The Maximum Spectral Radius of Non-Bipartite Graphs Forbidding Short Odd Cycles

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Abstract

It is well-known that eigenvalues of graphs can be used to describe structural properties and parameters of graphs. A theorem of Nosal and Nikiforov states that if G is a triangle-free graph with m edges, then $\lambda(G) \leqslant \sqrt{m}$, equality holds if and only if G is a complete bipartite graph. Recently, Lin, Ning and Wu [Combin. Probab. Comput. 30 (2021)] proved a generalization for non-bipartite triangle-free graphs. Moreover, Zhai and Shu [Discrete Math. 345 (2022)] presented a further improvement. In this paper, we present an alternative method for proving the improvement by Zhai and Shu. Furthermore, the method can allow us to give a refinement on the result of Zhai and Shu for non-bipartite graphs without short odd cycles.

Mathematics Subject Classifications: 05C50

1 Introduction

The present work can be viewed as the second paper of our previous project [29]. In this paper, we shall use the following standard notation; see e.g., the monograph [9]. We consider only simple and undirected graphs. Let G be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. We usually write n and m for the number of vertices and edges respectively. Let $N(v)$ or $N_G(v)$ be the set of neighbors of v, and $d(v)$ or $d_G(v)$ be the degree of a vertex v in G. For a subset $S \subseteq V(G)$, we write $e(S)$ for the number of edges with two endpoints in S. Let $K_{s,t}$ be the complete bipartite graph with parts of sizes s and t. We write C_n and P_n for the cycle and path on n vertices respectively. We denote by $t(G)$ the number of triangles in G.

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1.1 The classical extremal graph problems

We say that a graph G is F-free if it does not contain an isomorphic copy of F as a subgraph. Apparently, every bipartite graph is C_{2k+1}-free for every integer $k \geqslant 1$. The *Turán number* of a graph F is the maximum number of edges in an n-vertex F-free graph, and it is usually denoted by $\mathrm{ex}(n, F)$. A graph on n vertices with no subgraph F and with $\mathrm{ex}(n, F)$ edges is called an *extremal graph* for F. As is known to all, the Mantel theorem [36] asserts that if G is an n-vertex graph with at least $\lfloor \frac{n^2}{4} \rfloor$ edges, then either there exist three edges in G that form a triangle or $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, the balanced complete bipartite graph.

Theorem 1 (Mantel, 1907)**.** *Let G be an n-vertex graph. If G is triangle-free, then $e(G) \leqslant e(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) = \lfloor \frac{n^2}{4} \rfloor$, equality holds if and only if $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

Mantel's theorem has many interesting applications and generalizations in the literature; see, e.g., [1, pp. 269–273] and [5, pp. 294–301] for standard proofs, [6, 8] for generalizations, and [16, 47] for recent comprehensive surveys. In particular, Mantel's Theorem 1 was refined in the sense of the following stability form.

Theorem 2 (Erdős)**.** *Let G be an n-vertex triangle-free graph. If G is not bipartite, then $e(G) \leqslant \lfloor \frac{(n-1)^2}{4} \rfloor + 1$.*

It is said that this stability result attributes to Erdős; see [9, Page 306, Exercise 12.2.7]. The bound in Theorem 2 is best possible and the extremal graphs are not unique. To show that the bound is sharp for all integers n, we take two vertex sets X and Y with $|X| = \lfloor \frac{n}{2} \rfloor$ and $|Y| = \lceil \frac{n}{2} \rceil$. We take two vertices $u, v \in Y$ and join them, then we put every edge between X and $Y \setminus \{u, v\}$. We partition X into two parts X_1 and X_2 *arbitrarily* (this shows that the extremal graph is not unique), then we connect u to every vertex in X_1, and v to every vertex in X_2; see Figure 1 This yields a graph G which contains no triangle and $e(G) = \lfloor \frac{n^2}{4} \rfloor - \lfloor \frac{n}{2} \rfloor + 1 = \lfloor \frac{(n-1)^2}{4} \rfloor + 1$. Note that G has a cycle C_5, so G is not bipartite.

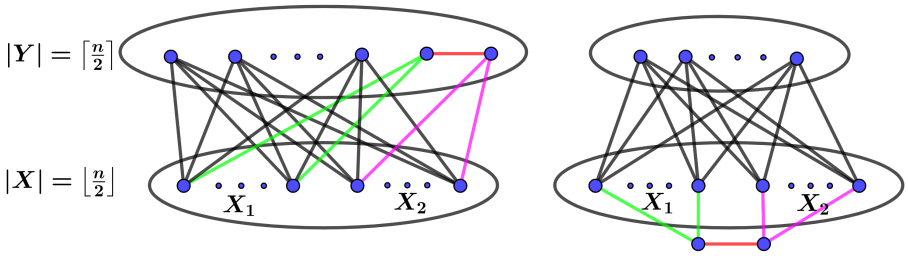


Figure 1: Extremal graphs in Theorem 2.

1.2 The spectral extremal graph problems

There are various matrices that are associated with a graph, such as the adjacency matrix, the incidence matrix, the distance matrix, the Laplacian matrix and signless Laplacian

matrix. One of the main problems of algebraic graph theory is to determine the combinatorial properties of a graph that are reflected from the algebraic properties of its associated matrices. Let G be a simple graph on n vertices. The *adjacency matrix* of G is defined as $A(G) = [a_{ij}]_{n\times n}$ where $a_{ij} = 1$ if two vertices v_i and v_j are adjacent in G, and $a_{ij} = 0$ otherwise. We say that G has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ if these values are eigenvalues of the adjacency matrix $A(G)$. We denote by $\lambda_i(G)$ the *i-th largest eigenvalue* of G. Let $\lambda(G)$ be the maximum value in absolute among all eigenvalues of G, which is known as the *spectral radius* of a graph G.

There is a rich history on the study of bounding the eigenvalues of a graph in terms of various parameters; see [2] for eigenvalues and expanders, [11, 15] for eigenvalues and diameters, [21] for spectral radius and genus, [3] for spectral radius and cut vertices, [12, 34] for regularity and eigenvalues, [13, 33] for non-regularity and spectral radius, [7] for spectral radius and cliques, [4, 52] for chromatic number and eigenvalues, [35, 17, 40] for independence number and eigenvalues, [14, 46] for matching, edge-connectivity and eigenvalues, [18] for spanning trees and eigenvalues, [48, 30] for eigenvalues of outerplanar and planar graphs, and [49] for the Colin de Verdière parameter, excluded minors and the spectral radius.

Let G be a graph on n vertices with m edges. Let $A(G)$ be the adjacency matrix of G. It is well-known that

$$\frac{2m}{n} \leqslant \lambda(G) \leqslant \sqrt{2m}. \tag{1}$$

Indeed, the lower bound is guaranteed by Rayleigh's inequality $\lambda(G) \geqslant \boldsymbol{e}^T A(G)\boldsymbol{e} = \frac{2m}{n}$, where $\boldsymbol{e} = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T \in \mathbb{R}^n$. The upper bound can be seen by invoking the fact that $\lambda(G)^2 \leqslant \sum_{i=1}^n \lambda_i^2 = \mathrm{tr}(A^2(G)) = \sum_{i=1}^n d_i = 2m$. This upper bound was further improved by Hong [20] as

$$\lambda(G) \leqslant \sqrt{2m - n + 1}. \tag{2}$$

We recommend the readers to [22] and [37] for further extensions. The classical extremal graph problems usually study the maximum or minimum number of edges that the extremal graphs can have. Correspondingly, we can study the extremal spectral problem. We denote by $\mathrm{ex}_\lambda(n, F)$ the largest eigenvalue of the adjacency matrix in an n-vertex graph that contains no copy of F, that is,

$$\mathrm{ex}_\lambda(n, F) := \max\big\{\lambda(G) : |G| = n \text{ and } F \nsubseteq G\big\}.$$

In 1970, Nosal [45] determined the largest spectral radius of a triangle-free graph in terms of the number of edges, which states that if G is a triangle-free graph, then $\lambda(G) \leqslant \sqrt{m}$. This result improved both inequalities (1) and (2) conditionally. In order to state this result more accurately, we combine with some contributions of Nikiforov's works [37, 38, 40], which determined the extremal graphs attaining the equality and also provided the spectral version of Theorem 1. Thus we write it as in the following complete form. Note that when we consider the result on a graph with respect to the given number of edges, we shall ignore the possible isolated vertices if there are no confusions.

Theorem 3 (Nosal–Nikiforov)**.** *Let G be a graph on n vertices with m edges. If G is triangle-free, then*

$$\lambda(G) \leqslant \sqrt{m}, \tag{3}$$

equality holds if and only if G is a complete bipartite graph. Moreover, we have

$$\lambda(G) \leqslant \lambda(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}), \tag{4}$$

equality holds if and only if G is a balanced complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

Theorem 3 implies that if G is a bipartite graph, then $\lambda(G) \leqslant \sqrt{m}$, equality holds if and only if G is a complete bipartite graph. On the one hand, inequality (3) implies the classical Mantel Theorem 1. Indeed, applying the Rayleigh inequality, we have $\frac{2m}{n} \leqslant \lambda(G) \leqslant \sqrt{m}$, which yields $m \leqslant \lfloor \frac{n^2}{4} \rfloor$. On the other hand, combining (3) with Mantel's theorem, we obtain $\lambda(G) \leqslant \sqrt{m} \leqslant \sqrt{\lfloor n^2/4 \rfloor} = \lambda(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})$. So inequality (3) in Theorem 3 can imply inequality (4), which is usually called the spectral Mantel theorem.

Theorem 3 stimulated the developments of two aspects in spectral extremal graph theory. On the one hand, it is natural to consider the spectral extremal problems for graphs with given number of vertices. In view of this perspective, various extensions and generalizations on inequality (4) have been obtained in the literature; see, e.g., [51, 38, 19, 25] for extensions on K_{r+1}-free graphs with given order; see [7, 40] for relations between cliques and spectral radius and [41, 10, 27] for surveys. Very recently, Lin, Ning and Wu [31, Theorem 1.4] proved a generalization on (4) for non-bipartite triangle-free graphs and provided a spectral version of Theorem 2; see [29] for an alternative proof and refinement of spectral Turán theorem, and [28] for more stability theorems on spectral graph problems. In addition, Lin and Guo [32] proved an extension of non-bipartite graphs without short odd cycles. This result was also independently proved by Li, Sun and Yu [26, Theorem 1.6] using a different method.

On the other hand, the inequality (3) in Theorem 3 boosted the great interests of studying the maximum spectral radius of graphs in terms of the number of edges, instead of the given number of vertices; see [37] for an extension on K_{r+1}-free graphs, [39] for an analogue of C_4-free graphs, [53] for further extensions on $K_{2,r+1}$-free graphs, and similar results of C_5-free and C_6-free graphs as well, [42] for an extension on B_k-free graphs, where B_k denotes the book graph consisting of k triangles sharing a common edge, and [31, 54] for refinements on non-bipartite triangle-free graphs. In this paper, we will focus mainly on the extremal spectral problems for graphs with given number of edges, which is becoming increasingly an important and popular topic in recent research on spectral graph theory.

