Vizing's and Shannon's Theorems for Defective Edge Colouring

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Abstract

We call a multigraph (k, d)-edge colourable if its edge set can be partitioned into k subgraphs of maximum degree at most d and denote as $\chi'_d(G)$ the minimum k such that G is (k, d)-edge colourable. We prove that for every odd integer d, every multigraph G with maximum degree Δ is $(\lceil \frac{3\Delta-1}{3d-1} \rceil, d)$-edge colourable and this bound is attained for all values of Δ and d. An easy consequence of Vizing's Theorem is that, for every (simple) graph G, $\chi'_d(G) \in \{\lceil \frac{\Delta}{d} \rceil, \lceil \frac{\Delta+1}{d} \rceil\}$. We characterize the values of d and Δ for which it is NP-complete to compute $\chi'_d(G)$. These results generalize classic results on the chromatic index of a graph by Shannon, Holyer, Leven and Galil and extend a result of Amini, Esperet and van den Heuvel.

Mathematics Subject Classifications: 05C15, 05C70

1 Introduction

Graphs in this paper are finite, undirected, and without loops, but may have multiple edges. A graph is *simple* if it has no parallel edges. Let G be a graph. We denote by $\Delta(G)$ the maximum degree of G. An *edge colouring of G with defect d* is a colouring of its edges so that each vertex is incident with at most d edges of the same colour. We say that G is *k-edge colourable with defect d*, or simply *(k, d)-edge colourable*, if G admits an edge colouring with defect d using (at most) k colours. In other words, the edge set can be partitioned into at most k subgraphs of maximum degree at most d. The *d-defective chromatic index* of G is the minimum k such that G is (k, d)-edge colourable and is denoted by $\chi'_d(G)$. So $\chi'_1(G)$ is the usual chromatic index.

This notion is called *frugal edge colouring* in [2] and *improper edge colouring* in [6]. We follow the vocabulary of the analogous concept of defective vertex colouring, a now well established notion. See [16] for a nice dynamic survey on defective vertex colouring.

Our first result is the following.

Theorem 1. *Let $d, \Delta \geqslant 1$ and let G a graph with maximum degree Δ. If d is even, then $\chi'_d(G) = \lceil \frac{\Delta}{d} \rceil$, and if d is odd, then $\chi'_d(G) \leqslant \lceil \frac{3\Delta - 1}{3d-1} \rceil$, and this bound is tight for all Δ and d.*

The case $d = 1$ corresponds to the classic result of Shannon [14] on chromatic index stating that for every graph G, $\chi'_1(G) \leqslant \lfloor \frac{3\Delta(G)}{2} \rfloor$ (observe that $\lceil \frac{3\Delta-1}{2} \rceil = \lfloor \frac{3\Delta}{2} \rfloor$ for all Δ). When d is even, the result is almost trivial in our context (see Theorem 9), and was already known in the more general context of list colouring [6, 2]. When d is odd, a proof that $\chi'_d(G) \leqslant \lceil \frac{3\Delta}{3d-1} \rceil$ in the context of list colouring is announced in [2], but seems to contain a flaw and actually holds only in the case where Δ is divisible by $3k - 1$. See Section 5 for more on the list colouring context.

Vizing's celebrated theorem on edge colouring [15] states that for every simple graph G, $\chi'_1(G) \in \{\Delta(G), \Delta(G) + 1\}$, and Holyer [8], and Leven and Galil [11] proved that deciding if $\chi'_1(G) = \Delta(G)$ is NP-complete even restricted to d-regular simple graphs as soon as $d \geqslant 3$. We generalize both results by proving that for every simple graph G, $\chi'_d(G) \in \{\lceil \frac{\Delta}{d} \rceil, \lceil \frac{\Delta+1}{d} \rceil\}$ (which is easily implied by Vizing's Theorem) and we characterize the values of Δ and d for which the problem is NP-complete. More precisely, we prove that, for given Δ and d, the problem of determining $\chi'_d(G)$ for a Δ-regular simple graph is NP-complete if and only if d is odd and $\Delta = kd$ for some integer $k \geqslant 3$. See Theorems 17 and 19.

We give some definitions and preliminary results in Section 2. We prove the generalization of Shannon's Theorem in Section 3 and the proof of the generalization of Vizing's Theorem in Section 4. Finally, in Section 5, we conjecture a generalisation of of Theorem 1 for list colouring and a generalisation of the Goldberg-Seymour Conjecture.

2 Definitions and preliminaries

Let G be a graph. The *order* (resp. *size*) of G is its number of vertices (resp. edges). It is *regular* if there is an integer k such that every vertex of G has degree k. In this case we can also say it is *k-regular*. We say that G is *k-edge-connected* if it remains connected whenever (strictly) fewer than k edges are removed. If $u, v \in V(G)$, we denote as $G + uv$ the graph $(V(G), E(G) \cup \{uv\})$ (recall that in this paper graphs can have multiple edges, so if there is already an edge between u and v, another one is added). Similarly, $G - uv = (V(G), E(G) \setminus \{uv\})$.

The following gives a trivial lower bound on the d-defective chromatic index that turns out to be tight whenever d is even (see Theorem 9).

