On the Girth of Three-Dimensional Algebraically Defined Graphs with Multiplicatively Separable Functions

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Abstract

For a field $\mathbb{F}$ and functions $f, g, h, j\colon \mathbb{F} \to \mathbb{F}$, we define $\Gamma_{\mathbb{F}}(f(X)h(Y), g(X)j(Y))$ to be a bipartite graph where each partite set is a copy of $\mathbb{F}^3$, and a vertex (a, a_2, a_3) in the first partite set is adjacent to a vertex $[x, x_2, x_3]$ in the second partite set if and only if

$$a_2 + x_2 = f(a)h(x) \quad \text{and} \quad a_3 + x_3 = g(a)j(x).$$

In this paper, we completely classify all such graphs by girth in the case $h = j$ (subject to some mild restrictions on h). We also present a partial classification when $h \neq j$ and provide some applications.

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1 Introduction

A *three-dimensional algebraically defined graph* $\Gamma_{\mathcal{R}}(f_2(X,Y), f_3(X,Y))$, or simply an *algebraically defined graph*, is a bipartite graph constructed using a ring $\mathcal{R}$ and two bivariate functions $f_2, f_3\colon \mathcal{R}^2 \to \mathcal{R}$. Each partite set is a copy of $\mathcal{R}^3$, where vertices are labeled as (a, a_2, a_3) in the first partite set and $[x, x_2, x_3]$ in the second. Two vertices are adjacent, denoted by $(a, a_2, a_3) \sim [x, x_2, x_3]$, if their coordinates satisfy the equations $a_i + x_i = f_i(a, x)$ for $i \in \{2, 3\}$.

The graphs $\Gamma_{\mathcal{R}}(f_2, \ldots, f_n)$, in the case when $\mathcal{R}$ is a finite field, were introduced in Viglione [26] and Lazebnik and Viglione [21], where their connectivity was studied. For the definition of graphs defined by systems of equations, their origins, properties and numerous applications, see Lazebnik and Woldar [22] and references therein. Subsequently,

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Dmytrenko, Lazebnik, and Williford [5] studied algebraically defined graphs where $\mathcal{R}$ is a finite field $\mathbb{F}_q$ of odd order and f_2 and f_3 are monomials (such graphs are aptly named monomial graphs). They conjectured that all such monomial graphs of girth at least eight are isomorphic to $\Gamma_{\mathbb{F}_q}(XY, XY^2)$. This work was expanded upon by Kronenthal [16], and the conjecture was ultimately proven by Hou, Lappano, and Lazebnik [10]. In addition, Kronenthal and Lazebnik [17] and Kronenthal, Lazebnik, and Williford [18] studied families of polynomial graphs over algebraically closed fields of characteristic zero and applied some of their techniques to graphs over finite fields; these results were recently extended by Cheng, Tang, and Xu [2]. Moreover, Kodess, Kronenthal, Manzano-Ruiz, and Noe [12] classified monomial graphs over the real numbers, and Ganger, Golden, Kronenthal, and Lyons [6] studied a two-dimensional analogue. A number of questions related to connectivity, diameter, and isomorphisms of similarly constructed directed graphs, as well as a peculiar result on the number of roots of certain polynomials in finite fields, were considered by Kodess [11], Kodess and Lazebnik [13, 14], Kodess, Lazebnik, Smith, and Sporre [15], and Coulter, De Winter, Kodess, and Lazebnik [3]. For more related results, see the survey paper by Lazebnik, Sun, and Wang [19].

In this paper, as in many of the aforementioned articles, we study three-dimensional undirected algebraically defined graphs $\Gamma_{\mathcal{R}}(f_2(X,Y), f_3(X,Y))$; but here, we extend our investigation to arbitrary multiplicatively separable bivariate functions f_2 and f_3, and we also allow $\mathcal{R}$ to be an arbitrary field $\mathbb{F}$. By multiplicatively separable functions, we mean $f_2(X,Y) = f(X)h(Y)$ and $f_3(X,Y) = g(X)j(Y)$ for some functions $f, g, h, j\colon \mathbb{F} \to \mathbb{F}$. We now summarize our main results as follows:

- When $h = j$, we completely classify graphs $\Gamma_{\mathbb{F}}(f(X)h(Y), g(X)h(Y))$ by girth under some mild restrictions on h (see Theorem 5 for $\mathbb{F} = \mathbb{R}$ and Theorem 7 for general $\mathbb{F}$).