In 2021, Lin, Ning and Wu [31] proved the following improvement on Theorem 3 by using tools from doubly stochastic matrix theory; see [42] for a simpler proof by using elementary numerical inequalities. Let P_n be the path on n vertices, and C_n be the cycle on n vertices. Given two graphs G and H, we write $G \cup H$ for the disjoint union of G and H. In other words, $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. For

simplicity, we write kG for the disjoint union of k copies of G. The blow-up of a graph G is a new graph obtained from G by replacing each vertex $v \in V(G)$ with an independent set I_v, and for two vertices $u, v \in V(G)$, we add all edges between I_u and I_v whenever $uv \in E(G)$.

Theorem 4 (Lin–Ning–Wu, 2021)**.** *Let G be a triangle-free graph with m edges. Then*

$$\lambda_1^2(G) + \lambda_2^2(G) \leqslant m,$$

equality holds if and only if G is a blow-up of a member of $\mathcal{G}$ in which

$$\mathcal{G} = \{P_2 \cup K_1, 2P_2 \cup K_1, P_4 \cup K_1, P_5 \cup K_1\}.$$

A conjecture of Bollobás and Nikiforov [7, Conjecture 1] states that if G is a K_{r+1}-free graph with m edges, then

$$\lambda_1^2(G) + \lambda_2^2(G) \leqslant \left(1 - \frac{1}{r}\right) 2m.$$

Theorem 4 confirmed the case $r = 2$; see [42, 26] for recent progress. This conjecture of Bollobás and Nikiforov remains open for the case $r \geqslant 3$. We remark here that $\lambda_1^2(G) + \lambda_2^2(G) \leqslant m$ does not hold for the C_4-free graphs G. Indeed, take $G = K_{1,m-1}^+$, the graph obtained from the star $K_{1,m-1}$ by adding an edge into its independent set. For example, setting $m = 20$, we have $\lambda_1(K_{1,19}^+) \approx 4.425$ and $\lambda_2(K_{1,19}^+) = 0.890$, while $\lambda_1^2 + \lambda_2^2 \approx 20.372 > 20$.

With the help of Theorem 4, Lin, Ning and Wu [31, Theorem 1.3] further proved the following refinement on (3) in Nosal's theorem for non-bipartite triangle-free graphs with given number of edges.

Theorem 5 (Lin–Ning–Wu, 2021)**.** *Let G be a triangle-free graph with m edges. If G is non-bipartite, then*

$$\lambda(G) \leqslant \sqrt{m-1},$$

equality holds if and only if $m = 5$ and $G = C_5$.

In 2022, Zhai and Shu [54] proved a further improvement on Theorem 5. Before stating their result, we need to introduce the extremal graph firstly. For every integer $m \geqslant 3$, we denote by $\beta(m)$ the largest root of

$$Z(x) := x^3 - x^2 - (m-2)x + m - 3.$$

It is not difficult to show that for $m \geqslant 6$, we have

$$\sqrt{m-2} < \beta(m) < \sqrt{m-1}. \tag{5}$$

Furthermore, one can verify that $\lim_{m\to\infty}(\beta(m) - \sqrt{m-2}) = 0$. On the other hand, if m is odd, let $SK_{2,\frac{m-1}{2}}$ be the graph obtained from $K_{2,\frac{m-1}{2}}$ by subdividing an edge; see Figure

2 for two different drawings of $SK_{2,\frac{m-1}{2}}$. In particular, for $m = 5$, we have $SK_{2,2} = C_5$. Clearly, $SK_{2,\frac{m-1}{2}}$ is a triangle-free graph on $n = \frac{m-1}{2} + 3$ vertices with m edges, and it is non-bipartite as it contains a copy of C_5. The characteristic polynomial of $SK_{2,\frac{m-1}{2}}$ is

$$\det(xI_n - A(SK_{2,\frac{m-1}{2}})) = x^{\frac{m-5}{2}}(x^2 + x - 1)\left(x^3 - x^2 - (m-2)x + m - 3\right).$$

Therefore, if m is odd, then $\beta(m)$ is the largest eigenvalue of $SK_{2,\frac{m-1}{2}}$.

For convenience, we denote

$$H(x) := (x^2 + x - 1)Z(x) = x^5 - mx^3 + (2m-5)x - m + 3. \tag{6}$$

So $\beta(m)$ is also the largest root of $H(x)$.

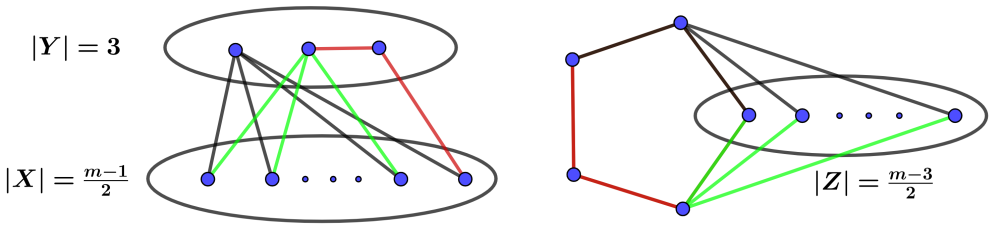


Figure 2: Two drawings of the graph $SK_{2,\frac{m-1}{2}}$.

The improvement of Zhai and Shu [54] on Theorem 5 can be stated as below.

Theorem 6 (Zhai–Shu, 2022)**.** *Let G be a graph of size m. If G is triangle-free and non-bipartite, then*

$$\lambda(G) \leqslant \beta(m),$$

equality holds if and only if m is odd and $G = SK_{2,\frac{m-1}{2}}$.

The way that Lin, Ning and Wu [31] proved Theorem 5 is original, and the line of the proof of Zhai and Shu [54] for Theorem 6 is technical. This paper is organized as follows. In Section 2, we shall present an alternative proof of Theorem 6. The present proof is different from the original proof in [54]. Our proof uses and develops the ideas in both [31] and [43], we shall make use of the information of all eigenvalues of graphs, instead of the second largest eigenvalue only. This proof could introduce the main ideas of the approach of our paper, without some technicalities that arise in the other cases, i.e., it can help us to deal with the extremal spectral problem for graphs without short odd cycles. In Section 3, by applying the ideas of the proof of Theorem 6, we will give further refinement on Theorem 6. In Section 4, we will conclude this paper with some possible open problems for interested readers. This paper can be regarded as a supplement of our previous article [29]. Both of these two papers provide extensions and generalizations on the results involving eigenvalues and triangles.

2 Alternative proof of Theorem 6

Recall that Theorem 6 is an improvement on Theorem 5, since $\beta(m) < \sqrt{m-1}$, where $\beta(m)$ is the largest root of $x^3 - x^2 - (m-2)x + m - 3 = 0$. The proof of Theorem 5 is succinct and relies on Theorem 4, which implies that if G is non-bipartite and $\lambda_1^2(G) + \lambda_2^2(G) \geqslant m$, then G contains a triangle. Combining the condition in Theorem 5, we know that if G satisfies $\lambda_1(G) \geqslant \sqrt{m-1}$, then $\lambda_2(G) < 1$. This bound on the second largest eigenvalue provided great convenience to characterize the local structure of G. For instance, combining $\lambda_2(G) < 1$ with Cauchy's interlacing theorem, we obtain that the shortest odd cycle of G is C_5. However, it is not sufficient to apply Theorem 4 for the proof of Theorem 6. Indeed, if G is a graph satisfying $\lambda(G) \geqslant \beta(m)$, then invoking the fact that $\lim_{m\to\infty}(\beta(m) - \sqrt{m-2}) = 0$, we get only that $\lambda_2(G) < 2$. Nevertheless, this bound is invalid for our purpose to describe the local structure of G. The original proof of Zhai and Shu [54] for Theorem 6 is innovative and avoided the use of Theorem 4, thus it made more detailed structure analysis of graphs; see [54] for more details.

In what follows, we shall provide an alternative proof of Theorem 6. Our proof grows out partially from the original proof [31] of Theorem 5. To overcome the obstacle mentioned above, we shall make full use of the information of all eigenvalues of graphs, instead of the second largest eigenvalue merely. By applying Cauchy's interlacing theorem of all eigenvalues, we will find some forbidden induced subgraphs and refine the structure of the desired extremal graph. A key idea relies on the eigenvalue interlacing theorem and a counting lemma [43], which established the relation between eigenvalues and the number of triangles of a graph.

The main steps of the proof can be outlined as below. It introduces the main ideas of the approach of this paper for treating the problem involving short odd cycles.

- ☆ First of all, applying the forthcoming Lemmas 7, 8 and 9, we will show that G can not contain the odd cycle C_{2k+1} as an induced subgraph for every $k \geqslant 3$, that is, C_5 is the shortest odd cycle in G; see Claim 10.

- ✯ Upon more computations, we will prove that more substructures, e.g., the graphs H_1, H_2, H_3 in Figure 3, are also forbidden as induced subgraphs in G by applying Lemmas 7, 8 and 9 again; see Claim 11.

- ★ Let S be the set of vertices of a copy of C_5 in G. Using the above informations of local structure of G, we will show that every vertex outside of S has exactly two neighbors in S; see Claim 12.

- ★ Combining with the previous steps, we will prove that G is isomorphic to the subdivision of the complete bipartite graph $K_{a,b}$ by subdividing an edge, where $a, b \geqslant 2$ are integers satisfying $m = ab + 1$. Finally, we will show that $\lambda(SK_{a,b})$ is at most $\beta(m)$, equality holds if and only if $a = 2$ or $b = 2$.

The following lemma is usually referred to as the eigenvalue interlacing theorem, also known as the Cauchy, Poincaré, or Sturm interlacing theorem. It states that the eigenvalues of a principal submatrix of a Hermitian matrix interlace those of the underlying matrix; see, e.g., [55, pp. 52–53] and [56, pp. 269–271]. It is worth noting that this eigenvalue interlacing theorem provides a useful technique to extremal combinatorics and plays a significant role in two breakthrough works [23, 24].

Lemma 7 (Eigenvalue Interlacing Theorem)**.** *Let H be an $n \times n$ Hermitian matrix partitioned as*

$$H = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix},$$

where A is an $m \times m$ principal submatrix of H for some $m \leqslant n$. Then for every $1 \leqslant i \leqslant m$,

$$\lambda_{n-m+i}(H) \leqslant \lambda_i(A) \leqslant \lambda_i(H).$$

Recall that $t(G)$ denotes the number of triangles in G. It is well-known that the value of (i,j)-entry of $A^k(G)$ is equal to the number of walks of length k in G starting from vertex v_i to v_j. Since each triangle of G contributes 6 closed walks of length 3, we can count the number of triangles and obtain

$$t(G) = \frac{1}{6}\sum_{i=1}^{n} A^3(i,i) = \frac{1}{6}\mathrm{Tr}(A^3) = \frac{1}{6}\sum_{i=1}^{n} \lambda_i^3. \tag{7}$$

The second lemma needed in this paper is a triangle counting lemma in terms of both the eigenvalues and the size of a graph, it could be seen from [43]. This could be viewed as a useful variant of (7) by using $\sum_{i=1}^{n} \lambda_i^2 = \mathrm{tr}(A^2) = \sum_{i=1}^{n} d_i = 2m$.

