Frobenius allowable gaps of Generalized Numerical Semigroups

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Submitted: Sep 23, 2021; Accepted: Oct 4, 2022; Published: Oct 21, 2022

Abstract

A generalized numerical semigroup is a submonoid S of $\mathbb{N}^d$ for which the complement $\mathbb{N}^d \setminus S$ is finite. The points in the complement $\mathbb{N}^d \setminus S$ are called gaps. A gap F is considered Frobenius allowable if there is some relaxed monomial ordering on $\mathbb{N}^d$ with respect to which F is the largest gap. We characterize the Frobenius allowable gaps of a generalized numerical semigroup. A generalized numerical semigroup that has only one maximal gap under the natural partial ordering of $\mathbb{N}^d$ is called a Frobenius generalized numerical semigroup. We show that Frobenius generalized numerical semigroups are precisely those whose Frobenius gap does not depend on the relaxed monomial ordering. We estimate the number of Frobenius generalized numerical semigroup with a given Frobenius gap $F = (F^{(1)}, \dots, F^{(d)}) \in \mathbb{N}^d$ and show that it is close to $\sqrt{3}^{(F^{(1)}+1)\cdots(F^{(d)}+1)}$ for large d. We define notions of quasi-irreducibility and quasi-symmetry for generalized numerical semigroups. While in the case of $d = 1$ these notions coincide with irreducibility and symmetry, they are distinct in higher dimensions.

Mathematics Subject Classifications: 20M14, 05A16, 11D07

1 Introduction

A *numerical semigroup* S is a subset of the natural numbers that contains 0, is closed under addition and has a finite complement $\mathbb{N} \setminus S$. The numbers in $\mathbb{N} \setminus S$ are called *gaps* and the largest gap is called the *Frobenius number* $\mathrm{F}(S)$. There is a big literature on numerical semigroups, see [1, 13] for a general reference.

A *generalized numerical semigroup* S is a subset of $\mathbb{N}^d$ that contains 0, is closed under addition and has a finite complement $\mathbb{N}^d \setminus S$. Generalized numerical semigroups have been studied in several recent papers [4, 6, 5, 3, 7, 9]. The points in the complement are called *gaps* and the collection of all gaps is denoted by $\mathcal{H}(S) = \mathbb{N}^d \setminus S$. The number of gaps is called the *genus*, $\mathrm{g}(S) = |\mathcal{H}(S)|$. Failla, Peterson and Utano gave this definition of generalized numerical semigroup in [7]. They also studied the question of counting generalized numerical semigroups by genus and generalized the notion of the semigroup tree.

We have a *natural partial ordering* on $\mathbb{N}^d$. Given $x, y \in \mathbb{N}^d$, let $x^{(i)}, y^{(i)}$ be the i^{th} component of x, y respectively. Define $x \leqslant y$, if $x^{(i)} \leqslant y^{(i)}$ for each $1 \leqslant i \leqslant d$. However, this is not enough to define the Frobenius gap of S, as $\mathcal{H}(S)$ could have more than one maximal element under the natural partial ordering. Failla et al. [7] extend the notion of Frobenius gap to generalized numerical semigroups with the help of *relaxed monomial orderings* on $\mathbb{N}^d$.

Definition 1. A total order $\prec$ on the elements of $\mathbb{N}^d$ is called a relaxed monomial order if it satisfies:
i) If $v, w \in \mathbb{N}^d$ and $v \prec w$, then $v \prec w + u$ for any $u \in \mathbb{N}^d$.
ii) If $v \in \mathbb{N}^d$ and $v \neq 0$, then $0 \prec v$.

Note that $u \prec v$ implies that $u \neq v$. We will write $u \preccurlyeq v$ to mean either $u \prec v$ or $u = v$. Given a relaxed monomial order $\prec$ on $\mathbb{N}^d$ and a generalized numerical semigroup $S \subseteq \mathbb{N}^d$, its *Frobenius gap* is defined as

$$\mathrm{F}_{\prec}(S) = \max_{\prec}(\mathcal{H}(S)).$$

Of course, different relaxed monomial orders can lead to different gaps becoming the Frobenius gap of S. Cisto, Failla, Peterson and Utano in [4] define a gap of S to be *Frobenius allowable* if it is the Frobenius gap with respect to some relaxed monomial ordering. The collection of all Frobenius allowable gaps of S is denoted by $\mathrm{FA}(S)$ and number of Frobenius allowable gaps is denoted by $\tau(S)$. It is clear that all Frobenius allowable gaps must be maximal elements of $\mathcal{H}(S)$ under the natural partial ordering. In [4], the authors ask whether all maximal elements of $\mathcal{H}(S)$ under the natural partial ordering are Frobenius allowable. They prove this (see [4, Proposition 4.5, 4.13]) in the special case when $\mathcal{H}(S)$ has exactly one or two maximal elements under the natural partial ordering. We answer their question in the general case.

Theorem 2. *Given a generalized numerical semigroup $S \subseteq \mathbb{N}^d$, the Frobenius allowable gaps of S are precisely the maximal elements of $\mathcal{H}(S)$ under the natural partial ordering, that is,*

$$\mathrm{FA}(S) = \mathit{Maximals}_{\leqslant}(\mathcal{H}(S)).$$

Cisto et al. in [4] define a *Frobenius generalized numerical semigroup* to be a generalized numerical semigroup S for which $\mathcal{H}(S)$ has exactly one maximal gap under the natural partial ordering. Theorem 2 shows that this property is equivalent to the Frobenius gap of S being independent of the choice of relaxed monomial ordering. The authors of [4, 6] study certain families of generalized numerical semigroups which they show are Frobenius generalized numerical semigroups.

If one fixes a point $F \in \mathbb{N}^d \setminus \{0\}$ with $d \geqslant 2$, then it is seen that there are infinitely many generalized numerical semigroups for which F is Frobenius allowable. However, the number of Frobenius generalized numerical semigroups with a given Frobenius gap F is clearly finite. We denote this by

$$N(F) = \#\{S \subseteq \mathbb{N}^d \mid S \text{ is a Frobenius generalized numerical semigroup}, \mathrm{F}(S) = F\}.$$

In the case of numerical semigroups, that is, $d = 1$, Backelin [2] estimates $N(F)$ and proves that

$$2^{\lfloor \frac{F-1}{2} \rfloor} \leqslant N(F) \leqslant 4 \times 2^{\lfloor \frac{F-1}{2} \rfloor}.$$

We build on the work of Backelin and make the first systematic study of counting Frobenius generalized numerical semigroups with a given Frobenius gap.

Given a point F in $\mathbb{N}^d$, let

$$\|F\| = \prod_{i=1}^{d} (F^{(i)} + 1).$$

So $\|F\|$ is the number of points in the box $\{x \in \mathbb{N}^d \mid 0 \leqslant x \leqslant F\}$. We trivially know that $N(F) \leqslant 2^{\|F\|}$. For large d we prove that $N(F)$ is close to $\sqrt{3}^{\|F\|}$. We use the notation $F - 1 = (F^{(1)} - 1, \ldots, F^{(d)} - 1)$.

Theorem 3. *Given $\epsilon > 0$, there is $M > 0$ such that for every $d > M$ and $F \in \mathbb{N}^d$, we have*

$$\left(\sqrt{3} - \epsilon\right)^{\|F-1\|} \leqslant N(F) \leqslant \sqrt{3}^{\|F\|}.$$

Given a generalized numerical semigroup $S \subsetneq \mathbb{N}^d$, a gap $P \in \mathcal{H}(S)$ is called a *pseudo-Frobenius gap* of S if for every nonzero element x of S, $x + P$ is also an element of S. The collection of all pseudo-Frobenius gaps of S is denoted by $\mathrm{PF}(S)$. And the number of pseudo-Frobenius gaps of S is called its *type*, $\mathrm{t}(S)$. We have a partial ordering $\preccurlyeq_S$ on $\mathbb{N}^d$ in which $x \preccurlyeq_S y$, whenever $y - x \in S$. It is easy to see that the pseudo-Frobenius gaps of S are precisely the maximal elements of $\mathcal{H}(S)$ under this partial ordering, that is,

$$\mathrm{PF}(S) = \mathrm{Maximals}_{\preccurlyeq_S}(\mathcal{H}(S)).$$

The family of irreducible numerical semigroups have received considerable attention in the literature. A numerical semigroup is called *irreducible* if it cannot be expressed as the intersection of two numerical semigroups properly containing it. Several characterizations of irreducible numerical semigroups are known. The authors of [8] prove that given a numerical semigroup S with $\mathrm{F}(S) = F$, the following are equivalent:

- S is irreducible.
- S is maximal (with respect to set theoretic inclusion) among all numerical semigroups that do not contain F.

