Some unexpected properties of Littlewood-Richardson coefficients

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Abstract

We define a family of partitions called *near-rectangular*. We introduce and give evidence for a conjectural identity between Littlewood-Richardson coefficients, when one partition is near-rectangular. That is, if λ is a near-rectangular partition and μ any partition, the irreducible decompositions of $V_n(\lambda) \otimes V_n(\mu)$ and $V_n(\lambda)^* \otimes V_n(\mu)$ coincide up to some unknown bijection. Here, $V_n(\lambda)$ denotes the irreducible $\mathrm{GL}_n(\mathbb{C})$-representation corresponding to λ. This conjecture is proved if μ is also near-rectangular. We also report some computational evidence for the conjecture.

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1 Introduction

Let $\Lambda_n = \{\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0 \,:\, \lambda_i \in \mathbb{Z}\}$ be the set of partitions with at most n parts. The irreducible polynomial representations of $\mathrm{GL}_n(\mathbb{C})$ are parametrized by Λ_n. For $\lambda \in \Lambda_n$, let $V_n(\lambda)$ be the associated irreducible polynomial representation of $\mathrm{GL}_n(\mathbb{C})$. The Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ is the tensor product multiplicity

$$V_n(\lambda) \otimes V_n(\mu) = \bigoplus_{\nu \in \Lambda_n} c^{\nu}_{\lambda\mu} V_n(\nu).$$

Let $V_n(\lambda)^*$ be the $\mathrm{GL}_n(\mathbb{C})$-representation dual to $V_n(\lambda)$. The tensor products $V_n(\lambda) \otimes V_n(\mu)$ and $V_n(\lambda)^* \otimes V_n(\mu)$ are not generally isomorphic as $\mathrm{GL}_n(\mathbb{C})$-representations. Nevertheless, we pose:

Problem 1. Compare $V_n(\lambda) \otimes V_n(\mu)$ and $V_n(\lambda)^* \otimes V_n(\mu)$ as $\mathrm{GL}_n(\mathbb{C})$-representations.

Coquereaux and Zuber [CZ11] prove that the sums of the multiplicities of these two representations coincide. Let det be the one-dimensional determinant representation of $\mathrm{GL}_n(\mathbb{C})$. For $\lambda \in \Lambda_n$, set $\lambda^* = \lambda_1 - \lambda_n \geqslant \lambda_1 - \lambda_{n-1} \geqslant \cdots \geqslant \lambda_1 - \lambda_2 \geqslant 0$. Then, $V_n(\lambda^*)$ is isomorphic to $V_n(\lambda)^* \otimes \det^{\otimes \lambda_1}$. Coquereaux-Zuber's states

$$\sum_{\nu \in \Lambda_n} c^{\nu}_{\lambda\mu} = \sum_{\nu \in \Lambda_n} c^{\nu}_{\lambda^*\mu}. \tag{1}$$

Let us further compare $\{c^{\nu}_{\lambda\mu} : \nu \in \Lambda_n\}$ and $\{c^{\nu}_{\lambda^*\mu} : \nu \in \Lambda_n\}$ as multisets. For instance, for $n = 3$, $\lambda = (5,3)$ and $\mu = (6,3)$, $V_3(\lambda) \otimes V_3(\mu)$ decomposes as

$$\begin{aligned}
&V_3(7,5,5) + V_3(7,7,3) + V_3(8,8,1) + V_3(9,4,4) + V_3(9,8) + V_3(10,7)\\
&+V_3(11,6) + V_3(11,3,3) + V_3(11,4,2) + V_3(11,5,1) + V_3(6,6,5)\\
&+2V_3(7,6,4) + 2V_3(8,5,4) + 2V_3(8,7,2) + 2V_3(9,7,1) + 2V_3(10,4,3)\\
&+2V_3(10,5,2) + 2V_3(10,6,1) + 3V_3(8,6,3) + 3V_3(9,5,3) + 3V_3(9,6,2),
\end{aligned}$$

while $V_3(\lambda^*) \otimes V_3(\mu)$ decomposes as

$$\begin{aligned}
&V_3(7,7,2) + V_3(8,4,4) + V_3(10,3,3) + V_3(8,8) + V_3(9,7) + V_3(10,6)\\
&+V_3(11,3,2) + V_3(11,4,1) + V_3(6,5,5) + V_3(6,6,4) + 2V_3(7,5,4)\\
&+2V_3(7,6,3) + 2V_3(8,7,1) + 2V_3(9,4,3) + 2V_3(9,6,1) + 2V_3(10,4,2)\\
&+2V_3(10,5,1) + 3V_3(8,5,3) + 3V_3(8,6,2) + 3V_3(9,5,2) + V_3(11,5).
\end{aligned}$$

Notice the multiplicities in the two expansions are the same: 11 occurrences of "1", 7 occurrences of "2" and 3 occurrences of "3" in both cases. Is this always true? In fact, in [CZ14], Coquereaux and Zuber give an affirmative answer for $\mathrm{GL}_3(\mathbb{C})$. Using a computer, we are able to compute explicitly, for $n = 3$, the function

$$\begin{array}{ccl}
(\Lambda_n^2) \times \mathbb{N} & \longrightarrow & \mathbb{N}\\
(\lambda, \mu, c) & \longmapsto & \mathrm{Nb}_n(c^{\bullet}_{\lambda\mu} > c) := \#\{\nu \in \Lambda_n : c^{\nu}_{\lambda\mu} > c\}.
\end{array}$$

Using these calculations, we obtain a new proof of Coquereaux-Zuber's result [CZ14]. In Section 5 we prove:

Proposition 1. *The function*

$$\begin{array}{cccl}
\mathrm{Nb}_3(c^{\bullet}_{\lambda\mu} > c) : & \Lambda_3 \times \Lambda_3 \times \mathbb{N} & \longrightarrow & \mathbb{N}\\
& (\lambda, \mu, c) & \longmapsto & \#\{\nu \in \Lambda_n : c^{\nu}_{\lambda\mu} > c\}
\end{array}$$

is piecewise polynomial of degree 2 with respect to a fan with 7 maximal cones. Moreover,

$$\mathrm{Nb}_3(c^{\bullet}_{\lambda\mu} > c) = \mathrm{Nb}_3(c^{\bullet}_{\lambda^*\mu} > c). \tag{2}$$

This piecewise polynomiality roughly means that the cone generated by $\Lambda_3 \times \Lambda_3 \times \mathbb{N}$ decomposes into 7 cones, such that on the integer points of each the function $\mathrm{Nb}_3(c^{\bullet}_{\lambda\mu} > c)$ is polynomial. See Section 2 for a precise definition of a piecewise quasi-polynomial function.

We now suggest a generalization to $\mathrm{GL}_n(\mathbb{C})$ for any n. Define a partition $\lambda \in \Lambda_n$ to be *near-rectangular* if $\lambda = \lambda_1\lambda_2^{n-2}\lambda_n$ for some integers $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_n$; that is, if $\lambda_2 = \cdots = \lambda_{n-1}$.

Conjecture 2. Let λ and μ in Λ_n. If λ is near-rectangular then

$$\forall c \in \mathbb{N} \qquad \#\{\nu \in \Lambda_n \,:\, c^\nu_{\lambda\mu} = c\} = \#\{\nu \in \Lambda_n \,:\, c^\nu_{\lambda^*\mu} = c\}.$$

Equivalently, we conjecture that, if λ is near-rectangular, there is a bijection $\varphi : \Lambda_n \longrightarrow \Lambda_n$, depending on λ and μ, such that

$$\forall \nu \in \Lambda_n \qquad c^\nu_{\lambda\mu} = c^{\varphi(\nu)}_{\lambda^*\mu}.$$

Since any partition of length at most 3 is near-rectangular, both the result of [CZ14] and the last assertion of Proposition 1 are equivalent to

Corollary 3. *Conjecture 2 holds for $n = 3$.*

We have verified Conjecture 2 on millions of examples for $\mathrm{GL}_4(\mathbb{C})$, $\mathrm{GL}_5(\mathbb{C})$, $\mathrm{GL}_6(\mathbb{C})$, and $\mathrm{GL}_{10}(\mathbb{C})$: see Section 9.2 for details. In addition, in Section 9.2 we offer an example showing that the assumption on λ cannot be dispensed with.

In literature, one finds numerous properties relating different Littlewood-Richardson coefficients. For example, there are symmetries (see [BR20] and references therein), stabilities (see [BOR15]), and reductions (see [CM11]). Moreover, there are various combinatorial models for the Littlewood-Richardson coefficients (see [Ful97, Lit95, Zel81, KT99, Vak06, Cos09]). As far as we can tell, none of these results are especially useful in attacking Conjecture 2. In addition, we remark in Section 5 that even for $n = 3$ the map $(\lambda, \mu, \nu) \longrightarrow (\lambda^*, \mu, \varphi(\nu))$ cannot be linear. This observation makes Conjecture 2 even more surprising.

A partition λ of length l parametrizes representations $V_n(\lambda)$ of $\mathrm{GL}_n(\mathbb{C})$ for any $n \geqslant l$. It is a classical result (see e.g. [Ful97]) of stability that $c^\nu_{\lambda\mu}$ does not depend on n. Our second result is a similar stability result, but for near-rectangular partitions of arbitrarily large length. Indeed, fix two near-rectangular partitions $\lambda = \lambda_1\, \lambda_2^{n-2} \lambda_n$ and $\mu = \mu_1\, \mu_2^{n-2} \mu_n$. We prove that the decomposition of $V_n(\lambda) \otimes V_n(\mu)$ does not depend on $n \geqslant 4$, but only on the six integers $\lambda_1, \lambda_2, \lambda_n, \mu_1, \mu_2$ and μ_n:

Proposition 4. *Let $n \geqslant 4$. Let $\lambda = \lambda_1\, \lambda_2^{n-2}$ and $\mu = \mu_1\, \mu_2^{n-2}$ be two near-rectangular partitions.*[1] *Let ν be a partition with at most n parts. Then, $c^\nu_{\lambda\mu} = 0$ unless $\nu = \nu_1\, \nu_2\, (\lambda_2 + \mu_2)^{n-4}\, \nu_{n-1} \nu_n$, for four integers ν_1, ν_2, ν_{n-1} and ν_n such that $\nu_1 \geqslant \nu_2 \geqslant \lambda_2 + \mu_2 \geqslant \nu_{n-1} \geqslant \nu_n$. In this case, set*

$$M = \max(0, \lambda_2 + \mu_1 - \nu_1, -\mu_2 + \nu_n)$$

and

$$m = \min(\lambda_1 + \mu_1 - \nu_1, \lambda_2 + \mu_1 - \nu_2, -\lambda_2 - \mu_2 + \nu_{n-1} + \nu_n, -\mu_2 + \nu_{n-1}).$$

Then

$$c^\nu_{\lambda\mu} = \begin{cases} m - M + 1 & \text{if } m \geqslant M, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, this value does not depend on $n \geqslant 4$.