Lemma 8. *(see [43]) Let G be a graph on n vertices with m edges. If $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ are all eigenvalues of G, then*

$$t(G) = \frac{1}{6}\sum_{i=2}^{n}(\lambda_1 + \lambda_i)\lambda_i^2 + \frac{1}{3}(\lambda_1^2 - m)\lambda_1.$$

For convenience, we next introduce a function.

Lemma 9. *Let $f(x) := (\sqrt{m-2} + x)x^2$. If $a \leqslant x \leqslant b \leqslant 0$, then*

$$f(x) \geqslant \min\{f(a), f(b)\}.$$

Proof. Since $f(x)$ is monotonically increasing when $x \in (-\infty, -\frac{2}{3}\sqrt{m-2})$, and monotonically decreasing when $x \in [-\frac{2}{3}\sqrt{m-2}, 0]$. Thus the desired statement holds immediately. □

It is the time to show an alternative proof of Theorem 6.

Proof of Theorem 6. Suppose that G contains no triangle and G is non-bipartite such that $\lambda(G) \geqslant \beta(m)$. We shall prove that m is odd and $G = SK_{2,\frac{m-1}{2}}$. Without loss of generality, we may assume that G has the maximum value of spectral radius. First of all, we can see that G must be connected. Otherwise, we can choose G_1 and G_2 as two different components, where G_1 attains the spectral radius of G, by identifying two vertices from G_1 and G_2 respectively, we get a new graph with larger spectral radius, which is a contradiction[1]. It is not hard to verify the desired theorem for $m \leqslant 10$, since we can consider whether $C_7 \subseteq G$ or $C_9 \subseteq G$ by a standard case analysis. Next, we shall consider the case $m \geqslant 11$. The proof proceeds by the following three claims.

Claim 10. *The shortest odd cycle in G has length 5.*

Proof of Claim 10. Since G is non-bipartite, let s be the length of a shortest odd cycle in G. Since G is triangle-free and non-bipartite, we have $s \geqslant 5$ and $\lambda(G) < \sqrt{m}$ by Theorem 3. Moreover, a shortest odd cycle $C_s \subseteq G$ must be an induced odd cycle. By computation, we know that the eigenvalues of C_s are $2\cos\frac{2\pi k}{s}$, where $k = 0, 1, \ldots, s-1$. Since C_s is an induced copy in G, we know that $A(C_s)$ is a principal submatrix of $A(G)$. Lemma 7 implies that $\lambda_{n-s+i}(G) \leqslant \lambda_i(C_s) \leqslant \lambda_i(G)$ for every $i \in [s]$, where λ_i means the i-th largest eigenvalue. We next show that $s = 5$. We denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the eigenvalues of G for simplicity.

Next we will show that C_7 is not an induced subgraph of G. By the monotonicity of $\cos x$, one can see in the proof that for odd integer $s \geqslant 7$, C_s can not be an induced subgraph of G. Suppose on the contrary that C_7 is an induced odd cycle of G, then $\lambda_2 \geqslant \lambda_2(C_7) = 2\cos\frac{2\pi}{7} \approx 1.246$ and $\lambda_3 \geqslant \lambda_3(C_7) = 2\cos\frac{12\pi}{7} \approx 1.246$. Evidently, we get

$$f(\lambda_2) \geqslant f(1.246) \geqslant 1.552\sqrt{m-2} + 1.934,$$

and

$$f(\lambda_3) \geqslant f(1.246) \geqslant 1.552\sqrt{m-2} + 1.934.$$

Our goal is to get a contradiction by applying Lemma 8 and showing $t(G) > 0$. It is not sufficient to obtain $t(G) > 0$ by using the positive eigenvalues of C_7 only. Next, we are going to consider the negative eigenvalues of C_7. For $i \in \{4, 5, 6, 7\}$, we know that $\lambda_i(C_7) < 0$. The Cauchy interlacing theorem yields $\lambda_{n-3} \leqslant \lambda_4(C_7) = -0.445$, $\lambda_{n-2} \leqslant \lambda_5(C_7) = -0.445$, $\lambda_{n-1} \leqslant \lambda_6(C_7) = -1.801$ and $\lambda_n \leqslant \lambda_7(C_7) = -1.801$. To apply Lemma 9, we need to find the lower bounds on λ_i for each $i \in \{n-3, n-2, n-1, n\}$. We know from (5) that $\lambda_1 \geqslant \beta(m) > \sqrt{m-2}$, and then $\lambda_n^2 \leqslant 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2 + \lambda_{n-1}^2) < 2m - (m - 2 + 6.744) = m - 4.744$, which implies $-\sqrt{m-4.744} < \lambda_n \leqslant -1.801$. By Lemma 9, we get

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-4.744}), f(-1.801)\} > \sqrt{m-2}$$

[1]There is another way to get a contradiction. We delete an edge within G_2, and then add an edge between G_1 and G_2. This operation will also lead to a new graph with larger spectral radius.

for every $m \geqslant 11$. Similarly, we have $\lambda_{n-1}^2 + \lambda_n^2 \leqslant 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2) < m - 1.501$. Combining with $\lambda_{n-1}^2 \leqslant \lambda_n^2$, we get $-\sqrt{(m-1.501)/2} < \lambda_{n-1} \leqslant -1.801$. By Lemma 9, we obtain

$$f(\lambda_{n-1}) \geqslant \min\{f(-\sqrt{(m-1.501)/2}), f(-1.801)\} > \sqrt{m-2}$$

for every $m \geqslant 9$. Note that $\sqrt{m} > \lambda_1 \geqslant \beta(m) > \sqrt{m-2}$. By Lemma 8, we get

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_n) + f(\lambda_{n-1})) - \frac{2}{3}\lambda_1 \\ &> \frac{1}{6}(5.104\sqrt{m-2} - 4\sqrt{m} + 3.868) > 0. \end{aligned}$$

This is a contradiction. Similarly, we can prove by applying the monotonicity of $\cos x$ that C_s can not be an induced subgraph of G for each odd integer $s \geqslant 7$. Thus we get $s = 5$. □

Let $S = \{u_1, u_2, u_3, u_4, u_5\}$ be the set of vertices of a copy of C_5 in G. We define the graphs H_1, H_2 and H_3 as in Figure 3. The eigenvalues of these graphs can be seen in Table 1. To avoid unnecessary calculations, we did not attempt to get the best bound on the size of graph in the proof. Next, we consider the case $m \geqslant 514$ in the remaining proof.

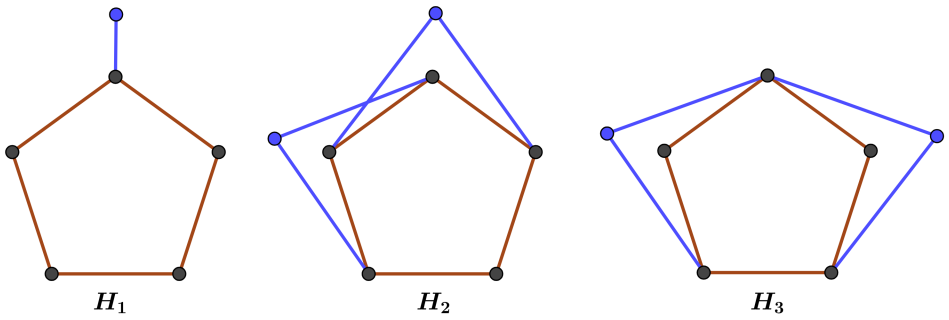


Figure 3: Some forbidden induced subgraphs in G.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
C_7	2	1.246	1.246	−0.445	−0.445	−1.801	−1.801
H_1	2.115	1	0.618	−0.254	−1.618	−1.860	
H_2	2.641	1	0.723	0.414	−0.589	−1.775	−2.414
H_3	2.681	1	0.642	0	0	−2	−2.323

Table 1: Eigenvalues of forbidden induced subgraphs.

Claim 11. *G does not contain any graph of $\{H_1, H_2, H_3\}$ as an induced subgraph.*

Proof of Claim 11. Suppose on the contrary that G contains H_i as an induced subgraph for some $i \in \{1,2,3\}$. To obtain a contradiction, we shall show $t(G) > 0$ by applying Lemma 8. We first consider the case that H_1 is an induced subgraph in G. The Cauchy interlacing theorem implies $\lambda_{n-6+i}(G) \leqslant \lambda_i(H_1) \leqslant \lambda_i(G)$ for every $i \in \{1,2,\ldots,6\}$. We denote $\lambda_i = \lambda_i(G)$ for short. Obviously, we have

$$f(\lambda_2) \geqslant f(1) = \sqrt{m-2} + 1,$$

and

$$f(\lambda_3) \geqslant f(0.618) \geqslant 0.381\sqrt{m-2} + 0.236.$$

We next consider the negative eigenvalues of G. The Cauchy interlacing theorem implies $\lambda_{n-2} \leqslant \lambda_4(H_1) = -0.254$ and $\lambda_{n-1} \leqslant \lambda_5(H_1) = -1.618$ and $\lambda_n \leqslant \lambda_6(H_1) = -1.860$. Moreover, we get from (5) that $\lambda_1 \geqslant \beta(m) > \sqrt{m-2}$ and $\lambda_n^2 \leqslant 2m-(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_{n-2}^2+\lambda_{n-1}^2) \leqslant 2m-(m-2+4.064) = m-2.064$, which implies $-\sqrt{m-2.064} < \lambda_n \leqslant -1.860$. By Lemma 9, we have

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-2.064}), f(-1.860)\} > 0.031\sqrt{m-2}.$$

Secondly, since $\lambda_{n-1}^2 + \lambda_n^2 \leqslant 2m - (\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_{n-2}^2) < m + 0.553$ and $\lambda_{n-1}^2 \leqslant \lambda_n^2$, we get $-\sqrt{(m+0.553)/2} < \lambda_{n-1} \leqslant -1.618$. By Lemma 9, we get

$$f(\lambda_{n-1}) \geqslant \min\{f(-\sqrt{(m+0.553)/2}), f(-1.618)\} > 2.617\sqrt{m-2} - 4.235$$

for every $m \geqslant 12$. Moreover, we have $-\sqrt{(m+0.618)/3} < \lambda_{n-2} \leqslant -0.254$ and then

$$f(\lambda_{n-2}) \geqslant \min\{f(-\sqrt{(m+0.618)/3}), f(-0.254)\} > 0.064\sqrt{m-2} - 0.016$$

for every $m \geqslant 4$. By Lemma 8, we get that for $m \geqslant 514$,

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_{n-2}) + f(\lambda_{n-1}) + f(\lambda_n)) - \frac{2}{3}\lambda_1 \\ &> \frac{1}{6}(4.093\sqrt{m-2} - 4\sqrt{m} - 2.015) > 0, \end{aligned}$$

which is a contradiction.

