On the hamiltonian property hierarchy of 3-connected planar graphs

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Abstract

The prism over a graph G is the Cartesian product of G with the complete graph K_2. The graph G is prism-hamiltonian if the prism over G has a Hamilton cycle. A good even cactus is a connected graph in which every block is either an edge or an even cycle and every vertex is contained in at most two blocks. It is known that good even cacti are prism-hamiltonian. Indeed, showing the existence of a spanning good even cactus has become the most common technique in proving prism-hamiltonicity. Špacapan [S. Špacapan. A counterexample to prism-hamiltonicity of 3-connected planar graphs. J. Combin. Theory Ser. B, 146:364–371, 2021] asked whether having a spanning good even cactus is equivalent to having a hamiltonian prism for 3-connected planar graphs. In this article we answer his question in the negative, by showing that there are infinitely many 3-connected planar prism-hamiltonian graphs that have no spanning good even cactus. In addition, we prove the existence of an infinite class of 3-connected planar graphs that have a spanning good even cactus but no spanning good even cactus with maximum degree three.

Mathematics Subject Classifications: 05C10, 05C38, 05C45

1 Introduction

All graphs in this article are finite and simple. A *Hamilton cycle* (respectively, *a Hamilton path*) in a graph G is a cycle (respectively, a path) that contains all vertices of G. A graph is *hamiltonian* if it has a Hamilton cycle. A *k-walk* is a spanning closed walk that visits every vertex at most k times, and a *k-tree* is a spanning tree with maximum degree at

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most k. Clearly, a graph has a k-walk if it has a k-tree. It was shown in [14] that every k-walk contains some $(k+1)$-tree. In particular, 1-walk and 2-tree are the same notion of Hamilton cycle and Hamilton path, respectively. The *prism* over a graph G is the Cartesian product of G and the complete graph K_2 on two vertices, denoted by $G\square K_2$. We say G is *prism-hamiltonian* if $G\square K_2$ is hamiltonian. The following hierarchy concerning hamiltonian properties is well-known:

$$\text{Hamilton cycle} \Rightarrow \text{Hamilton path} \Rightarrow \text{Hamilton cycle in prism} \Rightarrow \text{2-walk} \Rightarrow \text{3-tree}$$

The existence of spanning subgraphs in the above hierarchy has been extensively studied for various graph classes; we refer to [4, 5, 18, 17] for results on related topics. In this article, we investigate the hierarchy of 3-connected planar graphs.

In 1884, in an attempt to solve the Four Color Theorem (which was open then), Tait [22] gave a proof based on the truth of a hypothesis that every 3-connected 3-regular planar graph is hamiltonian. The proof, however, turned out to be false and the first counterexample to the hypothesis, that is, a non-hamiltonian 3-connected 3-regular planar graph, was constructed by Tutte [24] in 1946. Later, 3-connected 3-regular planar graphs with no Hamilton paths were also found [12].

On the positive side, Tutte [25] showed in 1956 that every 4-connected planar graph does have a Hamilton cycle. Tutte's result was strengthened by Thomassen [23] who proved that every 4-connected planar graph is hamiltonian-connected, that is, any two vertices are connected by some Hamilton path.

Barnette [1] showed that every 3-connected planar graph has a 3-tree. Confirming a conjecture of Jackson and Wormald [14] and strengthening Barnette's result, Gao and Richter [8] proved that every 3-connected planar graph has in fact a 2-walk (see also [9, 10]). Hence, it is natural to ask if all 3-connected planar graphs can reach the level of "Hamilton cycle in prism" in the hamiltonian property hierarchy. This question was formulated as a conjecture by Kaiser, Ryjáček, Král', Rosenfeld and Voss [15], which was also attributed to Rosenfeld and Barnette (see [11] and [20]).

Conjecture 1 ([15, Conjecture 1])**.** Every 3-connected planar graph has a hamiltonian prism.

To support this conjecture, the following graph classes were shown to be prism-hamiltonian: 3-connected 3-regular (not necessarily planar) graphs [19], Halin graphs [15], 3-connected bipartite planar graphs [2], near-triangulations [2] and 3-connected planar graphs with minimum degree at least four [21]. However, a recent breakthrough by Špacapan [20] showed that the conjecture does not hold in general. Based on Špacapan's technique, Ikegami, Maezawa and Zamfirescu [13] provided various classes of counterexamples with some special properties.

Theorem 2 ([20])**.** *There are infinitely many* 3*-connected planar non-prism-hamiltonian graphs.*

A *good cactus* is a connected graph in which every block is either an edge or a cycle and every vertex is contained in one or two blocks. A good cactus is *even* if it is bipartite (see

