, where m,n are length and width of a rectangular region in an infinite periodic graph. The proof of Theorem 70 can be applied to show that $f(m,n)$ has a limit as $m,n \to \infty$. One may replace $\gamma_{m,n}$ in the proof of Theorem 70 by $f(m,n)$ and obtain $\inf_{r,s} \frac{1}{rsk} f(r,s)$ as the limit. □

Remark 73. If a bounded function $f(m,n)$ is superadditive, where m,n are length and width of a rectangular region in an infinite periodic graph, then $f(m,n)$ has a limit as $m,n \to \infty$, and the limit equals $\sup_{r,s} \frac{1}{rsk} f(r,s)$.

Proof. Let $f(m,n)$ be a bounded superadditive function, where m,n are length and width of a rectangular region in an infinite periodic graph. Notice that $-f(m,n)$ is subadditive. By Corollary 72, $-f(m,n)$ has a limit as $m,n \to \infty$, and the limit equals $\inf_{r,s} \frac{1}{rsk}\big\{-f(r,s)\big\}$. Thus, $f(m,n)$ has a limit as $m,n \to \infty$, and the limit equals $\sup_{r,s} \frac{1}{rsk} f(r,s)$. □

A.2 Different Periodic Embeddings Yield the Same Domination Ratio

Theorem 74. *The domination ratio of an infinite periodic graph does not depend on the choice of periodic embedding.*

Proof. Let A and B be two periodic embeddings of an infinite graph G. Let $\gamma(G_A)$ and $\gamma(G_B)$ denote the domination ratio of G yielded by A and B respectively. The two periodic embeddings A and B provide two sets of (x,y) axes that may have different scales and angles between the x-axis and the y-axis. We can embed the infinite periodic graph in the plane such that the x-axis and the y-axis corresponding to periodic embedding A are orthogonal. Let coordinate-A and coordinate-B denote the coordinate system that correspond to the set of (x,y) axes provided by periodic embeddings A and B respectively. Recall that $R_G(m_1,m_2;n_1,n_2)$ denotes the subgraph of G induced by the vertices in the rectangle $[m_1,m_2) \times [n_1,n_2) \subset \mathbb{R}^2$. For simplicity, we denote $R_G(m_1,m_2;n_1,n_2)$ in coordinate-A and in coordinate-B by $R_A(m_1,m_2;n_1,n_2)$ and $R_B(m_1,m_2;n_1,n_2)$ respectively.

A rectangular region $R_B(0, m; 0, n)$ is a parallelogram in coordinate-A. Figure 31 illustrates the reasoning. Fix positive integers r, s. The origin in coordinate-B is in a $r \times s$ rectangle whose vertices have integer coordinates in coordinate-A. Let $R_A(\alpha r, (\alpha + 1)r; \beta s, (\beta + 1)s)$ denote the rectangular region that contains the origin in coordinate-B, where $\alpha, \beta \in \mathbb{Z}$. Similarly, let points $(m, 0), (m, n), (0, n)$ in coordinate-B be in rectangular regions:

$$R_A(\alpha r + \gamma r, \alpha r + \gamma r + r; \beta s + \delta s, \beta s + \delta s + s)$$

$$R_A(\alpha r + \gamma r + \theta r, \alpha r + \gamma r + \theta r + r; \beta s + \delta s + \lambda s, \beta s + \delta s + \lambda s + s)$$

$$R_A(\alpha r + \theta r, \alpha r + \theta r + r; \beta s + \lambda s, \beta s + \lambda s + s)$$

respectively, where $\alpha, \beta, \gamma, \delta, \theta, \lambda \in \mathbb{Z}$.