- In a more general setting, we partially classify graphs $\Gamma_{\mathbb{F}}(f(X)h(Y), g(X)j(Y))$ by girth (see Theorem 8 for $\mathbb{F} = \mathbb{R}$ and Theorem 10 for general $\mathbb{F}$).

- We characterize graphs $\Gamma_{\mathbb{F}_q}(f(X)h(Y), g(X)h(Y))$ of girth greater than four (see Theorem 12); for odd q, we also characterize such graphs of girth greater than six (see Theorem 13 and Corollary 14).

In Section 4, we discuss some applications of these results to several families of graphs over $\mathbb{R}$: $\Gamma_{\mathbb{R}}(X^{\frac{m}{n}}h(Y), a^X h(Y))$, $\Gamma_{\mathbb{R}}(f(X)h(Y), (\sin X)h(Y))$, and $\Gamma_{\mathbb{R}}(X^mY^n, a^{X+Y})$.

It is important to note that all results in this paper can be extended. Indeed, due to isomorphisms $(\mathcal{I}_1)$ and $(\mathcal{I}_2)$ in Lemma 2 (see Section 2), results about graphs $\Gamma_{\mathbb{F}}(f(X)h(Y), g(X)j(Y))$ can be restated to apply to graphs $\Gamma_{\mathbb{F}}(h(X)f(Y), j(X)g(Y))$, $\Gamma_{\mathbb{F}}(g(X)j(Y), f(X)h(Y))$, and $\Gamma_{\mathbb{F}}(j(X)g(Y), h(X)f(Y))$. Furthermore, isomorphisms $(\mathcal{I}_3)$ and $(\mathcal{I}_4)$ allow us to generalize our class of functions beyond those that are multiplicatively separable. For example, applying Theorem 17 and these four isomorphisms yields that $\Gamma_{\mathbb{R}}(a^{X+Y} - 2X^mY^n + 6\cos Y - 1, X^mY^n + 5X^3)$ has girth six for any positive integers m and n and any $a \in \mathbb{R}$ such that $a > 0$ and $a \neq 1$.

There is another important extension that we can apply. In Section 3, all statements specifically about $\mathbb{R}$ can be generalized to any ordered field $\mathbb{K}$. We have written statements using $\mathbb{R}$ instead of $\mathbb{K}$ to parallel results in the literature.

While Theorems 5 and 8 subsume the classification obtained in [12], those results are still useful. Indeed, since the results of [12] deal specifically with monomials, [12] features a classification in which the girth of any monomial graph can be instantly determined by examination. In contrast, the results of this paper are of importance because they apply to a substantially broader family of graphs, but due to this generality, determining which category a given graph falls under may require a bit of analysis.

The study of algebraically defined graphs can be motivated by incidence geometry. In two dimensions, it is known (see Dmytrenko [4] and Lazebnik and Thomason [20]) that every graph $\Gamma_{\mathbb{F}_q}(f)$ with girth greater than four can be completed to a projective plane of order q (although not all projective planes of order q can be constructed in this way). The three-dimensional analogue is motivated by the construction of generalized quadrangles; for additional details, see e.g., [17, 2].

2 Preliminary tools and notation

We shall soon see that the existence of a $2k$-cycle depends only on the first coordinates of its vertices. We say that a $2k$-cycle is of *type*

$$(a_1, a_2, \ldots, a_k; x_1, x_2, \ldots, x_k)$$

provided the first coordinates of its consecutive vertices are given by a_1, x_1, a_2, x_2, ..., a_k, x_k. Note that there are many $2k$-cycles of each prescribed type.

The following result gives necessary and sufficient conditions for the existence of a $2k$-cycle in a three-dimensional algebraically defined graph.

Lemma 1. ([4]; see also [12]) *A $2k$-cycle exists in $\Gamma_{\mathcal{R}}(f_2, f_3)$ if and only if there exist $a_j, x_j \in \mathcal{R}$, $1 \leqslant j \leqslant k$, such that $a_j \neq a_{j+1}$ and $x_j \neq x_{j+1}$ for all $1 \leqslant j \leqslant k$ (here, $a_{k+1} = a_1$ and $x_{k+1} = x_1$), and for $i \in \{2, 3\}$,*

$$\Delta_k(f_i)(a_1, a_2, \ldots, a_k; x_1, x_2, \ldots, x_k) \coloneqq \sum_{j=1}^{k} f_i(a_j, x_j) - f_i(a_{j+1}, x_j) = 0. \tag{1}$$