Lemma 2. *For every graph G, $\chi'_d(G) \geqslant \lceil \frac{\Delta(G)}{d} \rceil$.*

Proof. At least $\lceil \frac{\Delta(G)}{d} \rceil$ colours are needed to colour the edges incident to a vertex of degree $\Delta(G)$. □

Lemma 3. *Let k, d, Δ be integers. If every $(\Delta+1)$-regular graph is (k, d)-edge colourable, then every Δ-regular graph is also (k, d)-edge colourable.*

Proof. Let G be a Δ-regular graph. Take two disjoint copies G' and G'' of G and add an edge between each vertex $v \in V(G')$ and its copy in G''. The obtained graph H is $(\Delta + 1)$-regular and contains G as a subgraph, so $\chi'_d(G) \leqslant \chi'_d(H) \leqslant k$. □

Factors in graphs

A *k-factor* of G is a k-regular spanning subgraph of G. We sometimes consider a k-factor F as its edge set $E(F)$. We recall this theorem from Petersen [12], one of the very first fundamental results in graph theory:

Theorem 4. *[12] Let Δ be an even integer. A Δ-regular graph admits a k-factor for every even integer $k \leqslant \Delta$.*

An *Euler tour* of a graph G is a closed walk in G that traverses every edge of G exactly once. It is a well-known fact that a graph admits an Euler tour if and only if it is connected and all its vertices have even degree. The next two lemmas use this fact to prove the existence of factors. This idea was already used by Petersen to prove his theorem.

Lemma 5. *Let G be a connected $2k$-regular graph with an even number of edges. Then the edges of G can be partitioned into two k-factors.*

Proof. We number the edges $e_1, e_2, \ldots, e_{2t}$ of G along an Euler tour C and we let $A = \{e_1, e_3, \ldots, e_{2t-1}\}$ and $B = \{e_2, e_4, \ldots, e_{2t}\}$. Since consecutive edges of C are numbered with different parities and its first and last edges have distinct parities, A and B are both k-regular. □

Lemma 6. *Let G be a connected $2k$-regular graph with an odd number of edges, and let $e \in E(G)$. There exist two graphs $G_A = (V, A)$ and $G_B = (V, B)$ such that $E(G) = A \cup B \cup \{e\}$, $\Delta(G_A) \leqslant k$ and $\Delta(G_B) \leqslant k$.*

Proof. The proof is the same as for the previous Lemma, except that we do not assign the last edge of the Euler tour, and we choose e to be this last edge. □

The next theorem roughly says that, in a Δ-regular graph, one can find a k-factor as soon as k is even and is relatively small compared to Δ. It was first proved in [9]. See also Theorem 3.10 (v) in [1]. The version stated here is a simplified version of the original theorem.

Theorem 7. *[9] Let Δ be an odd integer and G a 2-edge connected Δ-regular graph. Let $e \in E(G)$. Let k be an even integer with $k \leqslant \frac{2\Delta}{3}$. Then G has a k-factor containing e.*

Shannon graphs

Given an integer k, the *Shannon graph* $Sh(k)$ is the graph made of three vertices connected by $\lfloor \frac{k}{2} \rfloor$, $\lfloor \frac{k}{2} \rfloor$ and $\lceil \frac{k}{2} \rceil$ edges respectively. See Figure 1. Observe that

- $\Delta(Sh(k)) = k$,
- when k is even, $Sh(k)$ is k-regular and has $\frac{3k}{2}$ edges and,
- when k is odd, $Sh(k)$ has two vertices of degree k and one vertex of degree $k-1$ and has $\frac{3k-1}{2}$ edges.

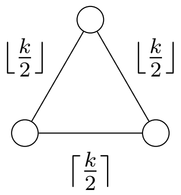

Figure 1: The Shannon graph $Sh(k)$.

Lemma 8. *Let $k, d \geqslant 1$ with d odd. Then $\chi'_d(Sh(k)) = \lceil \frac{3k-1}{3d-1} \rceil$.*

Proof. Consider an ordering $(e_i)_{1 \leqslant i \leqslant |E(Sh(k))|}$ of the edges of $Sh(k)$ such that for any $1 \leqslant i \leqslant |E(Sh(k))| - 2$, e_i, e_{i+1} and e_{i+2} form a triangle. Such an ordering can be obtained by setting e_1 to be any edge with both extremities of degree k and then setting, for $i = 2, \ldots, |E(Sh(k))| - 1$, e_{i+1} to be any unnumbered edge coming right after e_i in clockwise order. The following statement is easily proven using induction: *For every odd integer ℓ such that $1 \leqslant \ell \leqslant \frac{2|E(Sh(k)|-1}{3}$, every contiguous subsequence of $(e_i)_{1 \leqslant i \leqslant |E(Sh(k))|}$ of length $\frac{3\ell-1}{2}$ induces a graph of maximum degree ℓ.*