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[1] Nothing is lost with the hypothesis $\lambda_n = \mu_n = 0$ since $V_n(\lambda_1 \lambda_2^{n-2} \lambda_n) \simeq \det^{\lambda_n} \otimes V_n((\lambda_1 - \lambda_n)(\lambda_2 - \lambda_n)^{n-2})$.

Proposition 4 positively answers [PW22, Question 2] by giving a much stronger result. It also makes is possible to check a particular case of Conjecture 2. Indeed, with the help of a computer, we computed $\mathrm{Nb}_4(c^\bullet_{\lambda\mu} > c)$ for λ and μ near-rectangular. Let $\Lambda_n^{\mathrm{nr}} = \{\lambda_1\lambda_2^{n-2}\lambda_n : \lambda_1 \geqslant \lambda_2 \geqslant \lambda_n\}$ be the set of near-rectangular partitions of length at most n.

Proposition 5. *The function*

$$\begin{array}{rccl} \mathrm{Nb}_4(c^\bullet_{\lambda\mu} > c) : & \Lambda_4^{\mathrm{nr}} \times \Lambda_4^{\mathrm{nr}} \times \mathbb{N} & \longrightarrow & \mathbb{N} \\ & (\lambda, \mu, c) & \longmapsto & \#\{\nu \in \Lambda_n : c^\nu_{\lambda\mu} > c\} \end{array}$$

is piecewise polynomial of degree 3 with respect to a fan with 36 maximal cones. Moreover,

$$\mathrm{Nb}_4(c^\bullet_{\lambda\mu} > c) = \mathrm{Nb}_4(c^\bullet_{\lambda^*\mu} > c). \tag{3}$$

The 36 polynomial functions and cones are given in Section 7.3. As a consequence of Propositions 4 and 5, we obtain

Corollary 6. *Let $n \geqslant 4$. Conjecture 2 holds for $\mathrm{GL}_n(\mathbb{C})$, whenever μ is near-rectangular.*

A weaker version of Conjecture 2 is

Conjecture 7. If $\lambda \in \Lambda_n$ is near-rectangular then

$$\#\{\nu \in \Lambda_n : c^\nu_{\lambda\mu} \neq 0\} = \#\{\nu \in \Lambda_n : c^\nu_{\lambda^*\mu} \neq 0\}.$$

Equivalently, we wonder whether, for $\lambda \in \Lambda_n^{\mathrm{nr}}$,

$$\forall \mu \in \Lambda_n \qquad \mathrm{Nb}_n(c^\bullet_{\lambda\mu} > 0) = \mathrm{Nb}_n(c^\bullet_{\lambda^*\mu} > 0).$$

For $n = 4$ and λ near-rectangular, we computed $\mathrm{Nb}_4(c^\bullet_{\lambda\mu} > 0)$ and checked Conjecture 7. Here we report on this computation as follows (see Section 7.4 for details).

Proposition 8. *The function*

$$\begin{array}{rccl} \mathrm{Nb}_4(c^\bullet_{\lambda\mu} > 0) : & \Lambda_4^{\mathrm{nr}} \times \Lambda_4 & \longrightarrow & \mathbb{N} \\ & (\lambda, \mu) & \longmapsto & \#\{\nu \in \Lambda_n : c^\nu_{\lambda\mu} > 0\} \end{array}$$

is piecewise quasi-polynomial of degree 3 with respect to a fan with 205 maximal cones. The only congruence condition occurring is the parity of $\lambda_1 + |\mu|$ where, for any partition $\nu = (\nu_1 \geqslant \cdots \geqslant \nu_n)$, $|\nu| = \nu_1 + \cdots + \nu_n$. Moreover,

$$\mathrm{Nb}_4(c^\bullet_{\lambda\mu} > 0) = \mathrm{Nb}_4(c^\bullet_{\lambda^*\mu} > 0). \tag{4}$$

This symmetry, along with the complete duality $(\lambda, \mu) \longmapsto (\lambda^*, \mu^*)$, gives an action of $(\mathbb{Z}/2\mathbb{Z})^2$ on $\Lambda_4^{\mathrm{nr}} \times \Lambda_4$. Then this group acts on the 205 pairs (cone, quasi-polynomial) with 83 orbits.

This work is based on numerous computer aided computations with Barvinok [VSB+07], Normaliz [BIS] and SageMath [The20]. Details on these computations can be found on the webpage of the second author [Res20, Supplementary material].

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Remark 9. After a version of this work was posted on ArXiv, Darij Grinberg offered a solution to our main conjecture in [Gri21]. Therein he defines a piecewise linear involution φ from $\mathbb{Z}^n$ to $\mathbb{Z}^n$ satisfying

$$\forall \nu \in \Lambda_n \qquad c^{\nu}_{\lambda\mu} = c^{\varphi(\nu)}_{\lambda^*\mu}$$

if λ is near-rectangular, thus proving our conjecture. An amazing fact is that this bijection does not necessarily map a partition to a partition: if $\varphi(\nu)$ is not a partition then $c^{\nu}_{\lambda\mu}$ simply vanishes, allowing φ to work.

2 Piecewise quasi-polynomial functions

Let d be a positive integer. Let N be a free abelian group of rank d; set $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$. We denote by M the dual lattice of N, and by $M_{\mathbb{Q}}$ the dual vector space of $N_{\mathbb{Q}}$.

A closed half-space in $N_{\mathbb{Q}}$ is defined by an inequality $m \geqslant 0$ where m is a non-zero element of $M_{\mathbb{Q}}$. A (polyhedral convex) *cone* σ in $N_{\mathbb{Q}}$ is the intersection of finitely many closed half- spaces. A *face* of σ is an intersection $\sigma \cap (m = 0)$, for $m \in M$ such that the half-space $(m \geqslant 0)$ contains σ. A *fan* in $N_{\mathbb{Q}}$ is a finite set Σ of cones, such that:

1. if $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$;
2. if $\sigma, \tau \in \Sigma$ then $\sigma \cap \tau$ is a face of σ.

The *support* $|\Sigma|$ of the fan Σ is the union of its cones.

Let σ be a cone. A function $g : \sigma \cap N \longrightarrow \mathbb{R}$ is said to be *polynomial* if there exists a polynomial function p in $\mathrm{Sym}(M_{\mathbb{Q}})$ such that $g(t) = p(t)$, for any $t \in \sigma \cap N$. A function $g : \sigma \cap N \longrightarrow \mathbb{R}$ is said to be *quasi-polynomial* if there exist a d-dimensional lattice $\Lambda \subset N$, a set $\{\lambda_i\}$ of coset representatives of N/Λ, and polynomial functions p_i in $\mathrm{Sym}(M_{\mathbb{Q}})$ such that $g(t) = p_i(t)$, for any $t \in (\lambda_i + \Lambda) \cap \sigma$. The finite quotient group N/Λ is called the *congruence condition*.

A function $g : \sigma \cap N \longrightarrow \mathbb{R}$ is said to be *piecewise polynomial* (resp. *piecewise quasi-polynomial*) if there exists a fan Σ such that $|\Sigma| = \sigma$ and the restriction of g on the ineger points of any maximal dimensional cone in Σ is polynomial (resp. quasi-polynomial).

Observe that if there exists a smaller cone $\sigma' \subset \sigma$ on g is piecewise polynomial and vanishes outside σ', then it is also piecewise polynomial on σ. Indeed, the complement of σ' in σ can be subdivided in subcones such that g is zero on each.

3 Generalities on the function $\mathrm{Nb}_n(c_{\lambda\mu}^{\bullet} > c)$

Recall that, for $\lambda, \mu \in \Lambda_n$ and $c \in \mathbb{N}$, we set

$$\mathrm{Nb}_n(c_{\lambda\mu}^{\bullet} > c) = \#\{\nu \in \Lambda_n \,:\, c_{\lambda\mu}^{\nu} > c\}.$$

Let $1^n \in \Lambda_n$ denote the partition with n parts equal to 1. Then, $V_n(1^n) = \det$ is the one dimensional representation of $\mathrm{GL}_n(\mathbb{C})$ given by the determinant. Set $\Lambda_n^0 = \{\lambda \in \Lambda_n \,:\, \lambda_n = 0\}$. For $\lambda \in \Lambda_n$, set $\bar{\lambda} = \lambda - \lambda_n^n \in \Lambda_n^0$, the partition obtained by substracting λ_n to each part of λ. Then $V_n(\lambda) = V_n(\bar{\lambda}) \otimes \det^{\otimes \lambda_n}$, which gives

$$c_{\lambda\mu}^{\nu} = c_{\bar{\lambda}\bar{\mu}}^{\nu-(\lambda_n+\mu_n)^n}, \qquad \text{and hence} \qquad \mathrm{Nb}_n(c_{\lambda\mu}^{\bullet} > c) = \mathrm{Nb}_n(c_{\bar{\lambda}\bar{\mu}}^{\bullet} > c). \tag{5}$$

Since $V_n(\lambda) \otimes V_n(\mu) \simeq V_n(\mu) \otimes V_n(\lambda) \simeq (V_n(\lambda^*) \otimes V_n(\mu^*))^* \otimes \det^{\otimes(\lambda_1+\mu_1)}$, the function $\mathrm{Nb}_n(c_{\lambda\mu}^{\bullet} > c)$ satisfies

$$\mathrm{Nb}_n(c_{\lambda\mu}^{\bullet} > c) = \mathrm{Nb}_n(c_{\mu\lambda}^{\bullet} > c) = \mathrm{Nb}_n(c_{\lambda^*\mu^*}^{\bullet} > c) = \mathrm{Nb}_n(c_{\mu^*\lambda^*}^{\bullet} > c). \tag{6}$$

Set

$$\mathrm{Horn}_n = \{(\lambda, \mu, \nu) \in (\Lambda_n)^3 \,:\, c_{\lambda\mu}^{\nu} \neq 0\}.$$

By a result from Brion and Knop (see [É92]), Horn_n is a finitely generated semigroup. Knutson-Tao's saturation Theorem [KT99] shows that Horn_n is the set of integer points in a convex cone, the Horn cone. This cone is polyhedral and the minimal list of inequalities defining it is known (see *e.g.* [Ful00, Bel01, KTW04]). These inequalities contain the dominance inequalities and the Weyl inequalities

$$\nu_{i+j-1} \leqslant \lambda_i + \mu_j \quad \text{ whenever } i+j-1 \leqslant n. \tag{7}$$

The remaining inequalities are all of the form

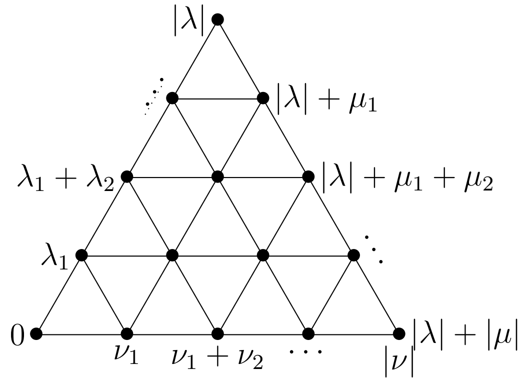
$$\sum_{k\in K} \nu_k \leqslant \sum_{i\in I} \lambda_i + \sum_{j\in J} \mu_j, \tag{8}$$

for some triples (I, J, K) of subsets of $\{1, \dots, n\}$ of the same cardinality.