- For every gap $x \in \mathcal{H}(S)$, either $2x = \mathrm{F}$ or $\mathrm{F} - x \in S$.
- Either $\mathrm{PF}(S) = \{F\}$, or $\mathrm{PF}(S) = \{\frac{F}{2}, F\}$.

A numerical semigroup with $\mathrm{PF}(S) = \{F\}$ is called *symmetric* and one with $\mathrm{PF}(S) = \{\frac{F}{2}, F\}$ is called *pseudo-symmetric*. A numerical semigroup has type 1 if and only if it is symmetric.

The authors of [4] define a generalized numerical semigroup to be *irreducible* if it cannot be written as an intersection of two generalized numerical semigroups properly containing it. They prove that irreducible generalized numerical semigroups are always Frobenius generalized numerical semigroups. They prove that the following are equivalent for a generalized numerical semigroup S:

- S is irreducible.
- There is a gap $F \in \mathcal{H}(S)$ such that for every gap $x \in \mathcal{H}(S)$, either $2x = F$ or $F - x \in S$.
- There is a gap $F \in \mathcal{H}(S)$ such that either $\mathrm{PF}(S) = \{F\}$ or $\mathrm{PF}(S) = \{\frac{F}{2}, F\}$.
- There is a gap $F \in \mathcal{H}(S)$ such that S is maximal (with respect to set theoretic inclusion) among generalized numerical semigroups that do not contain F.

They call a generalized numerical semigroup *symmetric* if $\mathrm{PF}(S) = \{F\}$ and *pseudo-symmetric* if $\mathrm{PF}(S) = \{\frac{F}{2}, F\}$. This shows that a generalized numerical semigroup is symmetric if and only if it has $\mathrm{t}(S) = 1$. The authors of [4] also show that every generalized numerical semigroup can be written as a finite intersection of irreducible generalized numerical semigroup.

By Theorem 2 we know that Frobenius allowable gaps of S are maximal in $\mathcal{H}(S)$ under the natural partial ordering $\leqslant$. Therefore, they are also maximal under $\preccurlyeq_S$ and hence are pseudo-Frobenius gaps. Therefore, we see that $\tau(S) \leqslant \mathrm{t}(S)$. We call a generalized numerical semigroup S *quasi-symmetric* if $\tau(S) = \mathrm{t}(S)$. In the case of Frobenius generalized numerical semigroup, $\tau(S) = 1$. Hence, a Frobenius generalized numerical semigroup S will be quasi-symmetric when $\mathrm{t}(S) = 1$. Thus the property of being symmetric is equivalent to being both quasi-symmetric and a Frobenius generalized numerical semigroup. The notions of symmetry and quasi-symmetry of course coincide in the case of numerical semigroups. In Theorem 4, we show that quasi-symmetric generalized numerical semigroups are characterized by a property similar to that of symmetric numerical semigroups.

Theorem 4. *Given a generalized numerical semigroup $S \subseteq \mathbb{N}^d$, $\tau(S) \leqslant \mathrm{t}(S)$. Moreover, equality holds if and only if S satisfies the property that for every $x \in \mathcal{H}(S)$, there is some Frobenius allowable gap F for which $F - x \in S$.*

We call a generalized numerical semigroup S *quasi-irreducible* if for every gap $x \in \mathcal{H}(S)$, either $2x \in \mathrm{FA}(S)$ or there is some $F \in \mathrm{FA}(S)$ for which $F - x \in S$. From Theorem 4, we see that all quasi-symmetric generalized numerical semigroups are quasi-irreducible. In Theorem 10 and Proposition 11 we prove the following.

Theorem 5. *Given a generalized numerical semigroup S, the following are equivalent:*

- *S is quasi-irreducible, that is, for every gap $x \in \mathcal{H}(S)$, either $2x \in \mathrm{FA}(S)$ or there is some $F \in \mathrm{FA}(S)$ for which $F - x \in S$.*
- *S is maximal in the collection of all generalized numerical semigroups S' for which $\mathrm{FA}(S) \subseteq \mathcal{H}(S')$.*
- *For all $P \in \mathrm{PF}(S)$, either $P \in \mathrm{FA}(S)$ or $2P \in \mathrm{FA}(S)$.*

We also study the bounds on the type of a Frobenius generalized numerical semigroup. For numerical semigroups it is known that

$$\frac{\mathrm{g}(S)}{\mathrm{F}(S) + 1 - \mathrm{g}(S)} \leqslant \mathrm{t}(S) \leqslant 2\,\mathrm{g}(S) - \mathrm{F}(S).$$

The first inequality was proved in [8], and the second in [11]. Numerical semigroups that satisfy $\mathrm{t}(S) = 2\,\mathrm{g}(S) - \mathrm{F}(S)$ are called *almost-symmetric*. In [6], the authors extended the second inequality. They prove that if S is a Frobenius generalized numerical semigroup, then

$$\mathrm{t}(S) \leqslant 2\,\mathrm{g}(S) + 1 - \|\,\mathrm{F}(S)\|.$$

If equality holds, then they call Frobenius generalized numerical semigroup *almost-symmetric*. The authors of [6] come up with a number of equivalent conditions for a Frobenius generalized numerical semigroup to be almost-symmetric. We give another property that is equivalent to almost-symmetry. The special case of Proposition 6 for numerical semigroups was proved in [14].

Proposition 6. *A Frobenius generalized numerical semigroup S with Frobenius gap F is almost-symmetric if and only if*

$$T(S) = \{x \in \mathbb{N}^d \mid F - x \in \big((\mathbb{Z}^d \setminus S) \cup \{0\}\big)\}$$

is a generalized numerical semigroup.

We also extend the first inequality as follows:

Theorem 7. *Given a Frobenius generalized numerical semigroup $S \subseteq \mathbb{N}^d$ we have*

$$\frac{\mathrm{g}(S)}{\|\,\mathrm{F}(S)\| - \mathrm{g}(S)} \leqslant \mathrm{t}(S).$$

2 Frobenius Allowable Gaps

In this section we will prove Theorem 2. We fix a generalized numerical semigroup $S \subseteq \mathbb{N}^d$ and an element h of $\mathcal{H}(S)$, which is maximal under the natural partial ordering. We construct an explicit relaxed monomial order on $\mathbb{N}^d$ with respect to which h becomes the Frobenius gap of S.

Theorem 2. Given a generalized numerical semigroup $S \subseteq \mathbb{N}^d$, the Frobenius allowable gaps of S are precisely the maximal elements of $\mathcal{H}(S)$ under the natural partial ordering.

Proof. Let h be a maximal element of $\mathcal{H}(S)$ under the natural partial ordering. We reorder the coordinates so that $h^{(1)}, h^{(2)}, \ldots, h^{(k)}$ are all non zero and $h^{(k+1)}, h^{(k+2)}, \ldots, h^{(d)}$ are zero. We then define a function ϕ on $\mathbb{N}^d$:

$$\phi(x) = \min_{1 \leqslant i \leqslant k} \left(\frac{x^{(i)}}{h^{(i)}} \right).$$

We now define $\prec$ as follows, suppose $x, y \in \mathbb{N}^d$.

- If $\phi(x) < \phi(y)$, then $x \prec y$.
- If $\phi(x) = \phi(y)$ and there is a $j \in \{0, 1 \ldots, d-1\}$ such that $x^{(i)} = y^{(i)}$ for $1 \leqslant i \leqslant j$ and $x^{(j+1)} < y^{(j+1)}$, then $x \prec y$.

