

# Cohen–Macaulay Growing Graphs

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## Abstract

We introduce a new family of simple graphs, so called, *growing graphs*. We investigate ways to modify a given simple graph  $G$  combinatorially to obtain a *growing graph*. One may obtain infinitely many growing graphs from a single simple graph. We show that a *growing graph* obtained from any given simple graph is Cohen–Macaulay and every Cohen–Macaulay chordal graph is a *growing graph*. We also prove that under certain conditions, a graph is *growing* if and only if its clique complex is grafted and give several equivalent conditions in this case. Our work is inspired by and generalizes a result of Villarreal on the use of whiskers and the work of Faridi on grafting of simplicial complexes.

**Mathematics Subject Classifications:** 13C14, 13F55, 05E40

## 1 Introduction

Throughout,  $G(V(G), E(G))$  denotes a simple graph, which is a graph without any loops or multiple edges. We fix here the vertex set  $V(G) = \{1, 2, \dots, n\}$ , and simply write  $G$  not specifying its vertex and edge sets. We identify each vertex  $i$  with the variable  $x_i$  and consider the ideal

$$I(G) = (x_i x_j : \{i, j\} \in E(G)) \subset S = K[x_1, \dots, x_n].$$

The ideal  $I(G)$  is usually referred to as the *edge ideal* of  $G$ . A graph  $G$  is called Cohen–Macaulay over the field  $K$  if  $S/I(G)$  is Cohen–Macaulay. Using the Stanley–Reisner correspondence, one can associate a simplicial complex  $\Delta_G$  to  $G$ , whose faces are subsets of  $V(G)$ ; namely independent sets of  $G$ . In this case, the Stanley–Reisner ideal of  $\Delta_G$

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coincides with the edge ideal of  $G$ , i.e.  $I_{\Delta_G} = I(G)$ . Classifying all Cohen–Macaulay graphs is notoriously intractable, and thus it is natural to study some special classes of graphs. Of particular interest are the classes of *trees*, *bipartite graphs* and *chordal graphs*. For instance, Villarreal [22] gave classification of all Cohen–Macaulay trees. Herzog and Hibi [16] classified all Cohen–Macaulay bipartite graphs later Herzog, Hibi and Zheng [18] did for all Cohen–Macaulay chordal graphs. The classification and construction of Cohen–Macaulay graphs is one of the central problems and enjoys rich literature, for example [3, 9, 10, 11, 15, 20, 21, 22] and [24].

Our paper complements this work, asking: Given an arbitrary simple graph  $G$ , how can one modify  $G$  to obtain a Cohen–Macaulay graph? A primary inspiration for this paper is Villarreal’s theorem from [22]. He showed that if  $G$  is a graph, and  $H$  is the graph formed by adding a *whisker* to every vertex of  $G$ , then  $H$  is Cohen–Macaulay. Cook and Nagel [9] extended this work and introduced *fully clique–whiskering*. They proved that  $G'$  obtained by *fully clique–whiskering* is Cohen–Macaulay. Francisco and Ha [15] investigated whiskering for obtaining families of sequentially Cohen–Macaulay graphs. Faridi [12] further generalized it for simplicial complexes and introduced *grafting* of simplicial complexes. She proved that the *facet ideal* of a *grafted simplicial complex* is Cohen–Macaulay, see also [12, Section 7].

In this paper, we introduce a construction of *growing graphs* see Section 3. Interestingly, the celebrated constructions mentioned in [9], [19] and [22] are particular cases of our construction. Moreover, corresponding to any simple graph  $G$  and a given clique partition of the vertex set of  $G$ , one may obtain infinitely many Cohen–Macaulay graphs by using our construction. At the end of this section, we give a complete characterization of all Cohen–Macaulay chordal graphs by adding one more equivalent condition to a well known characterization of Cohen–Macaulay chordal graphs due to Herzog, Hibi and Zheng [18].

In the final section, we discuss Cohen–Macaulay modifications, which are discussed in [1],[2], [4], [5] and [6]. Here we link the concept of growing graphs[Definition 3.1] to the grafted simplicial complexes[Definition 4.7], defined by Sara Faridi in [12].

## 2 Preliminaries

In this section, we give a rather quick background of basic terminology and recall some important results useful to apprehend the proceeding sections.

Given a subset  $W$  of  $V(G)$ , we define the *induced subgraph* of  $G$  on  $W$  to be the subgraph  $G_W$  on  $W$  consisting of those edges  $\{i, j\} \in E(G)$  with  $\{i, j\} \subset W$ .

*Definition 2.1.* A *chord* of a cycle  $C$  is an edge  $\{i, j\}$  of  $G$  such that  $i$  and  $j$  are vertices of  $C$  with  $\{i, j\} \notin E(C)$ . A graph is said to be *chordal* graph if each of its cycles of length  $> 3$  has a chord. In particular a tree, which is a graph with no cycle, is a chordal graph.

A subset  $C \in V(G)$  is called a *vertex cover* of  $G$  if  $C \cap E \neq \emptyset$  for all  $E \in E(G)$ . A vertex cover  $C$  is *minimal* if no proper subset of  $C$  is a vertex cover of  $G$  and if all minimal vertex covers of  $G$  have same cardinality, then we say that  $G$  is *unmixed*. All

Cohen–Macaulay graphs are unmixed but not vice versa.

A *simplicial complex*  $\Delta$  on the vertex set  $V = \{v_1, \dots, v_n\}$  is a collection of subsets of  $V$  such that  $\{v_i\} \in \Delta$  for all  $i$  and,  $F \in \Delta$  implies that all subsets of  $F$  are also in  $\Delta$ . The elements of  $\Delta$  are called *faces* and the maximal faces under inclusion are called *facets* of  $\Delta$ . We denote by  $\mathcal{F}(\Delta)$  the set of facets of  $\Delta$ . The *dimension* of a face  $F$  is  $\dim F = |F| - 1$ , where  $|F|$  denotes the cardinality of  $F$ . The *dimension* of  $\Delta$ ,  $\dim(\Delta)$ , is defined as:

$$\dim(\Delta) = \max\{\dim F : F \in \Delta\}.$$

The *pure  $d$ -skeleton* of a simplicial complex  $\Delta$  is the subcomplex of  $\Delta$  generated by all faces of dimension  $d$  and is denoted by  $\Delta^d$ . A notable process of associating a simplicial complex with a given graph is the following.

*Definition 2.2.* The *clique complex*  $\Delta(G)$  of a finite graph  $G$  on  $V(G)$  is a simplicial complex whose faces are the cliques of  $G$ , where a subset  $C$  of  $V(G)$  is called a *clique* of  $G$  if the induced subgraph  $G_C$  is complete.