Thus, colouring the first $\frac{3d-1}{2}$ edges of $(e_i)_{1 \leqslant i \leqslant |E(Sh(k))|}$ in one colour, the following $\frac{3d-1}{2}$ in a second colour and so on, yields a colouring with at most $\lceil \frac{|E(Sh(k))|}{\frac{3d-1}{2}} \rceil$ colours such that each colour class induces a subgraph with maximum degree at most d, and each colour class except at most one has $\frac{3d-1}{2}$ edges. Since every subgraph of $Sh(k)$ with maximum degree d (recall that d is odd) has at most $\frac{3d-1}{2}$ edges, this colouring is an optimal d-defective edge colouring and thus:

$$\chi'_d(Sh(k)) = \left\lceil \frac{|E(Sh(k))|}{\frac{3d-1}{2}} \right\rceil = \begin{cases} \lceil \frac{3k}{3d-1} \rceil = \lceil \frac{3k-1}{3d-1} \rceil & \text{if } k \text{ is even,} \\ \lceil \frac{3k-1}{3d-1} \rceil & \text{if } k \text{ is odd.} \end{cases}$$

□

3 Generalization of Shannon's Theorem

The goal of this section is to prove Theorem 1. In view of Lemma 2, for even d, it suffices to prove the upper bound $\chi'_d(G) \leqslant \lceil \frac{\Delta}{d} \rceil$. Moreover, for both even and odd d, it is enough to prove the result for Δ-regular graphs. Indeed, if G is not Δ-regular, then we can build a Δ-regular graph G' containing G as a subgraph as follows: take two copies of G, and for each vertex v of G, add $\Delta - d(v)$ edges between the two copies of v. Then $\chi'_d(G) \leqslant \chi'_d(G')$. So it suffices to prove the following two results. The case where d is even was already known, but we give the proof anyway for completeness.

Theorem 9. *[6, 2] Let $d, \Delta \geqslant 1$ with d even. For every Δ-regular graph G, $\chi'_d(G) = \lceil \frac{\Delta}{d} \rceil$.*

Proof. If Δ is even, then G has a $\min\{d, \Delta\}$-factor by Theorem 4, and it follows inductively that $\chi'_d(G) \leqslant \lceil \frac{\Delta(G)}{d} \rceil$. If Δ is odd, then $\Delta + 1$ is even and (by the previous sentence) every $(\Delta + 1)$-regular graph is (k, d)-edge colourable, where $k = \lceil \frac{\Delta+1}{d} \rceil = \lceil \frac{\Delta}{d} \rceil$; hence $\chi'_d(G) \leqslant \lceil \frac{\Delta}{d} \rceil$ by Lemma 3. Equality holds in both cases by Lemma 2. □

Theorem 10. *Let $d, \Delta \geqslant 1$ with d odd. For every Δ-regular graph G, $\chi'_d(G) \leqslant \lceil \frac{3\Delta-1}{3d-1} \rceil$.*

Proof. If $d = 1$, then the result follows from the classic result of Shannon, and so we may assume that $d \geqslant 3$. By Lemma 3, it is enough to prove it for values of Δ such that $\lceil \frac{3\Delta-1}{3d-1} \rceil < \lceil \frac{3(\Delta+1)-1}{3d-1} \rceil$, that is, for $\Delta \in \{(i+1)d - \lceil \frac{i}{3} \rceil \mid i \geqslant 0\} = \{d, 2d-1, 3d-1, 4d-1, 5d-1, 6d-2, \dots\}$. We call such integers *special*. In particular, we have $\Delta \geqslant d \geqslant 3$.

Let G be a counterexample that minimizes Δ and has minimum order. That is, Δ is special, G is Δ-regular, $\chi'_d(G) = \lceil \frac{3\Delta-1}{3d-1} \rceil + 1$, every Δ-regular graph with fewer vertices than G is $(\lceil \frac{3\Delta-1}{3d-1} \rceil, d)$-edge colourable, and for every special integer $\Delta' < \Delta$, every Δ'-regular graph is $(\lceil \frac{3\Delta'-1}{3d-1} \rceil, d)$-edge colourable.

Claim 11. *If G has a cut edge e, then at least one connected component of $G - e$ is isomorphic to $Sh(\Delta)$.*

Proof of claim. Set $e = ab$ and let A and B be the two connected components of $G - e$ containing a and b respectively. Assume for contradiction that neither A nor B is isomorphic to $Sh(\Delta)$. Vertices of A have degree Δ in A except for a, which has degree $\Delta - 1$; hence, Δ is odd. If $|V(A)| = 1$, then a has degree 1, a contradiction with the fact that $\Delta \geqslant 3$. If $|V(A)| = 3$, then A is isomorphic to $Sh(\Delta)$, a contradiction. We can thus assume $|V(A)| \geqslant 5$.

Let G_A be the graph obtained from G by replacing A by $Sh(\Delta)$ as in Figure 2. G_A is Δ-regular (because Δ is odd) and has strictly fewer vertices than G. Hence, by minimality of G, G_A admits an edge colouring c_A with defect d using at most $\lceil \frac{3\Delta-1}{3d-1} \rceil$ colours. We define symmetrically G_B and c_B. We may assume, by properly permuting colours in G_B, that $c_B(e) = c_A(e)$. We can now obtain an edge colouring of G with defect d using at most $\lceil\frac{3\Delta-1}{3d-1}\rceil$ colours by assigning colour $c_A(e)$ to e, colour $c_A(e')$ to any edge e' in B, and colour $c_B(e')$ to any edge e' in A, a contradiction. $\square$

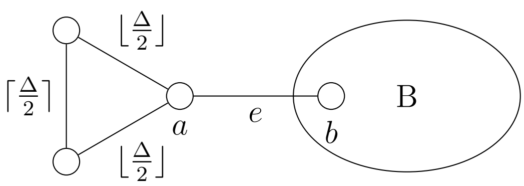

Figure 2: The graph G_A.