This is clearly a total ordering. It is also clear that $\phi(h) = 1$. Moreover, given some $x \in \mathcal{H}(S)$ other than h, we know that $h \nleqslant x$, as h is maximal in $\mathcal{H}(S)$. This means that there is a i for which $x^{(i)} < h^{(i)}$. In this case $h^{(i)} > 0$ and $i \leqslant k$. It follows that $\frac{x^{(i)}}{h^{(i)}} < 1$ and hence $\phi(x) < 1 = \phi(h)$. Therefore $x \prec h$. This shows that h is the maximum of $\mathcal{H}(S)$ with respect to $\prec$. The only thing that remains to be shown is that $\prec$ is a relaxed monomial order.

We know that $\phi(0) = 0$. If $v \in \mathbb{N}^d$ is non-zero, then there is some j for which $v^{(j)} > 0$. Consider the smallest such j. If $\phi(v) > 0$, then $0 \prec v$ since $\phi(0) = 0 < \phi(v)$. On the other hand if $\phi(v) = 0$, then for $1 \leqslant i \leqslant j-1$, we have $0^{(i)} = 0 = v^{(i)}$ and $0^{(j)} = 0 < v^{(j)}$. Therefore we still get $0 \prec v$.

Next suppose we have $u, v, w \in \mathbb{N}^d$ such that $v \prec w$. We know that $w^{(i)} \leqslant (w+u)^{(i)}$ for each i. Moreover, this implies that $\phi(w) \leqslant \phi(w+u)$. Combining all of this, we see that $w \preccurlyeq w+u$ and hence $v \prec w+u$.

Therefore, $\prec$ is indeed a relaxed monomial ordering and $\mathrm{F}_{\prec}(S) = h$. We see that h is Frobenius allowable and this completes the proof. $\square$

We also have a notion of a monomial order which is stronger than a relaxed monomial order.

Definition 8. A total order $\prec$, on the elements of $\mathbb{N}^d$ is called a monomial order if it satisfies:
i) If $v, w \in \mathbb{N}^d$ and $v \prec w$, then $v + u \prec w + u$ for any $u \in \mathbb{N}^d$.
ii) If $v \in \mathbb{N}^d$ and $v \neq 0$, then $0 \prec v$.

It is clear that all monomial orders are also relaxed monomial orders. However, as noted in [4], the converse is not true. In particular, the relaxed monomial order we constructed in the proof of Theorem 2 is not a monomial order. To see this consider the case when $d = 2$, $h = (1, 1)$. In this case $(1, 4) \prec (2, 2)$ but $(1, 4) + (2, 0) \succ (2, 2) + (2, 0)$.

So it is not a monomial order. Of course, it is a relaxed monomial order, so we have $(1,4) \prec (2,2) + (2,0)$.

In [12], the author proved that a general monomial order on $\mathbb{N}^d$ can be obtained in terms of d linearly independent vectors in $\mathbb{R}^d$ as follows. Given a monomial order $\prec$, we can find d linearly independent vectors $v_1, v_2, \ldots, v_d \in \mathbb{R}^d_{\geqslant 0}$, such that for any $v, w \in \mathbb{N}^d$ we have $v \prec w$ if and only if there is some $k \in \{1, 2, \ldots d\}$ with the property that for each $1 \leqslant i \leqslant k-1$, we have $\langle v, v_i \rangle = \langle w, v_i \rangle$ and $\langle v, v_k \rangle < \langle w, v_k \rangle$.

Given a generalized numerical semigroup $S \subseteq \mathbb{N}^d$ and a Frobenius allowable gap F, one could ask if there is a monomial order $\prec$ such that $F = \mathrm{F}_\prec(S)$. This is not always the case. For example let $d = 2$ and consider

$$S = \mathbb{N}^2 \setminus \{(0,1), (0,2), (0,3), (1,0), (2,0), (3,0), (1,1)\}.$$

This is closed under addition and hence is a generalized numerical semigroup. The gap $(1,1)$ is maximal among the gaps in the natural partial ordering. So $(1,1)$ is Frobenius allowable. However, there is no monomial order $\prec$ for which $\mathrm{F}_\prec(S) = (1,1)$. This is because given any $v_1, v_2 \in \mathbb{R}^2_{\geqslant 0}$, either $\langle (3,0), v_1 \rangle > \langle (1,1), v_1 \rangle$ or $\langle (0,3), v_1 \rangle > \langle (1,1), v_1 \rangle$.

Recall that a generalized numerical semigroup is called a Frobenius generalized numerical semigroup if $\mathcal{H}(S)$ has a unique maximal gap under the natural partial ordering. Theorem 2 allows us to classify which generalized numerical semigroups are Frobenius generalized numerical semigroups.

Theorem 9. *Given a generalized numerical semigroup S, the following are equivalent:*
i) S is a Frobenius generalized numerical semigroup, that is, $\mathcal{H}(S)$ has a unique maximal element under the natural partial ordering.
ii) The Frobenius gap of S is independent of the relaxed monomial ordering on $\mathbb{N}^d$, that is, $\tau(S) = 1$.
iii) $\mathrm{PF}(S)$ has a unique maximal element with respect to the natural partial ordering.

Proof. By Theorem 2 we know that i) and ii) are equivalent. It is known that the maximal members of $\mathcal{H}(S)$ under the natural partial ordering are pseudo-Frobenius gaps. This means that the maximal members of $\mathcal{H}(S)$ and $\mathrm{PF}(S)$ under the natural partial ordering are exactly the same. This shows that i) and iii) are equivalent. ☐

3 Quasi-irreducible generalized numerical semigroups

Recall that the type of a generalized numerical semigroup is the number of pseudo-Frobenius gaps it has, that is,

$$\mathrm{t}(S) = |\,\mathrm{PF}(S)| = \#\{P \in \mathcal{H}(S) \mid P + (S \setminus \{0\}) \subseteq S\}.$$

And $\tau(S)$ is the number of Frobenius allowable gaps of S. Since all Frobenius allowable gaps are pseudo-Frobenius, we have $\tau(S) \leqslant \mathrm{t}(S)$. We start this section by characterizing quasi-symmetric generalized numerical semigroups, that is, those generalized numerical semigroup for which $\tau(S) = \mathrm{t}(S)$.

Theorem 4. Given a generalized numerical semigroup $S \subseteq \mathbb{N}^d$, $\tau(S) \leqslant \mathrm{t}(S)$. Moreover, equality holds if and only if S satisfies the property that for every $x \in \mathcal{H}(S)$, there is some Frobenius allowable gap F for which $F - x \in S$.

Proof. We already know that $\mathrm{FA}(S) \subseteq \mathrm{PF}(S)$, so $\tau(S) \leqslant \mathrm{t}(S)$. We now prove the next part of the theorem. First suppose that S satisfies the given property. In this case, consider some $x \in \mathcal{H}(S)$ that is not Frobenius allowable. We know that there must be some Frobenius allowable gap F for which $F - x \in S$. We know that $F \neq x$, as x is not Frobenius allowable. Therefore, $F - x$ is a nonzero element of S and $x + (F - x) \notin S$. This shows that x is not a pseudo-Frobenius gap of S. We can conclude that $\tau(S) = \mathrm{t}(S)$.

We now prove the other direction. Suppose that $\tau(S) = \mathrm{t}(S)$. Consider $x \in \mathcal{H}(S)$. We know that $\mathrm{PF}(S) = \mathrm{Maximals}_{\preccurlyeq_S}(\mathcal{H}(S))$, so there is some $P \in PF(S)$ for which $x \preccurlyeq_S P$. This means $P - x \in S$. Moreover, since $\tau(S) = \mathrm{t}(S)$, we see that $P \in \mathrm{FA}(S)$. □

We note that if a generalized numerical semigroup has type 1, then it must have $\tau(S) = 1$, that is, it must be a Frobenius generalized numerical semigroup. Moreover, it must also be quasi-symmetric and hence must satisfy the condition of Theorem 4. These generalized numerical semigroups are studied in [4] and are called symmetric generalized numerical semigroups. A generalized numerical semigroup is symmetric if and only if $\tau(S) = 1$ and it is quasi-symmetric.

Recall that a generalized numerical semigroup S is called quasi-irreducible if for every $x \in \mathcal{H}(S)$, either $2x$ is Frobenius allowable or there is some Frobenius allowable gap F for which $F - x \in S$. Clearly all quasi-symmetric generalized numerical semigroups are quasi-irreducible.