Proposition 10. *Fix $n \geqslant 0$. The function*

$$\begin{array}{rccl} \mathrm{Nb}_n(c_{\lambda\mu}^{\bullet} > 0) : & \Lambda_n \times \Lambda_n & \longrightarrow & \mathbb{N} \\ & (\lambda, \mu) & \longmapsto & \#\{\nu \in \Lambda_n \,:\, c_{\lambda\mu}^{\nu} > 0\}. \end{array}$$

is piecewise quasi-polynomial.



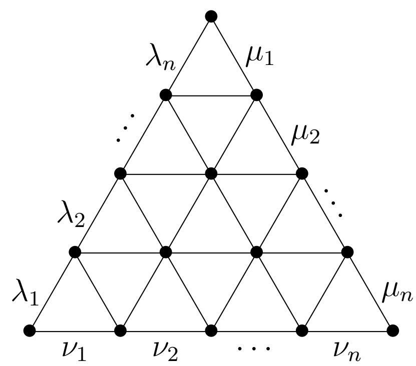


Figure 1: Hives with boundary conditions

Proof. We have

$$\mathrm{Nb}_n(c^{\bullet}_{\lambda\mu} > 0) = \#\left(\mathrm{Horn}_n \cap (\{(\lambda,\mu)\} \times \Lambda_n)\right). \tag{9}$$

Consider the Horn cone $\mathrm{Horn}_n^{\mathbb{Q}} \subset \mathbb{Q}^{3n}$ generated by Horn_n. By the discussion above, $\mathrm{Horn}_n^{\mathbb{Q}}$ is defined by the Horn inequalities (8). Knutson-Tao's saturation Theorem [KT99] asserts that Horn_n is precisely the set of integer points (that is, belonging to $(\Lambda_n)^3$) in $\mathrm{Horn}_n^{\mathbb{Q}}$. Now, equality (9) describes $\mathrm{Nb}_n(c^{\bullet}_{\lambda\mu} > 0)$ as the number of integer points in the affine section of the Horn cone obtained by fixing (λ, μ).

Since each inequality (8) depends linearly on (λ, μ), Proposition 10 then follows from the general theory of multivariate Ehrhart polynomial functions (see *e.g.* [BBDL$^+$19, Theorem 1.1] or [Stu95]). □

4 The hive model

For later use, we shortly review the hive model that expresses the Littlewood-Richardson coefficients as the number of integer points in polyhedra.

Fix an integer $n \geqslant 2$, and an equilateral triangle T_n of side length n. Subdivide T_n in n^2 equilateral triangles of side length 1 (called *unitary triangles*) as on Figure 1. We then have $\frac{(n+1)(n+2)}{2}$ bullet vertices (called simply *vertices*), and $\frac{3n(n+1)}{2}$ sides of length 1 (called *edges*).

Label the vertices of T_n with real numbers. Note that each interior edge is the diagonal of a unique rhombus as illustrated in Figure 2. For each such rhombus, consider the following so-called *rhombus inequality* (using the notation of Figure 2):

$$b + c \geqslant a + d. \tag{10}$$

A *hive* is defined to be such a labeling that satisfies Inequality (10) for each one of the $3\frac{n(n-1)}{2}$ rhombi. Furthermore, a hive is *integral* if all its labels are integers.

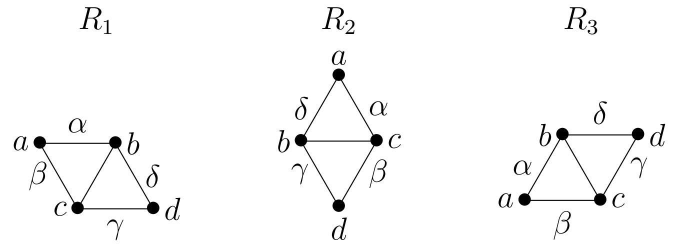


Figure 2: Rhombi

For $\lambda, \mu, \nu \in \Lambda_n$ such that $|\nu| = |\lambda| + |\mu|$, we can define a labeling of the boundary of T_n as in the left of Figure 1.

Theorem 11. *(see [KT99, Appendix]) Let λ, μ and ν in Λ_n such that $|\nu| = |\lambda| + |\mu|$. Then, $c^{\nu}_{\lambda\mu}$ is the number of integral hives whose boundary is labeled by λ, μ, ν.*

Alternatively, one can define a hive as a labeling of the $\frac{3n(n+1)}{2}$ edges such that the edge labels (using the notation as in Figure 2) satisfy

$$\beta \geqslant \delta, \qquad \text{or equivalently} \qquad \alpha \geqslant \gamma, \tag{11}$$

for each one of the $\frac{3n(n-1)}{2}$ rhombi in T_n.

The correspondence between the two definitions of hive is obtained by labeling each edge with the difference between the labels on its vertices, with the orientation illustrated on the right of Figure 1. In other words, to get the label of an edge, one subtracts the label of its leftmost vertice from the label of its rightmost one. Moreover, as shown in Figure 1, the entries of the three partitions are the labels of the edges on the boundary of T_n.

5 The case of $\mathbf{GL_3}(\mathbb{C})$

It is known that the function $(\Lambda_n)^3 \longrightarrow \mathbb{N}$, $(\lambda, \mu, \nu) \longmapsto c^{\nu}_{\lambda\mu}$ is piecewise polynomial (see [Ras04]) of degree $\frac{n^2-3n+2}{2}$. For $n = 3$, we have a more precise statement:

Proposition 12. *Let $\lambda = (\lambda_1, \lambda_2, 0)$, $\mu = (\mu_1, \mu_2, 0)$, and $\nu = (\nu_1, \nu_2, \nu_3)$ in Λ_3 be such that $|\nu| = |\lambda| + |\mu|$. Then $c^{\nu}_{\lambda\mu}$ is the number of integer points in the interval*

$$I = [\max(\mu_1 - \lambda_2, \mu_2, \nu_1 - \lambda_1, \mu_1 - \nu_3, \nu_2 - \lambda_2, \mu_1 + \mu_2 - \nu_2), \min(\mu_1, \nu_1 - \lambda_2, \mu_1 + \mu_2 - \nu_3)]\,.$$

Proposition 12 is well known and can easily be checked using the hive model. Indeed, once λ, μ and ν are fixed, a hive only depends on the label x of the unique central vertex of T_3. By definition, x has to satisfy 9 rhombus inequalities, and hence it has to belong to an interval I'. It is straightforward to verify that I can be obtained from I' via translation by $\lambda_1 + \lambda_2$.

Let c be any nonnegative integer. By Proposition 12,

$$c^{\nu}_{\lambda\mu} > c \iff \varphi - \psi \geqslant c$$

for all linear forms φ and ψ appearing inside the min and max function respectively in the definition of I. Namely, $c^{\nu}_{\lambda\mu} > c$ if and only if

$$\begin{array}{ll}
\lambda_1 - \lambda_2 - c \geqslant 0, & \lambda_2 - c \geqslant 0, \\
\mu_1 - \mu_2 - c \geqslant 0, & \mu_2 - c \geqslant 0, \\
\nu_1 - \nu_2 - c \geqslant 0, & \nu_2 - \nu_3 - c \geqslant 0, \\
\lambda_1 + \mu_1 - \nu_1 - c \geqslant 0, & \lambda_1 + \mu_1 - \nu_2 - \nu_3 - c \geqslant 0, \\
\lambda_1 + \mu_2 - \nu_2 - c \geqslant 0, & \lambda_1 + \lambda_2 + \mu_1 - \nu_1 - \nu_3 - c \geqslant 0, \\
\lambda_1 - \nu_3 - c \geqslant 0, & \lambda_1 + \lambda_2 + \mu_2 - \nu_2 - \nu_3 - c \geqslant 0, \\
\lambda_2 + \mu_1 - \nu_2 - c \geqslant 0, & \lambda_1 + \mu_1 + \mu_2 - \nu_1 - \nu_3 - c \geqslant 0, \\
\mu_1 - \nu_3 - c \geqslant 0, & \lambda_2 + \mu_1 + \mu_2 - \nu_2 - \nu_3 - c \geqslant 0, \\
\lambda_2 + \mu_2 - \nu_3 - c \geqslant 0, & \lambda_1 + \lambda_2 + \mu_1 + \mu_2 - \nu_1 - \nu_2 - c \geqslant 0,
\end{array} \tag{12}$$

and

$$|\nu| = |\lambda| + |\mu|. \tag{13}$$

Note that, for $c = 0$, we recover the 6 inequalities saying that λ, μ and ν are dominant, the 6 Weyl inequalities (7) and the 6 others inequalities (8) of the Horn cone (see *e.g.* [Ful00]).

Let us now compute the function mapping (λ, μ, c) to $\mathrm{Nb}_3(c^{\bullet}_{\lambda\mu} > c)$, the number of solutions of the system (12) whose unknowns are the 3 entries of ν. Our method is restating this problem in the langage of vector partition functions as in [Stu95].