The following result characterizes all Cohen–Macaulay chordal graphs.

*Theorem 2.3.* [18, Theorem 2.1] Let  $K$  be a field and  $G$  be a chordal graph on the vertex set  $[n]$ . Let  $F_1, \dots, F_m$  be the facets of  $\Delta(G)$  which admit a free vertex (belonging to only one facet). Then the following are equivalent:

1.  $G$  is Cohen–Macaulay;
2.  $G$  is Cohen–Macaulay over  $K$ ;
3.  $G$  is unmixed;
4.  $[n]$  is the disjoint union of  $F_1, \dots, F_m$ .

Given a simplicial complex  $\Delta$  on the vertex set  $[n]$ . For  $F \subseteq \{v_1, \dots, v_n\}$  let

$$x_F = \prod_{v_i \in F} x_i$$

The *non-face ideal* or the *Stanley–Reisner ideal* of  $\Delta$ , denoted by  $I_\Delta$ , is an ideal of  $S = K[x_1, \dots, x_n]$  generated by square-free monomials  $x_F$ , where  $F \notin \Delta$ .

The *facet ideal* of  $\Delta$  is the ideal  $I(\Delta)$  in  $S$  minimally generated by the square-free monomials  $x_F$  with  $F \in \mathcal{F}(\Delta)$ . Thus if  $\Delta = \langle F_1, \dots, F_q \rangle$ , then

$$I(\Delta) = (x_{F_1}, \dots, x_{F_q}).$$

Here, we recall some terminologies from [12]. Let  $\Delta$  be a simplicial complex on  $[n]$  with facets  $F_1, \dots, F_q$ . A *vertex cover* for  $\Delta$  is a subset  $A$  of  $[n]$ , with the property that  $F_i \cap A \neq \emptyset$  for each facet  $F_i$ . A *minimal vertex cover* of  $\Delta$  is a vertex cover such that no proper subset of  $A$  is a vertex cover. A simplicial complex  $\Delta$  is *unmixed* if all of its minimal vertex covers have the same cardinality.

*Theorem 2.4.* [14, Corollary 1.12] Let  $\Delta$  be a simplicial complex on  $[n]$ . If the facet ideal  $I(\Delta)$  is Cohen–Macaulay, then  $\Delta$  is unmixed.

There are simple examples which show that the converse is not true in general. Suppose that  $\Delta$  is a simplicial complex. A facet  $F$  of  $\Delta$  is called a *leaf* if either  $F$  is the only facet of  $\Delta$ , or there exists a facet  $G \in \mathcal{F}(\Delta) \setminus \{F\}$ , such that

$$F \cap F' \subseteq F \cap G$$

for every  $F' \in \mathcal{F}(\Delta) \setminus \{F\}$ . The set of all such  $G$  is denoted by  $\mathcal{U}_\Delta(F)$  and called the *universal set* of  $F$  in  $\Delta$ . If  $G \in \mathcal{U}_\Delta(F)$  and  $F \cap G \neq \emptyset$ , then  $G$  is called a *joint* of  $F$ .

A vertex of a simplicial complex  $\Delta$  is *free* if it belongs to exactly one facet of  $\Delta$ . Note that every leaf has at least one free vertex. Suppose that  $\Delta$  is a connected simplicial complex.  $\Delta$  is called a *simplicial tree* if all simplicial complexes  $\Gamma$  with  $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$  have a leaf. Such  $\Gamma$  is known as a subcomplex of  $\Delta$ . It is also known that a simplicial tree has at least two leaves, see [12, Lemma 4.1]. A nice property of simplicial trees is recorded as follows.

*Theorem 2.5.* [12, Corollary 8.3] Let  $\Delta$  be a simplicial tree over a set of vertices  $x_1, \dots, x_n$ , and let  $K$  be a field. Then the quotient ring  $K[x_1, \dots, x_n]/I(\Delta)$  is Cohen–Macaulay if and only if  $\Delta$  is unmixed.

Before moving forward, we need to recall few more notions from [24]. A *clutter*  $\mathcal{C}$  on a vertex set  $V(\mathcal{C})$  is a collection of subsets of  $V(\mathcal{C})$  such that  $e_1 \not\subseteq e_2$  for any two distinct members  $e_1, e_2 \in \mathcal{C}$ . The members of  $\mathcal{C}$  are called *circuits* of  $\mathcal{C}$ . A subset  $A \subset V(\mathcal{C})$ , is called *independent set* of  $\mathcal{C}$  if it contains no circuit. *The independence complex* of  $\mathcal{C}$  is denoted and defined as

$$\mathcal{I}(\mathcal{C}) = \{A \subset V(\mathcal{C}) : A \text{ is an independent set of } \mathcal{C}\}$$

Clutters can be linked with simplicial complexes via independence complex of  $\mathcal{C}$ . Let  $\mathcal{C}$  be a clutter and  $v \in V(\mathcal{C})$ , then

$$\mathcal{C} \setminus \{v\} = \{e : e \text{ is a circuit of } \mathcal{C} \text{ with } v \notin e\}$$

is called the *deletion* of  $\mathcal{C}$  and

$$\mathcal{C}/\{v\} = \text{Min}\{e \setminus \{v\} : e \text{ is a circuit in } \mathcal{C}\}$$

is called the *contraction* of  $\mathcal{C}$ . Note that both  $\mathcal{C} \setminus \{v\}$  and  $\mathcal{C}/\{v\}$  are clutters on the vertex set  $V(\mathcal{C}) \setminus \{v\}$ . A clutter  $\mathcal{D}$  obtained from  $\mathcal{C}$  after repeated deletion and/or contraction is called a *minor* of  $\mathcal{C}$ . It can easily be seen that for any two distinct vertices  $v$  and  $u$  in  $V(\mathcal{C})$ , we have  $(\mathcal{C} \setminus v) \setminus u = (\mathcal{C} \setminus u) \setminus v$ ,  $(\mathcal{C}/v)/u = (\mathcal{C}/u)/v$  and  $(\mathcal{C} \setminus v)/u = (\mathcal{C}/u) \setminus v$ .

Let  $\mathcal{C}$  be a clutter. A vertex  $v$  of  $\mathcal{C}$  is *simplicial* if for every two distinct circuits  $e_1$  and  $e_2$  of  $\mathcal{C}$  that contain  $v$ , there exists a third circuit  $e_3$  such that  $e_3 \subset (e_1 \cup e_2) \setminus \{v\}$ . A clutter  $\mathcal{C}$  is *chordal* if every minor of  $\mathcal{C}$  has a simplicial vertex. It is obvious that every minor of a chordal clutter is again chordal.