A subset $D \subseteq \mathbb{N}^d$ is called an anti-chain if whenever we have $x, y \in D$ with $x \leqslant y$, then $x = y$.

Theorem 10. *Let D be a finite subset of $\mathbb{N}^d \setminus \{0\}$ that is an anti-chain with respect to the natural partial ordering. Consider the collection of all generalized numerical semigroups $S \subseteq \mathbb{N}^d$ for which $D \subseteq \mathcal{H}(S)$. The maximal members of this collection are precisely the quasi-irreducible generalized numerical semigroups S with $\mathrm{FA}(S) = D$.*

Proof. First suppose that we have a quasi-irreducible generalized numerical semigroup S with $\mathrm{FA}(S) = D$. Assume for the sake of contradiction that S is not maximal in the collection. This means that there is some generalized numerical semigroup $S' \supsetneq S$ with $D \subseteq \mathcal{H}(S')$. Consider some $x \in S' \setminus S$. Since $x \in \mathcal{H}(S)$, we know that either $2x \in \mathrm{FA}(S) = D$ or there is some $F \in D$ for which $F - x \in S$. We know that $2x \in S'$, so $2x$ cannot be in D. However, if there is some $F \in D$ for which $F - x \in S$, then $F - x \in S'$ and hence $F = (F - x) + x \in S'$. This is also impossible. Therefore, we get a contradiction and S must be maximal in the collection.

We now prove the other direction, consider some generalized numerical semigroup S which is maximal in the collection. Let

$$S_1 = S \cup \{a \in \mathbb{N}^d \mid D \cap (a + \mathbb{N}^d) = \emptyset\}.$$

It is clear that S_1 is a generalized numerical semigroup and $D \cap S_1 = \emptyset$, so S_1 is in the collection. Also $S \subseteq S_1$, so the maximality of S implies that $S = S_1$. Now consider some $F \in \mathrm{FA}(S)$. Since $S = S_1$, F is not in S_1 and hence $D \cap (F + \mathbb{N}^d) \neq \emptyset$. Theorem 2 implies that $F \in D$. This shows that $\mathrm{FA}(S) \subseteq D$. Next consider some $x \in D$. We know that $x \in \mathcal{H}(S)$, so there must be some $F \in \mathrm{FA}(S)$ for which $x \leqslant F$. But then $F, x \in D$. Since D is an anti-chain, this implies $x = F$. Therefore $\mathrm{FA}(S) = D$.

Next consider
$$X = \{x \in \mathcal{H}(S) \mid 2x \notin D, D \cap (x + S) = \emptyset\}.$$
If X is empty, then S will be quasi-irreducible. Therefore assume for the sake of contradiction that X is non-empty. Consider some $x \in X$ that is maximal with respect to the natural partial ordering. Let
$$S_2 = S \cup \{x\}.$$
We will show that S_2 is closed under addition. Consider a non-zero s in S. By maximality of x, we know that $x + s \notin X$. Therefore, either $2(x+s) \in D$ or $D \cap ((x+s)+S) \neq \emptyset$ or $x+s \in S$. We wish to show that $x+s \in S$, so we will show that the other two possibilities are impossible.

- If $2(x+s) \in D$, then $2x + 2s \notin x + S$, that is, $x + 2s \notin S$. Let $y = x + 2s$. Then, $2y = 2(x+s) + 2s > 2(x+s)$. Since D is an anti-chain, this means that $2y \notin D$. Next, $D \cap (y + S) = D \cap (x + 2s + S) \subseteq D \cap (x + S) = \emptyset$. This means that $y \in X$, but $x < y$ and this contradicts the maximality of x.

- Note that $D \cap ((x+s)+S) \subseteq D \cap (x+S) = \emptyset$.

Therefore, we have shown that for any non-zero $s \in S$, $x + s$ is also an element of S. Next by the maximality of x, we also know that $2x \notin X$. Therefore, either $4x \in D$ or $D \cap (2x + S) \neq \emptyset$ or $2x \in S$. We wish to show that $2x \in S$, so we will rule out the other two possibilities.

- Suppose $D \cap (2x + S) \neq \emptyset$. Since $x \in X$, we know that $2x \notin D$ and hence $D \cap (2x + (S \setminus \{0\})) \neq \emptyset$. But we have already seen that $x + (S \setminus \{0\}) \subseteq S$, which implies $D \cap (2x + (S \setminus \{0\})) \subseteq D \cap (x + S) = \emptyset$. This is a contradiction.

- If $4x \in D$, then we know that $4x \notin x + S$, that is, $3x \in \mathcal{H}(S)$. Since D is an anti-chain and $4x \in D$, we know that $2(3x) \notin D$. Note that D being an anti-chain also implies $3x \notin D$, therefore
$$D \cap (3x + S) = D \cap (3x + (S \setminus \{0\})) \subseteq D \cap (2x + S) = \emptyset.$$
This means that $3x \in X$ and it contradicts the maximality of x.

We therefore conclude that $2x \in S$. This shows that S_2 is closed under addition and hence is a generalized numerical semigroup. Since $D \cap (x + S) = \emptyset$, we know that $x \notin D$. Therefore $S_2 \cap D = \emptyset$. This means that S_2 is in the collection and $S \subsetneq S_2$. This contradicts the maximality of S in the collection. Therefore, X must be empty and hence S is quasi-irreducible with $\mathrm{FA}(S) = D$. □

The special case of this theorem when $|D| = 1$ was proved in [4], they call such generalized numerical semigroups irreducible and study them. A generalized numerical semigroup is irreducible if and only if $\tau(S) = 1$ and it is quasi-irreducible. We now prove the second half of Theorem 5. The special case of this when $\tau(S) = 1$ was also proved in [4].

Proposition 11. *A generalized numerical semigroup S is quasi-irreducible if and only if it satisfies the property that for every $P \in \mathrm{PF}(S)$ either $P \in \mathrm{FA}(S)$ or $2P \in \mathrm{FA}(S)$.*

Proof. First suppose that S is quasi-irreducible. Consider some $P \in \mathrm{PF}(S)$. Since $P \in \mathcal{H}(S)$, we know that either $2P \in \mathrm{FA}(S)$ or there is some $F \in \mathrm{FA}(S)$ for which $F - P \in S$. We have nothing to prove in the first case. So suppose that there is some $F \in \mathrm{FA}(S)$ for which $F - P \in S$. This means $P \preccurlyeq_S F$. But $P \in \mathrm{PF}(S) = \mathrm{Maximals}_{\preccurlyeq_S}(\mathcal{H}(S))$, so this forces $P = F$ and hence $P \in \mathrm{FA}(S)$.

Conversely, suppose all pseudo Frobenius gaps of S satisfy the given property. Given $x \in \mathcal{H}(S)$, we know that there is some $P \in \mathrm{PF}(S)$ for which $x \preccurlyeq_S P$. Say $P - x = s \in S$. Now we know that either $P \in \mathrm{FA}(S)$ or $2P \in \mathrm{FA}(S)$. If $P \in \mathrm{FA}(S)$, then $P - x \in S$. If $2P \in \mathrm{FA}(S)$ and $s = 0$, then $2x \in \mathrm{FA}(S)$. Finally, if $2P \in \mathrm{FA}(S)$ and $s \neq 0$, then $2P - x = P + s \in S$. Therefore, S is quasi-irreducible. □

Corollary 12. *If S is a quasi-irreducible generalized numerical semigroup, then*

$$\tau(S) \leqslant \mathrm{t}(S) \leqslant 2\tau(S).$$

In the case of Frobenius generalized numerical semigroups, the characterization is actually a bit stronger. The authors of [4] prove that if S is a Frobenius generalized numerical semigroup and at least one coordinate of its Frobenius gap is odd, then S is irreducible if and only if $\mathrm{PF}(S) = \{\mathrm{F}(S)\}$. In this case the generalized numerical semigroup is symmetric. On the other hand if S is a Frobenius generalized numerical semigroup and all coordinates of its Frobenius gap are even, then S is irreducible if and only if $\mathrm{PF}(S) = \{\frac{\mathrm{F}(S)}{2}, \mathrm{F}(S)\}$. In this case the generalized numerical semigroup is pseudo-symmetric.