Start with the 18×8 matrix H whose rows are given by the coefficients of the 18 inequalities (12). Set

$$\Lambda = \{(\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2, \nu_3, c) \in \mathbb{Z}^8 \,:\, |\nu| = |\lambda| + |\mu|\}$$

and

$$\Lambda^+ = \{(\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2, \nu_3, c) \in \Lambda \,:\, \lambda, \mu, \nu \text{ dominant and } c \geqslant 0\}.$$

Let $\widetilde{\mathrm{Horn}}_3$ denote the set of points in Λ^+ that satisfy the inequalities (12).

To get nonnegative variables, let us consider the following change of coordinates

$$\begin{array}{lll}
a_1 = \lambda_1 - \lambda_2 - c, & b_1 = \mu_1 - \mu_2 - c, & c_1 = \nu_1 - \nu_2 - c, \\
a_2 = \lambda_2 - c, & b_2 = \mu_2 - c, & c_2 = \nu_2 - c.
\end{array}$$

Then, $\widetilde{\mathrm{Horn}}_3$ identifies with $\widetilde{\mathrm{Horn}}'_3 = \{X \in \mathbb{N}^7 \mid AX \geqslant 0\}$, where

$$A = \begin{pmatrix} -1 & -2 & -1 & -2 & 1 & 3 & -3 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & -1 & -2 & 1 & 2 & -2 \\ 0 & 1 & 1 & 1 & 0 & -1 & 1 \\ -1 & -2 & 0 & -1 & 1 & 2 & -2 \\ -1 & -1 & -1 & -1 & 1 & 2 & -2 \\ 0 & -1 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & 0 & 1 & 1 & -1 \\ 1 & 2 & 1 & 2 & -1 & -2 & 2 \end{pmatrix}.$$

Set $\tilde{A} = (A| - I_{13})$ in such a way that

$$\begin{array}{rcl} \widetilde{\mathrm{Horn}}_3' & \simeq & \{(X,Y) \in \mathbb{N}^7 \times \mathbb{N}^{13} \,|\, AX = Y\} \\ & \simeq & \{X \in \mathbb{N}^{20} \,|\, \tilde{A}X = 0\}. \end{array}$$

We now consider the affine section of $\widetilde{\mathrm{Horn}}_3'$ obtained by fixing λ and μ, as in Identity (9). Thus, up to our changes of variables, the function $(\lambda, \mu, c) \mapsto \mathrm{Nb}_3(c^{\bullet}_{\lambda\mu} > c)$ is the map

$$\begin{array}{rcl} \mathbb{N}^5 & \longrightarrow & \mathbb{N} \\ Y & \longmapsto & \#\{X \in \mathbb{N}^{15} \,:\, \tilde{B}X = -CY\}, \end{array}$$

where $\tilde{B} = (B \,| - I_{13})$, B is the matrix formed by columns 5 and 6 of the matrix A, and C is the matrix formed by the other columns of A.

Note that $\tilde{B}$ is not unimodular: the least common multiple of the maximal minors is not 1, but 6. There are 83 such nonzero minors. As a result, [Stu95] implies that $(\lambda, \mu, c) \mapsto \mathrm{Nb}_3(c^{\bullet}_{\lambda\mu} > c)$ is piecewise quasi-polynomial with respect to a fan whose the maximal cones are obtained by intersecting some of 83 explicit simplicial cones. We used [VSB+07], an implementation of Barvinok algorithm [Bar94], to compute this function. Surprisingly, we got only polynomial functions and only 7 maximal cones. In fact, the program produced 36 cones that turned out can be glued to give the 7 described in Proposition 13.

Proposition 13. *Let us write the partitions λ, μ in terms of fundamental weights: $\lambda = k_1\varpi_1 + k_2\varpi_2 = (k_1 + k_2)k_2$ and $\mu = l_1\varpi_1 + l_2\varpi_2 = (l_1 + l_2)l_2$. Then,* $\mathrm{Nb}_3(c^{\bullet}_{\lambda\mu} > c) = 0$ *unless*

$$c \leqslant \min(k_1, k_2, l_1, l_2). \tag{14}$$

Moreover, the set of $(c, k_1, k_2, l_1, l_2) \in \mathbb{N}^5$ satisfying (14) decomposes into 7 cones $C_1, \ldots, C_7$ on which $\mathrm{Nb}_3(c^{\bullet}_{\lambda\mu} > c)$ is given by polynomial functions $P_1, \ldots, P_7$ respectively. Five of these seven pairs (C_i, P_i) are kept unchanged by swapping k_1 and k_2. The two others are swapped by this operation.

In particular, Conjecture 2 holds for $\mathrm{GL}_3(\mathbb{C})$.

Proof. In the basis of fundamental weights, we are interested in the function

$$\begin{array}{rccl} \psi : & \mathbb{N}^5 & \longrightarrow & \mathbb{N} \\ & (k_1, k_2, l_1, l_2, c) & \longmapsto & \#\{\nu \in \Lambda_3 \,|\, c^{\nu}_{k_1\varpi_1+k_2\varpi_2, l_1\varpi_1+l_2\varpi_2} > c\} \end{array}.$$

Notice that swapping k_1 and k_2 corresponds to replacing λ by λ^*. Define now the following seven polynomial functions in (k_1, k_2, l_1, l_2, c):

$$P_1 = 2\,c^2 - c(k_1 + k_2 + l_1 + l_2 + 2) - \frac{1}{2}(k_1 + k_2 - l_1 - l_2)^2 + k_1k_2 + l_1l_2 + \frac{1}{2}(k_1 + k_2 + l_1 + l_2) + 1,$$

$$P_2 = 3c^2 - 3c(k_1 + k_2 + 1) + \frac{1}{2}(k_1 + k_2)^2 + k_1k_2 + \frac{3}{2}(k_1 + k_2) + 1,$$

$$P_3 = 3c^2 - 3c(l_1 + l_2 + 1) + \frac{1}{2}(l_1 + l_2)^2 + l_1l_2 + \frac{3}{2}(l_1 + l_2) + 1,$$

$$P_4 = \frac{5}{2}\,c^2 - c\left(2\,k_1 + 2\,k_2 + l_1 + \frac{5}{2}\right) + k_1k_2 + (k_1 + k_2)(l_1 + 1) - \frac{l_1}{2}(l_1 - 1) + 1,$$

$$P_5 = \frac{5}{2}\,c^2 - c\left(2\,k_1 + 2\,k_2 + l_2 + \frac{5}{2}\right) + k_1k_2 + (k_1 + k_2)(l_2 + 1) - \frac{l_2}{2}(l_2 - 1) + 1,$$

$$P_6 = \frac{5}{2}\,c^2 - c\left(k_1 + 2l_1 + 2l_2 + \frac{5}{2}\right) + l_1l_2 + (l_1 + l_2)(k_1 + 1) - \frac{k_1}{2}(k_1 - 1) + 1,$$

$$P_7 = \frac{5}{2}\,c^2 - c\left(k_2 + 2l_1 + 2l_2 + \frac{5}{2}\right) + l_1l_2 + (l_1 + l_2)(k_2 + 1) - \frac{k_2}{2}(k_2 - 1) + 1.$$

Notice that $P_1, \dots, P_5$ are symmetric in k_1, k_2, whereas P_6 and P_7 are interchanged when swapping k_1 and k_2. Moreover, notice that under the involution corresponding to swapping λ and μ, – i.e. swapping (k_1, k_2) and (l_1, l_2) –, P_3, P_6, P_7 are the images of P_2, P_4, P_5 respectively. Now in the case where $k_1, k_2, l_1, l_2 \geqslant c \geqslant 0$, the function ψ is given by the following piecewise polynomial function:

Cones of polynomiality	Polynomial giving ψ
$C_1 : k_1 + k_2 \geqslant \max(l_1, l_2) + c, \quad l_1 + l_2 \geqslant \max(k_1, k_2) + c$	P_1
$C_2 : k_1 + k_2 \leqslant \min(l_1, l_2) + c$	P_2
$C_3 : l_1 + l_2 \leqslant \min(k_1, k_2) + c$	P_3
$C_4 : l_1 + c \leqslant k_1 + k_2 \leqslant l_2 + c$	P_4
$C_5 : l_2 + c \leqslant k_1 + k_2 \leqslant l_1 + c$	P_5
$C_6 : k_1 + c \leqslant l_1 + l_2 \leqslant k_2 + c$	P_6
$C_7 : k_2 + c \leqslant l_1 + l_2 \leqslant k_1 + c$	P_7

One can then see that the cones C_1 to C_5 are stable when swapping k_1 and k_2 whereas the cones C_6 and C_7 are swapped when k_1 and k_2 are. Thus, for all $k_1, k_2, l_1, l_2, c \geqslant 0$,

$$\psi(k_1, k_2, l_1, l_2, c) = \psi(k_2, k_1, l_1, l_2, c).$$

This completes the proof of Proposition 13. $\square$

Remark 14. The last part of Proposition 13 asserts that there exists a bijection $(\Lambda_3^0)^2 \times \Lambda_3 \longrightarrow (\Lambda_3^0)^2 \times \Lambda_3$, $(\lambda, \mu, \nu) \longmapsto (\lambda^*, \mu, \tilde{\nu})$ such that

$$c_{\lambda\mu}^{\nu} = c_{\lambda^*\mu}^{\tilde{\nu}}.$$

One could hope for such a bijection to be linear. Unfortunately, it **cannot**.

One can check as follows. Identify $(\Lambda_3^0)^2 \times \Lambda_3$ as a subset of $\mathbb{Z}^7$ canonically. Assume that φ is the restriction of some linear map $\tilde{\varphi}$ on $\mathbb{Z}^7$. Endow $\mathbb{Z}^7$ with its standart basis. The matrix of $\tilde{\varphi}$ is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \nu^1 & \nu^2 & \nu^3 & \nu^4 & \nu^5 & \nu^6 & \nu^7 \end{pmatrix}$$

for some $\nu^1, \ldots, \nu^7$ in $\mathbb{Z}^3$. Since $\tilde{\varphi}(1,0,0,0,1,0,0) = (1,1,0,0,\nu^1+\nu^5)$ has to correspond to some nonzero Littlewood-Richardson coefficient, we have $\nu_1 + \nu_5 = (1,1,0)$. Similarly, $\tilde{\varphi}(0,0,1,1,1,1,0)$ has to be $(0,0,1,1,1,1,0)$, and $\nu^3+\nu^4+\nu^5+\nu^6 = (1,1,0)$. Now, the image of $\tilde{\varphi}(1,0,1,1,1,1,1)$ has to be a ray of the Horn cone. We deduce that this image is $(1,1,1,1,2,2,0)$, and thus $\nu^1+\nu^3+\nu^4+\nu^5+\nu^6+\nu^7 = (2,2,0)$.

