Crowns in linear 3-graphs of minimum degree 4

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Abstract

A 3-graph is a pair $H = (V, E)$ of sets, where elements of V are called points or vertices and E contains some 3-element subsets of V, called edges. A 3-graph is called *linear* if any two distinct edges intersect in at most one vertex.

There is a recent interest in extremal properties of 3-graphs containing no *crown*, three pairwise disjoint edges and a fourth edge which intersects all of them. We show that every linear 3-graph with minimum degree 4 contains a crown. This is not true if 4 is replaced by 3.

Mathematics Subject Classifications: 05B07, 05C35, 05D05

1 Introduction

A 3-*graph* is a pair $H = (V, E)$ of sets, where elements of V are called points or vertices and E contains some 3-element subsets of V, called edges. If not clear from the context, we use the notation $V(H)$ and $E(H)$ for V and E respectively. We restrict ourselves to the important family of *linear* 3-graphs where *any two distinct edges intersect in at most one vertex.* In the remainder of this paper we use the term 3-graph for linear 3-graph.

The number of edges containing a point $v \in V(H)$ is the *degree* of v and is denoted by $d(v)$ or $d_H(v)$. We denote by $\delta(H)$ the minimum degree of H. Similar notations are used for graphs (2-uniform linear hypergraphs). We use $[k]$ to denote $\{1, \ldots, k\}$.

Let F be a fixed 3-graph. A 3-graph, H, is called F-*free* if H has no subgraph isomorphic to F. The *(linear) Turán number of* F, $\mathrm{ex}_\ell(n, F)$, is the maximum number of edges in an F-free 3-graph on n vertices.

Turán and Ramsey numbers of several linear 3-graphs have been studied by Gyárfás and Sárközy [5] and the acyclic case by Gyárfás, Ruszinkó and Sárközy [6]. The behavior of $\mathrm{ex}_\ell(n, F)$ is interesting even if F has three or four edges. A famous theorem of Ruzsa and Szemerédi [7] is that $\mathrm{ex}_\ell(n, T) = o(n^2)$ if T is the *triangle.* For the *Pasch configuration*, P, $\mathrm{ex}_\ell(n, P) = \frac{n(n-1)}{6}$ for infinitely many n since there are P-free Steiner triple systems (see [2]). For the *fan*, F, we have $\mathrm{ex}_\ell(n, F) = \frac{n^2}{9}$ if n is divisible by 3 (see [4]). Figure 1 shows these 3-graphs (drawn with the convention that edges are represented as straight line segments).

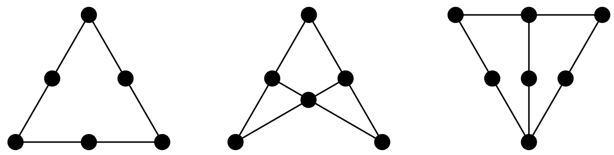

Figure 1: Triangle, Pasch configuration, and Fan

There is a recent interest in the Turán number of the *crown*, C (Figure 2).

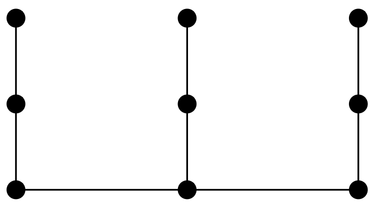

Figure 2: The crown C

The descriptive name crown was coined in [1], the list of small configurations in [2] refers to it as C_{13}. We call the horizontal edge of the crown the *base* and the vertical edges *jewels.*

The crown is the only 3-tree with at most four edges whose Turán number left open in [6] with the following bounds:

$$6\left\lfloor\frac{n-3}{4}\right\rfloor \leqslant \mathrm{ex}_\ell(n,C) \leqslant 2n. \tag{1}$$

The construction for the lower bound in (1) (for the case $n \equiv 3 \pmod 4$) is the following. Choose three vertices $\{a,b,c\}$, and define edges

$$\{a,x_i,y_i\},\ \{a,z_i,w_i\},\ \{b,x_i,w_i\},\ \{b,y_i,z_i\},\ \{c,x_i,z_i\},\ \{c,y_i,w_i\}$$

where $i = 1,2,\dots,\lfloor (n-3)/4\rfloor$ and x_i, y_i, z_i, and w_i are distinct vertices.

In this construction, all but three vertices have degree 3. This poses the question whether raising the minimum degree of a 3-graph H from 3 to 4 ensures a crown. We prove in this note (extracted from [1]) that the answer is affirmative.

