An irrational Turán density via hypergraph Lagrangian densities

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Abstract

Baber and Talbot asked whether there is an irrational Turán density of a single hypergraph. In this paper, we show that the Lagrangian density of a 4-uniform matching of size 3 is an irrational number. Sidorenko showed that the Lagrangian density of an r-uniform hypergraph F is the same as the Turán density of the extension of F. Therefore, our result gives a positive answer to the question of Baber and Talbot. We also determine the Lagrangian densities of a class of r-uniform hypergraphs on n vertices with $\Theta(n^2)$ edges. As far as we know, for every hypergraph F with known hypergraph Lagrangian density, the number of edges in F is less than the number of its vertices.

Mathematics Subject Classifications: 05D05

1 Introduction

For a positive integer r and a set V, let $\binom{V}{r}$ denote the family of all r-subsets of V. An r-*uniform graph* or r-*graph* G consists of a vertex set $V(G)$ and an edge set $E(G) \subseteq \binom{V(G)}{r}$. We sometimes write the edge set of G as G. Let $e(G)$ $(v(G))$ denote the number of edges (vertices) of G. Given an r-graph F, an r-graph G is called F-*free* if it does not contain a copy of F as a subgraph. The *Turán number* $ex(n, F)$ is the maximum number of edges in an F-free r-graph on n vertices. The *Turán density* of F is defined as $\pi(F) = \lim_{n\to\infty} ex(n, F)/\binom{n}{r}$; such a limit is known to exist. For 2-graphs (simple graphs), Erdős-Stone-Simonovits determined the asymptotic values of Turán numbers of all graphs except bipartite graphs. Very few results are known for hypergraphs and a recent survey on this topic can be found in Keevash's survey paper [16].

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Chung and Graham [5] proposed the conjecture that the Turán density of a finite family of r-graphs is a rational number. Baber and Talbot [1], and Pikhurko [21] disproved this conjecture by showing that there are a family of three 3-graphs and a finite family of r-graphs with irrational Turán densities, respectively. Baber and Talbot [1] asked whether there exists a single hypergraph whose Turán density is an irrational number. Let M_t^r be the r-graph formed by t disjoint edges. The *extension* of F, denoted by H^F is obtained as follows: for each pair of vertices v_i and v_j not contained in any edge of F, we add a set B_{ij} of $r-2$ new vertices and the edge $\{v_i, v_j\} \cup B_{ij}$, where the B_{ij}'s are pairwise disjoint over all such pairs $\{i, j\}$. In this paper, we show that the Turán density of the extension of M_3^4 is an irrational number. This result gives a positive answer to the question of Baber and Talbot. We remark that Yan and Peng [30] independently proved the existence of an irrational Turán density of a single 3-graph.

Theorem 1. *Let F be the extension of M_3^4, then*

$$\pi(F) = \frac{207 - 33\sqrt{33}}{32}.$$

Lagrangian has been a very important tool to study hypergraph Turán problems. Denote $[n] = \{1, 2, \ldots, n\}$. Let G be an r-graph on vertex set $V \subseteq [n]$. Define the *Lagrangian function* of G as $w(G, \boldsymbol{x}) = \sum_{e \in G} \prod_{i \in e} x_i$, where $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\} \in [0, \infty)^n$. Let

$$\Delta = \{\boldsymbol{x} \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = 1, x_i \geqslant 0 \text{ for every } i \in [n]\},$$

the *Lagrangian* of G is defined to be $\lambda(G) = \max_{\boldsymbol{x} \in \Delta} w(G, \boldsymbol{x})$. In fact, $\lambda(G)$ can be regarded as the density of the densest blow-up of G. The value x_i is called the *weight* of the vertex i. We call a weighting $\boldsymbol{x} \in \Delta$ *optimal* if $\lambda(G) = w(G, \boldsymbol{x})$. We first present a classic result for simple graphs given by Motzkin and Straus [18] in 1965, when they gave a new proof of Turán's classical result on Turán densities of complete graphs. Let K_t^r denote the *complete r-graph* on t vertices.

Theorem 2. ([18]) *If G is a simple graph in which a maximum complete subgraph has t vertices, then*

$$\lambda(G) = \lambda(K_t^2) = \frac{1}{2}\left(1 - \frac{1}{t}\right).$$

The Lagrangian for hypergraphs was developed independently by Sidorenko [22], and Frankl and Füredi [9], generalising work of Motzkin and Straus [18], and Zykov [31]. The *Lagrangian density* of F is defined to be

$$\pi_\lambda(F) = r! \sup\{\lambda(G) : G \text{ is } F\text{-free}\}.$$

The Lagrangian density problem has been studied in the recent years, strongly connected to the Turán problems. In fact, the Turán density of F is no larger than the Lagrangian density of F, equality holds when F *covers pairs*, that is, every pair of vertices is contained in some edge of F. The following relation between Lagrangian densities and Turán densities is implied by Theorem 2.6 in [22] (see Proposition 5.6 in [3] and Corollary 1.8 in [23] for the explicit statement).