For example, a 6-cycle of type $(a, b, c; x, y, z)$ exists in $\Gamma_{\mathcal{R}}(f_2, f_3)$ if and only if there exist distinct $a, b, c \in \mathcal{R}$ and distinct $x, y, z \in \mathcal{R}$ such that for $i \in \{2, 3\}$,

$$\Delta_3(f_i)(a, b, c; x, y, z) = f_i(a, x) - f_i(b, x) + f_i(b, y) - f_i(c, y) + f_i(c, z) - f_i(a, z) = 0. \tag{2}$$

We end this section with the following isomorphisms between algebraically defined graphs. For the proof of these results, see, e.g., [20, p. 1549] or [17, Proposition 2.2 on p. 190].

Lemma 2. *Let $\mathbb{F}$ be a field and $f_2, f_3 \colon \mathbb{F}^2 \to \mathbb{F}$. Further, let $c \in \mathbb{F} \setminus \{0\}$, $d \in \mathbb{F}$, and $u, v : \mathbb{F} \to \mathbb{F}$. Then*

$$\Gamma_{\mathbb{F}}(f_2(X,Y), f_3(X,Y)) \cong \Gamma_{\mathbb{F}}(f_2(Y,X), f_3(Y,X)), \tag{$\mathcal{I}_1$}$$

$$\Gamma_{\mathbb{F}}(f_2(X,Y), f_3(X,Y)) \cong \Gamma_{\mathbb{F}}(f_3(X,Y), f_2(X,Y)), \tag{$\mathcal{I}_2$}$$

$$\Gamma_{\mathbb{F}}(f_2(X,Y), f_3(X,Y)) \cong \Gamma_{\mathbb{F}}(f_2(X,Y), cf_3(X,Y) + df_2(X,Y)), \text{ and} \tag{$\mathcal{I}_3$}$$

$$\Gamma_{\mathbb{F}}(f_2(X,Y), f_3(X,Y)) \cong \Gamma_{\mathbb{F}}(f_2(X,Y), f_3(X,Y) + u(X) + v(Y)). \tag{$\mathcal{I}_4$}$$

3 Main results

In this section, we present our main results. Theorems 5 and 8 discuss the girths of $\Gamma_{\mathbb{R}}(f(X)h(Y), g(X)h(Y))$ and $\Gamma_{\mathbb{R}}(f(X)h(Y), g(X)j(Y))$, respectively. Although both theorems are restricted to $\mathbb{R}$, they can be generalized to any ordered field. Moreover, they can be easily generalized to an arbitrary field $\mathbb{F}$ with suitable changes to the condition on the function h; such changes are specified in Theorems 7 and 10. We finish this section with several theorems when the field is $\mathbb{F}_q$. We also remind the reader that results in this section can be extended using Lemma 2 as described in Section 1.

We begin with a lemma that applies to all fields $\mathbb{F}$.

Lemma 3. *Let $f, g : \mathbb{F} \to \mathbb{F}$ and let $a, b, c \in \mathbb{F}$ be distinct. Then*

$$(f(b) - f(a))(g(c) - g(b)) = (f(c) - f(b))(g(b) - g(a)) \tag{3}$$

if and only if $g(a) = g(b) = g(c)$ or there exist $k, \ell \in \mathbb{F}$ such that a, b, and c are roots of $R(X) \coloneqq f(X) - kg(X) - \ell$.

Proof. If (3) holds and $g(a) = g(b) = g(c)$ fails, then assume without loss of generality that $g(a) \neq g(b)$. Let

$$k = \frac{f(b) - f(a)}{g(b) - g(a)} \quad \text{and} \quad \ell = \frac{f(a)g(b) - f(b)g(a)}{g(b) - g(a)} = \frac{f(c)g(b) - f(c)g(a) - f(b)g(c) + f(a)g(c)}{g(b) - g(a)}, \tag{4}$$

where the last equality follows from (3). We can easily verify that $R(a) = R(b) = 0$ and $R(c) = 0$ by using the first representation for ℓ and the second representation for ℓ, respectively.