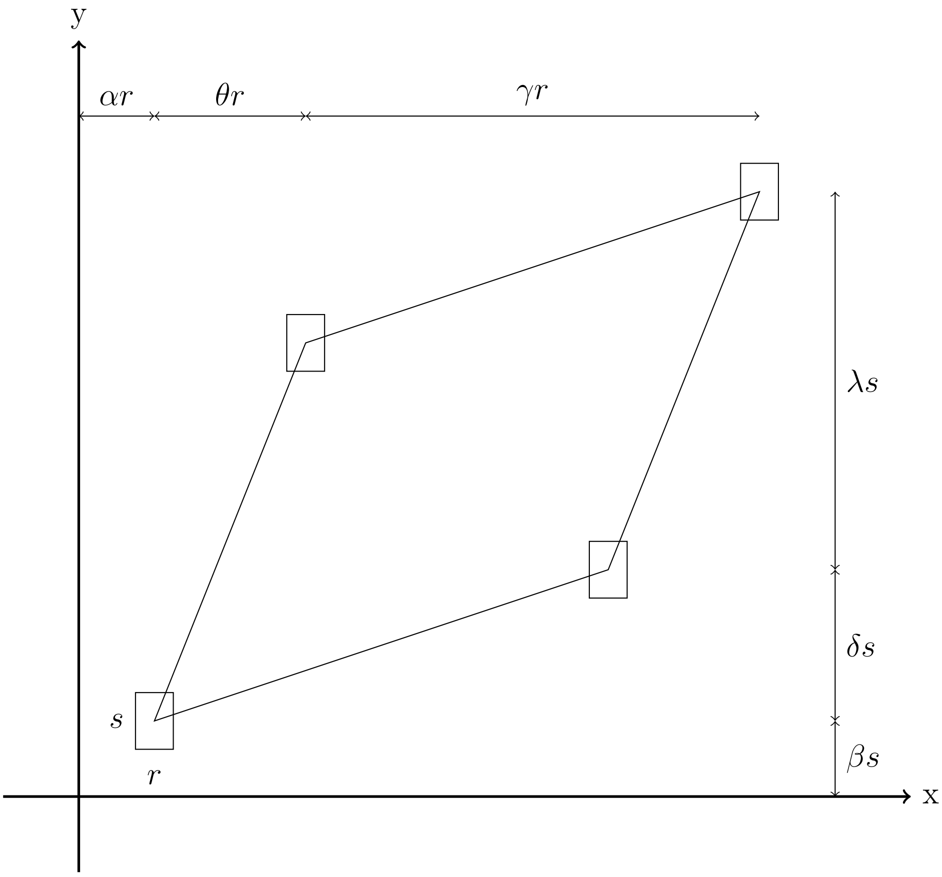

Figure 31: A rectangle $R_B(0, m; 0, n)$ is a parallelogram in coordinate-A.

Notice a that union of rectangles with length r and width s in coordinate-A has $R_B(0, m; 0, n)$ as a subgraph. Let k denote the minimum number of rectangles with length r and width s in coordinate-A whose union has $R_B(0, m; 0, n)$ as a subgraph. Recall that $\gamma_{m,n}(G)$ denotes the domination number of $R_G(0, m; 0, n)$, and $N_{m,n}(G)$ denotes the number of vertices in $R_G(0, m; 0, n)$. For simplicity, we denote $\gamma_{m,n}(G)$ and $N_{m,n}(G)$ in coordinate-A by $\gamma_{m,n}(A)$ and $N_{m,n}(A)$ respectively. Similarly, we denote $\gamma_{m,n}(G)$ and $N_{m,n}(G)$ in coordinate-B by $\gamma_{m,n}(B)$ and $N_{m,n}(B)$ respectively.