Proposition 3. ([22, 3, 23]) *Let F be an r-graph. Then (i) and (ii) hold.*
(i) $\pi(F) \leqslant \pi_\lambda(F)$;
(ii) $\pi(H^F) = \pi_\lambda(F)$. In particular, if F covers pairs, then $\pi(F) = \pi_\lambda(F)$.

The Lagrangian density problem has very few results as yet. We list some of them as follows. Let T be a tree or a forest on t vertices that satisfies Erdős and Sós' conjecture. Let F be an r-graph obtained by joining $r-2$ fixed vertices into every edge of T. Sidorenko [23] proved that $\pi_\lambda(F) = r!\lambda(K^r_{t+r-3}) = \binom{t+r-3}{r}\frac{r!}{(t+r-3)^r}$ for t large enough. Let H^r be the r-graph on $r+1$ vertices consisting of two edges sharing $r-1$ vertices. Sidorenko [22] showed that $\pi_\lambda(H^r) = r!\lambda(K^r_r) = \frac{r!}{r^r}$ for $r = 3$ and 4. Let M^r_t be the r-uniform matching with t disjoint edges (t-matching) and L^r_t be the r-uniform linear star with edge set $\{e \cup \{v_0\} : e \in M^{r-1}_t\}$. Hefetz and Keevash [13] determined the Lagrangian density of M^3_2. More generally, the authors [15] determined the Lagrangian density of M^3_t and L^4_t. Since $K^r_{v(F)-1}$ contains no copy of F, $\pi_\lambda(F) \geqslant r!\lambda(K^r_{v(F)-1})$ clearly. If the equality holds, then we call F *λ-perfect*. All hypergraphs in the above results are λ-perfect. While, Frankl and Füredi [10] proved that $\pi_\lambda(H^5) = 5!\frac{6}{11^4}$ and $\pi_\lambda(H^6) = 6!\frac{11}{12^4}$, Bene Watts, Norin and Yepremyan [2] proved that $\pi_\lambda(M^r_2) = (1-1/r)^{r-1}$ for $r \geqslant 4$, and thus, H^5, H^6, M^r_2 $(r \geqslant 4)$ are not λ-perfect. For more relevant Hypergraph Lagrangian (density) results, one may refer to [4, 6, 7, 8, 17, 24, 25, 26, 27, 28, 29] and so on.

Remark 4. If F is λ-perfect, then every spanning subgraph of F is λ-perfect.

It is interesting to study how dense a λ-perfect r-graph F can be. As far as we know, for every known λ-perfect r-graph F (in fact, for all known results), we have $e(F) < v(F)$. It is interesting to study λ-perfect F with $e(F) \geqslant v(F)$. We show that there is a class of λ-perfect r-graphs on n vertices with $\Theta(n^2)$ edges. Let r, s, t be integers, let F_s be the r-graph obtained by adding $r-2$ fixed vertices to every edge of a star with s edges, and let H_t be the r-graph obtained by adding $r-2$ fixed vertices to every edge of a complete graph on t vertices. That is,

$$F_s = \{\{v_1, v_2, \ldots, v_{r-1}\} \cup \{u_i\} : 1 \leqslant i \leqslant s\}$$

and

$$H_t = \{\{u_1, u_2, \ldots, u_{r-2}\} \cup \{w_i, w_j\} : 1 \leqslant i < j \leqslant t\}.$$

Let $F_{s,t} = F_s \cup H_t$ be the disjoint union of F_s and H_t. For positive $r \geqslant 2$, set

$$f_r(x) = (x+r-3)^{-r} \prod_{i \in [r-1]} (x+i-2).$$

Let M_r be the last (i.e., the rightmost) maximum of the function $f_r(x)$ on the interval $[2, \infty)$. We determine the Lagrangian densities of the disjoint union of F_s and H_t for various values of s and t.

Theorem 5. *Let r, s and t be positive integers satisfying $s+t+2r-4 \geqslant \frac{(t+r-2)r(r-1)}{2}$ and $s+t+r-1 \geqslant M_r$. Let G be an r-graph. If G is $F_{s,t}$-free, then $\lambda(G) \leqslant \lambda(K^r_{s+t+2r-4})$. In particular, $\pi_\lambda(F_{s,t}) = r!\lambda(K^r_{s+t+2r-4})$.*

Remark 6. Let r, s, t satisfy the conditions in Theorem 5, $\frac{r(r-1)}{2}t \leqslant s \leqslant c(r)t$ for some constant $c(r)$ depending on r. Then $F_{s,t}$ is λ-perfect with $e(F_{s,t}) = s + \binom{t}{2} = \Theta(n^2)$ if t is large enough for fixed r, where $n = v(F_{s,t})$.