Conversely, if $g(a) = g(b) = g(c)$, then (3) trivially holds. If $g(a) = g(b) = g(c)$ fails and $R(X) = f(X) - kg(X) - \ell$ has a, b, and c as roots, then assume without loss of generality again that $g(a) \neq g(b)$. From $R(a) = R(b)$ and $R(b) = R(c)$, we obtain $k = \frac{f(b)-f(a)}{g(b)-g(a)}$ and $f(c) - f(b) = k(g(c) - g(b))$, respectively. Substituting the representation for k into the last equation yields (3). $\square$

Remark 4. Although (3) is symmetric with respect to f and g, the corresponding necessary and sufficient condition in Lemma 3 is not. A result similar to Lemma 3 holds if one swaps the roles of f and g in the statement of the lemma: (3) holds if and only if $f(a) = f(b) = f(c)$ or there exist $\widehat{k}, \widehat{\ell} \in \mathbb{F}$ such that a, b, and c are roots of $\widehat{R}(X) \coloneqq g(X) - \widehat{k}f(X) - \widehat{\ell}$.

The proof is analogous, and the expressions for $\widehat{k}$ and $\widehat{\ell}$ are obtained from (4) by swapping the roles of f and g:

$$\widehat{k} = \frac{g(b) - g(a)}{f(b) - f(a)} \quad \text{and} \quad \widehat{\ell} = \frac{g(a)f(b) - g(b)f(a)}{f(b) - f(a)} = \frac{g(c)f(b) - g(c)f(a) - g(b)f(c) + g(a)f(c)}{f(b) - f(a)}.$$

In the following, we temporarily restrict our field to $\mathbb{R}$. A function $d\colon \mathbb{R} \to \mathbb{R}$ is said to satisfy the *intermediate value property* on a subset $A \subseteq \mathbb{R}$ if for any $a, b \in A$ and any y strictly between $d(a)$ and $d(b)$, there exists $x \in A$ such that $d(x) = y$. It is clear that d satisfies the intermediate value property on some subset of $\mathbb{R}$ if and only if the image of d contains a nonempty open interval; in this paper, we use the latter more explicit description in lieu of "intermediate value property". By Darboux's theorem, the derivative of a function differentiable on a closed interval has the intermediate value property on that interval. (Functions with this property are therefore also called Darboux functions.) We note that a function enjoying the intermediate value property need not be continuous; in fact, J.H. Conway constructed a function with this property that is discontinuous at every point of $\mathbb{R}$. See Oman [24] for more examples of such functions and a related discussion.

Theorem 5. *Let $f, g, h : \mathbb{R} \to \mathbb{R}$, and let $\Gamma = \Gamma_{\mathbb{R}}(f(X)h(Y), g(X)h(Y))$.*

1. *Γ has girth four if and only if at least one of the following two conditions is satisfied:*
   (a) *h is not injective;*
   (b) *there exist distinct $a, b \in \mathbb{R}$ such that $f(a) = f(b)$ and $g(a) = g(b)$.*
2. *Assume that the image of h contains a nonempty open interval $\mathfrak{I}$. Then Γ has girth six if and only if conditions 1a and 1b fail, and at least one of the following two conditions is satisfied:*
   (a) *there exist distinct $a, b, c \in \mathbb{R}$ such that $f(a) = f(b) = f(c)$ or $g(a) = g(b) = g(c)$;*
   (b) *there exist $k, \ell \in \mathbb{R}$ such that $f(X) - kg(X) - \ell$ has at least three distinct roots in $\mathbb{R}$.*
3. *Assume that the image of h contains a nonempty open interval. Then Γ has girth at most eight.*

Proof. Since Γ is a bipartite graph, Γ has no cycles of length less than four. The graph Γ contains a 4-cycle of type $S_1 = (a, b; x, y)$ if and only if $\Delta_2(f(X)h(Y))(S_1) = 0$ and $\Delta_2(g(X)h(Y))(S_1) = 0$, which yield $(h(x) - h(y))(f(a) - f(b)) = 0$ and $(h(x) - h(y))(g(a) - g(b)) = 0$, respectively. Statement 1 now follows immediately.

Statement 3 follows from the observation that Γ contains an 8-cycle of type $S_3 = (a, b, a, b; x, y, z, w)$. This can be shown by choosing distinct $x, y, z, w \in \mathbb{R}$ such that $h(x) - h(y) = h(w) - h(z)$ to obtain

$$\Delta_4(f(X)h(Y))(S_3) = (f(a) - f(b))(h(x) - h(y) + h(z) - h(w)) = 0 \quad \text{and}$$
$$\Delta_4(g(X)h(Y))(S_3) = (g(a) - g(b))(h(x) - h(y) + h(z) - h(w)) = 0.$$

To complete the proof, we now focus on statement 2.