Observe that, if a Δ-regular graph has a cut edge, then Δ must be odd. Moreover, if Δ is odd, then for every $(\frac{3\Delta-1}{3d-1}, d)$ edge colouring of $Sh(\Delta)$, there is a colour c such that the (unique) vertex of $Sh(\Delta)$ with degree $\Delta - 1$ is incident with at most $d-1$ edges coloured with c. This simple observation is used in the proof of the following claim.

Claim 12. *G has at most one cut edge.*

Proof of claim. Suppose for contradiction that G has two cut edges uv and $u'v'$. By Claim 11, we may assume that G is made of two disjoint copies of $Sh(\Delta)$ plus a graph A as in Figure 3. Note that $u = u'$ is possible.

Assume first that $u \neq u'$. Then $A + uu'$ is Δ-regular and has strictly fewer vertices than G. So, by minimality of G, $A + uu'$ admits a $(\lceil\frac{3\Delta-1}{3d-1}\rceil, d)$-edge colouring c_A. We can extend this colouring to G by giving colour $c_A(uu')$ to uv and $u'v'$ and then extending this colouring to the two copies of $Sh(\Delta)$ without any new colour (this is possible by the observation stated right before the claim). This leads to a $(\lceil\frac{3\Delta-1}{3d-1}\rceil, d)$-edge colouring of G, a contradiction.

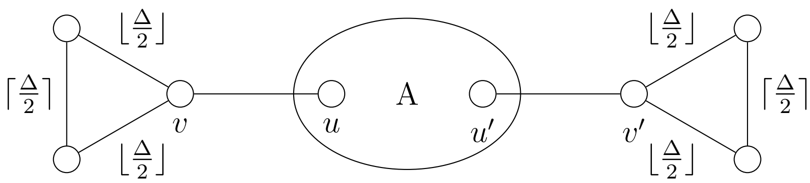

Figure 3: The graph G when $u \neq u'$.

Assume now that $u = u'$. We consider the graph G' obtained by replacing the two copies of $Sh(\Delta)$ by four new vertices $w,\ x,\ y,\ z$ as in Figure 4. It is easy to check that G' is Δ-regular and since G' has two vertices less then G, it is $(\lceil\frac{3\Delta-1}{3d-1}\rceil, d)$-edge colourable. This gives us a $(\lceil\frac{3\Delta-1}{3d-1}\rceil, d)$-edge colouring of A that can easily be extended to the two copies of $Sh(\Delta)$ without any new colour (this is again possible by the observation stated right before the claim), leading to a $(\lceil\frac{3\Delta-1}{3d-1}\rceil, d)$-edge colouring of G, a contradiction. $\square$

Claim 13. *G has a k-factor for every even integer $k \leqslant \frac{2\Delta}{3}$.*

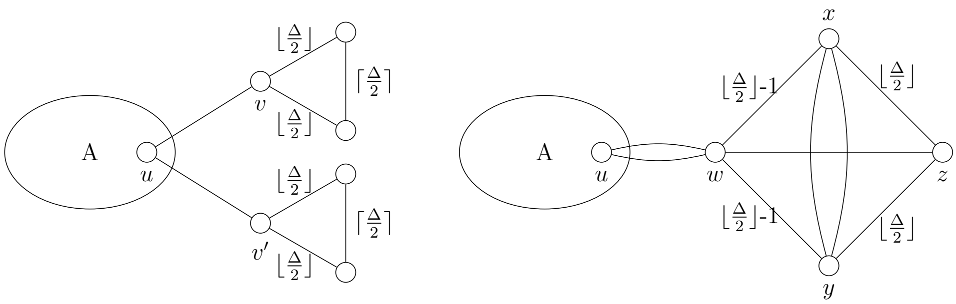

Figure 4: On the left: the graph G when $u = u'$, on the right: the graph G'.

Proof of claim. Let $k \leqslant \frac{2\Delta}{3}$ be an even integer. If Δ is even, the result holds by Theorem 4. So we may assume that Δ is odd. If G is 2-edge connected, then we are done by Theorem 7. So assume G has a cut edge uv. Let A, B be the two connected components of $G \setminus uv$ with $u \in V(A)$ and $v \in V(B)$. By Claim 12, G has no other cut edges and thus A and B are both 2-edge-connected. By Claim 11, either A or B is isomorphic to $Sh(\Delta)$. Without loss of generality, we suppose that it is B. Let w and x be the two other vertices of B. Let y be a neighbour of u in A. Consider $G' = G + uv + yw - uy - vw$ (see Figure 5). It is easy to check that G' is Δ-regular and 2-edge-connected (recall that $\Delta \geqslant 3$ and thus $\lfloor \frac{\Delta}{2} \rfloor \geqslant 1$). Applying Theorem 7 on G' with $e = wy$, G' has a k-factor F containing the edge wy. There exists an integer $s \leqslant k-1$ such that F contains s edges wx, and $k-s-1$ edges wv. So F must contain $k-s$ edges vx and thus F contains exactly one edge uv. Hence, $F - uv - yw + uy + vw$ is a k-factor of G. $\square$