The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we determine the Lagrangian density of M_3^4, which implies Theorem 1. In Section 4, we give a proof of Theorem 5. In the last section, we give some concluding remarks.

2 Preliminaries

By the definition of Lagrangian, it is easy to see the following fact.

Fact 7. *If $G' \subseteq G$, then $\lambda(G') \leqslant \lambda(G)$.*

An r-graph G is *dense* if and only if every proper subgraph G' of G satisfies $\lambda(G') < \lambda(G)$. This is equivalent to all optimal weightings of G are in the interior of Δ, which means no coordinate in an optimal weighting is zero.

Fact 8. ([12]) *If G is dense, then G covers pairs.*

Let G be an r-graph, $U \subseteq V(G)$ and $i, j \in V(G)$. Let $G - U = \{e \in G : e \cap U = \emptyset\}$ and $G[U] = \{e \in G : e \subseteq U\}$. The *link* of i in G, denoted by $L_G(i)$, is the hypergraph with edge set $\{e \in \binom{V(G)}{r-1} : e \cup \{i\} \in G\}$. Denote $L_G(j \setminus i) = L_{G-\{i\}}(j) \setminus L_G(i)$. We say G on vertex set $[n]$ is *left-compressed* if $L_G(j \setminus i) = \emptyset$ for every $1 \leqslant i < j \leqslant n$. The following Lagrangian result is useful for the proof of our result.

Lemma 9. ([15]) *Let G be an M_t^r-free r-graph. Then there exists an M_t^r-free, dense and left-compressed r-graph G' with $|V(G')| \leqslant |V(G)|$ such that $\lambda(G') \geqslant \lambda(G)$.*

Lemma 10. ([12]) *Let G be an r-graph on $[n]$ with at least an edge. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an optimal weighting on G. Then $r\lambda(G) = w(L_G(i), \boldsymbol{x}) < \lambda(L_G(i))$ for every $i \in [n]$ with $x_i > 0$.*

Lemma 11. ([12]) *Let G be an r-graph on vertex set $[n]$. If $L_G(i \setminus j) = L_G(j \setminus i)$, then there is an optimal weighting $\boldsymbol{x} \in \Delta$ such that $x_i = x_j$.*

The following Proposition follows from a result of Sidorenko in [23].

Proposition 12. ([23]) *Let s and t be two positive integers. Let S be an r-graph with edge set $\{\{v_1, v_2, \ldots, v_{r-1}, x\} : x \in [s+t+r-2]\}$ with $s+t+r-1 \geqslant M_r$. If an r-graph G satisfies $\lambda(G) > \lambda(K^r_{s+t+2r-4})$, then G contains a copy of S.*

3 Lagrangians of 4-graphs containing no three disjoint edges

In this section, we determine the maximum Lagrangian of M_3^4-free 4-graphs with the help of MATLAB for some calculations. Let $S_t^r(n)$ be the r-graph on n vertices with edge set $\{e \in \binom{[n]}{r} : e \cap [t] \neq \emptyset\}$. Denote the r-graph S_t^r on the infinite vertex set $V = \{1, 2, 3, \ldots\}$ with edge set $\{e \in \binom{V}{r} : e \cap [t] \neq \emptyset\}$.

Theorem 13. $\pi_\lambda(M_3^4) = 4!\lambda(S_2^4) = \frac{207-33\sqrt{33}}{32}$.

Note that Theorem 13 and Proposition 3 yield Theorem 1. First, we classify the left-compressed M_2^4-free 4-graphs on vertex $[n]$ into four types. For two positive integers m, n with $m < n$, denote $[m, n] = \{m, m+1, \ldots, n\}$. An edge $e = \{a_1, a_2, \ldots, a_r\}$ of an r-graph will be simply denoted by $a_1 a_2 \ldots a_r$. Define

$$\begin{cases} \mathcal{H}_1 = \{1ijk : 2 \leqslant i < j < k \leqslant n\}, \\ \mathcal{H}_2 = \{1ijk : 2 \leqslant i \leqslant 5, i < j < k \leqslant n\} \cup \{2345\}, \\ \mathcal{H}_3 = \{ijkl : 1 \leqslant i \leqslant 2, i < j \leqslant 5, j < k < l \leqslant n\}, \\ \mathcal{H}_4 = \binom{[6]}{4} \cup \{ijkl : 1 \leqslant i < j < k \leqslant 6 < l \leqslant n\}. \end{cases} \tag{1}$$