Combining these three constraints, we get $\nu^5 = \nu^7$, which contradicts the invertibility of φ.

Note also that the linear automorphisms of $(\Lambda_3)^3$ preserving the Littlewood-Richardson coefficients are proved to form a group of cardinality 288 (so big !) in [BR20].

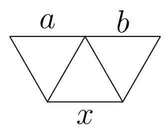
6 A stability result

In this section, we will focus on the case where λ and μ are near-rectangular. Using the hive model, we give a proof of Proposition 4.

Proof of Proposition 4. Using Theorem 11, we prove the proposition by counting the integral hives with boundary labels determined by λ, μ and ν as in Figure 1. Consider such an integral hive and focus on the labels of its edges. An edge is said to be *strictly interior* if it has no vertex on the sides of T_n.

Observe that for any unitary triangle, the label of the horizontal edge is the sum of the labels of its two other sides. This determines the labels of the three edges in the corners of T_n as in Figure 3.

Consider now a trapezoid



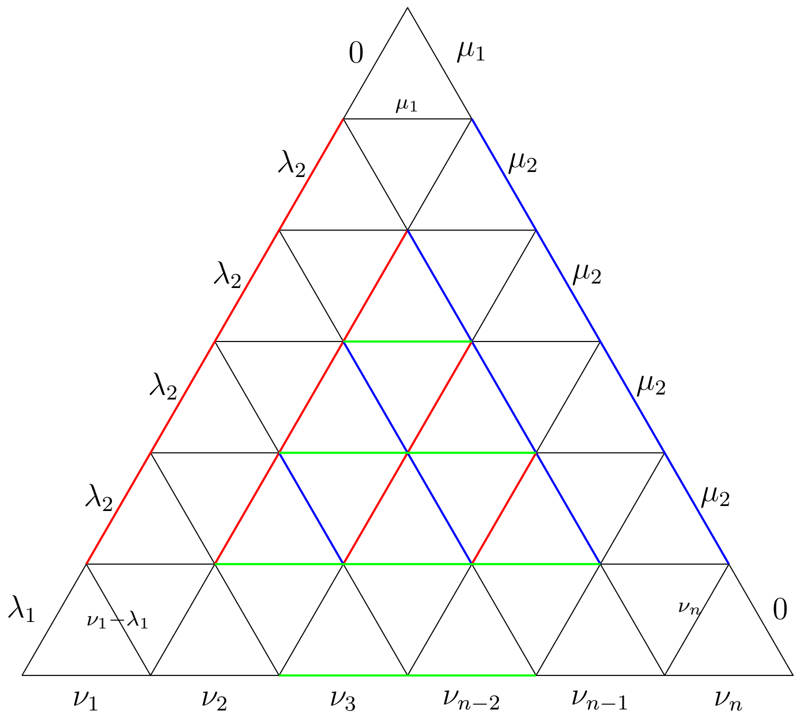


Figure 3: Hives for two near rectangular partitions

Inequalities (11) imply that $a \geqslant x \geqslant b$. In particular, if $a = b$ then $x = a$. Moreover, if $a = b$ then the labels on two parallel edges of the trapezoid are equal. The trapezoids with long side parallel to the two other sides of T_n have similar properties.

It then follows that the labels on the strictly interior edges parallel to the north-west (resp. north-east) side of T_n are equal to λ_2 (resp. μ_2). The relation between the 3 labels of any unitary triangle then implies that the labels on the strictly interior horizontal edges are equal to $\lambda_2 + \mu_2$. Now, the mentioned properties of the trapezoids imply that $\nu_3 = \cdots = \nu_{n-2} = \lambda_2 + \mu_2$. That is, ν must have the aforementioned form: $\nu = \nu_1\nu_2(\lambda_2 + \mu_2)^{n-4}\nu_{n-1}\nu_n$. Notice moreover that, even if $n = 4$, Inequalities (11) show immediately that one must still have $\nu_2 \geqslant \lambda_2 + \mu_2 \geqslant \nu_{n-1}$.

Let us now consider the labels of the edges with exactly one vertex on the boundary of T_n. By the properties of the trapezoids and unitary triangles as mentioned above, these labels depend on 8 values $a_0, a_1, \ldots, a_7$ as shown on Figure 4, and they are related by the following equations:

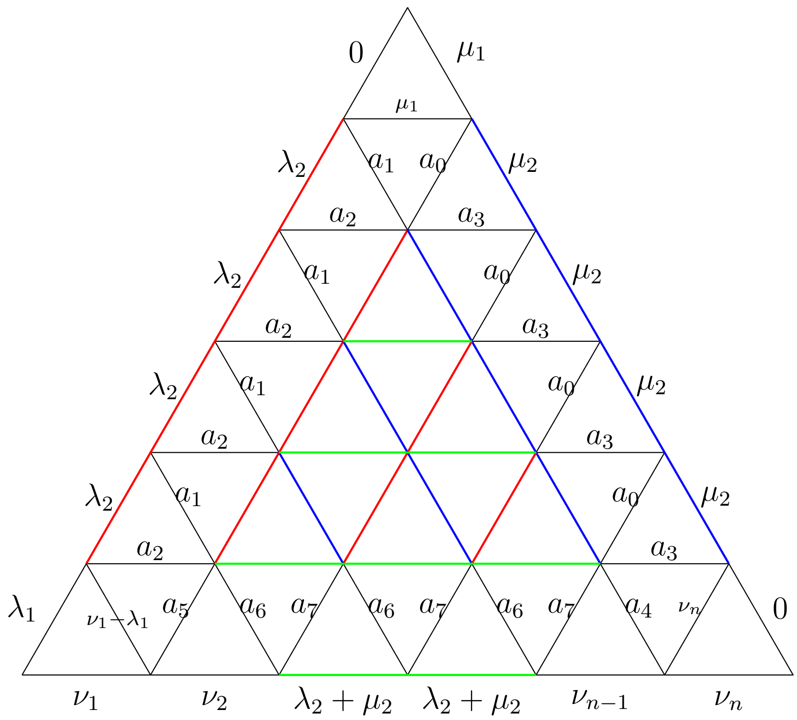


Figure 4: Hives for two near rectangular partitions 2

$$
\begin{cases}
a_0+a_1=\mu_1\\
a_0+\mu_2=a_3\\
\lambda_2+a_1=a_2\\
\nu_1-\lambda_1+a_5=a_2\\
a_4+\nu_n=a_3\\
a_4+a_7=\nu_{n-1}\\
a_5+a_6=\nu_2\\
a_6+a_7=\lambda_2+\mu_2
\end{cases}
\iff
\begin{cases}
a_1=\mu_1-a_0\\
a_2=\lambda_2+\mu_1-a_0\\
a_3=\mu_2+a_0\\
a_4=\mu_2-\nu_n+a_0\\
a_5=\lambda_1+\lambda_2+\mu_1-\nu_1-a_0\\
a_6=\lambda_2+2\mu_2-\nu_{n-1}-\nu_n+a_0\\
a_7=-\mu_2+\nu_{n-1}+\nu_n-a_0
\end{cases}
$$

In particular, the hive is entirely determined by the value of a_0. We can now look at all the rhombus inequalities that must be satisfied by these a_i's:

$$
\begin{array}{rcl}
a_0 & \geqslant & 0,\\
\lambda_2 & \geqslant & a_0,\\
a_1 & \geqslant & \mu_2,
\end{array}
\qquad
\begin{array}{rcl}
\nu_1 & \geqslant & a_2,\\
a_1 & \geqslant & \nu_1-\lambda_1,\\
a_2 & \geqslant & \nu_2,\\
\mu_2 & \geqslant & a_6,
\end{array}
\qquad
\begin{array}{rcl}
a_3 & \geqslant & \nu_n,\\
\nu_n & \geqslant & a_0,\\
\nu_{n-1} & \geqslant & a_3.
\end{array}
$$

Expressing these inequalities in terms of a_0, we obtain $m \geqslant a_0 \geqslant M$, where

$$
\begin{aligned}
&M=\max(0,\lambda_2+\mu_1-\nu_1,\nu_n-\mu_2), \quad \text{and}\\
&m=\min(\lambda_2,\mu_1-\mu_2,\lambda_1+\mu_1-\nu_1,\lambda_2+\mu_1-\nu_2,-\lambda_2-\mu_2+\nu_{n-1}+\nu_n,\nu_n,\nu_{n-1}-\mu_2).
\end{aligned}
$$

Observe finally that, by assumption, $\nu_{n-1} \leqslant \lambda_2+\mu_2$ and $\nu_2 \geqslant \lambda_2+\mu_2$. Thus

$$
\nu_{n-1}-\mu_2 \leqslant \lambda_2, \qquad \lambda_2+\mu_1-\nu_2 \leqslant \mu_1-\mu_2, \qquad -\lambda_2-\mu_2+\nu_{n-1}+\nu_n \leqslant \nu_n.
$$

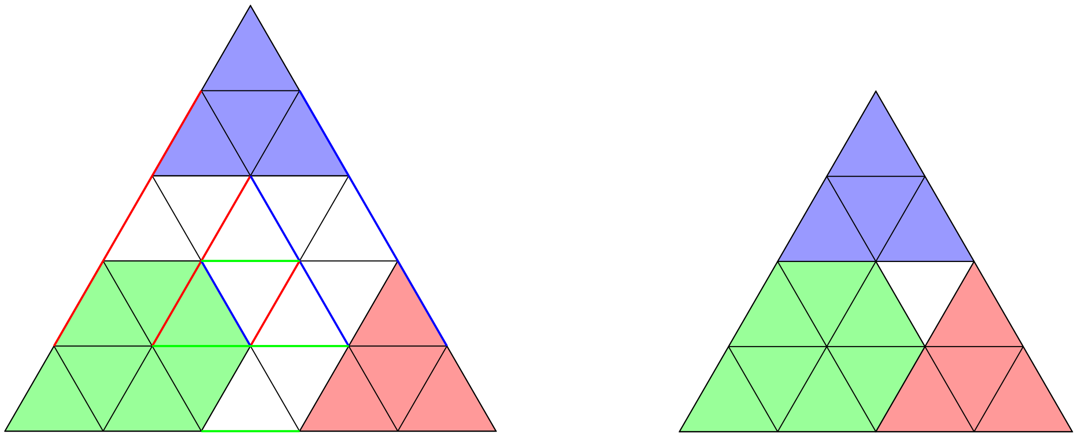


Figure 5: A bijection between sets of hives

We can rewrite the definition of m as

$$m = \min(\lambda_1 + \mu_1 - \nu_1, \lambda_2 + \mu_1 - \nu_2, -\lambda_2 - \mu_2 + \nu_{n-1} + \nu_n, -\mu_2 + \nu_{n-1}).$$

Since no integral hive exists if $M > m$, and $m - M + 1$ hives otherwise, the proof is complete. $\square$

The stability result of Proposition 4 can be interpreted as a proof of the existence of a bijection between sets of hives. Such a bijection can, for instance, be obtained as follows.