As a union of k rectangles with length r and width s in coordinate-A has $R_B(0,m;0,n)$ as a subgraph,

$$0 \leqslant kN_{r,s}(A) - N_{m,n}(B).$$

Notice that every rectangle in the union contains some vertices in $R_B(0,m;0,n)$, otherwise a union of less than k rectangles with length r and width s in coordinate-A has $R_B(0,m;0,n)$ as a subgraph, contradicting that k is the minimum number of $r \times s$ rectangles required. Since $2(\gamma+\theta+\delta+\lambda)$ rectangles with length r and width s can cover all vertices on the internal boundary of $R_B(0,m;0,n)$, at most $2(\gamma+\theta+\delta+\lambda)$ rectangles in the union contain vertices not in $R_B(0,m;0,n)$.

$$kN_{r,s}(A) - N_{m,n}(B) \leqslant 2(\gamma+\theta+\delta+\lambda)N_{r,s}(A),$$

$$0 \leqslant kN_{r,s}(A) - N_{m,n}(B) \leqslant 2(\gamma+\theta+\delta+\lambda)N_{r,s}(A),$$
$$N_{m,n}(B) \leqslant kN_{r,s}(A) \leqslant N_{m,n}(B) + 2(\gamma+\theta+\delta+\lambda)N_{r,s}(A),$$
$$1 \leqslant \frac{kN_{r,s}(A)}{N_{m,n}(B)} \leqslant 1 + \frac{2(\gamma+\theta+\delta+\lambda)N_{r,s}(A)}{N_{m,n}(B)},$$

where $N_{m,n}(B) = \Theta(mn)$ and $\gamma+\theta+\delta+\lambda = \Theta(m+n)$. Since $2N_{r,s}(A)$ is a fixed positive integer, as $m,n \to \infty$, we have

$$\frac{2(\gamma+\theta+\delta+\lambda)N_{r,s}(A)}{N_{m,n}(B)} \to 0.$$

Therefore, as $m,n \to \infty$,

$$\frac{kN_{r,s}(A)}{N_{m,n}(B)} \to 1.$$

Using subadditivity and the fact that domination number of a graph is no smaller than the domination number of its subgraph, we deduce that

$$k\gamma_{r,s}(A) \geqslant \gamma_{m,n}(B).$$

$$\frac{kN_{r,s}(A)}{N_{m,n}(B)} \times \frac{k\gamma_{r,s}(A)}{kN_{r,s}(A)} \geqslant \frac{\gamma_{m,n}(B)}{N_{m,n}(B)}.$$

As $m,n \to \infty$, $\frac{kN_{r,s}(A)}{N_{m,n}(B)} \to 1$. Therefore we have

$$\lim_{m,n\to\infty} \frac{k\gamma_{r,s}(A)}{kN_{r,s}(A)} \geqslant \lim_{m,n\to\infty} \frac{\gamma_{m,n}(B)}{N_{m,n}(B)}.$$

where the existence of the limit is proved in Theorem 70.

$$\lim_{m,n\to\infty} \frac{\gamma_{r,s}(A)}{N_{r,s}(A)} \geqslant \lim_{m,n\to\infty} \frac{\gamma_{m,n}(B)}{N_{m,n}(B)}.$$

Since $\lim_{m,n\to\infty} \frac{\gamma_{m,n}(B)}{N_{m,n}(B)} = \gamma(G_B)$ and $\frac{\gamma_{r,s}(A)}{N_{r,s}(A)}$ is independent of m, n, we have

$$\frac{\gamma_{r,s}(A)}{N_{r,s}(A)} \geqslant \gamma(G_B).$$

$$\inf_{r,s} \frac{\gamma_{r,s}(A)}{N_{r,s}(A)} \geqslant \gamma(G_B).$$

Since $\inf_{r,s} \frac{\gamma_{r,s}(A)}{N_{r,s}(A)} = \gamma(G_A)$, we have

$$\gamma(G_A) \geqslant \gamma(G_B).$$

Similarly, we can embed the infinite periodic graph on a plane such that the x-axis and the y-axis corresponding to the subgraph B are orthogonal. The same reasoning can be applied to show that $\gamma(G_A) \leqslant \gamma(G_B)$. Thus, $\gamma(G_A) = \gamma(G_B)$. □