If H_2 is an induced subgraph of G, then we get similarly that $\lambda_2 \geqslant 1, \lambda_3 \geqslant 0.723$ and $\lambda_4 \geqslant 0.414$. Then

$$\begin{aligned} f(\lambda_2) &\geqslant f(1) = \sqrt{m-2} + 1, \\ f(\lambda_3) &\geqslant f(0.723) \geqslant 0.522\sqrt{m-2} + 0.377, \end{aligned}$$

and

$$f(\lambda_4) \geqslant f(0.414) \geqslant 0.171\sqrt{m-2} + 0.07.$$

The negative eigenvalues of H_2 imply that $\lambda_{n-2} \leqslant -0.589$, $\lambda_{n-1} \leqslant -1.775$ and $\lambda_n \leqslant -2.414$. Due to $\lambda_n^2 \leqslant 2m - (\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2+\lambda_{n-2}^2+\lambda_{n-1}^2) < 2m-(m-2+5.191) = m-3.191$, we get $-\sqrt{m-3.191} \leqslant \lambda_n \leqslant -2.414$. Lemma 9 gives

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-3.191}), f(-2.414)\} > 0.5\sqrt{m-2}$$

for every $m \geqslant 8$. In addition, we have $-\sqrt{(m-0.041)/2} \leqslant \lambda_{n-1} \leqslant -1.775$ and

$$f(\lambda_{n-1}) \geqslant \min\{f(-\sqrt{(m-0.041)/2}), f(-1.775)\} > 2\sqrt{m-2}$$

for every $m \geqslant 17$. By Lemma 8, we obtain that for $m \geqslant 6$,

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_4) + f(\lambda_{n-1}) + f(\lambda_n)) - \frac{2}{3}\lambda_1 \\ &> \frac{1}{6}(4.193\sqrt{m-2} - 4\sqrt{m} + 1.447) > 0, \end{aligned}$$

which is also a contradiction.

If H_3 is an induced subgraph of G, then we get $\lambda_2 \geqslant 2$ and $\lambda_3 \geqslant 0.642$. Then

$$f(\lambda_2) \geqslant f(1) = \sqrt{m-2} + 1$$

and

$$f(\lambda_3) \geqslant f(0.642) \geqslant 0.412\sqrt{m-2} + 0.264.$$

Moreover, Cauchy's interlacing theorem gives $\lambda_{n-1} \leqslant -2$ and $\lambda_n \leqslant -2.323$. Since $\lambda_n^2 \leqslant 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_{n-1}^2) < 2m - (m - 2 + 5.412) = m - 3.412$, we get $-\sqrt{m-3.412} < \lambda_n \leqslant -2.323$. Then

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-3.412}), f(-2.323)\} \geqslant 0.7\sqrt{m-2}.$$

Similarly, we have $-\sqrt{(m+0.587)/2} < \lambda_{n-1} \leqslant -2$ and

$$f(\lambda_{n-1}) \geqslant \min\{f(-\sqrt{(m+0.587)/2}), f(-2)\} \geqslant 4\sqrt{m-2} - 8.$$

By Lemma 8, we obtain

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_{n-1}) + f(\lambda_n)) - \frac{2}{3}\lambda_1 \\ &> \frac{1}{6}(6.112\sqrt{m-2} - 4\sqrt{m} - 6.736) > 0, \end{aligned}$$

which is a contradiction. □

Let $N(S) := \cup_{u \in S} N(u) \setminus S$ be the union of neighborhoods of vertices of S. We denote by $d_S(v) = |N(v) \cap S|$ the number of neighbors of v in the set S.

Claim 12. *$V(G) = S \cup N(S)$ and $d_S(v) = 2$ for every $v \in N(S)$.*

Proof of Claim 12. First of all, we prove that $d_S(v) = 2$ for each vertex $v \in N(S)$. Without loss of generality, we may assume that $v \in N(u_1)$. If $d_S(v) \geqslant 3$, then there exists $i \in [5]$ such that $\{v, u_i, u_{i+1}\}$ forms a triangle in G, a contradiction. If $d_S(v) = 1$, then $S \cup \{v\}$ induces a copy of H_1, a contradiction. This implies that $d_S(v) = 2$ for every $v \in N(S)$. Next we prove that $V(G) = S \cup N(S)$. Otherwise, if there is a vertex $v' \in V(G) \setminus (S \cup N(S))$, then v' has distance at least 2 from S. We may assume

that $v'vu_1$ is an induced P_3 such that $v'u_i \notin E(G)$ for every $i \in [5]$. From the above discussion, we know from $vu_1 \in E(G)$ that $d_S(v) = 2$. By symmetry, we may assume that $N_S(v) = \{u_1, u_3\}$. Since G is triangle-free and $v'u_i \notin E(G)$ for every $i \in [5]$, we can see that $\{v', v, u_3, u_4, u_5, u_1\}$ induces a copy of H_1, a contradiction. Thus, we conclude that $V(G) = S \cup N(S)$ and $d_S(v) = 2$ for every $v \in N(S)$. □

Since $m \geqslant 11$, we can fix a vertex $v \in N(S)$ and assume that $N_S(v) = \{u_1, u_3\}$. For each $w \in V(G) \setminus (S \cup \{v\})$, since G contains no triangles and no H_3 as an induced subgraph, we know that $N_S(w) \neq \{u_3, u_5\}$ and $N_S(w) \neq \{u_4, u_1\}$. It is possible that $N_S(w) = \{u_1, u_3\}, \{u_2, u_4\}$ or $\{u_5, u_2\}$. Furthermore, if $N_S(w) = \{u_1, u_3\}$, then $wv \notin E(G)$, since G contains no triangles; if $N_S(w) = \{u_2, u_4\}$, then $wv \in E(G)$, since G contains no induced copy of H_2. We denote $N_{i,j} = \{w \in V(G) \setminus S : N_S(w) = \{u_i, u_j\}\}$. Note that G has no induced copy of H_3, there are at least one empty set in $\{N_{2,4}, N_{5,2}\}$. If $N_{2,4} = \varnothing$ and $N_{5,2} = \varnothing$, then $V(G) \setminus S = N_{1,3}$. Thus m is odd and $G = SK_{2,\frac{m-1}{2}}$. Without loss of generality, if $N_{2,4} \neq \varnothing$, then $V(G) \setminus S = N_{1,3} \cup N_{2,4}$. Moreover, $N_{1,3}$ and $N_{2,4}$ induce a complete bipartite subgraph in G. We denote $A = N_{1,3} \cup \{u_2, u_4\}$ and $B = N_{2,4} \cup \{u_3, u_1\}$. Clearly, we have $|A| = a \geqslant 2$ and $|B| = b \geqslant 2$. Then we observe that G is isomorphic to the subdivision of the complete bipartite graph $K_{a,b}$ by subdividing the edge u_1u_4 of $K_{a,b}$, and $m = e(G) = ab + 1$. By a direct computation, we get that $\lambda(G) \leqslant \beta(m)$, equality holds if and only if $a = 2$ or $b = 2$, and thus m is odd and $G = SK_{2,\frac{m-1}{2}}$. The detailed computations are stated below. The characteristic polynomial of $G = SK_{a,b}$ is

$$\begin{aligned}&\det(xI_n - A(SK_{a,b}))\\ &= x^{a+b-4}\left(x^5 - (ab+1)x^3 + (3ab - 2a - 2b + 1)x - 2ab + 2a + 2b - 2\right).\end{aligned}$$

Hence $\lambda(G)$ is the largest root of

$$F(x) := x^5 - mx^3 + (3m - 2 - 2a - 2\tfrac{m-1}{a})x - 2m + 2a + 2\tfrac{m-1}{a}.$$

Recall in (6) that $\beta(m)$ denotes the largest root of $H(x)$. We can easily verify that

$$H(x) - F(x) = (2a + 2\tfrac{m-1}{a} - m - 3)(x - 1),$$

which yields $H(x) \leqslant F(x)$ for every $x \geqslant 1$. Then we get $H(\lambda(G)) \leqslant F(\lambda(G)) = 0$, which implies $\lambda(G) \leqslant \beta(m)$. This completes the proof. □

Remark. Theorem 3 asserts that if G is a graph with $\lambda(G) \geqslant \sqrt{m}$, then either G contains a triangle, or G is a complete bipartite graph. Very recently, Ning and Zhai [43] proved an elegant spectral counting result, which states that if G is an m-edge graph with $\lambda(G) \geqslant \sqrt{m}$, then G has at least $\lfloor \frac{\sqrt{m}-1}{2} \rfloor$ triangles, unless G is a complete bipartite graph. Clearly, this saturation result is a generalization of Nosal's theorem as well as a spectral analogue of a result of Rademacher. A natural question is whether the counting result analogous to Theorem 6 is true. More precisely, if G is non-bipartite with $\lambda(G) \geqslant \beta(m)$, then it seems possible that G has at least $\Omega(\sqrt{m})$ triangles, unless $G = SK_{2,\frac{m-1}{2}}$.

Although we can see from the proof of Theorem 6 that many cases can yield the conclusion that G has at least $\Omega(\sqrt{m})$ triangles, the answer for the above question is surprisingly NEGATIVE. Taking $G = K^+_{1,m-1}$ as the graph obtained from the star $K_{1,m-1}$ by adding an edge into its independent set, we can see that G is not bipartite and $\lambda(K^+_{1,m-1}) > \sqrt{m-1} > \beta(m)$, while G has only one triangle and $G \neq SK_{2,\frac{m-1}{2}}$. Note that the graph $K^+_{1,m-1}$ has m edges and m vertices. Moreover, we can show that $\lambda(K^+_{1,m-1})$ is the largest root of the equation

$$x^3 - x^2 - (m-1)x + m - 3 = 0.$$

For $m = 4, 5, 6, 7, 8$, we can verify that $\lambda(K^+_{1,m-1}) > \sqrt{m}$; while for $m = 9$, we get $\lambda(K^+_{1,8}) = 3 = \lambda(K_{1,9})$. For $m \geqslant 11$, we can check that $\lambda(K^+_{1,m-1}) < \sqrt{m}$.

3 Graphs without short odd cycles

Let $S_3(K_{a,b})$ denote the graph obtained from the complete bipartite graph $K_{a,b}$ by replacing an edge with a five-vertex path P_5, that is, introducing three new vertices on an edge. Clearly, the shortest odd cycle in $S_3(K_{a,b})$ has length seven.

We next consider the further extension of Theorem 6 for graphs with given size and no short odd cycles. For each integer $m \geqslant 7$, we denote by $\gamma(m)$ the largest root of

$$L(x) := x^7 - mx^5 + (4m-14)x^3 - (3m-14)x - m + 5. \tag{8}$$

It is not difficult to check that

$$\sqrt{m-4} < \gamma(m) \leqslant \sqrt{m-3}. \tag{9}$$

Indeed, we observe that

$$L(\sqrt{m-4}) < x(x^6 - mx^4 + (4m-14)x^2 - (3m-14))\big|_{x=\sqrt{m-4}} = -m + 6 \leqslant 0,$$

which leads to $\sqrt{m-4} < \gamma(m)$. For every $m \geqslant 7$, we have

$$L(\sqrt{m-3}) = \sqrt{m-3}(m(m-11)+29) - m + 5 \geqslant 0,$$

equality holds only for $m = 7$. Combining with $L'(x) = 7x^6 - 5mx^4 + 3(4m-14)x^2 - (3m-14) \geqslant 0$ for every $x \geqslant \sqrt{m-3}$, we get $L(x) \geqslant L(\sqrt{m-3}) \geqslant 0$ for every $x \geqslant \sqrt{m-3}$, which implies $\gamma(m) \leqslant \sqrt{m-3}$.