Starting from a hive of size n ($n \geqslant 4$), consider the three areas colored on the left of Figure 5: the four triangles in the north corner, the four in the south-east one, and the seven in the south-west one. Then, send this hive to the one of size 4 obtained by keeping these three colored-areas (picture on the right). The proof of Proposition 4 implies that this map is well defined, and that it is a bijection.

Remark 15. Let α, β, γ be three partitions such that $c^{\gamma}_{\alpha,\beta} = 1$. By Fulton's Conjecture (see [KT99] or [Bel07, BKR12, Res11]), we have $c^{k\gamma}_{k\alpha,k\beta} = 1$, for any $k \geqslant 0$. Let $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ be a second triple of partitions. The stability result of [SS16] (see also [Par19, Pel19]) asserts that $c^{\tilde{\gamma}+k\gamma}_{\tilde{\alpha}+k\alpha,\tilde{\beta}+k\beta}$ does not depend on the integer k big enough.

Returning to the setting of Proposition 4, consider $\alpha = 1^{\lambda_2}$, $\beta = 1^{\mu_2}$ and $\gamma = 1^{\lambda_2+\mu_2}$ that satisfy $c^{\gamma}_{\alpha,\beta} = 1$. Set also $\tilde{\alpha} = (\lambda_1\lambda_2\lambda_2)'$, $\tilde{\beta} = (\mu_1\mu_2)'$ and $\tilde{\gamma} = (\nu_1\nu_2\nu_3\nu_4)'$, where $\square'$ denotes the conjugate partition. Since the Littlewood-Richardson coefficient is invariant under simultaneous conjugation of the three partitions, we get $c^{\nu}_{\lambda\mu} = c^{\tilde{\gamma}+k\gamma}_{\tilde{\alpha}+k\alpha,\tilde{\beta}+k\beta}$ for $k = n-2$. Thus, the stability result of [SS16] asserts that $c^{\nu}_{\lambda\mu}$ does not depend on n big enough. Proposition 4 asserts that, more precisely, this sequence is constant for $k \geqslant 2$.

7 The case of $\mathrm{GL}_4(\mathbb{C})$

This section is about $\mathrm{GL}_4(\mathbb{C})$. But Proposition 4 allows to extend several results to any $\mathrm{GL}_n(\mathbb{C})$ for $n \geqslant 4$.

7.1 The Horn cone

The set of points in Horn_n with λ and/or μ near-rectangular is the set of integer points on a face of this cone. Proposition 4 implies that the geometry of this face and the Littlewood-Richardson coefficients on it do not depend on $n \geqslant 4$. We denote by Horn_n^0 the set of points in Horn_n with the first two partitions λ and μ in Λ_n^0. Then $\mathrm{Horn}_n \simeq \mathbb{Z}^2 \times \mathrm{Horn}_n^0$. Moreover, Horn_n^0 is the set of integer points in some strongly convex cone. Hence it has a unique minimal set of generators, the *Hilbert basis* of the cone. Set

$$\mathrm{Horn}_4^{\mathrm{nr}^2} = \{(\lambda, \mu, \nu) \in \mathrm{Horn}_4 \,:\, \lambda \text{ and } \mu \text{ are near-rectangular}\}$$

and

$$\mathrm{Horn}_4^{\mathrm{nr}} = \{(\lambda, \mu, \nu) \in \mathrm{Horn}_4 \,:\, \lambda \text{ is near-rectangular}\}.$$

The inequalities defining the Horn cone $\mathrm{Horn}_n^{\mathbb{Q}}$ are well known (see Section 3). By convex geometry and explicit calculations, one can deduce the minimal lists of inequalities for $\mathrm{Horn}_4^{\mathrm{nr}^2}$ and $\mathrm{Horn}_4^{\mathrm{nr}}$. Softwares like Normaliz [BIS] also allow to make the computation.

Proposition 16. *Let λ, μ in Λ_4^0 and ν in Λ_4 such that λ and μ are near-rectangular. Then, $c_{\lambda\mu}^{\nu} \neq 0$ if and only if*

$$\begin{array}{cc}
\multicolumn{2}{c}{|\lambda| + |\mu| = |\nu|,} \\
\nu_1 \geqslant \nu_2, & \nu_4 \geqslant 0, \\
\multicolumn{2}{c}{\nu_3 + \nu_4 \geqslant \lambda_2 + \mu_2,} \\
\nu_1 + \nu_3 \geqslant \lambda_1 + \lambda_2 + \mu_2, & \nu_1 + \nu_3 \geqslant \lambda_2 + \mu_1 + \mu_2, \\
\multicolumn{2}{c}{\nu_2 \geqslant \lambda_2 + \mu_2 \geqslant \nu_3,} \\
\nu_3 \geqslant \lambda_2, & \nu_3 \geqslant \mu_2, \\
\lambda_1 + \mu_2 \geqslant \nu_2, & \lambda_2 + \mu_1 \geqslant \nu_2.
\end{array}$$

Remark 17. Proposition 4 also implies that $\nu_1 + \nu_4 \geqslant \lambda_2 + \mu_1$. This is a consequence of these 11 inequalities.

Proposition 18. *The cone generated by $\mathrm{Horn}_4^{\mathrm{nr}^2} \cap \mathrm{Horn}_4^0$ has 8 extremal rays generated by the triples (λ, μ, ν) associated to the following inclusions*

1. $V_4(1) \subset V_4(1) \otimes V_4(0)$ *(twice by permuting the factors);*
2. $V_4(1^3) \subset V_4(1^3) \otimes V_4(0)$ *(twice by permuting the factors);*
3. $V_4(1^2) \subset V_4(1) \otimes V_4(1)$*;*
4. $V_4(1^4) \subset V_4(1) \otimes V_4(1^3)$ *(twice by permuting the factors);*
5. $V_4(2^2 1^2) \subset V_4(1^3) \otimes V_4(1^3)$.

Each triple (λ, μ, ν) on one of these extremal rays indexes a Littlewood-Richardson coefficient with value 1. The minimal set of generators of the semigroup $\mathrm{Horn}_4^{\mathrm{nr}^2} \cap \mathrm{Horn}_4^0$ consists in these 8 triples.

We get similar descriptions for $\mathrm{Horn}_4^{\mathrm{nr}}$.

Proposition 19. *Let λ, μ in Λ_4^0 and ν in Λ_4 such that λ is near-rectangular. Then, $c_{\lambda\mu}^{\nu} \neq 0$ if and only if all of the following inequalities hold:*

$$
\begin{array}{ll}
\multicolumn{2}{c}{|\lambda| + |\mu| = |\nu|,} \\
\multicolumn{2}{c}{\nu_1 \geqslant \nu_2 \geqslant \nu_3 \geqslant \nu_4 \geqslant 0,} \\
\lambda_1 \geqslant \lambda_2 \geqslant 0, & \mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant 0, \\
\lambda_1 + \mu_1 \geqslant \nu_1, & \min(\lambda_2 + \mu_1, \lambda_1 + \mu_2) \geqslant \nu_2, \\
\min(\lambda_2 + \mu_2, \lambda_1 + \mu_3) \geqslant \nu_3, & \min(\lambda_1, \mu_1, \lambda_2 + \mu_3) \geqslant \nu_4, \\
\multicolumn{2}{c}{\nu_1 \geqslant \max(\lambda_1, \mu_1, \lambda_2 + \mu_2), \qquad \nu_2 \geqslant \max(\mu_2, \lambda_2 + \mu_3), \qquad \nu_3 \geqslant \max(\lambda_2, \mu_3) \qquad ,} \\
\nu_1 + \nu_2 \geqslant \max(\lambda_1 + \lambda_2 + \mu_2, \lambda_2 + \mu_1 + \mu_2), & \nu_1 + \nu_3 \geqslant \max(\lambda_1 + \lambda_2 + \mu_3, \lambda_2 + \mu_1 + \mu_3), \\
\nu_2 + \nu_3 \geqslant \lambda_2 + \mu_2 + \mu_3, & \nu_1 + \nu_4 \geqslant \lambda_2 + \mu_1, \\
\nu_2 + \nu_4 \geqslant \lambda_2 + \mu_2, & \nu_3 + \nu_4 \geqslant \lambda_2 + \mu_3.
\end{array}
$$

These are 32 inequalities, and each corresponds to a facet of the cone they define.

Proposition 20. *The cone generated by $\mathrm{Horn}_4^{\mathrm{nr}} \cap \mathrm{Horn}_4^0$ has 12 extremal rays generated by the triples (λ, μ, ν) associated to the following inclusions*

1. *$V_4(1) \subset V_4(1) \otimes V_4(0)$, $V_4(1) \subset V_4(1^3) \otimes V_4(0)$, $V_4(1) \subset V_4(0) \otimes V_4(1)$, $V_4(1^2) \subset V_4(0) \otimes V_4(1^2)$ and $V_4(1^3) \subset V_4(0) \otimes V_4(1^3)$;*
2. *$V_4(1^2) \subset V_4(1) \otimes V_4(1)$;*
3. *$V_4(1^4)$ is contained in $V_4(1) \otimes V_4(1^3)$ and $V_4(1^3) \otimes V_4(1)$;*
4. *$V_4(2^2 1^2)$ is contained in $V_4(1^3) \otimes V_4(1^3)$ and $V_4(21^2) \otimes V_4(1^2)$;*
5. *$V_4(1^3) \subset V_4(1) \otimes V_4(1^2)$;*
6. *$V_4(21^3) \subset V_4(1^3) \otimes V_4(1^2)$.*

Each triple (λ, μ, ν) on one of these extremal rays indexes a Littlewood-Richardson coefficient with value 1. The minimal set of genrators of the semigroup $\mathrm{Horn}_4^{\mathrm{nr}} \cap \mathrm{Horn}_4^0$ consists in the 12 primitive elements on these 12 extremal rays.