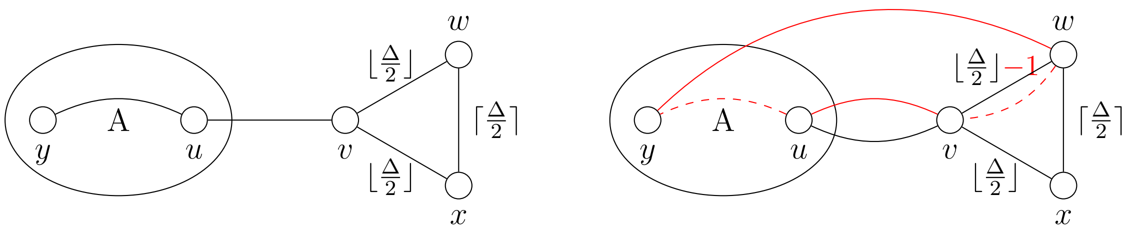

Figure 5: The graphs G and G'.

We are now ready to prove the theorem. We distinguish cases with respect to the value of Δ and the corresponding value of $\lceil \frac{3\Delta-1}{3d-1} \rceil$. Recall that Δ is a special integer, that is $\Delta \in \{(i+1)d - \lceil \frac{i}{3} \rceil \mid i \geqslant 0\} = \{d, 2d-1, 3d-1, 4d-1, 5d-2, 6d-2, \dots\}$.

Case 1: $\lceil \frac{3\Delta-1}{3d-1} \rceil = 1$, $\Delta = d$. The result holds trivially.

Case 2: $\lceil \frac{3\Delta-1}{3d-1} \rceil = 2$, $\Delta = 2d-1$. Since d is odd, $d-1$ is even, and $d-1 < \frac{4d-2}{3} = \frac{2\Delta}{3}$. So, by Claim 13, G has a $(d-1)$-factor, say F. Now, $G - F$ is d-regular, and thus $\chi'_d(G) \leqslant 2$. This proves case 2.

Case 3: $\lceil \frac{3\Delta-1}{3d-1} \rceil = 3$, $\Delta = 3d-1$. Since d is odd, Δ is even. By Theorem 4, G has a $2d$-factor F. By applying Lemma 5 on connected components of even size of F and Lemma 6 on connected components of odd size, we can extract two graphs G_A and G_B along with a matching M such that $E(F) = E(G_A) \cup E(G_B) \cup M$, $\Delta(G_A) \leqslant d$ and $\Delta(G_B) \leqslant d$. Now, $E(G)$ can be partitioned into $E(G_A)$, $E(G_B)$ and $E(G) \setminus (E(F) \setminus M)$. Since the graph induced by $E(G) \setminus (E(F) \setminus M)$ has maximum degree at most $3d-1-2d+1 = d$, each of these sets induce a graph with maximum degree at most d. This proves case 3.

Case 4: $\lceil \frac{3\Delta-1}{3d-1} \rceil = 4$, $\Delta = 4d-1$. Since $d \geqslant 3$, we have $2d < \frac{8d-2}{3} = \frac{2\Delta}{3}$. So, by Claim 13, G has a $2d$-factor, say A, and $B = G - A$ is a $(2d-1)$-factor of G. By applying Lemma 5 on connected components of A of even size and Lemma 6 on connected components of A of odd size, we get a partition of $E(A)$ into three sets A_1, A_2 and M such that $\Delta(A_1) \leqslant d$, $\Delta(A_2) \leqslant d$ and M is a matching.

It is now enough to prove that $\chi'_d(B \cup M) \leqslant 2$. Let C be a connected component of $B \cup M$. If every vertex of C is incident with an edge of M, then C has an even number of vertices and is $2d$-regular, so its number of edges is d times its number of vertices, which is even, and thus $\chi'_d(C) = 2$ by Lemma 5. Assume now that there exists a vertex of C that is not incident with an edge of M. Take two copies of C, and add an edge between the copies of each vertex of C not incident with an edge of M. The obtained graph has an even number of vertices and is $2d$-regular, so it is $(2,d)$-edge colourable by Lemma 5 and thus so is C. So each connected component of $B \cup M$ is $(2,d)$-edge colourable, and thus so is $B \cup M$. This proves case 4.