Lemma 14. *Let $\mathcal{F}$ be a left-compressed M_2^4-free 4-graph on vertex set $[n]$ with $n \geqslant 8$. Then $\mathcal{F}$ is contained in $\mathcal{H}_i$ for some $i \in [4]$.*

Proof. If $\mathcal{F}[[2, n]] = \emptyset$, then $\mathcal{F} \subseteq \mathcal{H}_1$. So assume that $\mathcal{F}[[2, n]] \neq \emptyset$, which yields $2345 \in \mathcal{F}$ since $\mathcal{F}$ is left-compressed. Thus $1678 \notin \mathcal{F}$ since otherwise $\{1678, 2345\}$ forms a copy of M_2^4 in $\mathcal{F}$, a contradiction. Now we divide it into three cases.

Case 1. $1578 \in \mathcal{F}$. Then $2346 \notin \mathcal{F}$ since otherwise $\{1578, 2346\}$ forms a copy of M_2^4 in $\mathcal{F}$, a contradiction. As $\mathcal{F}$ is left-compressed, then $\mathcal{F} \subseteq \mathcal{H}_2$.

Case 2. $1578 \notin \mathcal{F}$ and $3456 \notin \mathcal{F}$. As $\mathcal{F}$ is left-compressed, then $\mathcal{F} \subseteq \mathcal{H}_3$.

Case 3. $1578 \notin \mathcal{F}$ and $3456 \in \mathcal{F}$. Then $1278 \notin \mathcal{F}$ since otherwise $\{1278, 3456\}$ forms a copy of M_2^4 in $\mathcal{F}$, a contradiction. As $\mathcal{F}$ is left-compressed, therefore there is no edge e in $\mathcal{F}$ such that $\{i, j\} \subset e$ with $7 \leqslant i < j \leqslant n$. Hence $\mathcal{F} \subseteq \mathcal{H}_4$. □

Lemma 15. *Let $t \geqslant 3$ and $n \geqslant 4t$ be two integers. Let $\mathcal{F}$ be a left-compressed and M_t^4-free 4-graph on vertex set $[n]$. If $\mathcal{F}[[2, n]]$ contains a copy of M_{t-1}^4, then $\mathcal{F}[[2, 4t-3]]$ contains a copy of M_{t-1}^4.*

Proof. Denote the set of all $(t-1)$-matchings in $\mathcal{F}[[2, n]]$ as Ω. Let

$$J = \{|(\cup_{e \in \mathcal{M}} e) \cap [2, 4t-3]| : \mathcal{M} \in \Omega\}.$$

Note that $\max J \leqslant 4t - 4$. So it is sufficient to prove that $\max J = 4t - 4$. Otherwise let $\mathcal{M} \in \Omega$ satisfying $|(\cup_{e \in \mathcal{M}} e) \cap [2, 4t-3]| = \max J < 4t - 4$. Then there exists $e \in \mathcal{M}$ such that $e \cap [4t-2, n] \neq \emptyset$. Denote $A = e \cap [4t-2, n]$ and $B = [2, 4t-3] \setminus (\cup_{f \in \mathcal{M}} f)$, clearly $|B| \geqslant |A|$. Let $B' \in \binom{B}{|A|}$ and $e' = (e \setminus A) \cup B'$. Then $e' \subset [2, 4t-3]$ and $e' \cap f = \emptyset$ for every $f \in \mathcal{M} \setminus \{e\}$. Since $\mathcal{F}$ is left-compressed, $e' \in \mathcal{F}$. Let $\mathcal{M}' = (\mathcal{M} \setminus e) \cup \{e'\}$. Hence $\mathcal{M}' \in \Omega$. But $|(\cup_{f \in \mathcal{M}'} f) \cap [2, 4t-3]| > |(\cup_{f \in \mathcal{M}} f) \cap [2, 4t-3]|$, which is a contradiction. □

Recall that $S_2^4(n) = \{e \in \binom{[n]}{4} : e \cap [2] \neq \emptyset\}$ and $S_2^4 = \{e \in \binom{\{1,2,3,\ldots\}}{4} : e \cap [2] \neq \emptyset\}$.

Lemma 16. $\lambda(S_2^4) = \frac{(69-11\sqrt{33})}{256} (\approx 0.02269457)$.