Example 13. One might wonder if this stronger characterization can be extended to the case when $\tau(S) > 1$. One might guess that if S is a quasi-irreducible generalized numerical semigroup, then

$$\mathrm{PF}(S) = \mathrm{FA}(S) \cup \{P \in \mathbb{N}^d \mid 2P \in \mathrm{FA}(S)\}.$$

However, this is not the case. Consider $S \subseteq \mathbb{N}^2$, with

$$\mathcal{H}(S) = \{(1,0),(2,0),(0,1),(1,1),(2,2),(1,3)\}.$$

It is seen that this is indeed a generalized numerical semigroup and $\mathrm{PF}(S) = \mathrm{FA}(S) = \{(2,2),(1,3)\}$. This means that S is quasi-symmetric and hence quasi-irreducible. However, $2(1,1) \in \mathrm{FA}(S)$ and $(1,1) \notin \mathrm{PF}(S)$.

4 Frobenius generalized numerical semigroups of small and large type

In this section we find bounds for the type of a Frobenius generalized numerical semigroup. It is known that for any Frobenius generalized numerical semigroup S

$$\mathrm{t}(S) \leqslant 2\,\mathrm{g}(S) + 1 - \|\,\mathrm{F}(S)\|.$$

A Frobenius generalized numerical semigroup is called almost-symmetric if $\mathrm{t}(S) = 2\,\mathrm{g}(S) + 1 - \|\,\mathrm{F}(S)\|$. We establish a new property in Proposition 6 that is equivalent to almost-symmetry. We then find the lower bound on the type of a Frobenius generalized numerical semigroup given its genus and Frobenius gap. We show that

$$\frac{\mathrm{g}(S)}{\|\,\mathrm{F}(S)\| - \mathrm{g}(S)} \leqslant \mathrm{t}(S).$$

Proposition 6. A Frobenius generalized numerical semigroup S with Frobenius gap F is almost-symmetric if and only if

$$T(S) = \{x \in \mathbb{N}^d \mid F - x \in \big((\mathbb{Z}^d \setminus S) \cup \{0\}\big)\}$$

is a generalized numerical semigroup.

Proof. Firstly note that

$$\mathbb{N}^d \setminus T(S) = \{F - s \mid s \in S \setminus \{0\}, s \leqslant F\}.$$

So $|\mathbb{N}^d \setminus T(S)| = \|F\| - \mathrm{g}(S) - 1$. Let $A = \{x \in \mathbb{N}^d \mid x + T(S) \subseteq T(S)\}$, we will show that $A = S \cup \mathrm{PF}(S)$. Note that $x \in A$ if and only if for every $y \in \mathbb{N}^d \setminus T(S)$ we have $y - x \notin T(S)$. This means that $x \in A$ if and only if for every $s \in S \setminus \{0\}$ with $s \leqslant F$ we have $F - s - x \notin T(S)$.

Now suppose $x \in S \cup \mathrm{PF}(S)$ and consider some $s \in S \setminus \{0\}$ with $s \leqslant F$. From this we see that $x + s \in S$ and $x + s \neq 0$. This means that $F - (F - s - x)$ is not in $(\mathbb{Z}^d \setminus S) \cup \{0\}$. Hence $F - s - x \notin T(S)$. We therefore conclude that $x \in A$.

Next suppose $x \in A$ and assume for the sake of contradiction that $x \notin S \cup \mathrm{PF}(S)$. Since $x \in \mathcal{H}(S)$ and $x \notin \mathrm{PF}(S)$, there must be some $s \in S \setminus \{0\}$ for which $x + s \notin S$. Now since $x + s \in \mathcal{H}(S)$ and S is a Frobenius generalized numerical semigroup, we have $x + s \leqslant F$. In particular this implies $s \in S \setminus \{0\}$ and $s \leqslant F$. Since $x \in A$, we see that $F - s - x \notin T(S)$. However, we have $F - s - x \in \mathbb{N}^d$ and $s + x \in (\mathbb{Z}^d \setminus S)$, which means that $F - s - x \in T(S)$. This is a contradiction. Therefore, $x \in S \cup \mathrm{PF}(S)$.

We have shown that

$$A = \{x \in \mathbb{N}^d \mid x + T(S) \subseteq T(S)\} = S \cup \mathrm{PF}(S).$$

Now since $\mathrm{t}(S)$ is the size of $\mathrm{PF}(S)$, we see that

$$\mathrm{t}(S) = |A \setminus S| = |\mathbb{N}^d \setminus S| - |\mathbb{N}^d \setminus A| = \mathrm{g}(S) - |\mathbb{N}^d \setminus A|.$$

Since $0 \in T(S)$, we know that $A \subseteq T(S)$. This implies that

$$\mathrm{t}(S) = \mathrm{g}(S) - |\mathbb{N}^d \setminus A| \leqslant \mathrm{g}(S) - |\mathbb{N}^d \setminus T(S)| = 2\,\mathrm{g}(S) + 1 - \|F\|.$$

Moreover, equality holds if and only if $T(S) = A$. This is equivalent to $T(S)$ being closed under addition, which is of course equivalent to $T(S)$ being a generalized numerical semigroup. $\square$

When S is a numerical semigroup, its canonical ideal is

$$S^* = \{x \in \mathbb{Z} \mid \mathrm{F}(S) - x \notin S\}.$$

Our construction of $T(S)$ is closely related as $T(S) = S^* \cup \{\mathrm{F}(S)\}$.

Theorem 7. Given a Frobenius generalized numerical semigroup $S \subseteq \mathbb{N}^d$ we have

$$\frac{\mathrm{g}(S)}{\|\,\mathrm{F}(S)\| - \mathrm{g}(S)} \leqslant \mathrm{t}(S).$$

Proof. Fix a relaxed monomial ordering $\prec$ on $\mathbb{N}^d$. Define a map ϕ from $\mathcal{H}(S)$ to S as follows:

$$\phi(x) = \max_{\prec}\{s \in S \mid x + s \in \mathcal{H}(S)\}.$$

Here we are taking the maximum of a nonempty finite set, so ϕ is well defined. Consider some nonzero $s \in S$, we know that $\phi(x) + s \in S$ and $\phi(x) \prec \phi(x) + s$. The maximality of $\phi(x)$ implies that $x + \phi(x) + s \in S$. This means that $x + \phi(x) \in \mathrm{PF}(S)$. Let B be the box

$$B = \{x \in \mathbb{N}^d \mid 0 \leqslant x \leqslant \mathrm{F}(S)\}.$$

So $|B| = \|\,\mathrm{F}(S)\|$. Since S is a Frobenius generalized numerical semigroup, we know that $\mathcal{H}(S) \subseteq B$. This means that $|B \cap S| = \|\,\mathrm{F}(S)\| - \mathrm{g}(S)$. Now we define a map ψ from $\mathcal{H}(S)$ to $(S \cap B) \times \mathrm{PF}(S)$ given by

$$\psi(x) = (\phi(x), x + \phi(x)).$$

This map is clearly injective, therefore $\mathrm{g}(S) \leqslant (\|\,\mathrm{F}(S)\| - \mathrm{g}(S))\,\mathrm{t}(S)$. $\square$

5 Lower bounds for the number of Frobenius generalized numerical semigroups

In this and the next section we will attempt to count the number of Frobenius generalized numerical semigroups with a given Frobenius gap in $\mathbb{N}^d$. In this section we will obtain a lower bound for $N(F)$. We denote $\overline{x} = \lfloor \frac{x+1}{2} \rfloor$.

First consider the case when $d = 1$, that is, the case of numerical semigroups. Given $F \in \mathbb{N}$, let $B = \{x \in \mathbb{N} \mid \frac{F}{2} < x < F\}$. So $|B| = \overline{F} - 1$. Now for any subset $X \subseteq B$, let $S(X) = \{0\} \cup X \cup \{x \mid x > F\}$. Then $S(X)$ is closed under addition and hence is a numerical semigroup. Moreover, distinct X lead to distinct numerical semigroups. We can therefore conclude that for $F \in \mathbb{N}$, $N(F) \geqslant 2^{\overline{F}-1}$. We will extend this technique to higher d, by choosing a large piece of the box where we can pick points almost independently.