Theorem 1. *Every 3-graph with minimum degree $\delta(H) \geqslant 4$ contains a crown.*

The upper bound of (1) was first improved to $\frac{5n}{3}$ by Fletcher [3] then, with an essential new idea, Tang, Wu, Zhang and Zheng [8] proved that the lower bound is essentially best. Note that Theorem 1 does not follow from this, since minimum degree 4 ensures only $\frac{4n}{3} < \frac{3n}{2}$ edges.

In Sections 2 and 3 we define our tools. In Section 4 we prove Theorem 1.

2 Link graphs of edges with $D(e) = \langle 4,4,4\rangle$

Definition 2 (Link graph of an edge)**.** Assume that H is a 3-graph and $e = \{a,b,c\} \in E(H)$. The *link graph*, $G(e)$, is the graph whose edges are the pairs $\{x,y\}$ for which there exists $\{x,y,z\} \in E(H)$ with $z \in \{a,b,c\}$. The set of vertices of $G(e)$ is defined as the subset of $V(H)$ covered by the edges of $G(e)$.

A *matching* in a graph is a set of pairwise disjoint edges. An edge-coloring of a graph is *proper* if the edges in each color class form a matching. Note that Definition 2 provides a proper 3-coloring of the edges of $G(e)$ with colors a, b, and c. We denote by $\varphi(x,y)$ the color of the edge $\{x,y\}$ in this coloring. Edges with colors a, b, and c will be labelled α, β, and γ, respectively and are colored blue, green and red in colored figures. Observe that a crown with base edge e exists in H if and only if $G(e)$ has three pairwise disjoint edges with different colors, which we call a *rainbow matching*.

For $e = \{a,b,c\} \in E(H)$, let $D(e)$ denote the degree vector $\langle d(a),d(b),d(c)\rangle$ with coordinates in non-increasing order.

Lemma 3. *If a crown-free 3-graph H has an edge $e = \{a,b,c\}$ such that $D(e) = \langle 4,4,4\rangle$, then $G(e)$ is isomorphic (up to permutation of colors) to one of the five graphs in Figure 3.*