Proof. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an optimal weighting of $S_2^4(n)$. By Lemma 11, we can assume that $x_1 = x_2 = a$, $x_3 = x_4 = \cdots = x_n$. Then $\lambda(S_2^4) = \lim_{n\to\infty} \lambda(S_2^4(n)) = f(a) := \frac{1}{2}a^2(1-2a)^2 + \frac{1}{3}a(1-2a)^3 = \frac{1}{6}a(1-2a)^2(2-a)$. Hence $f'(a) = -\frac{1}{3}((2a-1)(4a^2-7a+1))$. Let $f'(a) = 0$, we have $a = \frac{1}{2}$ or $\frac{7\pm\sqrt{33}}{8}$. It is not hard to see that $f(a) \leqslant f((7-\sqrt{33})/8) = \frac{(69-11\sqrt{33})}{256}$ for $0 < a < 1/2$. □

3.1 Proof of Theorem 13

Proof of Theorem 13. Let $\mathcal{F}$ be an M_3^4-free 4-graph on $[n]$. By Lemma 9, we may assume that $\mathcal{F}$ is left-compressed and dense. As $S_2^4(n)$ is M_3^4-free, therefore $\pi_\lambda(M_3^4) \geqslant \lim_{n\to\infty} 4!\lambda(S_2^4(n)) = 4!\lambda(S_2^4)$. We show the upper bound next. If $n \leqslant 11$, then $\mathcal{F} \subseteq K_{11}^4$. Therefore $\lambda(\mathcal{F}) \leqslant \lambda(K_{11}^4) = \binom{11}{4}\frac{1}{11^4} = \frac{30}{11^3} < \lambda(S_2^4)$. Now assume that $n \geqslant 12$. We divide the proof into two cases.

Case 1. $\mathcal{F}[[2, n]]$ is M_2^4-free. Clearly, $\mathcal{F}[[2, n]]$ is left-compressed on $[2, n]$. By Lemma 14, $\mathcal{F}[[2, n]]$ is contained in a copy of $\mathcal{H}_i$ for some $i \in [4]$. Denote

$$\mathcal{G} = \left\{ e \in \binom{[n]}{4} : 1 \in e \right\}.$$

Subcase 1.1. $\mathcal{F}[[2, n]]$ is contained in a copy of $\mathcal{H}_1$. Then $\mathcal{F} \subseteq \mathcal{G} \cup \{2ijk : 3 \leqslant i < j < k \leqslant n\} = S_2^4(n)$. Consequently, $\lambda(\mathcal{F}) \leqslant \lambda(S_2^4(n))$.

Subcase 1.2. $\mathcal{F}[[2, n]]$ is contained in a copy of $\mathcal{H}_2$. Then

$$\mathcal{F} \subseteq \mathcal{H}_{12} := \mathcal{G} \cup \{2ijk : 3 \leqslant i \leqslant 6, i < j < k \leqslant n\} \cup \{3456\}.$$

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an optimal weighting of $\mathcal{H}_{12}$. By Lemma 11, we can assume that $x_1 = a$, $x_2 = b$, $x_3 = x_4 = x_5 = x_6 = c$ and $x_7 + \cdots + x_n = d$ with $a + b + 4c + d = 1$. Then $\lambda(\mathcal{F}) \leqslant \lambda(\mathcal{H}_{12}) \leqslant \max f_{12}$ subject to $a+b+4c+d = 1$, where $f_{12} = a(b(6c^2+4cd+0.5d^2) + 4c^3 + 6c^2d + 2cd^2 + d^3/6) + b(4c^3 + 6c^2d + 2cd^2) + c^4$. Hence $f_{12} < 0.02$ by using MATLAB[1] (see Table 1).

Subcase 1.3. $\mathcal{F}[[2, n]]$ is contained in a copy of $\mathcal{H}_3$. Then

$$\mathcal{F} \subseteq \mathcal{H}_{13} := \left\{ e \in \binom{[n]}{4} : 1 \in e \right\} \cup \{ijkl : 2 \leqslant i \leqslant 3, i < j \leqslant 6, j < k < l \leqslant n\}.$$

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an optimal weighting of $\mathcal{H}_{13}$. By Lemma 11, we can assume that $x_1 = a$, $x_2 = x_3 = b$, $x_4 = x_5 = x_6 = c$ and $x_7+\cdots+x_n = d$ with $a+2b+3c+d = 1$. Then $\lambda(\mathcal{F}) \leqslant \lambda(\mathcal{H}_{13}) \leqslant \max f_{13}$ subject to $a+2b+3c+d = 1$, where $f_{13} = a(b^2(3c+d)+2b(3c^2+3cd+0.5d^2) + c^3 + 3c^2d + 1.5cd^2 + d^3/6) + b^2(3c^2 + 3cd + 0.5d^2) + 2b(c^3 + 3c^2d + 1.5cd^2)$. Hence $f_{13} < 0.021$ by using MATLAB (see Table 1).