Moreover, if m is odd, let $S_3(K_{2,\frac{m-3}{2}})$ be the graph obtained from the complete bipartite graph $K_{2,\frac{m-3}{2}}$ by subdividing an edge into a path of length 4, i.e., putting 3 new vertices on an edge; see Figure 4. In particular, for $m = 7$, we have $S_3(K_{2,2}) = C_7$. Clearly, $S_3(K_{2,\frac{m-3}{2}})$ has $n = \frac{m-3}{2} + 5$ vertices and m edges. Moreover, $S_3(K_{2,\frac{m-3}{2}})$ contains no copy of both C_3 and C_5, but it has a copy of C_7 and so it is non-bipartite. Upon computation, the characteristic polynomial of $S_3(K_{2,\frac{m-3}{2}})$ is given as

$$\det(xI_n - A(S_3(K_{2,\frac{m-3}{2}}))) = x^{\frac{m-7}{2}}\big(x^7 - mx^5 + (4m-14)x^3 - (3m-14)x - m + 5\big).$$

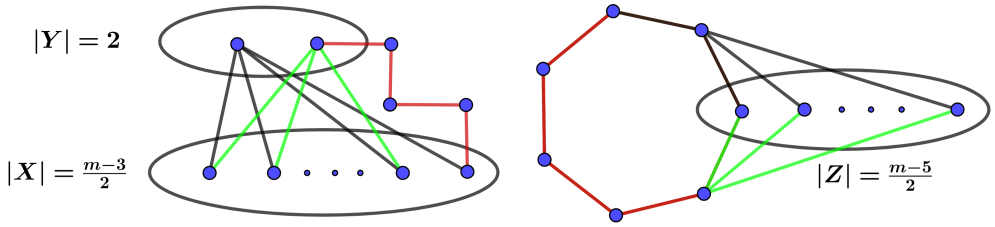


Figure 4: Two drawings of the graph $S_3(K_{2,\frac{m-3}{2}})$.

Hence, if m is odd, then $\gamma(m)$ is the largest eigenvalue of $S_3(K_{2,\frac{m-3}{2}})$.

Note that the extremal graph $SK_{2,\frac{m-1}{2}}$ in Theorem 6 contains a copy of C_5. In this section, we will prove a refinement on Theorem 6. To be more specific, we will determine the largest spectral radius for C_3-free and C_5-free non-bipartite graphs. To proceed, we need to introduce a lemma.

Lemma 13. *Let $a, b \geqslant 2$ and m be integers with $m = ab + 4$. If G is one of the m-edge graphs obtained from $S_3(K_{a,b})$ by adding an edge to one vertex, then $\lambda(G) < \gamma(m)$.*

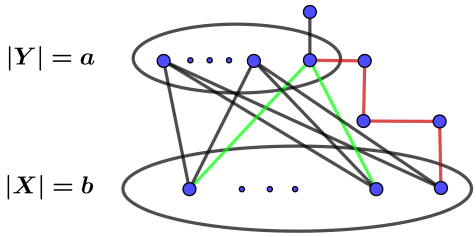


Figure 5: A possible graph in Lemma 13.

Proof. We know that G has 7 possible cases. We prove the above case in Figure 5 only, since the other cases can be proved in the same way. By computation, we obtain that $\lambda(G)$ is the largest root of $E(x)$, where $E(x)$ is defined as

$$\begin{aligned} E(x) := x^8 - (ab+4)x^6 + (6ab - 2a - 3b + 5)x^4 - (8ab - 5a - 7b + 5)x^2 \\ - (2ab - 2a - 2b + 2)x + ab - a - b + 1. \end{aligned}$$

Note that $m = ab + 4$ and

$$\begin{aligned} E(x) - xL(x) = (2a-3)(b-1)x^4 - (5a-7)(b-1)x^2 \\ - (ab - 2a - 2b + 3)x + ab - a - b + 1. \end{aligned}$$

We can verify that $x^2L(x) < E(x)$ for every $x \geqslant \sqrt{(a-1)b}$. Since $K_{a-1,b}$ is a subgraph of G, we know that $\lambda(G) \geqslant \lambda(K_{a-1,b}) = \sqrt{(a-1)b}$. Then $\lambda^2(G)L(\lambda(G)) < E(\lambda(G)) = 0$, which yields $L(\lambda(G)) < 0$ and $\lambda(G) < \gamma(m)$. □

The main result of this section is as follows.

Theorem 14. *Let G be a graph with m edges. If G does not contain any member of $\{C_3, C_5\}$ and G is non-bipartite, then*

$$\lambda(G) \leqslant \gamma(m),$$

equality holds if and only if m is odd and $G = S_3(K_{2,\frac{m-3}{2}})$.

Proof. Assume that G has no C_3 and C_5, and G is non-bipartite with $\lambda(G) \geqslant \gamma(m)$, we will show that m is odd and $G = S_3(K_{2,\frac{m-3}{2}})$. Similar with that in the proof of Theorem 6 in Section 2, it is sufficient to consider the case that G is connected. Since G is non-bipartite, we can assume that C_s is a shortest odd cycle of G. Note that a shortest odd cycle in G must be an induced subgraph. Since G is C_3-free and C_5-free, we have $s \geqslant 7$ and $\lambda(G) < \sqrt{m}$ by Theorem 3. In what follows, we shall show that $s = 7$. We denote

$$g(x) := (\sqrt{m-4} + x)x^2.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of G in decreasing order. Since G is non-bipartite and G has an induced odd cycle of length at least 7. For $m \leqslant 10$, we can do few case analysis whether $C_7 \subseteq G$ or $C_9 \subseteq G$. Thus it is easy to verify the required theorem for $m \leqslant 10$. Next, we shall consider the case $m \geqslant 11$ in the proof.

Claim 15. *A shortest odd cycle in G is C_7.*

Proof of Claim 15. Assume that C_9 is an induced odd cycle in G, the Cauchy interlacing theorem implies $\lambda_{n-9+i}(G) \leqslant \lambda_i(C_9) \leqslant \lambda_i(G)$ for every $i \in \{1, 2, \ldots, 9\}$. From the following Table 2, we can see that $\lambda_2, \lambda_3 \geqslant 1.532$. Then

$$g(\lambda_2), g(\lambda_3) \geqslant g(1.532) \geqslant 2.347\sqrt{m-4} + 3.596.$$

Moreover, we have $\lambda_4, \lambda_5 \geqslant 0.347$, which implies

$$g(\lambda_4), g(\lambda_5) \geqslant g(0.347) \geqslant 0.12\sqrt{m-4} + 0.041.$$

We next consider the negative eigenvalues of G. Note from (9) that $\lambda_1 \geqslant \gamma(m) > \sqrt{m-4}$. Since $\lambda_{n-3} \leqslant \lambda_{n-2} \leqslant \lambda_7(C_9) = -1$ and $\lambda_n \leqslant \lambda_{n-1} \leqslant \lambda_8(C_9) = -1.879$, we have $\lambda_n^2 \leqslant 2m - (\sum_{i=1}^{5} \lambda_i^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2 + \lambda_{n-1}^2) < 2m - (m - 4 + 10.465) = m - 6.465$. Thus $-\sqrt{m-6.465} \leqslant \lambda_n \leqslant -1.879$ and then

$$g(\lambda_n) \geqslant \min\{g(-\sqrt{m-6.465}), g(-1.879)\} > 0.8\sqrt{m-4},$$

where the last inequality holds for every $m \geqslant 11$. Similarly, we have $\lambda_{n-1}^2 + \lambda_n^2 \leqslant 2m - (\sum_{i=1}^{5} \lambda_i^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2) < m - 2.934$, which together with $\lambda_{n-1}^2 \leqslant \lambda_n^2$ yields $-\sqrt{(m-2.934)/2} < \lambda_{n-1} \leqslant -1.879$. Then for every $m \geqslant 11$, we have

$$g(\lambda_{n-1}) \geqslant \min\{g(-\sqrt{(m-2.934)/2}), g(-1.879)\} > 0.9\sqrt{m-4}.$$

Moreover, we can similarly get $-\sqrt{(m-1.934)/3} < \lambda_{n-2} \leqslant -1$ and

$$g(\lambda_{n-2}) \geqslant \min\{g(-\sqrt{(m-1.934)/3}), g(-1)\} \geqslant \sqrt{m-4} - 1.$$

The inequality $-\sqrt{(m-0.934)/4} < \lambda_{n-3} \leqslant -1$ implies

$$g(\lambda_{n-3}) \geqslant \min\{g(-\sqrt{(m-0.934)/4}), g(-1)\} \geqslant \sqrt{m-4} - 1.$$

Owing to $\sqrt{m} \geqslant \lambda(G) \geqslant \gamma(m) > \sqrt{m-4}$, by Lemma 8, we obtain

$$\begin{aligned} t(G) &> \frac{1}{6}\left(\sum_{i=2}^{5} g(\lambda_i) + g(\lambda_{n-5+i})\right) - \frac{4}{3}\lambda_1(G) \\ &> \frac{1}{6}\left(8.634\sqrt{m-4} - 8\sqrt{m} + 5.274\right) > 0, \end{aligned}$$

which is a contradiction. Therefore, the odd cycle C_9 can not be an induced subgraph in G. Similarly, we can show by using the monotonicity of $\cos x$ that C_s is not an induced subgraph of G for each $s \geqslant 11$. Consequently, we get $s = 7$. □

From Claim 15, we denote by $S = \{u_1, u_2, \ldots, u_7\}$ the set of vertices of a copy of C_7 in G. Next, we shall show that the following graphs are forbidden induced subgraphs in G, and compute their eigenvalues; see Figure 6 and Table 2.

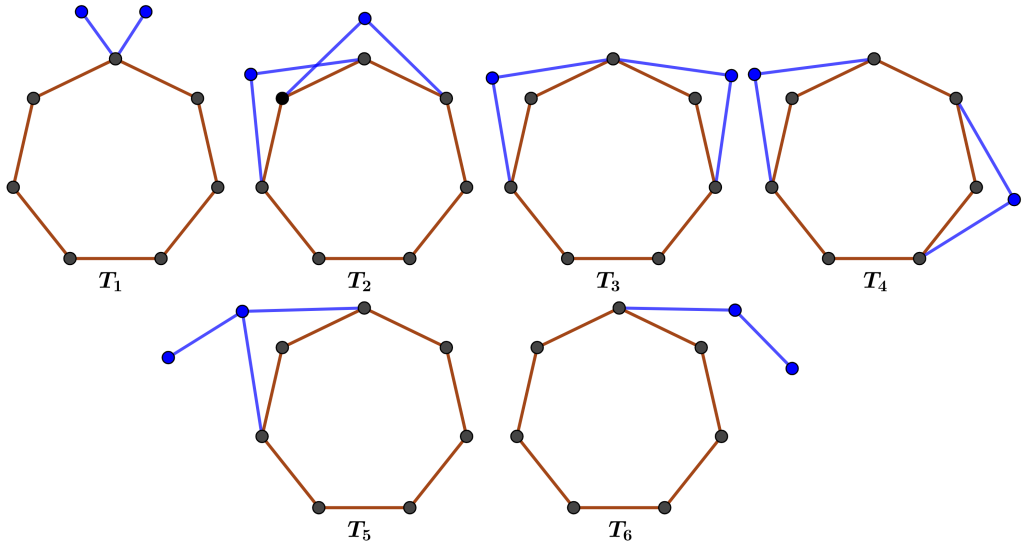


Figure 6: Some forbidden induced subgraphs in G.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9
C_9	2	1.532	1.532	0.347	0.347	-1	-1	-1.879	-1.879
T_1	2.223	1.568	1.247	0.288	0	-0.445	-0.919	-1.801	-2.161
T_2	2.573	1.453	1.441	0.566	-0.358	-0.485	-0.795	-1.871	-2.523
T_3	2.579	1.618	1.373	0	0	-0.451	-0.618	-2	-2.501
T_4	2.503	1.813	1.264	0	0	-0.470	-0.576	-2.191	-2.342
T_5	2.414	1.508	1.247	0.679	-0.414	-0.445	-0.825	-1.801	-2.362
T_6	2.124	1.540	1.247	0.807	-0.337	-0.445	-1.101	-1.801	-2.032

Table 2: Eigenvalues of graphs C_9 and T_i for $i \in \{1, 2, \ldots, 6\}$.