If Γ has girth six, then clearly, conditions 1a and 1b fail. Furthermore, as Γ contains a 6-cycle of type $S_2 = (a, b, c; x, y, z)$, we have

$$\Delta_3(f(X)h(Y))(S_2) = (f(a) - f(b))h(x) + (f(b) - f(c))h(y) + (f(c) - f(a))h(z) = 0 \quad (5)$$

and

$$\Delta_3(g(X)h(Y))(S_2) = (g(a) - g(b))h(x) + (g(b) - g(c))h(y) + (g(c) - g(a))h(z) = 0. \quad (6)$$

If $f(a) = f(b)$, then (5) reduces to $(f(a) - f(c))(h(y) - h(z)) = 0$, which implies $f(a) = f(c)$ since h is injective by the negation of condition 1a. Similarly, if $g(a) = g(b)$, then (6) implies $g(a) = g(c)$. Hence, condition 2a holds. If $f(a) \neq f(b)$ and $g(a) \neq g(b)$, then (5) yields

$$h(x) = \frac{(f(b) - f(c))h(y) + (f(c) - f(a))h(z)}{f(b) - f(a)},$$

which can be substituted into (6). After some algebra, we obtain

$$(h(y) - h(z))\big((f(b) - f(a))(g(c) - g(b)) - (f(c) - f(b))(g(b) - g(a))\big) = 0.$$

Since h is injective, we have (3), and condition 2b follows by Lemma 3.

Conversely, if conditions 1a and 1b fail and at least one of conditions 2a and 2b hold, then Γ has girth greater than four, and it remains to show that Γ has a 6-cycle. If condition 2a holds, then by $(\mathcal{I}_2)$, we may assume without loss of generality that $g(a) = g(b) = g(c)$, so (6) holds for any choice of x, y, and z. By the negation of condition 1b, $f(a)$, $f(b)$, and $f(c)$ are distinct, and hence

$$t \coloneqq \frac{f(b) - f(c)}{f(b) - f(a)} \in \mathbb{R} \setminus \{0, 1\}. \quad (7)$$

If $0 < t < 1$, then by choosing distinct $y, z \in \mathbb{R}$ such that $h(y), h(z) \in \mathfrak{I}$, there exists $x \in \mathbb{R} \setminus \{y, z\}$ such that

$$h(x) = t \cdot h(y) + (1 - t)h(z), \quad (8)$$

which implies (5). Similarly, if $t < 0$, by choosing distinct $x, y \in \mathbb{R}$ such that $h(x), h(y) \in \mathfrak{I}$, there exists $z \in \mathbb{R} \setminus \{x, y\}$ such that $h(z) = t'h(x) + (1 - t')h(y)$, where $t' = (1 - t)^{-1} \in (0, 1)$; if $t > 1$, after choosing distinct $x, z \in \mathbb{R}$ for which $h(x), h(z) \in \mathfrak{I}$, we will find $y \in \mathbb{R} \setminus \{x, z\}$ with $h(y) = t''h(x) + (1 - t'')h(z)$, where $t'' = t^{-1} \in (0, 1)$. In all cases, we obtain (8) and thus (5). Therefore, Γ has a 6-cycle of type $(a, b, c; x, y, z)$.

If condition 2a fails and condition 2b holds, then by Lemma 3, there exist distinct $a, b, c \in \mathbb{R}$ such that (3) holds. We claim that $f(a)$, $f(b)$, and $f(c)$ are distinct. If not, assume that $f(a) = f(b)$. Then (3) reduces to $(f(c) - f(b))(g(b) - g(a)) = 0$, which implies $f(b) = f(c)$ or $g(a) = g(b)$, contradicting the negation of condition 2a or the negation of condition 1b, respectively. A similar argument rules out $f(b) = f(c)$ and $f(c) = f(a)$.

Hence, (7) holds and, as shown in the previous paragraph, there exist distinct $x, y, z \in \mathbb{R}$ such that (8) and thus (5) are satisfied. Furthermore, (3) implies that

$$s := \frac{g(a) - g(b)}{f(a) - f(b)} = \frac{g(b) - g(c)}{f(b) - f(c)}, \text{ and so } s = \frac{g(a) - g(c)}{f(a) - f(c)}.$$

Hence, (6) follows from multiplying (5) by s, and therefore Γ has a 6-cycle of type $(a, b, c; x, y, z)$. $\square$