Subcase 1.4. $\mathcal{F}[[2, n]]$ is contained in a copy of $\mathcal{H}_4$. Then

$$\mathcal{F} \subseteq \mathcal{H}_{14} := \left\{ e \in \binom{[n]}{4} : 1 \in e \right\} \cup \binom{[2,7]}{4} \cup \{ijkl : 2 \leqslant i < j < k \leqslant 7 < l \leqslant n\}.$$

[1] MATLAB code URL: https://pan.baidu.com/s/1JKAZSpXximI2zkDMUVXJaQ?pwd=6a46

Table 1: Computed results by MATLAB

Function	Maximum value	Value of variables
f_{12}	0.01900025	$(a,b,c,d) \approx$(0.194817, 0.128138, 0.0576268, 0.446537)
f_{13}	0.02091778	$(a,b,c,d) \approx$(0.157263, 0.114602, 0.0710133, 0.400490)
f_{14}	0.01841879	$(a,b,c) \approx$(0.161020, 0.103510, 0.217918)
f_{21}	0.02179073	$(a,b,c) \approx$(0.125057, 0.0597065, 0.266589)
f_{22}	0.02255088	$(a,b,c,d) \approx$(0.115139, 0.104476, 0.0720990, 0.0359111)
f_{23}	0.02232719	$(a,b,c) \approx$(0.113147,0.0844620,0.182471)

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an optimal weighting of $\mathcal{H}_{14}$. By Lemma 11, we can assume that $x_1 = a$, $x_2 = \cdots = x_7 = b$ and $x_8 + \cdots + x_n = c$ with $a+6b+c = 1$. Then $\lambda(\mathcal{F}) \leqslant \lambda(\mathcal{H}_{14}) \leqslant \max f_{14}$ subject to $a+6b+c = 1$, where $f_{14} = a(20b^3 + 15b^2c + 3bc^2 + c^3/6) + 15b^4 + 20b^3c$. Hence $f_{14} < 0.019$ by using MATLAB (see Table 1).

Case 2. $M_2^4 \subseteq \mathcal{F}[[2,n]]$. By Lemma 15, $M_2^4 \subseteq \mathcal{F}[[2,9]]$. So $\{1,10,11,12\} \notin \mathcal{F}$.

Subcase 2.1. $\mathcal{F}[[3,n]]$ is contained in a copy of $\mathcal{H}_1$ or $\mathcal{H}_2$. Then

$$\mathcal{F} \subseteq \mathcal{H}'_{21} := \{ijkl : i \in [3], i < j \leqslant 9, j < k < l \leqslant n\} \cup \binom{[4,9]}{4}.$$

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an optimal weighting of $\mathcal{H}'_{21}$. By Lemma 11, we can assume that $x_1 = x_2 = x_3 = a$, $x_4 = \cdots = x_9 = b$ and $x_{10} + \cdots + x_n = c$ with $3a + 6b + c = 1$. Then $\lambda(\mathcal{F}) \leqslant \lambda(\mathcal{H}'_{21}) \leqslant \max f_{21}$ subject to $3a + 6b + c = 1$, where $f_{21} = a^3(6b + c) + 3a^2(15b^2 + 6bc + 0.5c^2) + 3a(20b^3 + 15b^2c + 3bc^2) + 15b^4$. Hence $f_{21} < 0.022$ by using MATLAB (see Table 1).

Subcase 2.2. $\mathcal{F}[[3,n]]$ is contained in a copy of $\mathcal{H}_3$. Then $\mathcal{F}$ is contained in the following hypergraph, which is denoted by $\mathcal{H}'_{22}$,

$$\{ijkl : i \in [2], i < j \leqslant 9, j < k < l \leqslant n\} \cup \{ijkl : 3 \leqslant i \leqslant 4, i < j \leqslant 7, j < k < l \leqslant n\}.$$

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an optimal weighting of $\mathcal{H}'_{22}$. By Lemma 11, we can assume that $x_1 = x_2 = a$, $x_3 = x_4 = b$, $x_5 = x_6 = x_7 = c$, $x_8 = x_9 = d$ and $x_{10} + \cdots + x_n = h$ with $2a+2b+3c+2d+h = 1$. Then $\lambda(\mathcal{F}) \leqslant \lambda(\mathcal{H}'_{22}) \leqslant \max f_{22}$ subject to $2a+2b+3c+2d+h = 1$, where $f_{22} = a^2(b^2 + 2b(3c + 2d + h) + 3c^2 + 3c(2d + h) + d^2 + 2dh + h^2/2) + 2a(b^2(3c + 2d + h) + 2b(3c^2 + 3c(2d + h) + d^2 + 2dh + h^2/2) + c^3 + 3c^2(2d + h) + 3c(d^2 + 2dh + h^2/2) + d^2h + dh^2) + b^2(3c^2 + 3c(2d + h) + d^2 + 2dh + h^2/2) + 2b(c^3 + 3c^2(2d + h) + 3c(d^2 + 2dh + h^2/2))$. Hence $f_{22} < 0.02256$ by using MATLAB (see Table 1).