Case 5: $\lceil \frac{3\Delta-1}{3d-1} \rceil \geqslant 5$, $\Delta \geqslant 5d-2$. Note that $3d-1$ is even since d is odd. Also, since $d \geqslant 3$, $3d-1 = \frac{9d-3}{3} < \frac{10d-4}{3} \leqslant \frac{2\Delta}{3}$. So, by Claim 13, G has a $(3d-1)$-factor, say F. By Case 3, F is $(3,d)$-edge colourable. As $G-F$ is $(\Delta-(3d-1))$-regular, and as $\Delta-(3d-1)$ is less than at least one special integer less than Δ, it follows from minimality of Δ that

$$\chi'_d(G-F) \leqslant \lceil \frac{3(\Delta-(3d-1))-1}{3d-1} \rceil,$$

and thus

$$\chi'_d(G) \leqslant 3 + \lceil \frac{3(\Delta-(3d-1))-1}{3d-1} \rceil = \lceil \frac{3\Delta-9d+3-1+9d-3}{3d-1} \rceil = \lceil \frac{3\Delta-1}{3d-1} \rceil.$$

This proves case 5 and the theorem. □

4 Simple graphs: Vizing's Theorem and NP-completeness

In this section, we will only consider simple graphs. Vizing [15] proved the following theorem:

Theorem 14 (Vizing's Theorem, [15]). *For every simple graph G with maximum degree Δ, $\chi'_1(G) \in \{\Delta, \Delta+1\}$.*

While there are only 2 possibilities, deciding between them was proven to be NP-complete even for regular simple graphs.

Theorem 15 (Holyer [8], Leven and Galil [11]). *For every $\Delta \geqslant 3$, it is NP-complete to decide if a Δ-regular simple graph G is Δ-edge colourable.*

Vizing's theorem easily implies its following generalization to d-defective edge colouring.

Corollary 16. *For every $d \geqslant 1$ and every simple graph G with maximum degree Δ, $\chi'_d(G) \in \{\lceil \frac{\Delta}{d} \rceil, \lceil \frac{\Delta+1}{d} \rceil\}$.*

Proof. The lower bound holds by Lemma 2. For the upper bound, consider an edge colouring of G with $\Delta(G)+1$ colours (it exists by Vizing's Theorem) and let $M_1, \dots, M_{\Delta(G)+1}$ be the classes of colours. By assigning colour 1 to $M_1 \cup \dots \cup M_d$, colour 2 to $M_{d+1} \cup \dots \cup M_{2d}$, etc, we obtain a $(\lceil \frac{\Delta+1}{d} \rceil, d)$ edge colouring of G. □

We point out that Vizing [15] also proved that for every (not necessarily simple) graph G with maximum degree Δ and edge multiplicity μ, $\chi'_1(G) \leqslant \Delta + \mu$ where the edge multiplicity is the maximum number of edges between two vertices. This directly implies that $\chi'_d(G) \leqslant \lceil \frac{\Delta+\mu}{d} \rceil$.

In the following cases one can distinguish between the two possibilities in Corollary 16.

Theorem 17. *Let $d, \Delta \geqslant 1$ and let G be a simple graph with maximum degree Δ. Then :*

(a) $\chi'_d(G) = \lceil \frac{\Delta}{d} \rceil$ if (i) d does not divide Δ, or (ii) d is even, or (iii) $\Delta = d$.

(b) If d is odd and $\Delta = 2d$, then $\chi'_d(G) = \lceil \frac{\Delta}{d} \rceil = 2$ if and only if every $2d$-regular connected component of G has an even number of vertices; otherwise $\chi'_d(G) = \lceil \frac{\Delta+1}{d} \rceil = 3$.

Proof. In (a), (i) follows from Corollary 16, since if d does not divide Δ, then $\lceil \frac{\Delta}{d} \rceil = \lceil \frac{\Delta+1}{d} \rceil$, (ii) is contained in Theorem 1 (even if G is not simple), and (iii) is obvious.

To prove (b), note first that a $2d$-regular component C of G with n vertices has dn edges, and d is odd; so the order and size of C are either both even or both odd.

Suppose first that every $2d$-regular component has even order and size. Take two disjoint copies of G and, for each vertex v of G, add $2d - d(v)$ edges between the two copies of v. The resulting (not necessarily simple) graph G' is $2d$-regular, and each of its connected components has even order and size (as the components of G of odd order were not $2d$-regular, they are included in components of even order in G'). Now, by Lemma 3, $\chi'_d(G) \leqslant \chi'_d(G') = 2$, and so $\chi'_d(G) = 2$.

Assume now that G has a $2d$-regular component C of odd order and size. Since d is odd, C does not admit a d-factor, and so C cannot be $(2, d)$-edge coloured. So, by Corollary 16, $\chi'_d(G) = \lceil \frac{2d+1}{d} \rceil = 3$. □

We now prove a generalization of Theorem 15 in the context of defective edge colouring. Before that, we need the following construction.

For all integers $k, d \geqslant 1$, we construct a simple graph $G_{kd,d}$ such that G is kd-regular and $\chi'_d(G) = k$. We can set $G_{d,d} = K_{d+1}$. Inductively, having defined $G_{kd,d}$, let $G_{(k+1)d,d}$

be the simple graph obtained by taking two disjoint copies of $G_{kd,d}$ and adding the edges of any d-regular bipartite simple graph between these two copies[1]. The obtained simple graph is clearly $(k+1)d$-regular, and we can $(k+1,d)$-edge colour it by taking a (k,d)-edge colouring for the two copies of $G_{kd,d}$ and add a new colour for the added edges, and finally by Lemma 2 it does not admit a (k,d)-edge colouring. Hence $\chi'_d(G_{(k+1)d,d}) = k+1$.