Figure 1). It is known that the prism of any good even cactus is hamiltonian. Therefore, one can assert that a graph is prism-hamiltonian if it has a spanning good even cactus. This strategy is in fact the most common approach used to prove the prism-hamiltonicity of many planar and non-planar graph classes; we refer to [19, 15, 2, 16, 3, 6, 7, 21] for examples.[1] It is worth noting that in [19, 15, 16, 3, 6] a more restrictive approach was adopted, namely to show the existence of a spanning good even cactus with maximum degree at most three, that is, any two cycles of the cactus are disjoint. This proof technique motivates us to refine the hamiltonian property hierarchy as follows:

$$\begin{gathered}\text{Hamilton cycle} \Rightarrow \text{Hamilton path}\\ \Rightarrow \text{spanning good even cactus with maximum degree at most three}\\ \Rightarrow \text{spanning good even cactus} \Rightarrow \text{Hamilton cycle in prism} \Rightarrow \text{2-walk} \Rightarrow \text{3-tree}\end{gathered}$$

The first two implications are known to be sharp for 3-connected planar graphs. For instance, the famous Herschel's graph on 11 vertices has a Hamilton path but no Hamilton cycle, and the graph given in Figure 2 has no Hamilton path but it has a spanning good even cactus with maximum degree three. As every 3-connected planar graph has a 2-walk, it follows from Špacapan's result [20] that the implication from "Hamilton cycle in prism" to "2-walk" is sharp as well. Špacapan [20] asked whether the implication from "spanning good even cactus" to "Hamilton cycle in prism" can be reversed for 3-connected planar graphs.

Problem 3 ([20, Problem 3.3])**.** Prove or disprove the following statement. Every 3-connected planar prism-hamiltonian graph has a spanning good even cactus.

The main purpose of this article is to show that every implication except the last one in the new hierarchy proposed above is sharp for 3-connected planar graphs. Inspired by Špacapan's counterexamples to Conjecture 1, we show that there are infinitely many 3-connected planar graphs that have a spanning good even cactus but no such spanning subgraph with maximum degree at most three (Theorem 5), and there are infinitely many 3-connected planar graphs that have a hamiltonian prism but no spanning good even cactus (Theorem 6), thereby answering the question raised by Špacapan.

We remark that "Hamilton cycle in prism" can be replaced by "spanning good cactus" in the hierarchy we consider above. As mentioned in [20], the partitioning result given in [8] assures that every 3-connected planar graph has a good cactus as a spanning subgraph. Another option is to replace "2-walk" by "Hamilton path in prism" as the existence of a Hamilton cycle in the prism obviously implies that of a Hamilton path in the prism, which in turn implies the existence of a 3-tree in the original graph. Note that there are 3-connected planar graphs that have no Hamilton path in the prism (see the remark after the proof of [20, Theorem 2.6]) and 3-connected planar graphs the prism over each of which has a Hamilton path but no Hamilton cycle (consider, for example, the

[1] Note that some graph classes in the given examples were not explicitly shown to have the property of having a spanning good even cactus, but one may justify it by modifying the corresponding original proof.

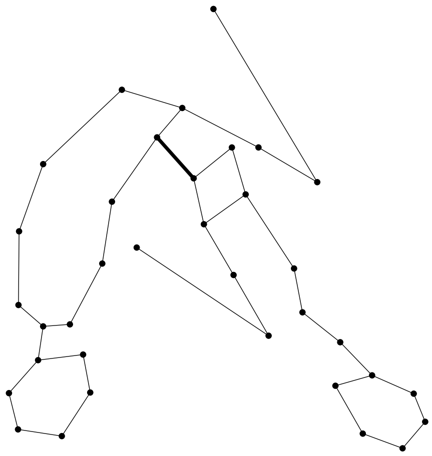

Figure 1: A good even cactus with maximum degree three, which becomes a good even cactus with maximum degree four if the thick edge is contracted.

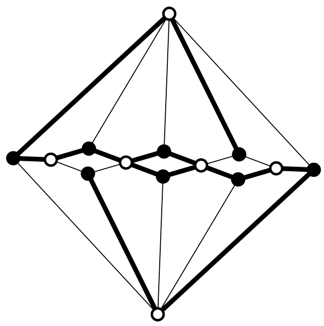

Figure 2: A variation of Herschel's graph which has no Hamilton path (as it has eight components when the six white vertices are removed) and has a spanning good even cactus with maximum degree at most three (thick edges).

graph Z_9 in the proof of [13, Theorem (iii)]). The discussion altogether establishes the following hierarchy regarding hamiltonian properties, in which every implication is sharp for 3-connected planar graphs:

$$\begin{gathered}\text{Hamilton cycle} \Rightarrow \text{Hamilton path} \\ \Rightarrow \text{spanning good even cactus with maximum degree at most three} \\ \Rightarrow \text{spanning good even cactus} \Rightarrow \text{Hamilton cycle in prism} \\ \Rightarrow \text{Hamilton path in prism} \Rightarrow \text{3-tree}\end{gathered}$$

The proofs of our main results will be given in the next section. We conclude this section with some terminology and notation.