*Examples 2.6.* [24] Following are some examples of chordal clutters:

1. If  $G$  is a graph, then  $G \setminus v$  is (up to singleton circuits) the induced subgraph  $G/N[v]$ , where  $N[v] = N(v) \cup \{v\}$  and  $N(v)$  denotes the set of adjacent vertices of  $v$  in  $G$ . Hence every chordal graph is also a chordal clutter.
2. The clutter with circuits  $\{1, 2, 3\}, \{1, 4, 5\}, \{2, 3, 4, 5\}, \{2, 3, 6\}, \{4, 5, 6\}$  has a simplicial vertex 1, and is chordal; but is not a chordal graph.

Here, we define an important class of simplicial complexes, called shellable simplicial complexes.

*Definition 2.7.* A simplicial complex is said to be *pure* if all facets are of equal dimension. Let  $\Delta$  be a  $d$ -dimensional pure simplicial complex. An ordering  $F_1, F_2, \dots, F_r$  of the facets of  $\Delta$  is a *shelling*, if the complex  $\langle F_1, \dots, F_{i-1} \rangle \cap F_i$  is pure of dimension  $d - 1$  for all  $1 < i \leq r$ . A simplicial complex admitting a shelling is called *shellable simplicial complex*.

To say  $F_1, F_2, \dots, F_r$  is a shelling order of  $\Delta$ ; it is equivalent to saying that for all  $F_i$  and all  $F_j < F_i$ , there exists  $x \in F_i \setminus F_j$  and  $F_k < F_i$  such that  $F_i \setminus F_k = \{x\}$ .

The shellability of simplicial complexes is one of the most important combinatorial property due to the following result.

*Theorem 2.8.* [17, Theorem 8.2.6] Let  $\Delta$  be pure shellable simplicial complex over ground set  $[n]$ . Then the Stanley–Reisner ring  $S/I_\Delta$  is Cohen–Macaulay over any field.

### 3 Main Construction

In this section, we illustrate our construction of *growing graphs*. The focus of this section is on the Cohen–Macaulayness of *growing graphs*, and subsequently recover the Villarreal’s theorem on Cohen–Macaulayness. The following definition is the foundation stone.

*Definition 3.1.* Let  $G$  be a simple graph on the vertex set  $[n]$  and  $A_1, \dots, A_r$  be a partition of  $[n]$  into disjoint subsets such that  $A_i$  is a clique in  $G$  ( $A_i$  can be empty set). For each  $i = 1, \dots, r$ , let  $B_i = \{y_{i,1}, \dots, y_{i,s_i}\}$  be a non-empty set. Define the graph  $G_{A_1, \dots, A_r}^{B_1, \dots, B_r}$  as follows:

$$G_{A_1, \dots, A_r}^{B_1, \dots, B_r} = G \cup \left( \bigcup_{i=1}^r \{F \subset A_i \cup B_i : |F| = 2\} \right). \quad (1)$$

where  $B_i \cap B_j = \emptyset$  for all  $i \neq j$ . We call the graph  $G_{A_1, \dots, A_r}^{B_1, \dots, B_r}$ , the growing graph associated to  $G$  with respect to  $B_1, \dots, B_r$ .

*Remark 3.2.* We shall call  $[n] = A_1 \cup \dots \cup A_r$  a clique partition of  $[n]$ . Corresponding to each partition of the vertex set, we have infinitely many choices for  $B_i$ ’s thus there are infinitely many growing graphs associated to any given graph. If some  $A_i = \emptyset$ , then the resultant growing graph will be disconnected.

Let  $G$  be a graph and  $G' = G_{A_1, \dots, A_r}^{B_1, \dots, B_r}$  be the growing graph associated to  $G$  with respect to  $B_1, \dots, B_r$ . Let  $\Delta$  and  $\Delta'$  be the independence complexes of  $G$  and  $G'$  respectively. Let

$$S = K[\{x_i : i = 1, \dots, n\} \cup \{y_{i,j} : i = 1, \dots, r, j = 1, \dots, s_i\}]$$

be the polynomial ring. Let us define the ordering on the variables:

$$x_1 > \cdots > x_n > y_{1,1} > \cdots > y_{1,s_1} > \cdots > y_{r,1} > \cdots > y_{r,s_r} \quad (2)$$

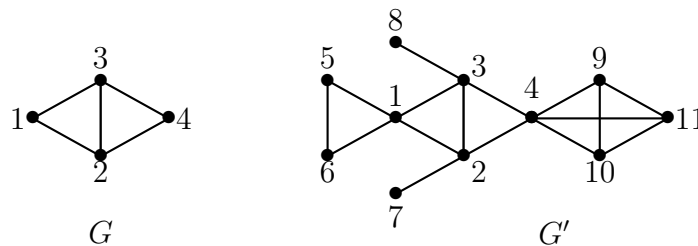
As the facets of  $\Delta'$  are maximal independent sets in  $G'$ , it is easy to see that the induced subgraph of  $G'$  on  $A_i \cup B_i$  is a complete graph. Thus in an independent set, we can select at most one element from  $A_i \cup B_i$  for all  $i$ . In other words, if  $T$  be a facet of  $\Delta'$ , then  $|T \cap (A_i \cup B_i)| = 1$  for all  $i$ . Let  $F = T \cap [n]$ , then  $F$  will be an independent set in  $G$  and hence a face of  $\Delta$ . Let us consider  $B = \cup_{i=1}^r B_i$  and  $F' = T \cap B$ , then  $T = F \cup F'$  and  $F \cap F' = \emptyset$ . It is easy to see that  $F'$  contains exactly one  $y_{j,k_j}$  for a given  $1 \leq k_j \leq s_j$ , where  $A_j \cap F = \emptyset$  holds. This observation is recorded in the following result.

*Proposition 3.3.* The independence complex  $\Delta'$  of the growing graph  $G'$  is pure. Every facet of  $\Delta'$  is of the form  $F \cup F'$ , where  $F$  is a face of  $\Delta$  and

$$F' = \bigcup_{j:A_j \cap F = \emptyset} \{y_{j,k_j} : \text{for exactly one } 1 \leq k_j \leq s_j\}.$$

Here, we explain this fact through the following example.