Given $F \in \mathbb{N}^d$, we denote by $S_F = \mathbb{N}^d \setminus \{x \in \mathbb{N}^d \mid 0 < x \leqslant F\}$. So S_F is a generalized numerical semigroup (note that $0 \in S_F$). In fact S_F is a Frobenius generalized numerical semigroup with Frobenius gap F.

Theorem 14. *Let $d_1 = \lceil \frac{d+1}{3} \rceil$. If $F \in \mathbb{N}^d$, then*

$$\left(3^{\frac{1}{2}\sum_{i=d_1}^{d-d_1}\binom{d}{i}}2^{\sum_{i=d-d_1+1}^{2d_1-1}\binom{d}{i}}\right)^{\overline{F^{(1)}}\cdots\overline{F^{(d)}}} \leqslant N(F).$$

Proof. For a subset $A \subseteq \{1, 2, \ldots, d\}$, consider the following box

$$B_A = \left\{ x \in \mathbb{N}^d \;\middle|\; \text{for } i \in A : \frac{F^{(i)}}{2} < x^{(i)} \leqslant F^{(i)};\ \text{for } i \notin A : 0 \leqslant x^{(i)} < \frac{F^{(i)}}{2} \right\}.$$

For each A, the size of the box is

$$|B_A| = \overline{F^{(1)}} \cdots \overline{F^{(d)}}.$$

Let B be the union of B_A for all subsets A with size $d_1 \leqslant |A| \leqslant d - d_1$. Since each B_A has the same size, the size of B is

$$|B| = \overline{F^{(1)}} \cdots \overline{F^{(d)}} \sum_{i=d_1}^{d-d_1} \binom{d}{i}. \tag{1}$$

Let C be the union of B_A for all subsets A with size $d - d_1 + 1 \leqslant |A| \leqslant 2d_1 - 1$. So C is disjoint from B and the size of C is

$$|C| = \overline{F^{(1)}} \cdots \overline{F^{(d)}} \sum_{i=d-d_1+1}^{2d_1-1} \binom{d}{i}. \tag{2}$$

Note that if $d \notin \{1, 3\}$, we have $1 \leqslant d_1 \leqslant d - d_1 \leqslant d$. For $d \in \{1, 3\}$, we have $B = \emptyset$, but Equation 1 is still satisfied. Also notice that if $d \equiv 2 \pmod 3$, then $C = \emptyset$, but Equation 2 still holds.

Next notice that for any $x \in B$, $F - x$ is also in B. And for any $x \in C$, $F - x \notin (B \cup C)$. A subset of $Y \subseteq B$ is called good if $x \in Y$ implies $F - x \notin Y$. Now, B consists of $\frac{|B|}{2}$ pairs of the form $x, F - x$. Choosing a good subset of B requires choosing at most one element from each such pair. There are three choices for each pair and hence there are $3^{\frac{1}{2}|B|}$ good subsets of B. For a good subset Y of B and any subset Z of C we let $X = Y \cup Z$ and define

$$S(X) = S(Y \cup Z) = S_F \cup X \cup (X + X).$$

Since Y was a good subset, we know that F is not in $S(X)$. It is therefore clear that F is the unique maximal element of $\mathbb{N}^d \setminus S(X)$ under the natural partial ordering.

We next show that $S(X)$ is closed under addition. Consider non-zero $x, y \in S(X)$. If at least one of them is in S_F, then $x+y$ is also in S_F. Therefore, suppose neither of them is in S_F. If both of them are in X, then $x+y$ is in $X+X$. The remaining cases are when one of them is in X and the other in $X+X$ or when both are in $X+X$. We can therefore write $x+y=\sum_{i=1}^{n} x_i$ with $x_i \in X$, $n \in \{3,4\}$. Say x_i is in B_{A_i} with $|A_i| \geqslant d_1$ for $1 \leqslant i \leqslant n$. Since

$$|A_1|+|A_2|+|A_3| \geqslant 3d_1 > d,$$

we know that A_1, A_2, A_3 cannot be pairwise disjoint. So say $t \in A_1 \cap A_2$. Therefore $x_1^{(t)}, x_2^{(t)}$ are both bigger than $\frac{F^{(t)}}{2}$ and hence $(x+y)^{(t)} > F^{(t)}$. This implies that $x+y \in S_F \subseteq S$. This shows that $S(X)$ is closed under addition. Therefore, $S(X)$ is a Frobenius generalized numerical semigroup with Frobenius gap F.

Finally we show that $S(X)$ are distinct for distinct X. This will follow from the fact that

$$S(X) \cap (B \cup C) = X.$$

Clearly $X \subseteq S(X) \cap (B \cup C)$. However if equality does not hold, then there will be some x in $(X+X) \cap (B \cup C)$. This means that $x = x_1 + x_2$ with $x_1, x_2 \in X$. Say $x_i \in B_{A_i}$, we know that $d_1 \leqslant |A_i| \leqslant 2d_1 - 1$. Now if $A_1 \cap A_2 \neq \emptyset$, then there will be some $t \in A_1 \cap A_2$. That will imply $x^{(t)} > F^{(t)}$ and contradict $x \in B \cup C$. On the other hand if $A_1 \cap A_2 = \emptyset$, then $|A_1 \cup A_2| = |A_1| + |A_2| \geqslant 2d_1 > 2d_1 - 1$. For each $i \in A_1 \cup A_2$ we have $x^{(i)} > \frac{F^{(i)}}{2}$. This means that x cannot be in any B_A with $|A| \leqslant 2d_1 - 1$ and this again contradicts $x \in B \cup C$. We therefore see that $(X+X) \cap (B \cup C) = \emptyset$ and hence $X = S(X) \cap (B \cup C)$. Therefore, $S(X)$ are distinct Frobenius generalized numerical semigroups for distinct X. Hence, the number of Frobenius generalized numerical semigroups we constructed is $3^{\frac{1}{2}|B|} 2^{|C|}$. $\square$

Corollary 15. *Let $d_1 = \lceil \frac{d+1}{3} \rceil$. If $F \in \mathbb{N}^d$, then*

$$\left(\left(\sqrt{3}\right)^{\frac{1}{2^d} \sum_{i=d_1}^{d-d_1} \binom{d}{i}} \times \left(2\right)^{\frac{1}{2^d} \sum_{i=d-d_1+1}^{2d_1-1} \binom{d}{i}} \right)^{\|F-1\|} \leqslant N(F).$$

For most d, the lower bound in Corollary 15 appears to be optimized. However, for $d=5$ this gives a lower bound of

$$\left(3^{\frac{5}{16}}\right)^{F^{(1)}F^{(2)}F^{(3)}F^{(4)}F^{(5)}} \leqslant N(F^{(1)}, F^{(2)}, F^{(3)}, F^{(4)}, F^{(5)}).$$

But we can improve the $3^{5/16}$ to $\sqrt{2}$.

Proposition 16. *For $F \in \mathbb{N}^5$*

$$\left(\sqrt{2}\right)^{\|F-1\|} \leqslant N(F).$$

Proof. We use the notations from the proof of Theorem 14. Let D be the union of all boxes B_A with $|A| \geqslant 3$. There are sixteen such boxes, so the size of D is

$$|D| = 16\overline{F^{(1)}} \ldots \overline{F^{(5)}}.$$

For an arbitrary subset X of D we define

$$S(X) = S_F \cup X.$$

We see that F is not in $S(X)$. It is therefore clear that F is the unique maximal element of $\mathbb{N}^d \setminus S(X)$ under the natural partial ordering.

We next show that $S(X)$ is closed under addition. Consider non-zero $x, y \in S(X)$. If at least one of them is in S_F, then $x + y$ is also in S_F. Therefore suppose that both of them are in X. Say $x \in B_{A_1}$ and $y \in B_{A_2}$ with $|A_i| \geqslant 3$. Since $|A_1| + |A_2| \geqslant 6 > 5$, we know that A_1 and A_2 cannot be disjoint. Say $t \in A_1 \cap A_2$, then $(x+y)^{(t)} > F^{(t)}$. This implies that $x + y \in S_F \subseteq S$. This shows that $S(X)$ is closed under addition. Therefore, $S(X)$ is a Frobenius generalized numerical semigroup with Frobenius gap F.