Let H be a graph and V be a vertex set (not necessarily a subset of $V(H)$). The subgraph of H induced by $V \cap V(H)$ is denoted by $H[V]$. For graphs H_1 and H_2, $H_1[H_2]$ means $H_1[V(H_2)]$. For any set U of vertices and edges, we use $H - U$ to denote the graph obtained from H by deleting the elements in U; we may also write $H - u$ instead of $H - \{u\}$ when $U = \{u\}$. The union $H_1 \cup H_2$ of graphs H_1 and H_2 is the graph $(V(H_1) \cup V(H_2), E(H_1) \cup E(H_2))$. Let H' be a subgraph of H and $E \subseteq E(H)$ be an edge set. We may denote by $H' \cup E$ the union of H' and the subgraph of H induced by E. Let u, v be two vertices in a connected graph H. The graph $H[u, v]$ is defined to be the minimal union of blocks of H such that $H[u, v]$ is connected and contains vertices u and v. For any graph H and any $v \in V(H)$, let H^i be a copy of H, we denote by v^i the duplicate of v in H^i.

A *cactus* Q is a connected graph such that every block of Q is either an edge or a cycle. For any $v \in V(Q)$, the *block degree* $b_Q(v)$ of v in Q is defined to be the number of blocks of Q that contain v. We call a block of Q that is an edge (a cycle) an *edge block* (a *cycle block*). We say that Q is *even* if every cycle block of it is an even cycle. A path P in Q is an *edge path* if no edge of P is contained in any cycle of Q. A cactus Q is *good* if $b_Q(v) \leqslant 2$ for any $v \in V(Q)$. Note that if we delete some vertex from a good even cactus, the new components are even cacti yet need not be good anymore. For this reason we introduce two more types of cacti as follows:

- Let P be an edge path in a cactus Q. We say that Q is a *P-good* cactus if (i) $b_Q(v) \leqslant 2$ for any vertex v that is not an internal vertex of P and (ii) $b_Q(v) \leqslant 3$ for any internal vertex of P.

- Let P_1 and P_2 be two edge paths in a cactus Q that have at most one common vertex. Then Q is a *$\{P_1, P_2\}$-good* cactus if (i) $b_Q(v) \leqslant 2$ for any vertex v that is neither an internal vertex of P_1 nor an internal vertex of P_2, (ii) $b_Q(v) \leqslant 3$ for any vertex v that is an internal vertex of either P_1 or P_2 (but not for both) and (iii) $b_Q(v) = 4$ for any vertex v that is an internal vertex for both paths P_1 and P_2.

A cactus Q is 1-*good* if there exists an edge path P in Q such that Q is P-good; 2-*good* if there exist two edge paths P_1 and P_2 in Q sharing at most one vertex such that Q is $\{P_1, P_2\}$-good.

We always assume that the complete graph K_2 is on $\{\alpha, \beta\}$. The prism $H \square K_2$ over H is defined to be the graph on $V(H \square K_2) := V(H) \times \{\alpha, \beta\}$ such that $(u, \gamma)(v, \delta)$ are adjacent if and only if (i) $uv \in E(H)$ and $\gamma = \delta$ or (ii) $u = v$ and $\gamma \neq \delta$. For any $\gamma \in V(K_2)$, we denote by $\bar{\gamma}$ the vertex of K_2 other than γ. Let S be a subgraph of $H \square K_2$. The *reflection* R of S is a graph defined as follows: (i) $(u, \gamma) \in V(R)$ if and only if $(u, \bar{\gamma}) \in V(S)$; (ii) $(u, \gamma)(v, \delta) \in E(R)$ if and only if $(u, \bar{\gamma})(v, \bar{\delta}) \in E(S)$.

2 Spanning good even cacti and prism-hamiltonicity

We first introduce three graphs that we need in our construction. The plane graph A is as depicted in Figure 3. Let n be any positive integer. We define C_n to be the cycle of length $2n + 3$ (see Figure 4(a)) and D_n to be the cactus that has exactly two blocks each of them is a cycle of length $2n + 3$ (see Figure 4(b)). We refer to Figures 3 and 4 for names of vertices in A, C_n and D_n that are not explicitly defined in the text.

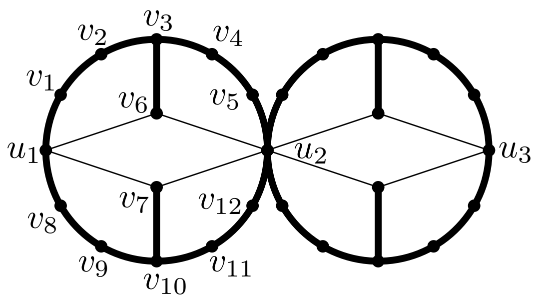

Figure 3: The plane graph A with endvertices u_1 and u_3. The thick edges induce a spanning good even cactus K_A of A with $b_{K_A}(u_1) = b_{K_A}(u_3) = 1$.

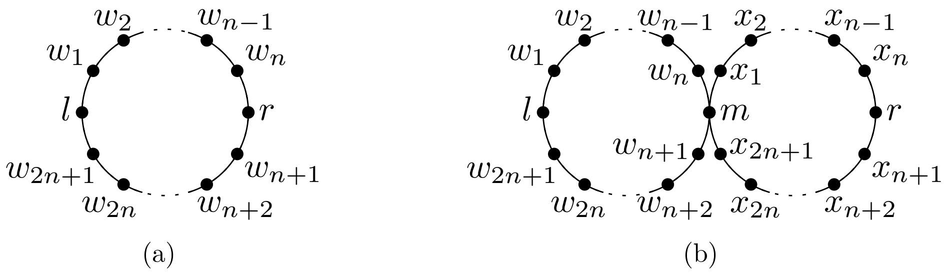

Figure 4: (a) The plane graph C_n with endvertices l and r. (b) The plane graph D_n with endvertices l and r.