Subcase 2.3. $\mathcal{F}[[3,n]]$ is contained in a copy of $\mathcal{H}_4$. Then $\mathcal{F}$ is contained in the following hypergraph, which is denoted by $\mathcal{H}'_{23}$,

$$\{ijkl : i \in [2], i < j \leqslant 9, j < k < l \leqslant n\} \cup$$
$$\{ijkl : 3 \leqslant i < j < k \leqslant 9, k < l \leqslant n\} \cup \tbinom{[3,9]}{4}.$$

Let $\boldsymbol{x}$ be an optimal weighting of $\mathcal{H}'_{23}$. By Lemma 11, we can assume that $x_1 = x_2 = a$, $x_3 = x_4 = \cdots = x_9 = b$ and $x_{10} + \cdots + x_n = c$ with $2a + 7b + c = 1$. Then $\lambda(\mathcal{F}) \leqslant \lambda(\mathcal{H}'_{23}) \leqslant \max f_{23}$ subject to $2a+7b+c = 1$, where $f_{23} = a^2(21b^2+7bc+0.5c^2)+2a(35b^3+21b^2c+3.5bc^2)+35b^4+35b^3c$. Hence $f_{23} < 0.0224$ by using MATLAB (see Table 1).

4 Proof for Theorem 5

Proof for Theorem 5. Suppose to the contrary that $\lambda(G) > \lambda(K^r_{s+t+2r-4})$. Assume that G is dense, otherwise replace G by a dense subgraph with equal Lagrangian. Denote $V(G) = [n]$. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an optimal weighting of G. Hence $x_i > 0$ for every $i \in [n]$ since G is dense. Let $W = \{v_1, v_2, \ldots, v_{r-1}\}$. Let S be an r-graph with edge set $\{W \cup \{i\} : i \in [s+t+r-2]\}$. Since $\lambda(G) > \lambda(K^r_{s+t+2r-4})$ and $v(S) = s+t+2r-3$, G contains a copy of S by Proposition 12. For convenience, assume that $S \subseteq G$.

Let $G_0 = G$ and u_0 be an arbitrary vertex in $V(G_0)$. Let G_1 be a dense subgraph of $L_{G_0}(u_0)$ (the link-graph of vertex u_0 in $V(G_0)$ with $\lambda(G_1) = \lambda(L_{G_0}(u_0))$. Since G_0 is dense with positive Lagrangian, we have $\lambda(G_1) \geqslant r\lambda(G_0)$ by Lemma 10. Similarly, for every $i \in \{0, 1, 2, \ldots, r-3\}$, we can find an arbitrary fixed vertex u_i in $V(G_i)$ such that $\lambda(G_{i+1}) \geqslant (r-i)\lambda(G_i)$, where G_{i+1} is a dense subgraph of $L_{G_i}(u_i)$ with $\lambda(G_{i+1}) = \lambda(L_{G_i}(u_i))$. Note that G_i is an $(r-i)$-graph.

Claim. G_{r-2} is a complete graph on $l \geqslant r+t-1$ vertices. Under the condition that the above Claim holds, for each $i \in \{0, 1, 2, \ldots, r-3\}$, since $V(G_{i+1})$ is a proper subset of $V(G_i)$, we have $v(G_i) \geqslant r+t-1$. By the arbitrariness of u_i, we can always choose $u_i \in V(G_i) \setminus W$.

Now we are going to prove the above Claim. As $\lambda(G_{i+1}) > (r-i)\lambda(G_i)$ for each $i \in \{0, 1, \ldots, r-3\}$, therefore

$$\lambda(G_{r-2}) > 3\lambda(G_{r-3}) > 3 \times 4\lambda(G_{r-4}) > \cdots > 3 \times 4 \times \cdots \times r\lambda(G_0). \tag{2}$$