Now assume d is odd and let H be obtained from $G_{kd,d}$ by subdividing one edge ab with a new vertex v of degree 2.

Lemma 18. *H has a (k,d)-edge colouring, and in every such colouring the edges av and bv have the same colour.*

Proof. Let $|V(G_{kd,d})| = n$. Let us first prove that n is even. Since $G_{kd,d}$ is kd-regular, in a (k,d)-edge colouring of $G_{kd,d}$, each vertex must be incident with exactly d edges of each colour. So every colour occurs on exactly $\frac{dn}{2}$ edges. Hence, $\frac{dn}{2}$ is an integer, and since d is odd, n is even.

Clearly H is (k,d)-edge colourable since $G_{kd,d}$ is. Every vertex of H except v has degree kd, so in a (k,d)-edge colouring of H, every vertex is incident with exactly d edges of each colour. Assume for contradiction that v is incident with edges of two different colours, and let c be one of these colours. In $H \setminus \{v\}$, c occurs on $\frac{d(n-1)+(d-1)}{2} = \frac{dn-1}{2}$ edges, and thus on $\frac{dn+1}{2}$ edges of H. But since n is even, $\frac{dn+1}{2}$ is not an integer, a contradiction. Thus av and bv must have the same colour. $\square$

In the proof of the following theorem, we will use many copies of H, all with the same values of k and d. We will use subscripts consistently: if $H_{u,i}$ is a copy of H, then it will contain a vertex $v_{u,i}$ of degree 2 with neighbours $a_{u,i}$ and $b_{u,i}$.

Note that the pairs d, Δ in the following theorem are precisely those that are not covered by Theorem 17.

Theorem 19. *Let d be an odd integer and $\Delta = kd$ for some integer $k \geqslant 3$. Then it is NP-complete to decide if a Δ-regular simple graph is (k,d)-edge colourable.*

Proof. The problem is clearly in NP. The case $d = 1$ is Theorem 15, and so we may assume that $d \geqslant 3$. We perform a reduction from the case $d = 1$. Let G be a k-regular simple graph.

We construct a simple graph G' containing G as follows: starting with G, for each vertex u of G add $\frac{k(d-1)}{2}$ disjoint copies $H_{u,i}$ of H for $i = 1, 2, \ldots, \frac{k(d-1)}{2}$ and identify each vertex $v_{u,i}$ with u. The graph G' is clearly simple and kd-regular. We will prove that $\chi'_1(G) = k$ if and only if $\chi'_d(G') = k$, and this will prove the theorem.

Suppose first that $\chi'_1(G) = k$. Starting with a $(k,1)$-edge colouring of G, we extend it to G' as follows: for each vertex $u \in V(G)$ and colour $c \in \{1, 2, \ldots, k\}$, give colour c to all edges $va_{u,i}$ and $vb_{u,i}$ with $\frac{(c-1)(d-1)}{2} + 1 \leqslant i \leqslant \frac{c(d-1)}{2}$, and extend this to a (k,d)-edge colouring of $H_{u,i}$, which is possible by Lemma 18. Now, for each colour $c \in \{1, 2, \ldots, k\}$,

[1]For example, naming $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ the vertices of the two copies of $G_{kd,d}$, add the edges $u_iv_i, u_iv_{i+1}, \ldots, u_iv_{i+d}$ for $i = 1, \ldots, n$, subscripts being taken modulo n. It gives a d-regular bipartite simple graph as soon as $n \geqslant d$.

u is incident to d edges coloured c: $d-1$ edges in $E(G') \setminus E(G)$ and one edge of G. So we have constructed a (k,d)-edge colouring of G'. Hence $\chi'_d(G') = k$.

Suppose now that $\chi'_d(G') = k$, and fix a (k,d)-edge colouring of G'. By Lemma 18, for each vertex $u \in V(G)$ and colour $c \in \{1, 2, \ldots, k\}$, u is incident to an even number of edges in $E(G') \setminus E(G)$ with colour c, and so (since d is odd) u must be incident to an odd number of edges of G with colour c. Since there are k colours and G is k-regular, u must be adjacent to exactly one edge of each colour, and so the (k,d)-edge colouring of G' contains a $(k,1)$-edge colouring of G. Hence $\chi_1(G) = k$. This completes the proof. □

5 Further work

Recall that graphs in this paper are allowed to have multiple edges.

List colouring

The *d-defective list chromatic index* of a graph G, denoted by $ch'_d(G)$, is defined as the minimum k such that, for any choice of list of k integers given to each edge, there is an edge colouring with defect d such that each edge receives a colour from its list. So $ch'_1(G)$ is the usual list chromatic index.