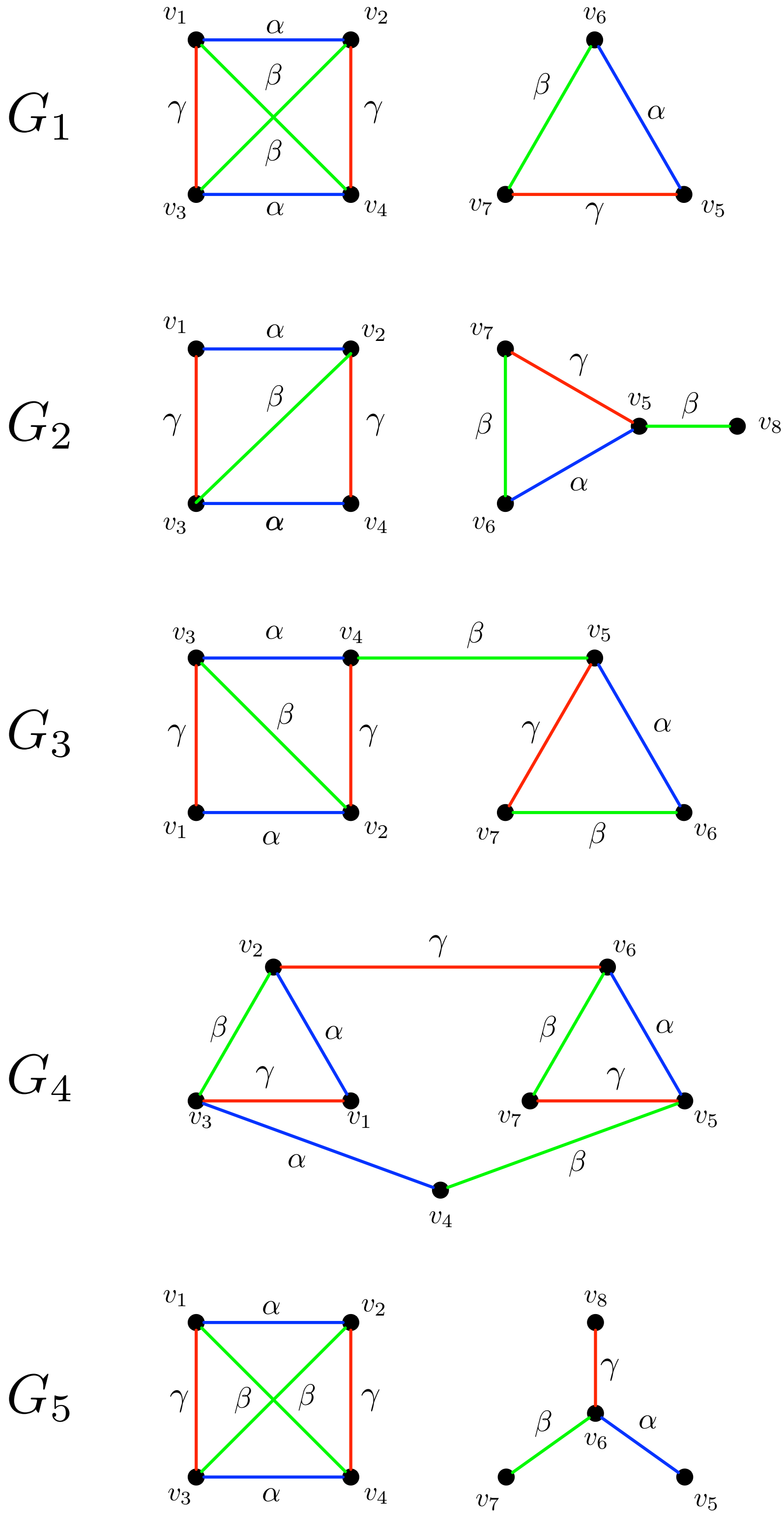

Figure 3: Link graphs $G_1, \ldots, G_5$

Proof. For $i \in [3]$, let M_i denote the vertex set of the matching of color a, b, and c in $G(e)$, respectively. Observe that for all $i, j \in [3]$, where $i \neq j$, M_i must intersect all the three edges in M_j. Otherwise, there exists an edge $f \in M_j$ not intersecting M_i and an edge $g \in M_k$, for $k \neq i$ or j, such that $g \cap f = \emptyset$. Then f, g, and some edge in M_i is a rainbow matching.

First, we show that $|M_1 \cap M_2| > 3$. It follows from the previous observation that $|M_1 \cap M_2| \geqslant 3$. Assume for contradiction that $|M_1 \cap M_2| = 3$. Then $S = M_1 \cap M_2$ cannot contain an edge from $M_1 \cup M_2$, otherwise we get a contradiction with our observation. Thus, M_1 and M_2 are matchings from S to $M_1 \setminus S$ and from S to $M_2 \setminus S$, respectively, where $M_1 \setminus S$ and $M_2 \setminus S$ are disjoint. If $e \in M_3$, then $e \in S$ since otherwise e would be part of a rainbow matching. However, it is impossible for there to be three disjoint edges in S since $|S| = 3$ by assumption.

Now, suppose $|M_1 \cap M_2| = 4$. Then, in S there is one edge of M_1 and one edge of M_2. If the two edges are disjoint, we have two disjoint α-β paths both with four vertices. One has two α-edges and one β-edge, and the other has the opposite. To avoid a rainbow matching, any γ-edge must intersect every edge in one of the paths. There are thus four possible locations for a γ-edge but any three of them gives G_2, proving the lemma. On the other hand, if the two edges in S intersect, we will get a contradiction. Indeed, in this case, the edges of $M_1 \cup M_2$ form two disjoint, alternating α-β paths with three and five vertices, respectively. Following the paths, label the three vertices v_1 through v_3, and the five vertices w_1 through w_5. The only possible location for a γ-edge that does not intersect v_2 and would not form a rainbow matching is $\{w_2, w_4\}$. However, then at least two γ-edges must intersect v_2, a contradiction.

We show that the next case implies that $G(e)$ is isomorphic (up to permutation of colors) to one of the G_i's.

Assume $|M_1 \cap M_2| = 5$. Then either $M_1 \cup M_2$ is an alternating α-β path on seven vertices, or it is a disjoint alternating α-β four-cycle and α-β path on three vertices.