7.2 Special case of self-dual representations

Let k and l be two nonnegative integers and $n \geqslant 4$. The $\mathrm{SL}_n(\mathbb{C})$-representations $V_n((2k)k^{n-2})$, $V_n((2l)l^{n-2})$ and hence $V_n((2k)k^{n-2}) \otimes V_n((2l)l^{n-2})$ are self-dual.

In [PW22, Section 8], conjectural values (for $n = 6$) are given for the numbers of isotypic components in $V_n((2k)k^{n-2}) \otimes V_n((2l)l^{n-2})$ and for the numbers of self-dual isotypic components. Here we prove and extend these formulas.

Corollary 21. *Let us assume, without loss, that $l \leqslant k$. The number of distinct isotypic components in $V_n((2k)k^{n-2}) \otimes V_n((2l)l^{n-2})$ is*

$$
\begin{cases} l^3 + 3l^2 + 3l + 1 & \text{if } 2l \leqslant k, \\ \frac{1}{3}k^3 - 2k^2 l + 4kl^2 - \frac{5}{3}l^3 - k^2 + 4kl - l^2 + \frac{2}{3}k + \frac{5}{3}l + 1 & \text{if } 2l \geqslant k. \end{cases}
$$

Proof. By Proposition 4, one may assume that $n = 4$. Then Proposition 16 implies that $\nu \in \mathbb{Z}^4$ is the highest weight of an isotypic component of $V_4((2k)k^2) \otimes V_4((2l)l^2)$ if and only if (recall that $l \leqslant k$) all of the following conditions hold:

$$\begin{array}{cc} 4(k+l) = |\nu|, & \nu_1 \geqslant \nu_2, \\ \nu_4 \geqslant 0, & \nu_3 + \nu_4 \geqslant k + l, \\ \multicolumn{2}{c}{\nu_1 + \nu_3 \geqslant 3k + l,} \\ \multicolumn{2}{c}{2n + m \geqslant \nu_2 \geqslant k + l \geqslant \nu_3 \geqslant k.} \end{array} \tag{15}$$

The corollary follows by explicit calculations that can be performed with [VSB+07]. □

Similarly, one gets the number of self-dual representations.

Corollary 22. *Let us assume, without loss, that $l \leqslant k$. The $\mathrm{SL}_n(\mathbb{C})$-representation $V_n((2k)k^{n-2}) \otimes V_n((2l)l^{n-2})$ contains $(l+1)^2$ distinct self-dual isotypic components.*

Proof. By Proposition 4, one may assume that $n = 4$. Then the set of self-dual isotypic components of $V_4((2k)k^2) \otimes V_4((2l)l^2)$ is obtained by adding the condition

$$\nu_1 + \nu_4 = \nu_2 + \nu_3,$$

to the conditions (15). The corollary follows by explicit calculations that can be performed with [VSB+07]. □

7.3 Computation of $\mathrm{Nb}_n(c^{\bullet}_{\lambda\mu} > c)$ for λ and μ near-rectangular

In this subsection, we report on the computation of the function

$$\begin{array}{rccl} \mathrm{Nb}_4(c^{\bullet}_{\lambda\mu} > c) : & (\Lambda_4^{\mathrm{nr}})^2 \times \mathbb{N} & \longrightarrow & \mathbb{N} \\ & (\lambda, \mu, c) & \longmapsto & \#\{\nu \in \Lambda_4 \,:\, c^{\nu}_{\lambda\mu} > c\}. \end{array}$$

By Proposition 4, this function determines $\mathrm{Nb}_n(c^{\bullet}_{\lambda\mu} > c)$ for any near-rectangular partitions λ and μ of length $n \geqslant 4$.

Since Propositions 12 and 4 give similar expressions for the Littlewood-Richardson coefficient, we can apply the strategy of Section 5.

We get that $\mathrm{Nb}_4(c^{\bullet}_{\lambda\mu} > c)$ is the number of points $\nu \in \Lambda_4$ such that $\lambda_1 + 2\lambda_2 + \mu_1 + 2\mu_2 = \nu_1 + \nu_2 + \nu_3 + \nu_4$ and

$$\begin{array}{ll} -\lambda_2 - \mu_2 + \nu_2 \geqslant 0, & \lambda_2 + \mu_2 - \nu_3 \geqslant 0, \\ \lambda_1 - \lambda_2 \geqslant c, & -\lambda_2 + \nu_3 \geqslant c, \\ \nu_1 - \nu_2 \geqslant c, & \nu_3 - \nu_4 \geqslant c, \\ \lambda_1 + \mu_1 - \nu_1 \geqslant c, & \lambda_2 + \mu_1 - \nu_2 \geqslant c, \\ -\lambda_2 - \mu_2 + \nu_3 + \nu_4 \geqslant c, & -\mu_2 + \nu_3 \geqslant c, \\ \lambda_1 + \mu_2 - \nu_2 \geqslant c, & \lambda_1 + \lambda_2 + \mu_2 - \nu_2 - \nu_4 \geqslant c, \\ \lambda_1 + \mu_1 + \mu_2 - \nu_1 - \nu_4 \geqslant c, & \lambda_2 + \mu_1 + \mu_2 - \nu_2 - \nu_4 \geqslant c. \end{array}$$

In particular, $\mathrm{Nb}_4(c^{\bullet}_{\lambda\mu} > c)$ is the number of integer points in some polytope depending linearly on the data (λ, μ, c). Therefore $\mathrm{Nb}_4(c^{\bullet}_{\lambda\mu} > c)$ is piecewise quasi-polynomial, and can be computed using Barvinok's algorithm. Surprisingly, here $\Lambda = (\Lambda_4^{\mathrm{nr}})^2 \times \mathbb{Z}$ and $\mathrm{Nb}_4(c^\bullet_{\lambda\mu} > c)$ is in fact piecewise polynomial.

As in Section 5, from this point on we use the basis of fundamental weights to write $\lambda = k_1\varpi_1 + k_2\varpi_1^*$ and $\mu = l_1\varpi_1 + l_2\varpi_1^*$. Thus the symmetry we want to observe corresponds once again to swapping k_1 and k_2. Consider the function

$$\begin{array}{rccl} \psi: & \mathbb{N}^5 & \longrightarrow & \mathbb{N} \\ & (k_1,k_2,l_1,l_2,c) & \longmapsto & \#\{\nu \in \Lambda_4 \mid c^\nu_{k_1\varpi_1+k_2\varpi_1^*, l_1\varpi_1+l_2\varpi_1^*} > c\} \end{array}.$$

We now give details about the results in Proposition 5:

Proposition 23. *We have $\psi(k_1,k_2,l_1,l_2,c) = 0$ unless*

$$c \leqslant \min(k_1,k_2,l_1,l_2). \tag{16}$$

Moreover, the set of $(k_1,k_2,l_1,l_2,c) \in \mathbb{Q}^5$ satisfying (16) decomposes into 36 cones $C_1,\dots,C_{36}$, such that on the integer points of each ψ is given by polynomial functions $P_1,\dots,P_{36}$. The first 12 of these pairs (C_i,P_i) are kept unchanged by swapping k_1 and k_2. The 24 other such pairs are pairwise swapped by this operation (for all $i \in \{7,\dots,18\}$, (C_{2i-1},P_{2i-1}) and (C_{2i},P_{2i}) are swapped).

In particular, Conjecture 2 holds for $\mathrm{GL}_4(\mathbb{C})$ and λ,μ near-rectangular.

To present these cones and polynomial functions as clearly as possible without writing all of them, let us define the two following involutions. Let s_1 be the involution corresponding to swapping k_1 and k_2, and s_2 corresponding to swapping (k_1,k_2) and (l_1,l_2). Then, the generated group $\langle s_1,s_2\rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts on the set of all pairs (C_i,P_i) with 8 orbits. Let us give below one representative for each one of these 8 orbits. The labelling is the one of the complete list [Res20, pol_and_cones_SL4nr2.txt], chosen so that the stability when swapping k_1 and k_2 is easier to see:

$$C_1: \quad l_1+l_2 \leqslant k_1+c, \quad l_1+l_2 \leqslant k_2+c,$$

$$P_1 = \left(-\frac{1}{2}\right)\cdot(-l_2+c-1)\cdot(-l_1+c-1)\cdot(-l_1-l_2+2c-2)$$

has a $\langle s_1,s_2\rangle$-orbit of size 2;

$$C_{16}: \quad l_1+l_2 \leqslant k_1+c, \quad l_1+l_2 \geqslant k_2+c, \quad k_2 \geqslant l_1, \quad k_2 \geqslant l_2,$$

$$P_{16} = P_1 - \binom{-k_2+l_1+l_2-c+2}{3}$$

has an orbit of size 4;

$$C_2: \quad l_1+l_2 \geqslant k_1+c, \quad l_1+l_2 \geqslant k_2+c, \quad k_1 \geqslant l_1, \quad k_1 \geqslant l_2, \quad k_2 \geqslant l_1, \quad k_2 \geqslant l_2,$$

$$P_2 = P_{16} - \binom{-k_1+l_1+l_2-c+2}{3}$$

has an orbit of size 2;

$$C_{19}: \quad l_1 + l_2 \geqslant k_1 + c, \quad k_1 \geqslant l_1, \quad k_2 \leqslant l_1, \quad k_2 \geqslant l_2,$$
$$P_{19} = P_2 + \binom{-k_2 + l_1 + 1}{3}$$

has an orbit of size 8;

$$C_{21}: \quad l_1 + l_2 \leqslant k_1 + c, \quad k_2 \leqslant l_1, \quad k_2 \geqslant l_2,$$
$$P_{21} = P_{16} + \binom{-k_2 + l_1 + 1}{3}$$

has an orbit of size 8;

$$C_{29}: \quad k_1 + k_2 \geqslant l_1 + l_2, \quad l_1 + l_2 \geqslant k_1 + c, \quad k_2 \leqslant l_1, \quad k_2 \leqslant l_2,$$
$$P_{29} = P_{19} + \binom{-k_2 + l_2 + 1}{3}$$

has an orbit of size 4;

$$C_{27}: \quad k_1 + k_2 \leqslant l_1 + l_2, \quad k_1 \geqslant l_1, \quad k_1 \geqslant l_2,$$
$$P_{27} = P_{29} + \binom{-k_1 - k_2 + l_1 + l_2 + 1}{3}$$

has an orbit of size 4; finally,

$$C_{36}: \quad l_1 + l_2 \leqslant k_1 + c, \quad k_2 \leqslant l_1, \quad k_2 \leqslant l_2,$$
$$P_{36} = P_{21} + \binom{-k_2 + l_2 + 1}{3}$$

also has an orbit of size 4.