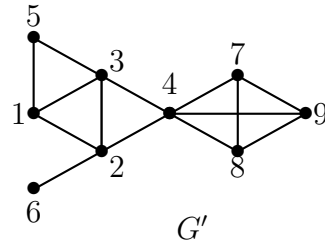
*Example 3.4.* Consider the graph  $G$  with vertex set  $V(G) = \{1, 2, 3, 4\}$  and edge set  $E(G) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$ . Consider the vertex partition as  $V(G) = \{1\} \cup \{2\} \cup \{3\} \cup \{4\}$  and take  $B_1 = \{5, 6\}, B_2 = \{7\}, B_3 = \{8\}, B_4 = \{9, 10, 11\}$  then  $G' = G_{A_1, \dots, A_4}^{B_1, \dots, B_4}$  is displayed as follows.



The following Table-A illustrates all facets of  $\Delta'$  correspondence to respective faces of  $\Delta$ .

Faces of $\Delta$	Corresponding facets of $\Delta'$
$\emptyset$	$\{5, 7, 8, 9\}, \{5, 7, 8, 10\}, \{5, 7, 8, 11\}, \{6, 7, 8, 9\}, \{6, 7, 8, 10\}, \{6, 7, 8, 11\}$
$\{1\}$	$\{1, 7, 8, 9\}, \{1, 7, 8, 10\}, \{1, 7, 8, 11\}$
$\{2\}$	$\{2, 5, 8, 9\}, \{2, 5, 8, 10\}, \{2, 5, 8, 11\}, \{2, 6, 8, 9\}, \{2, 6, 8, 10\}, \{2, 6, 8, 11\}$
$\{3\}$	$\{3, 5, 7, 9\}, \{3, 5, 7, 10\}, \{3, 5, 7, 11\}, \{3, 6, 7, 9\}, \{3, 6, 7, 10\}, \{3, 6, 7, 11\}$
$\{4\}$	$\{4, 5, 7, 8\}, \{4, 6, 7, 8\}$
$\{1, 4\}$	$\{1, 4, 7, 8\}$

Let us consider another partition of vertex set as  $V(G) = \{1, 3\} \cup \{2\} \cup \{4\}$  and  $B_1 = \{5\}, B_2 = \{6\}, B_3 = \{7, 8, 9\}$ , then  $G' = G_{A_1, \dots, A_3}^{B_1, \dots, B_3}$  is shown as follows.

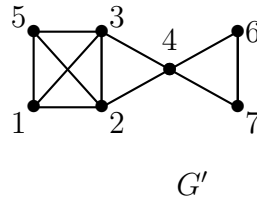


In this case, the illustrative Table-B is given.

Table 2: B

Faces of $\Delta$	Corresponding facets of $\Delta'$
$\emptyset$	$\{5, 6, 7\}, \{5, 6, 8\}, \{5, 6, 9\}$
$\{1\}$	$\{1\} \cup \{6, 7\}, \{1\} \cup \{6, 8\}, \{1\} \cup \{6, 9\}$
$\{2\}$	$\{2\} \cup \{5, 7\}, \{2\} \cup \{5, 8\}, \{2\} \cup \{5, 9\}$
$\{3\}$	$\{3\} \cup \{6, 7\}, \{3\} \cup \{6, 8\}, \{3\} \cup \{6, 9\}$
$\{4\}$	$\{4\} \cup \{5, 6\}$
$\{1, 4\}$	$\{1, 4\} \cup \{6\}$

Yet, another interesting case with the partition of vertex set  $V(G) = \{1, 2, 3\} \cup \{4\}$  with  $B_1 = \{5\}, B_2 = \{6, 7\}$ . In this case,  $G' = G_{A_1, \dots, A_2}^{B_1, \dots, B_2}$  is drawn as follows.



The corresponding Table-C of facets is presented as follows.

Table 3: C

Faces of $\Delta$	Corresponding facets of $\Delta'$
$\emptyset$	$\{5, 6\}, \{5, 7\}$
$\{1\}$	$\{1\} \cup \{6\}, \{1\} \cup \{7\}$
$\{2\}$	$\{2\} \cup \{6\}, \{2\} \cup \{7\}$
$\{3\}$	$\{3\} \cup \{6\}, \{3\} \cup \{7\}$
$\{4\}$	$\{4\} \cup \{5\}$
$\{1, 4\}$	$\{1, 4\} \cup \emptyset$

From this example, it can be noted that there are  $\prod_{j:A_j \cap F = \emptyset} |s_j|$  choices for  $F'$ , thus corresponding to each face  $F$  of  $\Delta$ , there is a block of facets of  $\Delta'$ . Also note that all new graphs  $G'$  are obtained from  $G$ . It is important to mention that none of them can be recovered from  $G$  by using techniques given in [22, 9, 19, 12].

*Theorem 3.5.* Let  $G$  be a graph on the vertex set  $[n]$  and  $A_1, \dots, A_r$  be a partition of  $[n]$  into disjoint subsets such that  $A_i$  is a clique in  $G$ . Let  $B_i = \{y_{i,1}, \dots, y_{i,s_i}\}$  and  $G' = G_{A_1, \dots, A_r}^{B_1, \dots, B_r}$  be a growing graph associated to  $G$  with respect to  $B_1, \dots, B_r$ . Let  $\Delta'$  be the independence complex of  $G'$  and  $S = K[\{x_i : i = 1, \dots, n\} \cup \{y_{i,j} : i = 1, \dots, r, j = 1, \dots, s_i\}]$  be the polynomial ring. Then,  $\Delta'$  is pure shellable of dimension  $r - 1$  and hence the ring  $S/I(G')$  is Cohen–Macaulay of dimension  $r$ .

*Proof.* Proposition 3.3 guarantees that the independence complex  $\Delta'$  of  $G'$  is pure and has dimension  $r - 1$ , thus it is sufficient to show that  $\Delta'$  is shellable. We prove the fact by presenting an order of the facets of  $\Delta'$  that appears as a shelling on  $\Delta'$ . Our claim is based on the following ordering:

Order the faces of  $\Delta$  in terms of increasing dimensions. If two faces have the same dimension, then order them by the ordering of variables defined in equation (2). Then, associated to each face  $F$  of  $\Delta$ , consider the block of facets of  $\Delta'$  associated to  $F$ , ordered as in (2).

From above discussion (see Example 3.4), we know that corresponding to every face of  $\Delta$ , there is a block of facets of  $\Delta'$ . Assume  $S$  and  $T$  are two distinct facets of  $\Delta'$  with  $S < T$ ; here arises two cases.