We now define our main construction. Let B be either C_n or D_n (which will be fixed throughout the construction). We take eight copies $A^1, \ldots, A^8$ of A and seven copies $B^1, \ldots, B^7$ of B, and form a connected graph $G^-(B)$ from these fifthteen fragments by identifying u_3^i with l^i and identifying r^i with u_1^{i+1} for every $i \in \{1, \ldots, 7\}$. We see $G^-(B)$ as a plane graph by inheriting the plane embeddings of A and B given by Figures 3 and 4. So the boundary walk around the unbounded face is the union of two edge-disjoint paths with endvertices u_1^1 and u_3^8, so that the vertices $w_1^1, \ldots, w_1^7$ and, if B is D_n, the vertices $x_1^1, \ldots, x_1^7$ will be contained in the "upper path" but not the "lower path". The graph $G(B)$ is obtained from $G^-(B)$ and two new vertices s and t by joining s to every vertex in the upper path and t to every vertex in the lower path. We will simply write G and G^- instead of $G(B)$ and $G^-(B)$ if it is clear from the context what B denotes or it causes no ambiguity. It is clear that G is planar. Moreover, it can be shown in exactly the same way as in the proof of [20, Lemma 2.5] that G is 3-connected. We conclude with the following lemma.

Lemma 4. *Let n be any positive integer and B be either C_n or D_n. The graph $G(B)$ constructed above is a 3-connected planar graph.*

The main goal of this article is to prove the following two results.

Theorem 5. *For any positive integer n, the 3-connected planar graph $G(C_n)$ has a spanning good even cactus but no spanning good even cactus with maximum degree three.*

Theorem 6. *For any positive integer n, the 3-connected planar graph $G(D_n)$ is prism-hamiltonian but has no spanning good even cactus.*

We note that the graphs $G(C_n)$ and $G(D_n)$ have unbounded maximum degree (when n tends to infinity). However, by adapting the approach of Ikegami et al. [13], graphs that have a bounded maximum degree and satisfy the condition given in Theorem 5 or Theorem 6 can be constructed.

The proofs of Theorems 5 and 6 will be given in Sections 2.2 and 2.3, respectively. Before that, we prepare in Section 2.1 some lemmas that we need in the proofs later.

2.1 Preliminaries

We first discuss some properties of the fragments we use in the construction regarding whether they can contain spanning cacti with specified block degree condition.

Lemma 7. *Let $I := A[\{u_1, u_2\} \cup \{v_1, \ldots, v_{12}\}]$. Let Q be a spanning even cactus of I that contains an edge path P with endvertices u_1 and u_2. If every vertex of Q that is not an internal vertex of P has block degree at most two in Q, then u_1 and u_2 have block degree two in Q.*

Proof. By symmetry, we may assume that P is contained in $I[\{u_1, u_2\} \cup \{v_7, \ldots, v_{12}\}]$. Suppose that $b_Q(u_1) = 1$ (reductio ad absurdum). It follows immediately that $u_1v_1 \notin E(Q)$, $u_1v_6 \notin E(Q)$, and v_1v_2 and v_2v_3 are two edge blocks of Q (as Q is a spanning cactus). If v_4 is contained in any cycle block of $Q \subset I - \{u_1v_1, u_1v_6\}$, then that block must be $v_3v_4v_5u_2v_6v_3$, which is however impossible since Q is an even cactus. By the same argument, we have that v_5 and v_6 are not in any cycle block of Q.

Note that Q has at least one edge of v_3v_6 and v_6u_2 and at least one edge of v_3v_4 and v_5u_2, since Q is a connected spanning subgraph of I. We consider the following two cases. If both v_3v_6 and v_6u_2 are in $E(Q)$, then (depending on v_3v_4 or v_5u_2 is an edge of Q) v_3 or u_2 has to have block degree at least three in Q, contradicting the given condition. If exactly one of v_3v_6, v_6u_2 is in $E(Q)$, then Q must contain $v_3v_4v_5u_2$ as an edge path and v_3 or u_2 will have block degree at least three in Q, again, a contradiction. We thus conclude that $b_Q(u_1) = b_Q(u_2) = 2$. ☐

Lemma 8. *Let Q be an even cactus of A that contains an edge path P with endvertices u_1 and u_3. If Q is P-good, then Q is not a spanning subgraph of A.*

Proof. Suppose that Q is a spanning subgraph of A (reductio ad absurdum). Let $I_1 := A[\{u_1, u_2\} \cup \{v_1, \ldots, v_{12}\}]$ and $I_2 := A - (\{u_1\} \cup \{v_1, \ldots, v_{12}\})$. Then, for $i \in \{1, 2\}$, $Q_i := Q[I_i]$ is a spanning even cactus of I_i having $P_i := P[I_i]$ as an edge path with endvertices u_i and u_{i+1}. It follows from the definition (of a P-good cactus) that every vertex of Q_i that is not an internal vertex of P_i is contained in at most two blocks of Q_i. Therefore we can apply Lemma 7 twice to conclude that $b_{Q_1}(u_2) = b_{Q_2}(u_2) = 2$ and hence $b_Q(u_2) = 4$, contradicting our assumption that Q is P-good. ☐