In the former case, label the vertices v_1 through v_7 along the path. Then the possible γ-edges that don't create a rainbow matching are $\{v_1, v_3\}$, $\{v_1, v_6\}$, $\{v_2, v_4\}$, $\{v_2, v_6\}$, $\{v_2, v_7\}$, $\{v_4, v_6\}$, and $\{v_5, v_7\}$. Apart from the symmetry (reflection of a point of the path through v_4), the non-intersecting triples of these edges are

$$\big\{\{v_1, v_3\}, \{v_2, v_4\}, \{v_5, v_7\}\big\},$$

$$\big\{\{v_1, v_3\}, \{v_2, v_6\}, \{v_5, v_7\}\big\},\ \big\{\{v_1, v_3\}, \{v_2, v_7\}, \{v_4, v_6\}\big\}.$$

The first triple gives G_3, and the last two triples give G_4.

In the latter case denote the alternating four-cycle and the alternating path by C_4 and P_3, respectively.

When $G(e)$ is disconnected, at least one γ-edge has to intersect C_4, otherwise all three γ-edges intersect P_3, which clearly results in a rainbow matching. Moreover, both vertices of such a γ-edge must intersect C_4, otherwise, this γ-edge together with two disjoint α-β paths contains a rainbow matching. Therefore, there are two ways of placing a γ-edge in C_4. If a γ-edge intersects P_3, it must intersect both edges of P_3, extending P_3 to either a triangle or to a four-edge star. If only one γ-edge intersects C_4 and two intersect P_3, we obtain G_2. If two γ-edges intersect C_4 and only one intersects P_3, it gives either G_1 or G_5.

When $G(e)$ is connected, some γ-edge connects C_4 and P_3. Observe that such a γ-edge has to intersect both edges in P_3. The only possibility for the remaining two γ-edges is that one is a diagonal of C_4 and the other extends P_3 to a triangle. This gives G_3.

Lastly, suppose $|M_1 \cap M_2| = 6$. Then $M_1 \cup M_2$ is an alternating α-β six-cycle. Any γ-edges intersecting the cycle in at most one vertex or along a long diagonal are in rainbow matchings. However, at most two γ-edges can be short diagonals without intersecting, which is a contradiction thus concludes the proof. □

3 Good quintuple lemma

As shown in Section 2, it is easy to recognize a crown with base edge e: We have to find a rainbow matching in $G(e)$. To recognize other crowns related to $G(e)$, we introduce the following definition:

Definition 4 (Good quintuple)**.** Let H be a 3-graph and $e \in H$. An ordered quintuple $Q = (x_1, x_2, x_3, x_4, x_5)$ of distinct vertices of $G(e)$ is *good* if

- $\{x_1, x_2\}$, $\{x_2, x_3\}$, and $\{x_4, x_5\}$ are edges of $G(e)$,
- $\varphi(x_1, x_2) = \varphi(x_4, x_5)$. ($\{x_1, x_2\} \cap \{x_4, x_5\} = \emptyset$ since φ is a proper coloring.)

Remark 5. The ordering of the vertices in $Q = (x_1, x_2, x_3, x_4, x_5)$ is important. Assume that Q is a good quintuple. Then the quintuple $(x_1, x_2, x_3, x_5, x_4)$ is still good. However, observe that $(x_2, x_1, x_3, x_4, x_5)$ is good if and only if $\{x_1, x_3\}$ is an edge in $G(e)$. On the other hand, $(x_3, x_2, x_1, x_4, x_5)$ is never good.

Remark 6. Observe (see Figure 3) that apart from $v_6 \in V(G_5)$, every vertex in each G_i is the first vertex of some good quintuple.

Lemma 7 (Good quintuple lemma)**.** *Assume H is a crown-free* 3*-graph and $Q = \{x_1, x_2, x_3, x_4, x_5\}$ is a good quintuple in $G(e)$ for some $e = \{a, b, c\} \in E(H)$. Then there is no edge $f \in E(H)$ such that $f \cap e = \emptyset$ and that $f \cap Q = \{x_1\}$.*

Proof. Without loss of generality, Q defines the edges $\{x_1, x_2, a\}$, $\{x_2, x_3, b\}$, and $\{x_4, x_5, a\}$ in H. Assume towards contradiction that edge $f = \{p, q, x_1\}$ where (from the assumptions) $p, q \notin Q \cup \{a, b, c\}$.