Remark 6. Let Γ have girth greater than four. Then the image $H = h(\mathbb{R})$ must be uncountable since h is injective. If H does not contain a nonempty open interval, then neither condition in statement 2 of Theorem 5 necessarily implies that Γ has girth six. For instance, suppose that $H = \left\{0.a_1a_2a_3 \ldots \in \left[\frac{1}{9}, \frac{2}{3}\right] : a_i \in \{1, 6\} \text{ for all } i \in \mathbb{N}\right\}$ and at least one of the conditions 2a and 2b is true. Suppose further that $f(a) \neq f(b)$ and $t := \frac{f(b)-f(c)}{f(b)-f(a)} = \frac{1}{5}$. Then there do not exist distinct $x, y, z \in \mathbb{R}$ that satisfy (8), implying that (5) fails and Γ has no cycle of length six. This can be seen by the following argument. For any distinct $y, z \in \mathbb{R}$, denote $h(y) = 0.a_1a_2a_3 \ldots$ and $h(z) = 0.b_1b_2b_3 \ldots$. Define $c = 0.c_1c_2c_3 \ldots := t \cdot h(y) + (1 - t)h(z)$. As h is injective, there exists $j \in \mathbb{N}$ such that $a_j \neq b_j$. Then $c_j \in \{2, 5\}$, and thus $c \notin H$. An example of a much "larger" H is considered in the Appendix.

As mentioned at the beginning of this section, we can broaden the scope of Theorem 5 by replacing $\mathbb{R}$ with an arbitrary field $\mathbb{F}$. The proof of statements 1 and 3 is identical to the one given above, and the proof of statement 2 can be slightly simplified once we assume that h is surjective.

Theorem 7. *Let $f, g, h : \mathbb{F} \to \mathbb{F}$, and let $\Gamma = \Gamma_{\mathbb{F}}(f(X)h(Y), g(X)h(Y))$.*

1. *Γ has girth four if and only if at least one of the following two conditions is satisfied:*
   (a) *h is not injective;*
   (b) *there exist distinct $a, b \in \mathbb{F}$ such that $f(a) = f(b)$ and $g(a) = g(b)$.*
2. *Assume that h is surjective. Then Γ has girth six if and only if conditions 1a and 1b fail, and at least one of the following two conditions is satisfied:*
   (a) *there exist distinct $a, b, c \in \mathbb{F}$ such that $f(a) = f(b) = f(c)$ or $g(a) = g(b) = g(c)$;*
   (b) *there exist $k, \ell \in \mathbb{F}$ such that $f(X) - kg(X) - \ell$ has at least three distinct roots in $\mathbb{F}$.*
3. *Assume that there exist distinct $x, y, z, w \in \mathbb{F}$ such that $h(x) - h(y) = h(w) - h(z)$. Then Γ has girth at most eight.*

In the following theorem, we consider the graphs $\Gamma_{\mathbb{R}}(f(X)h(Y), g(X)j(Y))$, removing the requirement that $h = j$. A function $d\colon \mathbb{R} \to \mathbb{R}$ is said to satisfy the *weak intermediate value property* on a subset $A \subseteq \mathbb{R}$ if for any $a, b \in A$, there exists $x \in A$ such that $d(x)$ is strictly between $d(a)$ and $d(b)$.

Theorem 8. *Let $f, g, h, j : \mathbb{R} \to \mathbb{R}$, and let $\Gamma = \Gamma_{\mathbb{R}}(f(X)h(Y), g(X)j(Y))$.*

1. *Γ has girth four if and only if at least one of the following four conditions is satisfied:*
   (a) *h and g are not injective;*
   (b) *j and f are not injective;*
   (c) *there exist distinct $x, y \in \mathbb{R}$ such that $h(x) = h(y)$ and $j(x) = j(y)$;*
   (d) *there exist distinct $a, b \in \mathbb{R}$ such that $f(a) = f(b)$ and $g(a) = g(b)$.*
2. *Γ has girth six if conditions 1a through 1d all fail, and at least one of the following two conditions is satisfied:*
   (a) *the image of h contains a nonempty open interval, and there exist distinct $a, b, c \in \mathbb{R}$ such that $g(a) = g(b) = g(c)$;*
   (b) *the image of h is a nonempty interval, f satisfies the weak intermediate value property, and j and g are not injective.*
3. *Γ has girth at most twelve.*

Remark 9. Similar to Lemma 3, statement 2 of this theorem can be "symmetrically reflected" by using $(\mathcal{I}_2)$ to swap the roles of f and g and those of h and j, or by using $(\mathcal{I}_1)$ to swap the roles of f and h and those of g and j. Also, note that statement 2 only provides sufficient conditions which are not necessary; see Theorem 17.