Lemma 9. *Let Q be a good even cactus of B that contains vertices l and r, where B is taken to be C_n or D_n for some positive integer n. If l and r have block degree one, then Q is not a spanning subgraph of B.*

Proof. Since all cycles of B are odd, $b_Q(l) = b_Q(r) = 1$ and $b_Q(v) \leqslant 2$ for all $v \in V(Q) \setminus \{l, r\}$, we have Q is a path with endvertices l and r, and hence Q does not span B. □

As mentioned before, one cannot guarantee that the components are good cacti after removing some vertices from a good cactus. However, we may characterize, in terms of good, 1-good and 2-good cacti, the components of the graph obtained from some spanning good cactus of $G(B)$ by deleting vertices s and t.

Lemma 10. *Let K be a good cactus with maximum degree at most three and s, t be two vertices in K. One of the following statements holds:*

(I) *$K - s - t$ is a vertex-disjoint union of at most four cacti, at most two of which are 1-good and the rest are good.*

(II) *$K - s - t$ is a vertex-disjoint union of at most three cacti, one of which is 2-good and the rest are good.*

Proof. As K is a good cactus with maximum degree at most three, no two distinct cycle blocks in K can intersect.

We consider the following cases.

If any one of s and t is not contained in any cycle block, $K - s - t$ has up to three components, at most one of which is a 1-good cactus and the rest are good cacti, which is included in Case (I).

If s and t are in two cycle blocks S_s and S_t in K, respectively, such that $S_s - s$ and $S_t - t$ are contained in separate components of $K - s - t$, then we are in Case (I) since $K - s - t$ is comprised of one $(S_s - s)$-good cactus, one $(S_t - t)$-good cactus and at most one good cactus.

If s and t are in two cycle blocks S_s and S_t in K, respectively, such that $S_s - s$ and $S_t - t$ are in the same component of $K - s - t$, then we are in Case (II) as $K - s - t$ is comprised of one $\{S_s - s, S_t - t\}$-good cactus and $b_K(s) + b_K(t) - 2 \leqslant 2$ good cacti.

Finally, suppose s and t are in some cycle block S in K. Let $k \leqslant 2$ be the number of components of $S - s - t$. Then $K - s - t$ is comprised of $b_K(s) + b_K(t) - 2 \leqslant 2$ good cacti and k 1-good cacti (each of which contains some component of $S - s - t$). So we are in Case (I). □

The following lemma can be proved analogously, thus we omit the proof.

Lemma 11. *Let K be a good cactus and s, t be two vertices in K. One of the following statements holds:*

(I) *$K - s - t$ is a vertex-disjoint union of at most four cacti, each of which is good or 1-good.*

(II) *$K - s - t$ is a vertex-disjoint union of at most three cacti, one of which is 2-good and each of the rest is good or 1-good.*

Suppose G has a spanning subgraph K that is a good even cactus. Let Q be a component of $K - s - t$. Let H be any copy of A or B in G^-. We say H is a *bag* of Q if Q contains the endvertices of H but not all vertices of H, where the endvertices of A (respectively, B) are u_1 and u_3 (respectively, l and r). For $i < 1$, we define l^i and r^i to be u_1^1; for $i > 7$, we define l^i and r^i to be u_3^8. We may choose $0 \leqslant a \leqslant b \leqslant 8$ such that Q is contained in $G^-[l^a, r^b]$ and, subject to this, $b - a$ is minimum. The following three lemmas give us lower bounds on the number of bags of Q.

Lemma 12. *If Q is a good even cactus, then it has at least* $\max\{b - a - 1, 0\}$ *bags.*

Proof. For every $a < i < b$, $Q[B^i]$ is an even cactus of B^i containing l^i and r^i. By the minimality of $b - a$, l^i is contained in some block of $Q[A^i]$ and hence has block degree one in $Q[B^i]$. Similarly, we have $b_{Q[B^i]}(r^i) = 1$. As vertices in $Q[B^i]$ other than l^i, r^i have block degree at most two, we may apply Lemma 9 to conclude that $Q[B^i]$ does not span B^i and hence B^i is a bag of Q. Collecting all these bags for every $a < i < b$, we conclude that Q has at least $\max\{b - a - 1, 0\}$ bags. $\square$

Lemma 13. *Suppose Q is a P-good even cactus for some edge path P in Q. We have that Q has at least* $\max\{b-a-2, 0\}$ *bags. Moreover, if Q has no bag and $b = a+2$, then P is contained in $G^-[r^a, l^b]$.*

Proof. Choose $a \leqslant a' \leqslant b' \leqslant b$ such that $P[A^{i+1}]$ is a path with endvertices r^i and l^{i+1} for any $a' \leqslant i < b'$, and, subject to this, $b' - a'$ is maximum.