Remark 24. One can observe that the polynomial functions P_i are expressed using each other. We exploit here the fact that the difference between two polynomial functions associated to two adjacent cones has a simple expression theoretically given by Paradan's Formula [Par04, BV09].

7.4 Computation of $\mathbf{Nb}_4(c^\bullet_{\lambda\mu} > 0)$ for $\boldsymbol{\lambda}$ near-rectangular

In this section, we report on the computation of the function

$$\begin{array}{rccl} \mathrm{Nb}_4(c^\bullet_{\lambda\mu} > 0): & \Lambda_4^{\mathrm{nr}} \times \Lambda_4 & \longrightarrow & \mathbb{N} \\ & (\lambda, \mu) & \longmapsto & \#\{\nu \in \Lambda_4 : c^\nu_{\lambda\mu} > 0\}. \end{array}$$

As we recalled in Proposition 10, $\mathrm{Nb}_4(c^\bullet_{\lambda\mu} > 0)$ is the number of integer points in an affine section of the Horn cone. The inequalities defining this cone are explicitly given in Proposition 19. One can compute explicitly the quasi-polynomial functions with the program [VSB+07]. The output is too big (even using symmetries) to be collected there, so we include the details in [Res20, Supplementary material] for interested reader.

Proposition 25. *The cone generated by $\Lambda_4^{\mathrm{nr}} \times \Lambda_4$ decomposes into 205 cones of non empty interior. On the integer points of 151 of them, the function $\mathrm{Nb}_4(c^\bullet_{\lambda\mu} > 0)$ is polynomial of degree 3, and on those of 54 other ones it is quasi-polynomial. The only congruence occurring is the parity of $\lambda_1 + |\mu|$.*

Moreover, for any pair (C, P) where C is one of the 205 cones and P the corresponding function, one can see that in this list there is also a pair (C', P') obtained by replacing λ by λ^* (in 57 cases, $(C', P') = (C, P)$). In particular, Conjecture 7 holds.

Under the action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, there are 61 *orbits of actual polynomial functions and* 22 *orbits of quasi-polynomial functions.*

Here we give three examples illustrating some of the variety of cases that one can observe. The function $\mathrm{Nb}_4(c^\bullet_{\lambda\mu} > 0)$ for $\lambda = k_1\varpi_1 + k_2\varpi_1^* \in \Lambda_4^{\mathrm{nr}} \cap \Lambda_4^0$ and $\mu = \mu_1\mu_2\mu_3 \in \Lambda_4^0$ satisfies:

- on the cone defined by $\mu_1 \geqslant k_1 + \mu_3$, $\mu_1 \geqslant k_2 + \mu_3$, $\mu_2 \geqslant \mu_3$, $k_1 + k_2 + \mu_3 \geqslant \mu_1 + \mu_2$, $\mu_3 \geqslant 0$, $\mathrm{Nb}_4(c^\bullet_{\lambda\mu} > 0)$ is equal to

$$P = \frac{\mu_3}{2} \cdot \Big(\mu_2(2\mu_1 - \mu_2 + 1) + 2(\mu_1 + 1) - (\mu_3 + 1)(k_1 + k_2 + \mu_1 - \mu_2 + 2)\Big)$$
$$-\frac{\mu_2 + 1}{6} \cdot \Big(3(k_1^2 + k_2^2) - 3(k_1 + k_2)(2\mu_1 + 1) + 3\mu_1^2 + 2\mu_2^2 - 3\mu_1 + 4\mu_2 - 6\Big),$$

  which is symmetric in (k_1, k_2).

- on the cone defined by $\mu_1 + \mu_2 \geqslant k_1 + k_2 + \mu_3$, $k_2 + \mu_1 \geqslant k_1 + \mu_2 + \mu_3$, $k_2 + \mu_3 \geqslant \mu_2$, $k_1 + \mu_1 \geqslant k_2 + \mu_2 + \mu_3$, $k_1 + \mu_3 \geqslant \mu_2$, $k_1 + k_2 \geqslant \mu_1$, $\mu_3 \geqslant 0$ (adjacent to the previous one), $\mathrm{Nb}_4(c^\bullet_{\lambda\mu} > 0)$ is

$$\begin{cases} P + \dfrac{1}{24}(k_1 + k_2 - \mu_1 - \mu_2 + \mu_3 - 1) & \\ \cdot(k_1 + k_2 - \mu_1 - \mu_2 + \mu_3 + 1) & \text{if } k_1 + k_2 + \mu_1 + \mu_2 + \mu_3 \text{ is odd,} \\ \cdot(-2\,k_1 - 2\,k_2 + 2\,\mu_1 + 2\,\mu_2 + 4\,\mu_3 + 3) & \\ & \\ P + \dfrac{1}{24}(k_1 + k_2 - \mu_1 - \mu_2 + \mu_3) & \\ \cdot\Big(2 + (k_1 + k_2 - \mu_1 - \mu_2 + \mu_3) & \text{if } k_1 + k_2 + \mu_1 + \mu_2 + \mu_3 \text{ is even,} \\ \cdot(-2\,k_1 - 2\,k_2 + 2\,\mu_1 + 2\,\mu_2 + 4\,\mu_3 + 3)\Big) & \end{cases}$$

  which is also symmetric in (k_1, k_2).

- on the cone defined by $\mu_1 \geqslant k_1$, $\mu_1 \geqslant k_2 + \mu_3$, $\mu_2 \geqslant \mu_3$, $k_2 \geqslant \mu_2$, $k_1 + \mu_3 \geqslant \mu_1$ (also adjacent to the first one), $\mathrm{Nb}_4(c^\bullet_{\lambda\mu} > 0)$ is

$$P + \binom{k_1 - \mu_1 + \mu_3 + 1}{3}.$$

  which is not symmetric in (k_1, k_2).

8 More checking of Conjecture 2

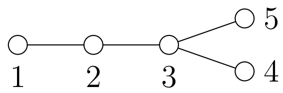
It is to be noted that we checked Conjecture 2 on a few other examples, using SageMath. See [Res20, test_Conj1.sage]:

- Conjecture 1 holds for $\mathrm{GL}_4(\mathbb{C})$ if $\max(\lambda_1 - \lambda_2, \lambda_2) \leqslant 20$ and $|\mu| \leqslant 40$.
- Conjecture 1 holds for $\mathrm{GL}_5(\mathbb{C})$ if $\max(\lambda_1 - \lambda_2, \lambda_2) \leqslant 20$ and $|\mu| \leqslant 30$.
- Conjecture 1 holds for $\mathrm{GL}_6(\mathbb{C})$ if $\max(\lambda_1 - \lambda_2, \lambda_2) \leqslant 10$ and $|\mu| \leqslant 30$.
- Conjecture 1 holds for $\mathrm{GL}_{10}(\mathbb{C})$ if $\max(\lambda_1 - \lambda_2, \lambda_2) \leqslant 10$ and $|\mu| \leqslant 15$.

9 Related questions

9.1 In type D_n

Here we consider questions similar to Conjecture 2 for simple groups of other types than A. The only types where there are irreducible representations that are not self-dual are D_n ($n \geqslant 4$) and E_6. Consider the type D_5.

D_5 

Let $(\varpi_1, \ldots, \varpi_5)$ be the list of fundamental weights. Then, $V(\varpi_4)^* \simeq V(\varpi_5)$ whereas $V(\varpi_1)$, $V(\varpi_2)$ and $V(\varpi_3)$ are self-dual. The natural generalization of near-rectangular partitions is then to consider the dominant weights in $\mathbb{N}\varpi_4 \oplus \mathbb{N}\varpi_5$. A natural generalization of Conjecture 7 would be: for $\lambda = a\varpi_4 + b\varpi_5 \in \mathbb{N}\varpi_4 \oplus \mathbb{N}\varpi_5$ and μ a dominant weight of D_5, do the two tensor products

$$V_{D_5}(a\varpi_4 + b\varpi_5) \otimes V_{D_5}(\mu) \qquad \text{and} \qquad V_{D_5}(b\varpi_4 + a\varpi_5) \otimes V_{D_5}(\mu)$$

contain the same number of isotypic components?

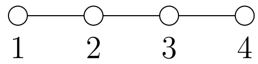
The answer is **no**, even assuming that $\mu \in \mathbb{N}\varpi_4 \oplus \mathbb{N}\varpi_5$ too. An example is $\lambda = 2\varpi_4 + \varpi_5$ and $\mu = \varpi_4 + 2\varpi_5$. The two tensor products have respectively 31 and 30 isotypic components, as checked using SageMath [The20]:

```
sage:  D5=WeylCharacterRing("D5",style="coroots")
sage:  len(D5(0,0,0,2,1)*D5(0,0,0,1,2))
31
sage:  len(D5(0,0,0,1,2)*D5(0,0,0,1,2))
30
```

9.2 In type A_{n-1}

The representations of $\mathrm{SL}_n(\mathbb{C})$ corresponding to near-rectangular partitions are of the form $V(a\varpi_1 + b\varpi_{n-1})$. Observe that (ϖ_1, ϖ_{n-1}) is a pair of mutually dual fundamental weights. One could hope that Conjecture 2 or 7 hold for any linear combination of a given pair of mutually dual fundamental weights. This is **not true** even for (ϖ_2, ϖ_3) and $n = 5$:

A_4



Indeed, for $\lambda = \varpi_2 + 2\varpi_3$ and $\mu = 3\varpi_2 + \varpi_3$, the numbers of isotypic components in $V(\lambda) \otimes V(\mu)$ and $V(\lambda)^* \otimes V(\mu)$ differ:

```
sage:  len(lrcalc.mult([3,3,2],[4,4,1],5))
34
sage:  len(lrcalc.mult([3,3,1],[4,4,1],5))
33
```

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