Borodin et al. [3] proved that Shannon bound holds for the list chromatic index, that is, for every graph G, $ch'_1(G) \leqslant \lceil \frac{3\Delta(G)}{2} \rceil$. It is then natural to ask if Theorem 1 extends to defective list edge colouring. As mention in the introduction, when d is even, it is proved in [6] (and a simpler proof is given in [2]) that for every graph G, $ch'_d(G) = \lceil \frac{\Delta(G)}{d} \rceil$. When d is odd, a proof that $ch'_d(G) \leqslant \lceil \frac{3\Delta}{3d-1} \rceil$ is announced in [2] but seems to have a flaw and actually holds only in the case where Δ is divisible by $3k-1$.

Conjecture 20. For every odd integer d and for every graph G, $ch'_d(G) \leqslant \lceil \frac{3\Delta-1}{3d-1} \rceil$

We finally mention the following stronger conjecture that corresponds to the infamous list edge colouring conjecture for $d = 1$ and is proved for bipartite graph in [6].

Conjecture 21. [7] For every graph G and every integer d, $ch'_d(G) = \chi'_d(G)$.

The Goldberg-Seymour Conjecture

Let $d \geqslant 1$ and G a graph. Observe that in any edge colouring of G with defect d, and for any $X \subseteq V(G)$, each colour class contains at most $\lfloor \frac{d|X|}{2} \rfloor$ edges, which leads to the following lower bound on the d-defective edge chromatic number of any graph G:

$$\chi'_d(G) \leqslant \Gamma_d(G) = \max\left\{ \left\lceil \frac{|E(G[X])|}{\lfloor \frac{d|X|}{2} \rfloor} \right\rceil \mid X \subseteq V(G),\ |X| \geqslant 2 \right\}.$$

The following was known as the Goldberg-Seymour Conjecture [5, 13] for almost 50 years. Recently, Chen, Jing and Zang [4] announced a proof (the paper is still under revision).

Theorem 22 (Golberg-Seymour [5, 13]). *For every graph G,*

$$\chi'_1(G) \leqslant \max\{\Gamma_1(G), \Delta(G)+1\}.$$

We think that the following generalization could hold.

Conjecture 23. Every graph G satisfies $\chi'_d(G) \leqslant \max\{\Gamma_d(G), \lceil\frac{\Delta(G)+1}{d}\rceil\}$.

An easy proof of the conjecture could start as follows. Let G be a counter-example to Conjecture 23, that is $\chi'_d(G) > \max\{\Gamma_d(G), \lceil\frac{\Delta(G)+1}{d}\rceil\}$ for some $d \geqslant 3$. By Theorem 22, $\chi'_1(G) \leqslant \max\{\Gamma_1(G), \Delta(G)+1\}$. As $\chi'_d(G) \leqslant \lceil\frac{\chi'_1(G)}{d}\rceil$, if $\chi'_1(G) \leqslant \Delta(G)+1$, then $\chi'_d(G) \leqslant \frac{\Delta(G)+1}{d}$, a contradiction. So may assume that $\Delta(G)+1 < \chi'_1(G) = \Gamma_1(G)$. This implies that $\chi'_d(G) \leqslant \lceil\frac{\Gamma_1(G)}{d}\rceil$. So it is enough to prove that $\lceil\frac{\Gamma_1(G)}{d}\rceil \leqslant \max\{\Gamma_d(G), \lceil\frac{\Delta(G)+1}{d}\rceil\}$.

Unfortunately this last inequality does not hold, for example on the following simple example. Consider the graph G made of three vertices connected by respectively 7, 7 and 2 edges. So $\Delta(G)+1 = 15$, $\Gamma_1(G) = \max\{\frac{2}{1}, \frac{7}{1}, \frac{16}{1}\} = 16$ and $\chi'_1(G) = 16$. Moreover, $\Gamma_3(G) = \max\{\frac{2}{3}, \frac{7}{3}, \frac{16}{4}\} = 4$. Hence,

$$6 = \left\lceil\frac{\Gamma_1(G)}{3}\right\rceil > \max\left\{\Gamma_3(G), \left\lceil\frac{\Delta(G)+1}{3}\right\rceil\right\} = \max\left\{4, \left\lceil\frac{15}{3}\right\rceil\right\} = 5.$$

The degree Ramsey number of stars

In this subsection, we briefly describe the link between the degree Ramsey number of stars and defective edge colouring. We are thankfull to Ross Kang for bringing this to our attention.

Let H, G be simple graphs. Let $H \to_s G$ means that every colouring of $E(H)$ with s colours produces a monochromatic copy of H. The *degree Ramsey number* of a simple graph G is $R_\Delta(G;s) = \min\{\Delta(H) : H \to_s G\}$. Observe that $H \to_s K_{1,d+1}$ means that $\chi'_d(H) \geqslant s+1$. Hence, $R_\Delta(K_{1,d+1};s) = \min\{\Delta(H) : \chi'_d(H) \geqslant s+1\}$.

It can be proved (with a little brain gymnastic) that the following result of Kinnersley, Milans and West is equivalent to corollary 16.

Theorem 24. *[10] If $s \geqslant 2$, then* $R_\Delta(K_{1,d+1};s) = \begin{cases} s\cdot d & \text{if } d \text{ is odd,} \\ s\cdot d+1 & \text{if } d \text{ is even.} \end{cases}$

It could be of interest to look at the degree Ramsey number of (multi)graphs.

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