Note that if $P[A^{i+1}]$ is a path with endvertices r^i, l^{i+1}, then $Q[A^{i+1}]$ is a $P[A^{i+1}]$-good even cactus of A^{i+1}. Therefore, by Lemma 8, $Q[A^{i+1}]$ does not span A^{i+1}, and hence A^{i+1} is a bag of Q.

By the maximality of $b' - a'$ and the fact that P is a path, we have that P is contained in $G^-[r^{a'-1}, l^{b'+1}]$. This and the minimality of $b - a$ imply that for any i with $a < i < a'$ or $b' < i < b$, $Q[B^i]$ is an even cactus of B^i containing l^i and r^i. Moreover, l^i and r^i have block degree one in $Q[B^i]$ while all other vertices in $Q[B^i]$ have block degree at most two. By Lemma 9, we assure that B^i is a bag of Q. Thus the number of bags of Q is at least $(b' - a') + (a' - a - 1) + (b - b' - 1) = b - a - 2$.

Suppose Q has no bag and $b = a + 2$, we claim that P is contained in $G^-[r^a, l^{a+2}]$. Otherwise, by symmetry, we may assume that P has one endvertex in $B^a - r^a$. Let p be the other endvertex of P. If p is in $G^-[l^{a+1}, r^{a+2}]$, then it follows from Lemma 8 that A^a is a bag of Q. If p is not in $G^-[l^{a+1}, r^{a+2}]$, then, by Lemma 9, B^{a+1} is a bag of Q. In any case it contradicts the assumption that Q has no bag. This thus justifies our claim. $\square$

Lemma 14. *If Q is a 2-good even cactus, then it has at least* $\max\{b - a - 3, 0\}$ *bags.*

Proof. The proof is similar to what we have done for the previous lemmas. Let P_1 and P_2 be two edge paths in Q having at most one common vertex such that Q is $\{P_1, P_2\}$-good. We may choose $a \leqslant a' \leqslant b' \leqslant a'' \leqslant b'' \leqslant b$ such that $(P_1 \cup P_2)[A^{i+1}]$ is a path with endvertices r^i and l^{i+1} for any i with $a' \leqslant i < b'$ or $a'' \leqslant i < b''$, and, subject to this, $b'' - a'' + b' - a'$ is maximum. Now, if $(P_1 \cup P_2)[A^{i+1}]$ is a path with endvertices r^i, l^{i+1}, then $Q[A^{i+1}]$ is an even cactus of A^{i+1} satisfying the block degree condition required by Lemma 8, from which it follows that $Q[A^{i+1}]$ does not span A^{i+1} and A^{i+1} is a bag of Q.

As a', b', a'', b'' are chosen with $b'' - a'' + b' - a'$ maximized and $P_1 \cup P_2$ is either a vertex-disjoint union of two paths or a tree that has at most one vertex of degree larger than two, we have that $P_1 \cup P_2$ is contained in $G^-[r^{a'-1}, l^{b'+1}] \cup G^-[r^{a''-1}, l^{b''+1}]$. For any i with $a < i < a'$ or $b' < i < a''$ or $b'' < i < b$, $Q[B^i]$ is a good even cactus of B^i containing vertices l^i and r^i of block degree one. Applying Lemma 9, we have that B^i is a bag of Q, and hence Q has at least $(b'-a')+(b''-a'')+(a'-a-1)+(a''-b'-1)+(b-b''-1) = b-a-3$ bags. ☐

2.2 Proof of Theorem 5

In this section we shall show that $G(C_n)$ has a spanning good even cactus, but it does not contain any spanning good even cactus with maximum degree three. Note that B will represent the fragment C_n throughout this section.

A spanning good even cactus K_A of A is depicted in Figure 3. We denote by K_A^i the corresponding copy of K_A in A^i. It is straightforward to verify that

$$\left(\bigcup_{i=1}^{8} K_A^i\right) \cup \left(\bigcup_{i=1,3} (B^i - l^i w_1^i)\right) \cup \left(\bigcup_{i=5,7} (B^i - w_{n+1}^i r^i)\right)$$
$$\cup \left(\bigcup_{i=2,4,6} (B^i - l^i v_1^i - w_{n+1}^i r^i)\right) \cup \{sl^1, sw_1^1, tl^3, sw_1^3, tw_{n+1}^5, sr^5, tw_{n+1}^7, tr^5\}$$

is a spanning good even cactus of G.

Thus it is left to show that G does not have any spanning good even cactus with maximum degree at most three. Suppose that G has a spanning good even cactus K with maximum degree at most three (reductio ad absurdum). Let $Q_1, \dots, Q_k$ be the components of $K - s - t$ that contain some vertex from $U_2 := \{u_2^1, \dots, u_2^8\}$. For every Q_j ($j \in \{1, \dots, k\}$), we choose $0 \leqslant a(j) \leqslant b(j) \leqslant 8$ such that Q_j is contained in $G^-[l^{a(j)}, r^{b(j)}]$ and, subject to this, $b(j) - a(j)$ is minimum. Since the union of $Q_1, \dots, Q_k$ contains all vertices in U_2, we have that

$$\sum_{j=1}^{k} (b(j) - a(j)) \geqslant |U_2| = 8.$$

We denote by q_1 and q_2 the numbers of 1-good and 2-good components among $Q_1, \dots, Q_k$.