Proof of Theorem 8. Since Γ is a bipartite graph, Γ has no cycles of length less than four. The graph Γ contains a 4-cycle of type $S_1 = (a, b; x, y)$ if and only if $\Delta_2(f(X)h(Y))(S_1) = 0$ and $\Delta_2(g(X)j(Y))(S_1) = 0$, which yield $(h(x) - h(y))(f(a) - f(b)) = 0$ and $(j(x) - j(y))(g(a) - g(b)) = 0$, respectively. Statement 1 now follows immediately.

It is straightforward to check that for any functions $f, g, h, j : \mathbb{R} \to \mathbb{R}$, Γ contains a closed walk of length 12 whose first coordinates of its consecutive vertices are given by $a, u, b, v, a, w, b, u, a, v, b, w$ (also see [4, p. 15–16]), which proves statement 3. To complete the proof, we now focus on statement 2.

If conditions 1a through 1d all fail and at least one of conditions 2a and 2b hold, then Γ has girth greater than four, and it remains to show that Γ has a 6-cycle of type $S_2 = (a, b, c; x, y, z)$. If condition 2a holds, then the equation

$$\Delta_3(g(X)j(Y))(S_2) = (g(a) - g(b))j(x) + (g(b) - g(c))j(y) + (g(c) - g(a))j(z) = 0 \quad (9)$$

holds for any choice of x, y, and z. By the same argument as given in the proof of Theorem 5 statement 2, (5) holds and Γ has a 6-cycle of type S_2.

If condition 2b holds instead, then there exist distinct $a, b \in \mathbb{R}$ and distinct $y, z \in \mathbb{R}$ such that $g(a) = g(b)$ and $j(y) = j(z)$, which again implies (9). By the negation of conditions 1a and 1b, f and h are injective. Since f satisfies the weak intermediate value property, there exists $c \in \mathbb{R}$ such that $f(c)$ is strictly between $f(a)$ and $f(b)$, thus $t := \frac{f(b)-f(c)}{f(b)-f(a)} \in (0, 1)$. As the image of h is a nonempty interval, there exists $x \in \mathbb{R} \setminus \{y, z\}$ such that (8) holds. Therefore, (5) holds and Γ has a 6-cycle of type S_2. $\square$

Similar to Theorems 5 and 7, Theorem 10 broadens the scope of Theorem 8 with an analogous proof by replacing $\mathbb{R}$ with an arbitrary field $\mathbb{F}$. In particular, under condition 2b, due to the negation of conditions 1a and 1b, f and h are injective. Hence, for all $c \in \mathbb{F} \setminus \{a, b\}$, the assumption under statement 2 that h is surjective implies the existence of $x \in \mathbb{F} \setminus \{y, z\}$ such that (8) holds for $t := \frac{f(b)-f(c)}{f(b)-f(a)}$.

Theorem 10. *Let $f, g, h, j : \mathbb{F} \to \mathbb{F}$, and let $\Gamma = \Gamma_{\mathbb{F}}(f(X)h(Y), g(X)j(Y))$.*

1. *Γ has girth four if and only if at least one of the following four conditions is satisfied:*
   - *(a) h and g are not injective;*
   - *(b) j and f are not injective;*
   - *(c) there exist distinct $x, y \in \mathbb{F}$ such that $h(x) = h(y)$ and $j(x) = j(y)$;*
   - *(d) there exist distinct $a, b \in \mathbb{F}$ such that $f(a) = f(b)$ and $g(a) = g(b)$.*
2. *Assume that h is surjective. Then Γ has girth six if conditions 1a through 1d all fail, and at least one of the following two conditions is satisfied:*
   - *(a) there exist distinct $a, b, c \in \mathbb{F}$ such that $g(a) = g(b) = g(c)$;*
   - *(b) j and g are not injective.*
3. *Γ has girth at most twelve.*

Remark 11. The assumption that h is surjective in Theorems 7 and 10 for a general field $\mathbb{F}$ seems very strong. However, if Γ has girth greater than four, then without loss of generality due to isomorphisms $(\mathcal{I}_1)$ and $(\mathcal{I}_2)$, h is injective. Hence, if $\mathbb{F}$ is a finite field, then it follows immediately that h is surjective, meaning that this assumption does not impose any additional restriction.