Claim 16. *Any graph of* $\{T_i : 1 \leqslant i \leqslant 6\}$ *can not be an induced subgraph in* G.

We denote by T_0 be the graph on 8 vertices obtained from C_7 by hanging an edge. Unfortunately, we can not prove that T_0 is not an induced subgraph of G by using similar calculations. This is slightly different from the proof of Theorem 6 in Section 2, and makes the forthcoming proof more complicated.

Proof of Claim 16. Our proof needs some tedious calculations similar with that in Claim 15. Suppose on the contrary that G contains T_i as an induced subgraph for some $i \in \{1, 2, 3, 4, 5, 6\}$. To obtain a contradiction, we shall show $t(G) > 0$ by applying Lemma 8. If T_1 is an induced subgraph of G, then Cauchy's interlacing theorem gives $\lambda_{n-9+i}(G) \leqslant \lambda(T_1) \leqslant \lambda_i(G)$ for every $i \in \{1, 2, \ldots, 9\}$. In particular, we have $\lambda_2 \geqslant 1.568$, $\lambda_3 \geqslant 1.247$ and $\lambda_4 \geqslant 0.288$. Then

$$g(\lambda_2) \geqslant g(1.568) \geqslant 2.458\sqrt{m-4} + 3.855,$$
$$g(\lambda_3) \geqslant g(1.247) \geqslant 1.555\sqrt{m-4} + 1.939,$$

and

$$g(\lambda_4) \geqslant g(0.288) \geqslant 0.082\sqrt{m-4} + 0.023.$$

In addition, the negative eigenvalues of T_1 imply that $\lambda_{n-3} \leqslant -0.445$, $\lambda_{n-2} \leqslant -0.919$, $\lambda_{n-1} \leqslant -1.801$ and $\lambda_n \leqslant -2.161$. We know from (9) that $\lambda_1 \geqslant \gamma(m) > \sqrt{m-4}$, which yields $\lambda_n^2 \leqslant 2m - (\sum_{i=1}^{4} \lambda_i^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2 + \lambda_{n-1}^2) < 2m - (m - 4 + 8.382) = m - 4.382$. Then $-\sqrt{m-4.382} < \lambda_n \leqslant -2.161$ and

$$g(\lambda_n) \geqslant \min\{g(-\sqrt{m-4.382}), g(-2.161)\} > 0.15\sqrt{m-4}.$$

Since $\lambda_{n-1}^2 + \lambda_n^2 \leqslant 2m - (\sum_{i=1}^{4} \lambda_i^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2) < 2m - (m - 4 + 5.139) = m - 1.139$ and $\lambda_{n-1}^2 \leqslant \lambda_n^2$, we get $-\sqrt{(m-1.139)/2} < \lambda_{n-1} \leqslant -1.801$ and

$$g(\lambda_{n-1}) \geqslant \min\{g(-\sqrt{(m-1.139)/2}), g(-1.801)\} \geqslant 3.243\sqrt{m-4} - 5.841.$$

Similarly, we have $-\sqrt{(m-0.294)/3} < \lambda_{n-2} \leqslant -0.919$ and

$$g(\lambda_{n-2}) \geqslant \min\{g(-\sqrt{(m-0.294)/3}), g(-0.919)\} \geqslant 0.844\sqrt{m-4} - 0.776.$$

By Lemma 8, we have

$$\begin{aligned} t(G) &> \frac{1}{6}(g(\lambda_2)+g(\lambda_3)+g(\lambda_4)+g(\lambda_n)+g(\lambda_{n-1})+g(\lambda_{n-2}))-\frac{4}{3}\lambda_1(G) \\ &> \frac{1}{6}(8.332\sqrt{m-4}-8\sqrt{m}-0.8)>0, \end{aligned}$$

which is a contradiction.

If T_2 is an induced subgraph of G, then Cauchy's interlacing theorem implies $\lambda_{n-9+i}(G) \leqslant \lambda_i(T_2) \leqslant \lambda_i(G)$ for every $i \in \{1,2,\ldots,9\}$. Since $\lambda_2 \geqslant 1.453$, $\lambda_3 \geqslant 1.441$ and $\lambda_4 \geqslant 0.566$, we obtain

$$\begin{aligned} g(\lambda_2) &\geqslant g(1.453) \geqslant 2.111\sqrt{m-4}+3.067, \\ g(\lambda_3) &\geqslant g(1.441) \geqslant 2.076\sqrt{m-4}+2.992, \end{aligned}$$

and

$$g(\lambda_4) \geqslant g(0.566) \geqslant 0.320\sqrt{m-4}+0.181.$$

On the other hand, the negative eigenvalues of T_2 can imply that $\lambda_{n-4} \leqslant -0.358$, $\lambda_{n-3} \leqslant -0.485$, $\lambda_{n-2} \leqslant -0.795$, $\lambda_{n-1} \leqslant -1.871$ and $\lambda_n \leqslant -2.523$. Due to $\lambda_1 > \sqrt{m-4}$, then we have $\lambda_n^2 \leqslant 2m-(\sum_{i=1}^{4}\lambda_i^2+\lambda_{n-i}^2) < 2m-(m-4+9.004) \leqslant m-5.004$, which yields $-\sqrt{m-5.004} < \lambda_n \leqslant -2.523$. Consequently, we get

$$g(\lambda_n) \geqslant \min\{g(-\sqrt{m-5.004}), g(-2.523)\} > 0.4\sqrt{m-4}.$$

Moreover, since $\lambda_{n-1}^2+\lambda_n^2 \leqslant 2m-(\sum_{i=1}^{4}\lambda_i^2+\lambda_{n-4}^2+\lambda_{n-3}^2+\lambda_{n-2}^2) < 2m-(m-4+5.503) = m-1.503$ and $\lambda_{n-1}^2 \leqslant \lambda_n^2$, we get $-\sqrt{(m-1.503)/2} < \lambda_{n-1} \leqslant -1.871$ and

$$g(\lambda_{n-1}) \geqslant \min\{g(-\sqrt{(m-1.503)/2}), g(-1.871)\} \geqslant 3.5\sqrt{m-4}-6.549.$$

Similarly, we have $-\sqrt{(m-0.871)/3} < \lambda_{n-2} \leqslant -0.795$ and

$$g(\lambda_{n-2}) \geqslant \min\{g(-\sqrt{(m-0.871)/3}), g(-0.795)\} \geqslant 0.632\sqrt{m-4}-0.502.$$

By Lemma 8, we have

$$\begin{aligned} t(G) &> \frac{1}{6}(g(\lambda_2)+g(\lambda_3)+g(\lambda_4)+g(\lambda_n)+g(\lambda_{n-1})+g(\lambda_{n-2}))-\frac{4}{3}\lambda_1(G) \\ &> \frac{1}{6}(9.039\sqrt{m-4}-8\sqrt{m}-0.811)>0, \end{aligned}$$

which is a contradiction.

If T_3 is an induced subgraph of G, then $\lambda_2 \geqslant 1.618$ and $\lambda_3 \geqslant 1.373$. We get

$$g(\lambda_2) \geqslant g(1.618) \geqslant 2.617\sqrt{m-4}+4.235,$$

and

$$g(\lambda_3) \geqslant g(1.373) \geqslant 1.885\sqrt{m-4}+2.588.$$

The negative eigenvalues of T_3 can give that $\lambda_{n-3} \leqslant -0.451$, $\lambda_{n-2} \leqslant -0.618$, $\lambda_{n-1} \leqslant -2$ and $\lambda_n \leqslant -2.501$. Since $\lambda_n^2 \leqslant 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2 + \lambda_{n-1}^2) < 2m - (m - 4 + 9.088) = m - 5.088$, then $-\sqrt{m-5.088} < \lambda_n \leqslant -2.501$ and

$$g(\lambda_n) \geqslant \min\{g(-\sqrt{m-5.088}), g(-2.501)\} > 0.5\sqrt{m-4}.$$

Since $\lambda_{n-1}^2 + \lambda_n^2 \leqslant 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2) < 2m - (m - 4 + 5.088) = m - 1.088$ and $\lambda_{n-1}^2 \leqslant \lambda_n^2$, we get $-\sqrt{(m-1.088)/2} < \lambda_{n-1} \leqslant -2$ and

$$g(\lambda_{n-1}) \geqslant \min\{g(-\sqrt{(m-1.088)/2}), g(-2)\} \geqslant 4\sqrt{m-4} - 8.$$

By Lemma 8, we have

$$\begin{aligned} t(G) &> \frac{1}{6}(g(\lambda_2) + g(\lambda_3) + g(\lambda_n) + g(\lambda_{n-1})) - \frac{4}{3}\lambda_1(G) \\ &> \frac{1}{6}(9.002\sqrt{m-4} - 8\sqrt{m} - 1.177) > 0, \end{aligned}$$

which is a contradiction.

If T_4 is an induced subgraph of G, then we get from Cauchy's interlacing theorem that $\lambda_2 \geqslant 1.813$ and $\lambda_3 \geqslant 1.264$. Then

$$g(\lambda_2) \geqslant g(1.813) \geqslant 3.286\sqrt{m-4} + 5.959,$$

and

$$g(\lambda_3) \geqslant g(1.264) \geqslant 1.597\sqrt{m-4} + 2.019.$$

Moreover, we have $\lambda_{n-3} \leqslant -0.470$, $\lambda_{n-2} \leqslant -0.576$, $\lambda_{n-1} \leqslant -2.191$ and $\lambda_n \leqslant -2.342$. Since $\lambda_n^2 \leqslant 2m - (\sum_{i=1}^{3} \lambda_i + \lambda_{n-i}) < 2m - (m - 4 + 10.237) = m - 6.237$, we get $-\sqrt{m-6.237} < \lambda_n \leqslant -2.342$ and

$$g(\lambda_n) \geqslant \min\{g(-\sqrt{m-6.237}), g(-2.342)\} \geqslant \sqrt{m-4}.$$

Since $\lambda_{n-1}^2 + \lambda_n^2 \leqslant 2m - (\sum_{i=1}^{3} \lambda_i + \lambda_{n-3} + \lambda_{n-2}) < 2m - (m - 4 + 5.437) = m - 1.437$ and $\lambda_{n-1}^2 \leqslant \lambda_n^2$, we get $-\sqrt{(m-1.437)/2} < \lambda_{n-1} \leqslant -2.191$ and

$$g(\lambda_{n-1}) \geqslant \min\{g(-\sqrt{(m-1.437)/2}), g(-2.191)\} \geqslant 4.8\sqrt{m-4} - 10.517.$$

By Lemma 8, we obtain

$$\begin{aligned} t(G) &> \frac{1}{6}(g(\lambda_2) + g(\lambda_3) + g(\lambda_n) + g(\lambda_{n-1})) - \frac{4}{3}\lambda_1(G) \\ &> \frac{1}{6}(10.683\sqrt{m-4} - 8\sqrt{m} - 2.539) > 0, \end{aligned}$$

which is a contradiction.