Let c_j be the number of bags of Q_j. Note that every bag H of Q_j does contain some component of $K - s - t$ that does not contain the endvertices of H. We have that c_j bags of Q_j contain (at least) c_j distinct components of $K - s - t$. Moreover, it is not hard to see that no distinct components from $Q_1, \dots, Q_k$ can have any bag in common. Therefore $K - s - t$ has at least $\sum_{j=1}^{k} (1 + c_j)$ components.

We consider the following two cases according to Lemma 10.

Case (I). If $K-s-t$ has at most four components such that at most two of them are 1-good even cacti and the rest are good even cacti, then, by Lemmas 12 and 13, $K-s-t$ has at least $\sum_{j=1}^{k}(1+c_j) \geqslant \sum_{j=1}^{k}(b(j)-a(j))-q_1 \geqslant 8-2=6$ components, which contradicts that $K-s-t$ has at most four components.

Case (II). If $K-s-t$ has at most three components such that one of them is a 2-good even cactus and the rest are good even cacti, then, by Lemmas 12 and 14, $K-s-t$ has at least $\sum_{j=1}^{k}(1+c_j) \geqslant \sum_{j=1}^{k}(b(j)-a(j))-2q_2 \geqslant 8-2=6$ components, which contradicts that $K-s-t$ has at most three components.

Therefore we conclude that G has no spanning good even cactus with maximum degree at most three, and this completes the proof of Theorem 5.

2.3 Proof of Theorem 6

In this section we show that there is a Hamilton cycle in the prism over $G(D_n)$, but there is no spanning subgraph of $G(D_n)$ that is a good even cactus. Note that B will be the fragment D_n throughout this section.

We first prove that $G(D_n)\Box K_2$ is hamiltonian. Here we assume n is odd, the case n is even can be dealt with analogously. The following result can be easily derived from the proof of [7, Theorem 2.3].

Proposition 15 ([7]). *Let Q be a good even cactus. The prism over Q has a Hamilton cycle that contains all edges $(v,\alpha)(v,\beta)$ with $v \in V(Q)$ and $b_Q(v)=1$.*

As shown in Figure 3, A has a spanning good even cactus in which u_1 and u_3 have block degree one. Hence it follows from Proposition 15 that there exists a Hamilton cycle H_A in $A\Box K_2$ containing the edges $(u_1,\alpha)(u_1,\beta)$ and $(u_3,\alpha)(u_3,\beta)$. In the prism over B, we define L_B, S_B and $\tilde{S}_B$ to be the graphs depicted in Figures 5(a), (b) and (c), respectively; and R_B and $\tilde{R}_B$ be the reflections of S_B and $\tilde{S}_B$, respectively. Each of the graphs L_B, S_B, $\tilde{S}_B$, R_B and $\tilde{R}_B$ is a union of vertex-disjoint paths that spans $B\Box K_2$. One can readily verify that

$$
\begin{aligned}
&(H_A^1-(u_3^1,\alpha)(u_3^1,\beta)) \cup \left(\bigcup_{i=2}^{7}(H_A^i-(u_1^i,\alpha)(u_1^i,\beta)-(u_3^i,\alpha)(u_3^i,\beta))\right) \cup (H_A^8-(u_1^8,\alpha)(u_1^8,\beta))\\
&\cup \left(\bigcup_{i=2,4,6} L_B^i\right) \cup S_B^1 \cup \{(s,\beta)(w_n^1,\beta),(t,\alpha)(x_{2n+1}^1,\alpha)\} \cup \tilde{S}_B^3 \cup \{(s,\beta)(w_n^3,\beta),(t,\beta)(x_{2n}^3,\beta)\}\\
&\qquad \cup R_B^5 \cup \{(s,\alpha)(w_n^5,\alpha),(t,\beta)(x_{2n+1}^5,\beta)\} \cup \tilde{R}_B^7 \cup \{(s,\alpha)(w_n^7,\alpha),(t,\alpha)(x_{2n}^7,\alpha)\}
\end{aligned}
$$

is a Hamilton cycle of $G\Box K_2$.

Now we show that G does not have any spanning good even cactus. Suppose that G has a spanning good even cactus K (reductio ad absurdum). As in the proof of Theorem 5, we consider the components $Q_1,\dots,Q_k$ of $K-s-t$ that contain some vertex from $U_2 := \{u_2^1,\dots,u_2^8\}$. For any component Q_j of $K-s-t$ ($j \in \{1,\dots,k\}$), we choose