Finally we see that $S(X)$ are distinct for distinct X. This follows from the fact that

$$S(X) \cap D = X.$$

Hence, the number of Frobenius generalized numerical semigroups we constructed is $2^{|D|}$. Finally notice that

$$2^{|D|} = 2^{16\overline{F^{(1)}}\ldots\overline{F^{(5)}}} \geqslant 2^{16\frac{F^{(1)}\ldots F^{(5)}}{32}} = \sqrt{2}^{\|F-1\|}. \qquad \square$$

Lemma 17. *For $d \in \mathbb{N}$, let $d_1 = \lceil \frac{d+1}{3} \rceil$. As $d \to \infty$ we have:*

$$\lim_{d\to\infty} \frac{1}{2^d} \sum_{i=d_1}^{d-d_1} \binom{d}{i} = 1.$$

Proof. Suppose $d \geqslant 4$. Hoeffding's inequality [10, Theorem 1] states that if $X_1, \ldots, X_d$ are independent random variables such that $0 \leqslant X_i \leqslant 1$ and $S_d = X_1 + \cdots + X_d$, then

$$\mathcal{P}\Big(S_d - \mathbb{E}(S_d) \leqslant -t\Big) \leqslant \exp\left(-2\frac{t^2}{d}\right).$$

We take $X_1, \ldots, X_d$ to be independent random variables with $\mathcal{P}(X_i = 0) = \mathcal{P}(X_i = 1) = \frac{1}{2}$ and take $t = \frac{d}{6}$. This means that $\mathbb{E}(S_d) = \frac{d}{2}$ and

$$\mathcal{P}\Big(S_d - \mathbb{E}(S_d) \leqslant -\frac{d}{6}\Big) = \mathcal{P}\Big(S_d \leqslant \frac{d}{3}\Big) = \sum_{0 \leqslant i \leqslant \frac{d}{3}} \frac{1}{2^d}\binom{d}{i} \leqslant \exp\left(-\frac{d}{18}\right).$$

Therefore, we see that

$$1 \geqslant \frac{1}{2^d} \sum_{i=d_1}^{d-d_1} \binom{d}{i} = 1 - 2\frac{1}{2^d}\sum_{i=0}^{d_1-1}\binom{d}{i} \geqslant 1 - 2\exp\left(-\frac{d}{18}\right).$$

We conclude that

$$\lim_{d\to\infty} \frac{1}{2^d} \sum_{i=d_1}^{d-d_1} \binom{d}{i} = 1. \qquad \square$$

Corollary 18. *Given $\epsilon > 0$, for sufficiently large d we have: for every $F \in \mathbb{N}^d$*

$$\left(\sqrt{3} - \epsilon\right)^{\|F-1\|} \leqslant N(F).$$

Proof. We see that

$$\lim_{d\to\infty} \left(\sqrt{3}\right)^{\frac{1}{2^d}\sum_{i=d_1}^{d-d_1}\binom{d}{i}} = \sqrt{3}.$$

Therefore for large d, we will have $\left(\sqrt{3}\right)^{\frac{1}{2^d}\sum_{i=d_1}^{d-d_1}\binom{d}{i}} \geqslant \sqrt{3} - \epsilon$. And for such d, we have

$$N(F) \geqslant \left(\left(\sqrt{3}\right)^{\frac{1}{2^d}\sum_{i=d_1}^{d-d_1}\binom{d}{i}}\right)^{\|F-1\|} \geqslant \left(\sqrt{3} - \epsilon\right)^{\|F-1\|}. \qquad \square$$

6 Upper bounds for the number of Frobenius generalized numerical semigroups

In this section we will obtain an upper bound for the number of Frobenius generalized numerical semigroups with a given Frobenius gap.

Lemma 19. *For any $F \in \mathbb{N}^d$ we have:*

$$N(F) \leqslant \sqrt{3}^{\|F\|}.$$

Proof. Consider the box $B = \{x \in \mathbb{N}^d \mid 0 \leqslant x \leqslant F\}$. This box has $\|F\|$ points in it. They are divided into $\lfloor\frac{\|F\|}{2}\rfloor$ pairs of the form $x, F - x$ and possibly a single point x with $x + x = F$. If S is a Frobenius generalized numerical semigroup with Frobenius gap F, then $S_F \subseteq S$ and $\frac{F}{2} \notin S$. Moreover, for each of the $\lfloor\frac{\|F\|}{2}\rfloor$ pairs we can either pick one of the two points or neither of them. This gives 3 choices for each pair. Therefore,

$$N(F) \leqslant 3^{\lfloor\frac{\|F\|}{2}\rfloor} \leqslant \sqrt{3}^{\|F\|}. \qquad \square$$

Combining Corollary 18 and Lemma 19, we get the following result.

Theorem 3. Given $\epsilon > 0$, there exists $M > 0$ such that for all $d > M$ and $F \in \mathbb{N}^d$

$$\left(\sqrt{3} - \epsilon\right)^{\|F-1\|} \leqslant N(F) \leqslant \sqrt{3}^{\|F\|}.$$

Theorem 3 shows that for large d the upper bound of $\sqrt{3}^{\|F\|}$ is close to the actual value. However, for small d this is not the case. For example for $d = 1$, [2] proves that for any $F \in \mathbb{N}$

$$N(F) \leqslant 4\sqrt{2}^F.$$

We therefore look for a stronger upper bound, specially for smaller d.

Given $P, F \in \mathbb{N}^d$ with $P < F$, we denote by $L(P, F)$ the number of Frobenius generalized numerical semigroups with Frobenius gap F that do not contain P.

Lemma 20. *Given $P, F \in \mathbb{N}^d$ with $P < F$, we have*

$$L(P,F) \leqslant \frac{\phi^2}{\sqrt{5}}\phi^{\|F\|}\left(\frac{\phi}{\sqrt[4]{5}}\right)^{\|F\|-\|P\|}.$$

Proof. Consider the box $B = \{x \in \mathbb{N}^d \mid x \leqslant F\}$. It has $\|F\|$ points. Consider a graph with points of B as vertices, such that x and y are connected by an edge when $x + y \in \{P, F\}$ and $x \neq y$ (see Example 21). For any Frobenius generalized numerical semigroup S with $\mathrm{F}(S) = F$ and $P \in \mathcal{H}(S)$, we know that $S \cap B$ will be a subset of the graph that does not contain any edges within it. We therefore count such subsets to get an upper bound for $L(P, F)$.

Let us analyze the degrees of vertices $x \in B$.

- If $x \notin \{\frac{F}{2}, \frac{P}{2}\}$ and $x \leqslant P$, then x has degree two. It has edges to $F - x$ and $P - x$.
- If $x \notin \{\frac{F}{2}, \frac{P}{2}\}$ and $x \nleqslant P$, then x has degree one. It has an edge to $F - x$.
- If $x = \frac{P}{2}$ (so $\frac{P}{2} \in \mathbb{N}^d$), then x has degree one. It has an edge to $F - x$.
- If $x = \frac{F}{2}$ and $\frac{F}{2} \leqslant P$, then x has degree one. It has an edge to $P - x$.
- If $x = \frac{F}{2}$ and $\frac{F}{2} \nleqslant P$, then x has degree zero.

Therefore, the number of degree one vertices is $\|F\| - \|P\| + \alpha$, for some $\alpha \in \{-1, 0, 1, 2\}$. Let β be the number of isolated vertices, so $\beta \in \{0, 1\}$. Ignoring the possible isolated vertex, denote the rest of the graph as G. So G has $\|F\| - \beta$ vertices and each vertex has degree one or two. Therefore, G must be a union of paths and cycles.

We will show that G cannot have any cycles. Assume for the sake of contradiction that G has a cycle $x_1, x_2, \ldots, x_n$. Let $x_{n+1} = x_1$. Since each x_i has degree two, we see that $\{x_{i-1} + x_i, x_i + x_{i+1}\} = \{P, F\}$. This means that the edges alternately sum to P and F. Therefore, n must be even, say $n = 2n_1$. Without loss of generality suppose $x_1 + x_2 = F$. Then for each $i \in \{1, 2, \ldots, n_1\}$, we have $x_{2i-1} + x_{2i} = F$ and $x_{2i} + x_{2i+1} = P$. This implies that

$$n_1 F = \sum_{j=1}^{2n_1} x_j = n_1 P.$$

But this is impossible since $P \neq F$. Therefore, we have a contradiction and G cannot have any cycles.