If T_5 is an induced subgraph of G, then Cauchy's interlacing theorem implies $\lambda_2 \geqslant 1.508$, $\lambda_3 \geqslant 1.247$ and $\lambda_4 \geqslant 0.679$. Then

$$g(\lambda_2) \geqslant g(1.508) \geqslant 2.274\sqrt{m-4} + 3.429,$$
$$g(\lambda_3) \geqslant g(1.247) \geqslant 1.555\sqrt{m-4} + 1.939,$$

and

$$g(\lambda_4) \geqslant g(0.679) \geqslant 0.461\sqrt{m-4} + 0.313.$$

The negative eigenvalues of T_5 imply that $\lambda_{n-4} \leqslant -0.414$, $\lambda_{n-3} \leqslant -0.445$, $\lambda_{n-2} \leqslant -0.825$, $\lambda_{n-1} \leqslant -1.801$ and $\lambda_n \leqslant -2.362$. Since $\lambda_n^2 \leqslant 2m - (\sum_{i=1}^{4} \lambda_i + \lambda_{n-i}) < 2m - (m - 4 + 8.583) = m - 4.583$, we get $-\sqrt{m-4.583} < \lambda_n \leqslant -2.362$ and

$$g(\lambda_n) \geqslant \min\{g(-\sqrt{m-4.583}), g(-2.362)\} \geqslant 0.25\sqrt{m-4}.$$

Since $\lambda_{n-1}^2 + \lambda_n^2 \leqslant 2m - (\sum_{i=1}^{4} \lambda_i^2 + \lambda_{n-4}^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2) < 2m - (m - 4 + 5.34) = m - 1.34$ and $\lambda_{n-1}^2 \leqslant \lambda_n^2$, we have $-\sqrt{(m-1.34)/2} < \lambda_{n-1} \leqslant -1.801$ and

$$g(\lambda_{n-1}) \geqslant \min\{g(-\sqrt{(m-1.34)/2}), g(-1.801)\} \geqslant 3.243\sqrt{m-4} - 5.841.$$

Similarly, we can get $-\sqrt{(m-0.659)/3} < \lambda_{n-2} \leqslant -0.825$ and

$$g(\lambda_{n-2}) \geqslant \min\{g(-\sqrt{(m-0.659)/3}), g(-0.825)\} \geqslant 0.68\sqrt{m-4} - 0.561.$$

By Lemma 8, we obtain

$$\begin{aligned} t(G) &> \frac{1}{6}(g(\lambda_2) + g(\lambda_3) + g(\lambda_4) + g(\lambda_n) + g(\lambda_{n-1}) + g(\lambda_{n-2})) - \frac{4}{3}\lambda_1(G) \\ &> \frac{1}{6}(8.463\sqrt{m-4} - 8\sqrt{m} - 0.721) > 0, \end{aligned}$$

which is a contradiction.

If T_6 is an induced subgraph of G, then Cauchy's interlacing theorem implies that $\lambda_2 \geqslant 1.540$, $\lambda_3 \geqslant 1.247$ and $\lambda_4 \geqslant 0.807$. Then

$$g(\lambda_2) \geqslant g(1.540) \geqslant 2.371\sqrt{m-4} + 3.652,$$
$$g(\lambda_3) \geqslant g(1.247) \geqslant 1.555\sqrt{m-4} + 1.939,$$

and

$$g(\lambda_4) \geqslant g(0.807) \geqslant 0.651\sqrt{m-4} + 0.525.$$

The negative eigenvalues of T_6 yield that $\lambda_{n-4} \leqslant -0.337$, $\lambda_{n-3} \leqslant -0.445$, $\lambda_{n-2} \leqslant -1.101$, $\lambda_{n-1} \leqslant -1.801$ and $\lambda_n \leqslant -2.032$. Since $\lambda_n^2 \leqslant 2m - (\sum_{i=1}^{4} \lambda_i^2 + \lambda_{n-i}^2) < 2m - (m - 4 + 9.345) = m - 5.345$, we get $-\sqrt{m-5.345} < \lambda_n \leqslant -2.032$ and

$$g(\lambda_n) \geqslant \min\{g(-\sqrt{m-5.345}), g(-2.032)\} \geqslant 0.65\sqrt{m-4}.$$

Since $\lambda_{n-1}^2+\lambda_n^2 \leqslant 2m-(\sum_{i=1}^4 \lambda_i^2+\lambda_{n-4}^2+\lambda_{n-3}^2+\lambda_{n-2}^2) < 2m-(m-4+6.101) = m-2.101$ and $\lambda_{n-1}^2 \leqslant \lambda_n^2$, we get $-\sqrt{(m-2.101)/2} < \lambda_{n-1} \leqslant -1.801$ and

$$g(\lambda_{n-1}) \geqslant \min\{g(-\sqrt{(m-2.101)/2}), g(-1.801)\} \geqslant 3.243\sqrt{m-4} - 5.841.$$

Similarly, we can get $-\sqrt{(m-0.889)/3} < \lambda_{n-2} \leqslant -1.101$ and

$$g(\lambda_{n-2}) \geqslant \min\{g(-\sqrt{(m-0.889)/3}), g(-1.101)\} \geqslant 1.212\sqrt{m-4} - 1.334.$$

By Lemma 8, we obtain

$$\begin{aligned} t(G) &> \frac{1}{6}(g(\lambda_2)+g(\lambda_3)+g(\lambda_4)+g(\lambda_n)+g(\lambda_{n-1})+g(\lambda_{n-2})) - \frac{4}{3}\lambda_1(G) \\ &> \frac{1}{6}(9.682\sqrt{m-4} - 8\sqrt{m} - 1.059) > 0, \end{aligned}$$

which is a contradiction. □

Claim 17. *$V(G) = S \cup N(S)$ and $d_S(v) \in \{1,2\}$ for each vertex $v \in N(S)$.*

Proof of Claim 17. For each $v \in N(S)$, without loss of generality, we may assume that $v \in N(u_1)$. If $d_S(v) \geqslant 3$, then we can find either a C_3 or C_5 in G, a contradiction. This implies that $d_S(v) \in \{1,2\}$ for every $v \in N(S)$. Next we prove that $V(G) = S \cup N(S)$. Otherwise, if there is a vertex $v' \in V(G) \setminus (S \cup N(S))$, then v' has distance at least 2 from S. Let $v'vu_1$ be a path of G such that $v'u_i \notin E(G)$ for every $i \in [7]$. Since $d_S(v) \in \{1,2\}$ and G is both C_3-free and C_5-free, we know by symmetry that either $N_S(v) = \{u_1\}$ or $N_S(v) = \{u_1, u_3\}$. If $N_S(v) = \{u_1\}$, then $\{v', v\} \cup S$ induces a copy of T_6, a contradiction. If $N_S(v) = \{u_1, u_3\}$, then $\{v', v\} \cup S$ induces a copy of T_5, which is a contradiction. Thus, we conclude that $V(G) = S \cup N(S)$ and $d_S(v) \in \{1,2\}$ for every $v \in N(S)$. □

From Claim 17, we assume that $V(G) \setminus S = V_1 \cup V_2$, where $V_i = \{v \in N(S) : d_S(v) = i\}$ for every $i = 1, 2$. Since T_1 is not an induced subgraph of G, we get $0 \leqslant |V_1| \leqslant 1$. Since $m \geqslant 11$, we get $V_2 \neq \varnothing$. We can fix a vertex $v \in V_2$ and assume that $N_S(v) = \{u_1, u_3\}$. For each $w \in V_2$, since G contains no triangles and no T_3 as induced subgraphs, we know that $N_S(w) \neq \{u_3, u_5\}$ and $N_S(w) \neq \{u_6, u_1\}$. Similarly, since G contains no pentagons and T_4 as induced subgraphs, we get $N_S(w) \neq \{u_4, u_6\}$ and $N_S(w) \neq \{u_5, u_7\}$. Therefore, it is possible that $N_S(w) = \{u_1, u_3\}, \{u_2, u_4\}$ or $\{u_7, u_2\}$. Furthermore, if $N_S(w) = \{u_1, u_3\}$, then $wv \notin E(G)$, since G contains no triangles; if $N_S(w) = \{u_2, u_4\}$, then $wv \in E(G)$, since G contains no induced copy of T_2. We denote $N_{i,j} = \{w \in V(G) \setminus S : N_S(w) = \{u_i, u_j\}\}$. Note that G has no induced copy of T_3, there are at least one empty set in $\{N_{2,4}, N_{7,2}\}$.

Case 1. If both $N_{2,4} = \varnothing$ and $N_{7,2} = \varnothing$, then $V_2 = N_{1,3}$ and $V(G) \setminus S = N_{1,3} \cup V_1$. If $|V_1| = 0$, then m is odd and $G = S_3(K_{2,\frac{m-3}{2}})$, as desired. If $|V_1| = 1$, then m is even and G is a graph obtained from $S_3(K_{2,\frac{m-4}{2}})$ by hanging an edge to one vertex. By setting $a = 2$ and $b = \frac{m-4}{2}$ in Lemma 13, we know that $\lambda(G) < \gamma(m)$.

Case 2. Without loss of generality, we may assume that $N_{2,4} \neq \varnothing$, then $V(G) \setminus S = N_{1,3} \cup N_{2,4} \cup V_1$. Moreover, $N_{1,3}$ and $N_{2,4}$ induce a complete bipartite subgraph in G. We denote $A = N_{1,3} \cup \{u_2, u_4\}$ and $B = N_{2,4} \cup \{u_3, u_1\}$. Clearly, we have $|A| = a \geqslant 2$ and $|B| = b \geqslant 2$. If $|V_1| = 0$, then G is isomorphic to the subdivision of $K_{a,b}$ by replacing the edge u_1u_4 of $K_{a,b}$ with a path of length 4, and $m = ab + 3$. Note that $\lambda(S_3(K_{a,b}))$ is the largest root of

$$\begin{aligned} Q(x) := x^7 - (ab+3)x^5 + (5ab - 2a - 2b + 2)x^3 \\ + (-5ab + 4a + 4b - 3)x - 2ab + 2a + 2b - 2. \end{aligned}$$

Recall in (8) that $\gamma(m)$ denotes the largest root of $L(x)$. We can easily verify that $L(x) \leqslant Q(x)$ for every $x \geqslant 1$, so we get $L(\lambda(G)) \leqslant Q(\lambda(G)) = 0$, which implies $\lambda(G) \leqslant \gamma(m)$, equality holds if and only if $a = 2$ or $b = 2$, and thus m is odd and $G = S_3(K_{2,\frac{m-3}{2}})$. If $|V_1| = 1$, then $m = ab + 4$ and G is obtained from $S_3(K_{a,b})$ by hanging an edge to one vertex. By Lemma 13 again, we get $\lambda(G) < \gamma(m)$. This completes the proof. □

4 Concluding remarks

We remark that the method stated in Sections 2 and 3 can further allow us to determine the largest spectral radius of non-bipartite graphs with no copy of C_3, C_5 and C_7, and so far as to C_9 by more careful computations. From this evidence, we propose the following conjecture for interested readers. Let $S_{2k-1}(K_{s,t})$ denote the graph obtained from the complete bipartite graph $K_{s,t}$ by replacing an edge with a path P_{2k+1} on $2k+1$ vertices, that is, introducing $2k-1$ new vertices on an edge. Clearly, the odd girth of $S_{2k-1}(K_{s,t})$ is $2k+3$.