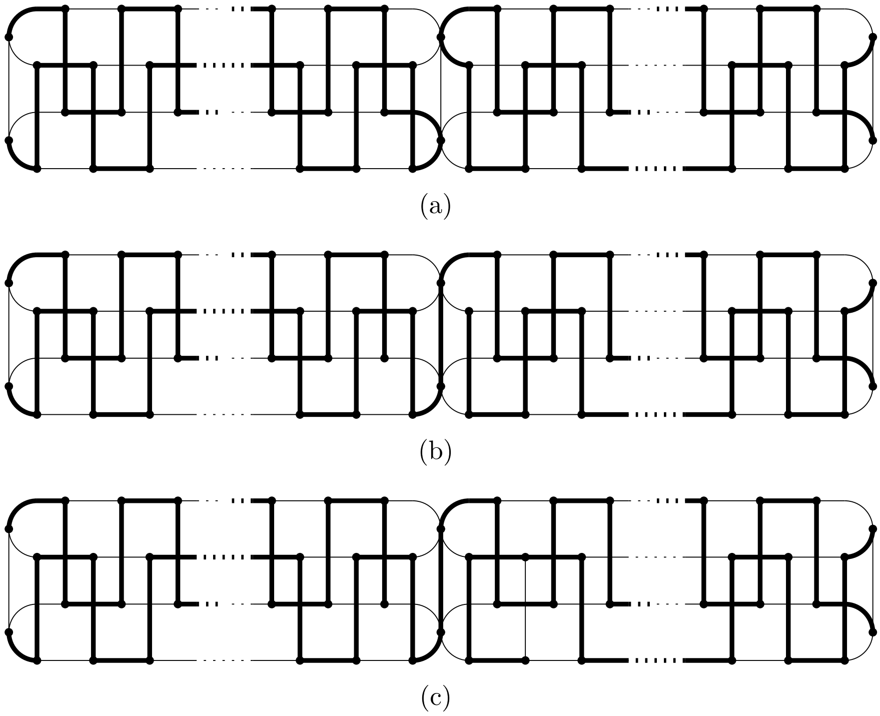

Figure 5: In each subfigure the thin and thick edges together represent the prism over D_n such that the vertices of $V(D_n) \times \{\alpha\}$ are placed above that of $V(D_n) \times \{\beta\}$; the leftmost and rightmost vertical edges denote $(l, \alpha)(l, \beta)$ and $(r, \alpha)(r, \beta)$, respectively; and each copy of D_n is embedded in the same way as depicted in Figure 4(b). (a) The graph L_{D_n} (thick edges) consists of one path with endvertices (l, α) and (l, β) and one with endvertices (r, α) and (r, β). (b) The graph S_{D_n} (thick edges) consists of one path with endvertices (l, α) and (w_n, β), one with endvertices (l, β) and (r, β) and one with endvertices (r, α) and (x_{2n+1}, α). (c) The graph S_{D_n} (thick edges) consists of one path with endvertices (l, α) and (w_n, β), one with endvertices (l, β) and (r, β) and one with endvertices (r, α) and (x_{2n}, β).

$0 \leqslant a(j) \leqslant b(j) \leqslant 8$ such that Q_j is contained in $G^-[l^{a(j)}, r^{b(j)}]$ and, subject to this, $b(j) - a(j)$ is minimum. Again, the inequality $\sum_{j=1}^{k}(b(j) - a(j)) \geqslant 8$ holds. Let q_1 and q_2 be the numbers of 1-good and 2-good cacti among $Q_1, \ldots, Q_k$, respectively.

As we have discussed in the previous section, $K - s - t$ has at least $\sum_{j=1}^{k}(1 + c_j)$ components, where c_j is the number of bags of Q_j. By Lemma 11, we have the following two cases.

Case (I). If $K-s-t$ consists of at most four even cacti which are good or 1-good, then, by Lemmas 12 and 13, $K-s-t$ has at least $\sum_{j=1}^{k}(1+c_j) \geqslant \sum_{j=1}^{k}(b(j)-a(j))-q_1 \geqslant 8-4 = 4$ components. Since there are at most four components in $K - s - t$, the equality must hold. In this case we must have that $q_1 = 4$. This implies that $k = 4$ and every component Q_j is 1-good. Then we have $c_j = 0$ for every j. In other words, no Q_j can have any bag. As $0 = c_j \geqslant b(j) - a(j) - 2$ holds for every j and $\sum_{j=1}^{4}(b(j) - a(j)) \geqslant 8$, we have that

$b(j) = a(j) + 2$ for every $j \in \{1, \dots, 4\}$. We may assume that $a(j) = 2(j-1)$ for any j. Let P_1, P_2 be the edge paths such that Q_1 and Q_2 are P_1-good and P_2-good, respectively. By Lemma 13, P_1 and P_2 are contained in $G^-[r^0, l^2]$ and $G^-[r^2, l^4]$, respectively. Note that $(Q_1 \cup Q_2)[B^2]$ consists of one or two disjoint good even cacti whose union spans B^2. Since B^2 has no even cycle, each of these cacti is a path with at least one of l^2 and r^2 as an endvertex. Clearly, one or two such paths cannot span B^2, a contradiction.
Case (II). If $K - s - t$ has at most three components such that one of them is a 2-good even cactus and the rest are good or 1-good even cacti, then, by Lemmas 12, 13 and 14, $K - s - t$ has at least $\sum_{j=1}^{k}(1 + c_j) \geqslant \sum_{j=1}^{k}(b(j) - a(j)) - q_1 - 2q_2 \geqslant 4$ components, which contradicts that $K - s - t$ has at most three components.

Hence we conclude that no spanning subgraph of G can be a good even cactus, and this completes the proof of Theorem 6.

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