*Case:1.* When  $S$  and  $T$  belongs to different blocks corresponding to different faces of  $\Delta$ : We can write  $S = F \cup F'$  and  $T = G \cup G'$  where  $F, G$  are different faces of  $\Delta$ , where

$$F' = \bigcup_{j:A_j \cap F = \emptyset} \{y_{j,k_j} : \text{for exactly one } 1 \leq k_j \leq s_j\}$$

and

$$G' = \bigcup_{j:A_j \cap G = \emptyset} \{y_{j,p_j} : \text{for exactly one } 1 \leq p_j \leq s_j\}.$$

As  $F \neq G$ , we must have some  $x_t \in G \setminus F$ . Let  $G_1 = G \setminus \{x_t\}$ , then  $G_1$  will also be a face of  $\Delta$ . As  $G_1 \subset G$  so  $\{j : A_j \cap G = \emptyset\} \subset \{j : A_j \cap G_1 = \emptyset\}$  thus

$$G' \subset G'_1 := \bigcup_{j:A_j \cap G_1 = \emptyset} \{y_{j,p_j} : \text{for exactly one } 1 \leq p_j \leq s_j\}.$$

In fact, if  $x_t \in A_p$  then  $G'_1 = G' \cup \{y_{p,k_p}\}$ , for exactly one  $1 \leq k_p \leq s_p$ . Let us take  $T_1 = G_1 \cup G'_1$ , then  $T_1 < T$  with  $T \setminus T_1 = \{x_t\}$ .

*Case:2.* When  $S$  and  $T$  belongs to same block corresponding to a face  $F$  of  $\Delta$ :

In this case,  $S = F \cup F'$  and  $T = F \cup F''$ . As  $S \neq T$ , we have  $F' \neq F''$  and  $T \setminus S \neq \emptyset$ . Let  $l$  be the least number, such that  $y_{l,k_l} \in F'' \setminus F'$ , thus  $y_{l,k_l} \in F''$  and  $y_{l,k_l} \notin F'$  which further implies the existence of a  $y_{l,k'_l} \in F'$  for some  $1 \leq k'_l \neq k_l \leq s_l$  with  $y_{l,k'_l} < y_{l,k_l}$  as  $S < T$ .

If  $y_{l,k_l}$  is the only vertex in  $F'' \setminus F'$ , take  $T_1 := S < T$  with  $T \setminus T_1 = \{y_{l,k_l}\}$  and we are done, otherwise suppose  $y_{m,k_m} \in F'' \setminus F'$ . We have ordered  $F'$  and  $F''$  as defined in



(2) and as we have assumed  $l$  to be least such number, thus the first  $l - 1$  vertices (w.r.t ordering (2)) in  $F'$  and  $F''$  will be the same. Thus  $F'$  and  $F''$  will be of the form,

$$F' = \{\dots, y_{l,k'_l}, \dots, y_{m,k'_m}, \dots\}$$

$$F'' = \{\dots, y_{l,k_l}, \dots, y_{m,k_m}, \dots\}$$

where  $y_{m,k'_m} \neq y_{m,k_m}$ . Let us consider,

$$F''' = \{\dots, y_{l,k'_l}, \dots, y_{m,k_m}, \dots\}$$

and suppose  $T_1 = F \cup F'''$ , then  $T_1$  will be a facet of  $\Delta'$  by Proposition 3.3 with  $T_1 < T$ . Note that  $y_{m,k_m} \notin F''' \setminus F''$  and  $y_{l,k_l} \in F''' \setminus F''$ , thus  $y_{m,k_m} \notin T \setminus T_1$  and  $y_{l,k_l} \in T \setminus T_1$ . If  $y_{l,k_l}$  is the only element in  $T \setminus T_1$ , we are done, otherwise repeat the same process. This process will eventually terminate in finitely many steps yielding a  $T_i < T$  such that  $T \setminus T_i = \{y_{l,k_l}\}$ , as required. □

Here, we give a descriptive definition of a growing graph.

*Definition 3.6.* A graph  $G$  is said to be *growing* if there exists some graph  $H$  such that  $G$  is a growing graph associated to  $H$ .

The following result shows that the whiskering of a graph, given by Villarreal in [22] is a particular case of our construction.

*Corollary 3.7.* [22] Suppose  $G$  is a graph and let  $G'$  be the graph obtained by adding a whisker at every vertex  $v \in G$ . Then the ideal  $I(G')$  is Cohen–Macaulay.

*Proof.* Suppose  $V(G) = \{v_1, \dots, v_n\}$  and consider the trivial clique partition of  $V(G)$  into singleton sets as,  $V(G) = \{v_1\} \cup \dots \cup \{v_n\}$ . If we take  $B_i = \{y_i\}$  for all  $1 \leq i \leq n$ , then  $G' = G_{A_1, \dots, A_n}^{B_1, \dots, B_n}$ , thus  $\Delta'$  is pure and  $I(G')$  is Cohen–Macaulay by Theorem 3.5. □

R. Woodroof [24] defined the concept of *clique-starring* (or *clique-whiskering*) in the Proposition 22. It means to add a new vertex  $w$  and connecting it with all vertices of some clique  $W$  of  $G$ , resulting graph is denoted by  $G^W$ . D. Cook and U. Nagel [9] generalized the the concept of clique-whiskering, and defined the term *fully clique-whiskering*.

Recall from [9] that a *clique vertex-partition*  $\pi$  of a graph  $G = (V, E)$  is a partition  $\pi = \{W_1, \dots, W_t\}$  of  $V$  such that each subgraph induced on  $W_i$  is a nonempty clique. Fully clique-whiskering by a clique vertex-partition  $\pi = \{W_1, \dots, W_t\}$  is  $G$  clique-whiskered at every clique of  $\pi$ ; it produces the graph

$$G^\pi = (V \cup \{w_1, \dots, w_t\}, E \cup \{vw_i | v \in W_i\})$$

The fully clique-whiskering is also a particular case of our construction.

*Corollary 3.8.* [9, Corollary 3.5] Let  $\pi = \{W_1, \dots, W_t\}$  be a clique vertex-partition of  $G$ . Then  $I(G^\pi)$  is Cohen–Macaulay.

*Proof.* If  $\pi = \{W_1, \dots, W_t\}$  be a clique vertex-partition of  $G$ . Let us take  $B_i = \{w_i\}$ , singleton sets for all  $i$ . Then  $G^\pi = H_{W_1, \dots, W_t}^{B_1, \dots, B_t}$  is a particular growing graph associated to  $G$  and hence Cohen–Macaulay by Theorem 3.5.  $\square$

Adding a whisker to each vertex is the same as saying that *attaching* the complete graph  $K_2$  to each vertex. Hibi *et al.*[19] further generalized the construction given by Cook and Nagel[9].