Conjecture 18. Let G be a graph with m edges. If G does not contain any member of $\{C_3, C_5, \ldots, C_{2k+1}\}$ and G is non-bipartite, then

$$\lambda(G) \leqslant \lambda(S_{2k-1}(K_{2,\frac{m-2k+1}{2}})),$$

equality holds if and only if m is odd and $G = S_{2k-1}(K_{2,\frac{m-2k+1}{2}})$.

Let B_k be the book graph, i.e., the graph obtained from k triangles by sharing a common edge. In particular, we have $B_1 = K_3$ and $B_2 = K_4^-$, the 4-vertex complete graph minus an edge. In 2021, Zhai, Lin and Shu [53, Conjecture 5.2] made the following conjecture: Let m be large enough and G be a B_k-free graph with m edges. Then

$$\lambda(G) \leqslant \sqrt{m},$$

equality holds if and only if G is a complete bipartite graph.

Soon after, Nikiforov [42] confirmed Zhai–Lin–Shu's conjecture by showing the following stronger theorem. Let $bk(G)$ denote the booksize of G, that is, the maximum number

of triangles with a common edge in G. Nikiforov [42] proved that if G is a graph with m edges and $\lambda(G) \geqslant \sqrt{m}$, then

$$bk(G) > \frac{1}{12}\sqrt[4]{m},$$

unless G is a complete bipartite graph (with possibly some isolated vertices). Since B_2 contains both C_3 and C_4 as a subgraph, the result of Nikiforov generalized Theorem 3.

We conclude this paper with the following problem and conjecture that the lower bound $bk(G) \geqslant c\sqrt{m}$ is true for some constant $c > 0$.

Conjecture 19. If G is a graph with m edges and $\lambda(G) \geqslant \sqrt{m}$, then

$$bk(G) \geqslant c\sqrt{m}$$

for some constant $c > 0$, unless G is a complete bipartite graph.

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References

[1] M. Aigner, G. M. Ziegler, Proofs from THE BOOK, 5th edition, Springer-Verlag, New-York, 2014. See Chapter 40.

[2] N. Alon, Eigenvalues and expanders, Combinatorica 6 (1986), 83–96.

[3] A. Berman, X.-D. Zhang, On the spectral radius of graphs with cut vertices, J. Combin. Theory Ser. B 83 (2001), 233–240.

[4] Y. Bilu, Tales of Hoffman: Three extensions of Hoffman's bound on the graph chromatic number, J. Combin. Theory Ser. B 96 (2006), 608–613.

[5] B. Bollobás, Extremal Graph Theory, Academic Press, New York, 1978.

[6] B. Bollobás, A. Thomason, Dense neighbourhoods and Turán's theorem, J. Combin. Theory Ser. B 31 (1981), 111–114.

[7] B. Bollobás, V. Nikiforov, Cliques and the spectral radius, J. Combin. Theory Ser. B 97 (2007), 859–865.

[8] J.A. Bondy, Large dense neighbourhoods and Turán's theorem, J. Combin. Theory Ser. B 34 (1983), 109–111.

[9] J.A. Bondy, U.S.R. Murty, Graph Theory, Vol. 244 of Graduate Texts in Mathematics, Springer, 2008.

[10] M.-Z. Chen, X.-D. Zhang, Some new results and problems in spectral extremal graph theory, J. Anhui Univ. Nat. Sci. 42 (2018), 12–25.

[11] F. Chung, Diameters and eigenvalues, J. Amer. Math. Soc. 2 (1989), 187–196.

[12] S.M. Cioabă, On the extreme eigenvalues of regular graphs, J. Combin. Theory Ser. B 96 (2006), 367–373.

[13] S.M. Cioabă, D.A. Gregory, V. Nikiforov, Extreme eigenvalues of nonregular graphs, J. Combin. Theory Ser. B 97 (2007), 483–486.

[14] S.M. Cioabă, D.A. Gregory, W.H. Haemers, Matchings in regular graphs from eigenvalues, J. Combin. Theory Ser. B 99 (2009), 287–297.

[15] S.M. Cioabă, E.R. van Dam, J.H. Koolen, J.-H. Lee, A lower bound for the spectral radius of graphs with fixed diameter, European J. Combin. 31 (2010), 1560–1566.

[16] Z. Füredi, M. Simonovits, The history of degenerate (bipartite) extremal graph problems, in Erdős Centennial, Bolyai Soc. Math. Stud., 25, János Bolyai Math. Soc., Budapest, 2013, pp. 169–264.

[17] C.D. Godsil, M. W. Newman, Eigenvalue bounds for independent sets, J. Combin. Theory Ser. B 98 (2008), 721–734.

[18] X. Gu, H.-J. Lai, P. Li, S. Yao, Edge-disjoint spanning trees, edge connectivity, and eigenvalues in graphs, J. Graph Theory 81 (2016), 16–29.

[19] B.D. Guiduli, Spectral extrema for graphs, Ph.D. Thesis, 105 pages, University of Chicago, December 1996. See http://people.cs.uchicago.edu/~laci/students/guiduli-phd.pdf

[20] Y. Hong, A bound on the spectral radius of graphs, Linear Algebra Appl. 108 (1988), 135–140.

[21] Y. Hong, Upper bounds of the spectral radius of graphs in terms of genus, J. Combin. Theory Ser. B 74 (1998), 153–159.

[22] Y. Hong, J.-L. Shu, K. Fang, A sharp upper bound of the spectral radius of graphs, J. Combin. Theory Ser. B 81 (2001), 177–183.

[23] H. Huang, Induced subgraphs of hypercubes and a proof of the Sensitivity Conjecture, Annals of Mathematics, 190 (2019), 949–955.

[24] Z. Jiang, J. Tidor, Y. Yao, S. Zhang, Y. Zhao, Equiangular lines with a fixed angle, Annals of Mathematics, 194 (2021), 729–743.

[25] L. Kang, V. Nikiforov, Extremal problem for the p-spectral radius of graphs, Electronic J. Combin. 21 (3) (2014), #P3.21.

[26] S. Li, W. Sun, Y. Yu, Adjacency eigenvalues of graphs without short odd cycles, Discrete Math. 345 (2022), 112633.

[27] Y. Li, W. Liu, L.H. Feng, A survey on spectral conditions for some extremal graph problems, Advances in Math. (China), 51 (2) (2022), 193–258.

[28] Y. Li, Y. Peng, New proofs of stability theorems on spectral graph problems, arXiv:2203.03142 (2022).

[29] Y. Li, Y. Peng, Refinement on spectral Turán's theorem, arXiv:2204.09194 (2022).

[30] H. Lin, B. Ning, A complete solution to the Cvetković–Rowlinson conjecture, J. Graph Theory 97 (3) (2021), 441–450.

[31] H. Lin, B. Ning, B. Wu, Eigenvalues and triangles in graphs, Combin. Probab. Comput. 30 (2) (2021), 258–270.

[32] H. Lin, H. Guo, A spectral condition for odd cycles in non-bipartite graphs, Linear Algebra Appl. 631 (2021), 83–93.

[33] B. Liu, J. Shen, X. Wang, On the largest eigenvalue of non-regular graphs, J. Combin. Theory Ser. B 97 (2007), 1010–1018.

[34] H. Lu, Regular graphs, eigenvalues and regular factors, J. Graph Theory 69 (2012), 349–355.

[35] M. Lu, H. Liu, F. Tian, Laplacian spectral bounds for clique and independence numbers of graphs, J. Combin. Theory Ser. B 97 (5) (2007), 726–732.

[36] W. Mantel, Problem 28, Solution by H. Gouwentak, W. Mantel, J. Teixeira de Mattes, F. Schuh and W. A. Wythoff. Wiskundige Opgaven, 10 (1907), 60–61.

[37] V. Nikiforov, Some inequalities for the largest eigenvalue of a graph, Combin. Probab. Comput. 11 (2002), 179–189.

[38] V. Nikiforov, Bounds on graph eigenvalues II, Linear Algebra Appl. 427 (2007), 183–189.

[39] V. Nikiforov, The maximum spectral radius of C_4-free graphs of given order and size, Linear Algebra Appl. 430 (2009), 2898–2905.

[40] V. Nikiforov, More spectral bounds on the clique and independence numbers, J. Combin. Theory Ser. B 99 (6) (2009), 819–826.

[41] V. Nikiforov, Some new results in extremal graph theory, Surveys in Combinatorics, London Math. Soc. Lecture Note Ser., 392, Cambridge Univ. Press, Cambridge, 2011, pp. 141–181.

[42] V. Nikiforov, On a theorem of Nosal, arXiv:2104.12171 (2021).

[43] B. Ning, M. Zhai, Counting substructures and eigenvalues I: triangles, arXiv:2112.12937 (2021).

[44] B. Ning, M. Zhai, Counting substructures and eigenvalues II: quadrilaterals, arXiv:2112.15279 (2021).

[45] E. Nosal, Eigenvalues of graphs, Master's thesis, University of Calgary, 1970.

[46] S. O, S.M. Cioabă, Edge-connectivity, eigenvalues, and matchings in regular graphs, SIAM J. Discrete Math. 24 (2010), 1470–1481.

[47] M. Simonovits, Paul Erdős' influence on extremal graph theory, in The Mathematics of Paul Erdős II, R.L. Graham, Springer, New York, 2013, pp. 245–311.

[48] M. Tait, J. Tobin, Three conjectures in extremal spectral graph theory, J. Combin. Theory Ser. B 126 (2017), 137–161.

[49] M. Tait, The Colin de Verdière parameter, excluded minors, and the spectral radius, J. Combin. Theory Ser. A 166 (2019), 42–58.

[50] P. Turán, On an extremal problem in graph theory, Mat. Fiz. Lapok 48 (1941), pp. 436–452. (in Hungarian).

[51] H. Wilf, Spectral bounds for the clique and indendence numbers of graphs, J. Combin. Theory Ser. B 40 (1986), 113–117.

[52] P. Wocjan, C. Elphick, New spectral bounds on the chromatic number encompassing all eigenvalues of the adjacency matrix, Electron. J. Combin. 20 (3) (2013), #P39.

[53] M. Zhai, H. Lin, J. Shu, Spectral extrema of graphs with fixed size: Cycles and complete bipartite graphs, European J. Combin. 95 (2021), 103322.

[54] M. Zhai, J. Shu, A spectral version of Mantel's theorem, Discrete Math. 345 (2022), 112630.

[55] X. Zhan, Matrix Theory, Graduate Studies in Mathematics, vol. 147, Amer. Math. Soc., Providence, RI, 2013.

[56] F. Zhang, Matrix Theory: Basic Results and Techniques, 2nd edition, Springer, New York, 2011.