Note that G_{r-2} is a simple graph. By Theorem 2, G_{r-2} is a complete graph. Denote $l = v(G_{r-2})$. Therefore $\lambda(G_{r-2}) = \lambda(K^2_l) = \frac{1}{2}\left(1 - \frac{1}{l}\right)$. Combined with $\lambda(G_0) > \lambda(K^r_{s+t+2r-4})$, inequality (2) yields

$$\frac{1}{2}\left(1 - \frac{1}{l}\right) > \frac{r!}{2}\binom{k}{r}\frac{1}{k^r} = \frac{(k-1)(k-2)\cdots(k-r+1)}{2k^{r-1}}, \tag{3}$$

where $k = s+t+2r-4$ and $\lambda(K^r_k) = \binom{k}{r}\frac{1}{k^r}$. Inequality (3) implies

$$\frac{1}{l} < 1 - \frac{(k-1)(k-2)\cdots(k-r+1)}{k^{r-1}} = -\sum_{i=2}^{r}(-k)^{1-i}\sum_{1\leqslant x_1<\cdots<x_{i-1}\leqslant r-1}\prod_{j=1}^{i-1}x_j.$$

Denote $u_i = k^{1-i}\sum_{1\leqslant x_1<\cdots<x_{i-1}\leqslant r-1}\prod_{j=1}^{i-1}x_j$, $2 \leqslant i \leqslant r$. Note that

$$u_{i+1} < k^{-i}\sum_{l=1}^{r-1} l \sum_{1\leqslant x_1<\cdots<x_{i-1}\leqslant r-1}\prod_{j=1}^{i-1}x_j = \frac{r(r-1)}{2k}u_i < u_i,$$

where the last inequality follows from $k \geqslant \frac{(t+r-2)r(r-1)}{2}$. Then we have $u_i > u_{i+1}$, which yields that $\sum_{i=2}^{r}(-1)^i u_i < u_2 = \frac{r(r-1)}{2k}$. Thus $\frac{1}{l} < \frac{r(r-1)}{2k}$, which combines with $k \geqslant \frac{(t+r-2)r(r-1)}{2}$ imply that $l > \frac{2k}{r(r-1)} \geqslant t+r-2$. So $l \geqslant t+r-1$.

Let $w_1, w_2, \ldots, w_t \in V(G_{r-2}) \setminus W$ be t different vertices. We know that $\{\{u_0, u_1, \ldots, u_{r-3}, w_i, w_j\} : 1 \leqslant i < j \leqslant t\}$ forms a copy of H_t in G. Since $|[s+t+r-2] \setminus \{w_1, w_2, \ldots, w_t, u_0, u_1, \ldots, u_{r-3}\}| \geqslant s$ and $\{w_1, w_2, \ldots, w_t, u_0, u_1, \ldots, u_{r-3}\} \cap W = \emptyset$, therefore $S - \{w_1, w_2, \ldots, w_t, u_0, u_1, \ldots, u_{r-3}\}$ contains a copy of F_s. Note that H_t and F_s are disjoint. So we get a contradiction that G is $F_{s,t}$-free. We complete the proof.

5 Concluding remarks

Since every spanning subgraph of an λ-perfect hypergraph is also λ-perfect, it is interesting to study those "dense" λ-perfect r-graphs. Let $f(n,r)$ be the maximum number of edges in all λ-perfect r-graphs on n vertices. Since every simple graph is λ-perfect by Theorem 2, we have $f(n,2) = \binom{n}{2}$ for all $n \geqslant 2$. Let K_4^{3-} be the 3-graph on 4 vertices with 3 three edges. Frankl and Füredi [11] showed that $\pi(K_4^{3-}) \geqslant 2/7$. Since K_4^{3-} covers pairs, therefore $\pi_\lambda(K_4^{3-}) = \pi(K_4^{3-}) \geqslant 2/7 > 2/9 = 3!\lambda(K_3^3)$. So K_4^{3-} is not λ-perfect, which implies that K_4^3 is not λ-perfect, either. On the other hand, $\pi_\lambda(\{123, 124\}) = 3!\lambda(\{123\}) = 2/9$ by Sidorenko [22]. Therefore $f(4,3) = 2$. It seems that it is hard to determine $f(n,r)$ even for special pair (n,r) when $n > r \geqslant 3$. Now we propose the following problem.

Problem 17. Let n, r be two integers with $n > r \geqslant 3$.
(i) Whether $\lim_{n\to\infty} f(n,r)/\binom{n}{r} = 0$?
(ii) Can we determine $f(n,r)$ for some special pair (n,r), such as $f(5,3)$?

Let us close this paper with a conjecture. Recall that $S_{t-1}^r(n) = \{e \in \binom{[n]}{r} : e \cap [t-1] \neq \emptyset\}$. Note that $S_{t-1}^r(n)$ and K_{rt-1}^r are two obvious maximum M_t^r-free 4-graphs. We propose the following conjecture.

Conjecture 18. Let $\mathcal{F}$ be an M_t^r-free r-graph. Then

$$\lambda(\mathcal{F}) \leqslant \max\{\lambda(K_{rt-1}^r), \lambda(S_{t-1}^r)\}.$$

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