*Corollary 3.9.* [19, Theorem 1.1] Let  $G$  be a finite simple graph on a vertex set  $V = \{x_1, \dots, x_n\}$ . Let  $k_1, \dots, k_n \geq 2$  be integers. Then the graph  $G'$  obtained from  $G$  by attaching the complete graph  $K_{k_i}$  to  $x_i$  for  $i = 1, \dots, n$  is Cohen–Macaulay.

*Proof.* Consider the trivial partition  $V(G) = \{x_1\} \cup \dots \cup \{x_n\}$  and take  $B_i = \{y_1, \dots, y_{k_i-1}\}$ , then the resulting growing graph is exactly  $G'$ .  $\square$

Herzog *et al.* [18] characterized Cohen–Macaulay chordal graphs as recorded earlier in Theorem 2.3. It can easily be observed [Corollary 3.10] that every Cohen–Macaulay chordal graph is in fact a growing graph associated to some graph. Thus, it characterizes all Cohen–Macaulay chordal graphs. By a *free vertex* in a simplicial complex, we mean a vertex that belongs to exactly one facet of the simplicial complex.

*Corollary 3.10.* Let  $K$  be a field and  $G$  be a chordal graph on the vertex set  $[n]$ . Let  $F_1, \dots, F_m$  be the facets of  $\Delta(G)$  that admit a free vertex. Then the following are equivalent:

1.  $G$  is Cohen–Macaulay;
2.  $G$  is Cohen–Macaulay over  $K$ ;
3.  $G$  is unmixed;
4.  $[n]$  is the disjoint union of  $F_1, \dots, F_m$ .
5.  $G$  is a growing graph.

*Proof.* (1) to (4) are equivalent by Theorem 2.3.

(4)  $\Rightarrow$  (5) : As,

$$[n] = F_1 \cup \dots \cup F_m$$

where  $F_i$ 's are cliques of  $\Delta(G)$  containing free vertices. Let  $A_i$  and  $B_i$  denote the non-free and free vertices of  $F_i$  respectively. Let  $A = \cup_{i=1}^r A_i$  and  $H := G_A$  be the induced graph. Then  $G = H_{A_1, \dots, A_m}^{B_1, \dots, B_m}$ .

(5)  $\Rightarrow$  (1) : Follows from Theorem 3.5.  $\square$

In particular, one may directly recover the following result.

*Corollary 3.11.* If  $G$  is a tree then the following are equivalent:

1.  $G$  is Cohen–Macaulay.

2.  $G$  is unmixed.
3.  $G$  is a growing graph.

*Remark 3.12.* In case of small graphs, it is easier to check whether a given graph is growing or not. It is pertinent to mention that there exist Cohen–Macaulay graphs that are not growing graphs. For example, 5–cycle is not growing graph but Cohen–Macaulay.

## 4 A Classification

Let  $I$  be a squarefree Cohen–Macaulay monomial ideal in  $S = K[x_1, \dots, x_n]$  for a field  $K$ . We denote the unique minimal system of monomial generators of  $I$  by  $G(I)$ . Let  $G(I) = \{u_1, \dots, u_m\}$ . We call a monomial ideal  $J$  a *modification of  $I$* , if  $G(J) = \{v_1, \dots, v_m\}$  and  $\text{supp}(v_i) = \text{supp}(u_i)$  for all  $i$ , where  $\text{supp}(u) = \{i \mid x_i \text{ divides } u\}$ . A monomial ideal  $J$  is called a *trivial modification of  $I$* , if there exist nonnegative integers  $a_1, \dots, a_n$  such that  $J$  is obtained from  $I$  by the substitution  $x_i \mapsto x_i^{a_i}$  for  $i = 1, \dots, n$ . If  $J$  is a trivial modification of  $I$ , then  $J$  is Cohen–Macaulay, since  $J = \varphi(I)S$  where  $\varphi: S \rightarrow S$  is the flat  $K$ -algebra homomorphism with  $\varphi(x_i) = x_i^{a_i}$  for all  $i$ .

Here arises a natural question: Under what conditions, squarefree Cohen–Macaulay monomial ideals does there exist at least one nontrivial Cohen–Macaulay modification, or do exist infinitely many nontrivial Cohen–Macaulay modifications? The answer to this question has been given for several classes of ideal in the following papers [1], [2], [4], [5], [6]. In this section, we discuss the conditions for which the facet ideal of the chordal simplicial complex [Definition 4.3] admits non trivial modifications.

Along the way, we modify the simplicial complex  $\Delta$  in such a way that the new simplicial complex  $\Delta_F$  shares some properties with original  $\Delta$ . Following definition is necessary for our construction.

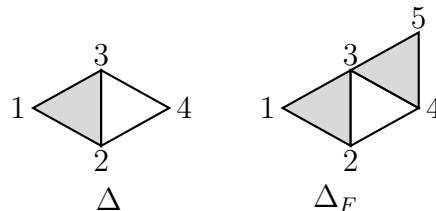
*Definition 4.1.* Let  $\Delta = \langle F_1, \dots, F_{i-1}, F_i, F_{i+1}, \dots, F_t \rangle$  be a simplicial complex on the vertex set  $[n]$ . We define the *modification of  $\Delta$  by  $F_i$* , denoted by  $\Delta_{F_i}$  with the vertex set  $[n+1]$  as

$$\Delta_{F_i} = \langle F_1, \dots, F_{i-1}, F'_i, F_{i+1}, \dots, F_t \rangle,$$

where,  $F'_i = F_i \cup \{n+1\}$ .

Following example explains our construction.

*Example 4.2.* If  $\Delta = \langle \{1, 2, 3\}, \{2, 4\}, \{3, 4\} \rangle$  is the simplicial complex on the vertex set  $[4]$ . Then  $\Delta_F = \langle \{1, 2, 3\}, \{2, 4\}, \{3, 4, 5\} \rangle$  is modification of  $\Delta$  by the facet  $F = \{3, 4\}$ .



*Definition 4.3.* A simplicial complex  $\Delta$  on  $[n]$  is said to be *chordal* if  $\mathcal{F}(\Delta)$  is a chordal clutter.

*Lemma 4.4.* Let  $G$  be a chordal graph, then  $\Delta(G)$  is a chordal simplicial complex.