Observe that

$$\{p, q, x_1\}, \{x_2, x_3, b\}, \{x_4, x_5, a\}$$

are pairwise disjoint edges and $\{x_1, x_2, a\}$ intersects all of them, thus we have a crown (with base $\{x_1, x_2, a\}$), a contradiction. □

4 Proof of Theorem 1

Suppose that Theorem 1 is not true, there exists a crown-free 3-graph H with $\delta(H) \geqslant 4$. Select an arbitrary edge $e = \{a, b, c\} \in E(H)$ and let H' be the 3-graph obtained from H by removing edges intersecting e until $D(e) = \langle 4, 4, 4 \rangle$ in H'. Then Lemma 3 can be applied to H' and we get that $G_i \subseteq G(e)$ for some $i \in [5]$. Further, note that every vertex v in G_i has degree at most three in H', thus we can select $f_v \in E(H)$ such that $v \in f_v$ and $f_v \cap e = \emptyset$. Selecting $v \neq v_6$, there exists a good quintuple Q with first vertex v in G_i (see Remark 6). We shall get a contradiction from Lemma 7, finding a good quintuple Q satisfying $f_v \cap Q = \{v\}$. This is obvious if $f_v \cap V(G_i) = \{v\}$, therefore in the subsequent cases we may assume that $f_v = \{v, p, q\}$ where $v, p \in V(G_i)$.

- $G(e) = G_1$. Set $v = v_1$ and from the symmetry of G_1 we may assume that $f_{v_1} = \{v_1, v_7, q\}$ (where $q \notin V(G_1)$). Then $Q = (v_1, v_2, v_3, v_5, v_6)$ is a good quintuple.

- $G(e) = G_2$. Set $v = v_1$ and (apart from symmetry) we have to consider either $f_{v_1} = \{v_1, v_7, q\}$ (where $q = v_8$ is possible) or $f_{v_1} = \{v_1, p, q\}$ where $p \in \{v_5, v_8\}$ and $q \notin V(G_2)$. In the former case $Q = (v_1, v_2, v_3, v_5, v_6)$ and in the latter $Q = (v_2, v_3, v_4, v_6, v_7)$ is a good quintuple.

- $G(e) = G_3$. Set $v = v_4$ and (up to symmetry) we have to consider either $f_{v_4} = \{v_4, v_6, q\}$ (where $q = v_1$ is possible) or $f_{v_4} = \{v_4, v_1, q\}$ (where $q \notin G(e)$). In both cases $Q = (v_4, v_5, v_7, v_2, v_3)$ is a good quintuple.

- $G(e) = G_4$. Set $v = v_2$. We have to consider three cases: either $f_{v_2} = \{v_2, v_4, q\}$ (where $q = v_7$ is possible), $f_{v_2} = \{v_2, v_7, q\}$ (where $q \neq v_4$), and $f_{v_2} = \{v_2, v_5, q\}$. In the first two cases $Q = (v_2, v_1, v_3, v_5, v_6)$ is a good quintuple, and in the last case $Q = (v_2, v_3, v_4, v_6, v_7)$ is a good quintuple.

- $G(e) = G_5$. Set $v = v_8$. Assume first $f = f_{v_8} \neq \{v_5, v_7, v_8\}$. Up to symmetry, we may consider $f_{v_8} = \{v_8, v_1, q\}$ (where $q = v_7$ is possible). In both cases, $Q = (v_8, v_6, v_5, v_2, v_4)$ is a good quintuple. Now, assume that $f = f_{v_8} = \{v_5, v_7, v_8\}$. In this case, $d_H(a) = d_H(b) = d_H(c) = 4$, i.e. $H = H'$ since otherwise we have an edge g intersecting e and intersecting $V(G_5)$ in at most one of $\{v_5, v_7, v_8\}$ and in none of $\{v_1, v_2, v_3, v_4\}$ (by linearity). Then g would be a jewel in a crown with base $e = \{a, b, c\}$ leading to contradiction. Since $d_{H'}(v_8) \geqslant 4$, there exists $f' = f'_{v_8} = \{v_8, p, q\} \in E(H)$ different from f_{v_8} and from $\{a, v_6, v_8\}$. Since $f' \cap e = \emptyset$ (from $H = H'$), we can select f' instead of f. Up to symmetry, we may assume that $p = v_1$ and $q \notin G(e)$, which we have already considered.

Since all cases ended by finding $v \in V(G_i)$, $f_v \in E(H)$, and a good quintuple Q such that $f_v \cap e = \emptyset$ and $f_v \cap Q = \{v\}$, we get a contradiction from Lemma 7, concluding the proof. □

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