Now we know that G is a disjoint union of paths and it has $\|F\| - \|P\| + \alpha$ vertices of degree one. Let $k = \frac{\|F\| - \|P\| + \alpha}{2}$, so G must be a union of k disjoint paths. Say the lengths of the paths are $n_1, n_2, \ldots, n_k$. Then $\sum_{i=1}^{k} n_i = \|F\| - \beta$.

We call a subset of the vertices good if the subset does not contain any edges. We claim that for a path graph with n vertices there are F_{n+2} good subsets. Here, F_k is the k^{th} Fibonacci number. This is easily seen for $n \in \{1, 2\}$. We proceed by induction. Suppose that $n \geqslant 3$ and this has been checked for $n - 1$, $n - 2$. Now consider a path graph of length n, call the vertices $x_1, \dots, x_n$ in order. If a good subset includes x_n, then it cannot include x_{n-1}, so there are $F_{(n-2)+2}$ good subsets that include x_n. On the other hand there are $F_{(n-1)+2}$ good subsets that do not have x_n. Therefore, the total number of good subsets is $F_n + F_{n+1} = F_{n+2}$. This completes the induction step.

We therefore see that the number of good subsets of G is

$$\prod_{i=1}^{k} F_{n_i+2} \leqslant \prod_{i=1}^{k} \frac{1}{\sqrt{5}} \phi^{n_i+2} = \left(\frac{\phi^2}{\sqrt{5}}\right)^k \phi^{\|F\|-\beta} \leqslant \frac{\phi^2}{\sqrt{5}} \left(\frac{\phi^2}{\sqrt{5}}\right)^{\frac{\|F\|-\|P\|}{2}} \phi^{\|F\|}. \qquad \square$$

Example 21. Suppose $d = 2$, $F = (3,3)$ and $P = (2,2)$. Then the graph G has 16 vertices and is a union of 4 paths.

$$\begin{array}{c}
(3,0) \text{ —— } (0,3) \\
(3,1) \text{ —— } (0,2) \text{ —— } (2,0) \text{ —— } (1,3) \\
(3,2) \text{ —— } (0,1) \text{ —— } (2,1) \text{ —— } (1,2) \text{ —— } (1,0) \text{ —— } (2,3) \\
(3,3) \text{ —— } (0,0) \text{ —— } (2,2) \text{ —— } (1,1)
\end{array}$$

We now obtain an improved upper bound of $N(F)$ by combining Lemma 19 and Lemma 20, while keeping in mind that Lemma 20 is more accurate when $\|F\| - \|P\|$ is small.

Lemma 22. *For any ϵ with $0 < \epsilon < 1$ and for any $F \in \mathbb{N}^d$ we have*

$$N(F) \leqslant \sqrt{3}^{(1-2\epsilon^d)\|F\|} + \frac{\phi^2}{\sqrt{5}} \epsilon^d \|F\| \phi^{\|F\|} \left(\frac{\phi}{\sqrt[4]{5}}\right)^{(1-(1-\epsilon)^d)\|F\|}.$$

Proof. Denote by B the box consisting of those $x \in \mathbb{N}^d$ with $x \leqslant F$. Let B_1 be the box of those x with $(1-\epsilon)F \leqslant x \leqslant F$. And let B_2 be the box of those x with $x \leqslant \epsilon F$. We divide the Frobenius generalized numerical semigroups with $\mathrm{F}(S) = F$ into two categories. The first one consisting of those that have at least one gap in B_1 (other than F) and the second one consisting of those that have no gaps in B_1 (except F).

First we count the first category generalized numerical semigroups. There are $\epsilon^d \|F\|$ points in B_1. For each $P \in B_1$ the number of first category generalized numerical semigroups with P as a gap is at most

$$\frac{\phi^2}{\sqrt{5}} \phi^{\|F\|} \left(\frac{\phi}{\sqrt[4]{5}}\right)^{\|F\|-\|P\|} \leqslant \frac{\phi^2}{\sqrt{5}} \phi^{\|F\|} \left(\frac{\phi}{\sqrt[4]{5}}\right)^{\|F\|(1-(1-\epsilon)^d)}.$$

Therefore, the total number of first category generalized numerical semigroups is at most

$$\frac{\phi^2}{\sqrt{5}} \epsilon^d \|F\| \phi^{\|F\|} \left(\frac{\phi}{\sqrt[4]{5}}\right)^{\|F\|(1-(1-\epsilon)^d)}.$$

We now count the second category generalized numerical semigroups. A second category generalized numerical semigroup must contain all of B_1 (except F) and hence cannot intersect B_2 (other than 0). There are $(1-2\epsilon^d)\|F\|$ points in $B \setminus (B_1 \cup B_2)$. They can be divided into pairs of the form $x, F-x$. A generalized numerical semigroup cannot have both the points from any of these pairs. Therefore, we have 3 choices for each pair and the total number of second category generalized numerical semigroups is at most

$$\sqrt{3}^{(1-2\epsilon^d)\|F\|}. \qquad \square$$

We now optimize the ϵ in Lemma 22.

Proposition 23. *Let ϵ_d be the solution of the equation:*

$$(1-\epsilon_d)^d \log\left(\frac{\phi}{\sqrt[4]{5}}\right) - \epsilon_d^d \log(3) = \log\left(\frac{\phi^2}{\sqrt[4]{5}\sqrt{3}}\right).$$

Let $b_d = \sqrt{3}^{(1-2\epsilon_d^d)}$. Then

$$N(F) \leqslant \frac{2\phi^2}{\sqrt{5}}\|F\| b_d^{\|F\|}.$$

7 Further Questions

While we have obtained a good estimate of $N(F)$ for large d, our upper bound is weak for smaller d. For $d \neq 5$, let $d_1 = \lceil \frac{d+1}{3} \rceil$ and

$$a_d = \left(\left(\sqrt{3}\right)^{\frac{1}{2^d}\sum_{i=d_1}^{d-d_1}\binom{d}{i}} \times \left(2\right)^{\frac{1}{2^d}\sum_{i=d-d_1+1}^{2d_1-1}\binom{d}{i}}\right).$$

For $d = 5$, let $a_5 = \sqrt{2}$. Let $b_1 = \sqrt{2}$, and for $d \geqslant 2$, let b_d be the constants from Proposition 23. We have shown that for each $F \in \mathbb{N}^d$

$$a_d^{\|F-1\|} \leqslant N(F) \leqslant O\left(\|F\| b_d^{\|F\|}\right).$$

The constants $\sqrt{2} \leqslant a_d \leqslant b_d < \sqrt{3}$ satisfy

$$\lim_{d\to\infty} a_d = \lim_{d\to\infty} b_d = \sqrt{3}.$$

Some of these constants are listed in Table 1 up to four decimal places.

Conjecture 24. For each $d \in \mathbb{N}_{>0}$, $N(F)$ is of the magnitude of $a_d^{\|F\|}$.

Another direction to extend this would be to consider an anti-chain A of k points in $\mathbb{N}^d$ (anti-chain with respect to natural partial ordering). And attempting to count the number of generalized numerical semigroups $S \subseteq \mathbb{N}^d$ for which $\mathrm{FA}(S) = A$.

d	a_d	b_d	d	a_d	b_d	d	a_d	b_d
1	1.4142	1.4142	6	1.4904	1.7311	11	1.5293	1.7320
2	1.3160	1.6630	7	1.5130	1.7319	12	1.5798	1.7320
3	1.4142	1.6968	8	1.4777	1.7320	13	1.5891	1.7320
4	1.4612	1.7173	9	1.5415	1.7320	14	1.5693	1.7320
5	1.4142	1.7275	10	1.5553	1.7320	15	1.6095	1.7320

Table 1: Constants for upper and lower bound

Acknowledgements

We would like to thank Nathan Kaplan for several helpful discussions.

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