*Proof.* A vertex  $v$  in  $G$  is simplicial if its neighbors form a complete subgraph, which means that  $v$  is simplicial in  $G$  if and only if  $v$  is free in the clique complex  $\Delta(G)$ . By [24] [Theorem 4.1], we know that a graph  $G$  is chordal if and only if every induced subgraph of  $G$  has a simplicial vertex which in turn is equivalent to say that every minor of  $\Delta(G)$  has a simplicial vertex. Hence the clique complex of a chordal graph is chordal simplicial complex, as required.  $\square$

With this terminology, we can state the following results,

*Theorem 4.5.* [2, Corollary 3.7] If  $\Delta = \langle F_1, \dots, F_t \rangle, t \geq 2$  is a chordal simplicial complex on  $[n]$ . Then, the following are equivalent:

1.  $I(\Delta)$  is Cohen–Macaulay.
2.  $\Delta$  is unmixed.
3.  $I(\Delta)$  has a nontrivial Cohen–Macaulay modification.
4.  $I(\Delta)$  has infinitely many nontrivial Cohen–Macaulay modifications.
5.  $I(\Delta_F)$  is Cohen–Macaulay, where  $F$  is a facet of  $\Delta$  containing a simplicial vertex.

Similarly, we can define the modification of  $G$  by assuming it as a 1-dimensional simplicial complex.

*Corollary 4.6.* [2, Corollary 3.8] Let  $K$  be a field and  $G$  be a chordal graph on the vertex set  $[n]$ . Let  $F_1, \dots, F_m$  be the facets of  $\Delta(G)$  which admit a free vertex. Then the following are equivalent:

1.  $G$  is Cohen–Macaulay;
2.  $G$  is Cohen–Macaulay over  $K$ ;
3.  $G$  is unmixed;
4.  $[n]$  is the disjoint union of  $F_1, \dots, F_m$ .
5.  $I(G)$  has a nontrivial Cohen–Macaulay modification.
6.  $I(G)$  has infinitely many nontrivial Cohen–Macaulay modifications.
7.  $I(G_e)$  is Cohen–Macaulay, where  $e$  is an edge of  $G$  containing a simplicial vertex.

Let us now discuss an important notion defined by S. Faridi in [12],

*Definition 4.7.* A simplicial complex  $\Delta$  is said to be a grafting of the simplicial complex  $\Delta' = \langle G_1, \dots, G_s \rangle$  with the simplices  $F_1, \dots, F_r$  (or we say that  $\Delta$  is grafted) if

$$\Delta = \langle F_1, \dots, F_r \rangle \cup \langle G_1, \dots, G_s \rangle$$

with the following properties:

1.  $V(\Delta') \subseteq V(F_1) \cup \dots \cup V(F_r)$ ;
2.  $F_1, \dots, F_r$  are all the leaves of  $\Delta$ ;
3.  $\{G_1, \dots, G_s\} \cap \{F_1, \dots, F_r\} = \emptyset$ ;
4. For  $i \neq j$ ,  $F_i \cap F_j = \emptyset$ ;
5. If  $G_i$  is a joint of  $\Delta$ , then  $\Delta \setminus \langle G_i \rangle$  is also grafted.

A well known property of grafted simplicial complexes is the following,

*Theorem 4.8.* [12, Theorem 8.2] Let  $\Delta$  be a grafted simplicial complex over a set of vertices labeled  $x_1, \dots, x_n$ , and let  $K$  be a field. Then  $K[x_1, \dots, x_n]/I(\Delta)$  is Cohen–Macaulay.

One can say even more for the case of simplicial trees:

*Theorem 4.9.* [8, Theorem 6.7] If  $\Delta$  is a simplicial tree, the following are equivalent:

1.  $\Delta$  is unmixed;
2.  $\Delta$  is grafted;
3.  $I(\Delta)$  is Cohen–Macaulay.

The following definition is compulsory for our construction,

*Definition 4.10.* A facet  $F$  of a simplicial complex  $\Delta$  is called a *reducible leaf* if

$$\{F \cap G_i : G_i \text{ is a facet of } \Delta\}$$

*is totally ordered with respect to set inclusion.*

The following lemma is crucial for our construction,

*Lemma 4.11.* [8, Lemma 6.8] A simplicial complex  $\Delta$  is grafted if and only if (1) for each vertex  $v$ , there exists a unique leaf  $F$  such that  $v \in F$ , and (2) all leaves of  $\Delta$  are reducible.

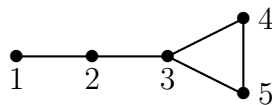
The main aim of this section is to interlink the two concepts; grafting of simplicial complexes and growing graphs. The first result towards this direction is the following immediate observation,

*Lemma 4.12.* If  $\Delta$  is grafted, then  $\Delta^1$  is a growing graph.

*Proof.* Let  $\Delta = \langle F_1, \dots, F_r \rangle \cup \langle G_1, \dots, G_r \rangle$  is a grafted simplicial complex, where  $F_i$  are leaves for all  $i$ . Let  $A_i$  and  $B_i$  denotes the set of non-free and free vertices of the facet  $F_i$  for all  $i$ , respectively. Consider  $\Delta' = \langle G_1, \dots, G_r \rangle$  and take  $H = \Delta'^1$ . Obviously  $V(H) = A_1 \cup \dots \cup A_r$  is a clique partition and  $\Delta^1 = H_{A_1, \dots, A_r}^{B_1, \dots, B_r}$  is the growing graph associated to  $H$ , as required.  $\square$

It is also worth mentioning that in graphs (i.e. 1-dimensional simplicial complexes), there exist growing graphs which are not grafted, so in some sense it is generalization to grafting. For example if  $G$  is a graph with  $V(G) = \{1, 2, 3, 4, 5\}$  and

$$E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$$



Then  $G$  is growing graph which is not grafted.

The converse of the Lemma 4.12 is not generally true. The following example illustrate this,

*Example 4.13.* Consider the graph  $G$  with  $V(G) = \{1, 2, 3, 4, 5, 6\}$  and

$$E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}, \{1, 5\}, \{2, 5\}, \{3, 6\}, \{4, 6\}\}$$



Then  $G$  is a growing graph but its clique complex  $\Delta(G)$  is not grafted because it has no leaves.

Now the question is, under what conditions this converse holds? For the answer we need some preparation,

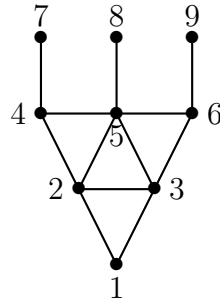
*Definition 4.14.* Let  $G$  be a graph and  $v \in G$ . Let  $N_G(v)$  denotes the set of all neighbors (vertices adjacent to  $v$ ) of  $v$  in  $G$  and  $N_G[v] = N_G(v) \cup \{v\}$ . A vertex  $v$  is called *reducible* if the collection,

$$\{N_G[v] \cap C_i : C_i \text{ is a maximal clique of } G\}$$

is totally ordered with respect to set inclusion.

*Example 4.15.* Consider the graph  $G$  with  $V(G) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$  and

$$E(G) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 6\}, \{4, 5\}, \{5, 6\}, \{4, 7\}, \{5, 8\}, \{6, 9\}\}$$



Note that the vertices 7, 8 and 9 are reducible but the vertex 1 is not because the collection  $\{\{2\}, \{3\}, \{2, 3\}\}$  is not totally ordered.

The following Lemma is important to answer the above question,

*Lemma 4.16.* Let  $G = H_{A_1, \dots, A_r}^{B_1, \dots, B_r}$  be a growing graph then  $F_i = A_i \cup B_i$  are all the leaves of  $\Delta(G)$ .

*Proof.* By definition, we have a clique partition of the vertex set of  $H$  as  $V(H) = A_1 \cup \dots \cup A_r$ . Define  $F_i = A_i \cup B_i$ , then  $F_i \cap F_j = \emptyset$  for all  $i \neq j$  because  $A_i \cap A_j = \emptyset$  and  $B_i \cap B_j = \emptyset$  for all  $i \neq j$ . By definition of growing graph,  $F_i$  will be maximal cliques in  $G$  and hence facets of  $\Delta(G) = \Delta$ . It is also clear that  $\bigcup_i F_i = V(G)$ , hence  $F_i$ 's make a partition of  $G$  which guarantees that each vertex  $v \in \Delta$  belongs to a unique  $F_i$ .

As  $F_i = A_i \cup B_i$  and  $A_i$  is a maximal clique or part of a maximal clique in  $H$ , say  $G_i$ . Then  $F_i \cap G_i = A_i$  while  $F_i \cap G_j \subseteq A_i$  for all  $j \neq i$ , which shows that  $F_i$  is a leaf of  $\Delta$ . If  $T$  is a facet of  $\Delta$  with  $T \neq F_i$  for any  $i$ , then  $V(T) \subset V(H) = A_1 \cup \dots \cup A_r$ . For any  $v \in V(T)$  we have  $v \in A_k$  for some  $1 \leq k \leq r$  which further implies  $v \in F_k$  and hence  $v$  is not free. We can conclude that  $T$  cannot be a leaf because it has no free vertices.  $\square$

Here we have the answer to our above posed question,

*Proposition 4.17.* If  $G$  is a growing graph and all simplicial vertices in  $G$  are reducible, then  $\Delta(G)$  is grafted.

*Proof.* Let  $G = H_{A_1, \dots, A_r}^{B_1, \dots, B_r}$  be a growing graph. By Lemma 4.16 and Lemma 4.11, it is enough to show that all  $F_i$ 's are reducible leaves. Let  $v$  be a simplicial vertex in  $G$  then its neighbors form a complete subgraph, which means that  $v$  is simplicial in  $G$  if and only if  $v$  is free in the clique complex  $\Delta(G)$  and  $v \in F_i$  for some  $i$ . Consider

$$\{N_G[v] \cap C_i = F_i \cap C_i \text{ for any maximal clique } C_i \in G\}.$$

As  $v$  is reducible, this collection is totally ordered, which further implies that  $F_i$  is reducible leaf of  $\Delta$ . Hence  $\Delta$  is grafted, as required.  $\square$

We are now in a position to prove the main result of this section,

*Theorem 4.18.* Let  $K$  be any field and  $G$  be a chordal graph on the vertex set  $[n]$  and all simplicial vertices in  $G$  are reducible. Let  $F_1, \dots, F_m$  be the facets of  $\Delta(G)$  which admit a free vertex. Then the following are equivalent:

1.  $G$  is Cohen–Macaulay;
2.  $G$  is Cohen–Macaulay over  $K$ ;
3.  $G$  is unmixed;
4.  $[n]$  is the disjoint union of  $F_1, \dots, F_m$ ;
5.  $I(G)$  has a nontrivial Cohen–Macaulay modification;
6.  $I(G)$  has infinitely many nontrivial Cohen–Macaulay modifications;
7.  $I(G_e)$  is Cohen–Macaulay, where  $e$  is an edge of  $G$  containing a simplicial vertex;
8.  $G$  is a growing graph;
9.  $\Delta(G)$  is grafted;

*Proof.* Corollary 3.10 and Corollary 4.6 shows that the statements (1) through (8) are equivalent. (8) implies (9) is followed by the Proposition 4.17, while (9) implies (8) follows from Lemma 4.12.

□

If the clique complex of a chordal graph becomes a simplicial tree, we can add several more equivalent conditions in the statement of Theorem 4.18,

*Corollary 4.19.* Let  $K$  be a field and  $G$  be a chordal graph on the vertex set  $[n]$  such that all simplicial vertices are reducible and  $\Delta(G)$  is a simplicial tree. Let  $F_1, \dots, F_m$  be the facets of  $\Delta(G)$  which admit a free vertex. Then the following are equivalent:

1.  $G$  is Cohen–Macaulay;
2.  $G$  is Cohen–Macaulay over  $K$ ;
3.  $G$  is unmixed;
4.  $[n]$  is the disjoint union of  $F_1, \dots, F_m$ ;
5.  $I(G)$  has a nontrivial Cohen–Macaulay modification;
6.  $I(G)$  has infinitely many nontrivial Cohen–Macaulay modifications;
7.  $I(G_e)$  is Cohen–Macaulay, where  $e$  is an edge of  $G$  containing a simplicial vertex;
8.  $G$  is a growing graph;
9.  $\Delta(G)$  is grafted;



10.  $I(\Delta(G))$  is Cohen–Macaulay;
11.  $I(\Delta(G))$  is Cohen–Macaulay over  $K$ ;
12.  $\Delta(G)$  is unmixed;
13.  $I(\Delta(G))$  has a nontrivial Cohen–Macaulay modification;
14.  $I(\Delta(G))$  has infinitely many nontrivial Cohen–Macaulay modifications;
15.  $I(\Delta(G_F))$  is Cohen–Macaulay, where  $F$  is an edge of  $\Delta(G)$  containing a simplicial vertex.

*Proof.* Conditions (1) through (9) are equivalent by Theorem 4.18. By Lemma 4.4, the simplicial complex  $\Delta$  is chordal, hence (10) through (15) are equivalent by Theorem 4.5. As  $\Delta(G)$  is simplicial tree, by Theorem 4.9, (9) through (12) are also equivalent, which completes the proof.  $\square$

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