Building off of Remark 11, we conclude our section with two theorems regarding the case when $\mathbb{F}$ is a finite field.

Theorem 12. *Let $\mathbb{F}_q$ be a finite field and let $f, g, h\colon \mathbb{F}_q \to \mathbb{F}_q$. Assume that there do not exist distinct $a, b \in \mathbb{F}_q$ satisfying $f(a) = f(b)$ and $g(a) = g(b)$ simultaneously. Then $\Gamma = \Gamma_{\mathbb{F}_q}(f(X)h(Y), g(X)h(Y))$ has girth greater than four if and only if Γ is isomorphic to $\Gamma_{\mathbb{F}_q}(f(X)Y, g(X)Y)$.*

Proof. By Theorem 7, Γ has girth greater than four if and only if h is injective and there do not exist distinct $a, b \in \mathbb{F}_q$ such that $f(a) = f(b)$ and $g(a) = g(b)$. Note that the injectivity of h implies its bijectivity. The result follows by noticing that

$$\begin{aligned}(a_1, a_2, a_3) &\mapsto (a_1, a_2, a_3)\\ [x_1, x_2, x_3] &\mapsto [h(x_1), x_2, x_3]\end{aligned}$$

defines a graph isomorphism from Γ to $\Gamma_{\mathbb{F}_q}(f(X)Y, g(X)Y)$. □

Theorem 13. *Let $\mathbb{F}_q$ be a finite field, where q is odd, and let $f, g, h\colon \mathbb{F}_q \to \mathbb{F}_q$. If $\Gamma = \Gamma_{\mathbb{F}_q}(f(X)h(Y), g(X)h(Y))$ has girth greater than six, then at least one of f and g is not injective.*

Proof. Assume by contradiction that both f and g are injective. Since Γ has girth greater than six, we claim that for all $a \in \mathbb{F}_q$, $\frac{f(a)-f(b)}{g(a)-g(b)}$ and $\frac{f(a)-f(c)}{g(a)-g(c)}$ are distinct in $\mathbb{F}_q \setminus \{0\}$ for all distinct $b, c \in \mathbb{F}_q \setminus \{a\}$. First, the injectivity of f and g guarantees that both fractions are well-defined nonzero values in $\mathbb{F}_q$. Next, if

$$\frac{f(a)-f(b)}{g(a)-g(b)} = \frac{f(a)-f(c)}{g(a)-g(c)} = k$$

for some $k \in \mathbb{F}_q$, then $f(a) - kg(a) = f(b) - kg(b) = f(c) - kg(c)$. Hence, condition 2b of Theorem 7 is satisfied, implying that Γ has girth at most six, which leads to a contradiction. Therefore, $\frac{f(a)-f(b)}{g(a)-g(b)}$ forms a permutation of $\mathbb{F}_q \setminus \{0\}$ when b varies in $\mathbb{F}_q \setminus \{a\}$. As a result, there exists a unique $b \in \mathbb{F}_q \setminus \{a\}$ such that $\frac{f(a)-f(b)}{g(a)-g(b)} = 1$. In other words, $\mathbb{F}_q$ can be partitioned into disjoint pairs $\{a, b\}$ such that $\frac{f(a)-f(b)}{g(a)-g(b)} = 1$, meaning that the cardinality of $\mathbb{F}_q$ is even, which contradicts the assumption that q is odd. $\square$

Corollary 14. *Let $\mathbb{F}_q$ be a finite field, where q is odd, and let $f, g, h\colon \mathbb{F}_q \to \mathbb{F}_q$. If f is injective and $\Gamma = \Gamma_{\mathbb{F}_q}(f(X)h(Y), g(X)h(Y))$ has girth greater than six, then Γ is isomorphic to $\Gamma_{\mathbb{F}_q}(XY, \tilde{g}(X)Y)$ for some noninjective $\tilde{g} : \mathbb{F}_q \to \mathbb{F}_q$.*

Proof. By Theorem 12, Γ is isomorphic to $\Gamma_{\mathbb{F}_q}(f(X)Y, g(X)Y)$. Note that the injectivity of f implies its bijectivity, and

$$\begin{aligned}(a_1, a_2, a_3) &\mapsto (f(a_1), a_2, a_3)\\ [x_1, x_2, x_3] &\mapsto [x_1, x_2, x_3]\end{aligned}$$

defines a graph isomorphism from $\Gamma_{\mathbb{F}_q}(f(X)Y, g(X)Y)$ to $\Gamma_{\mathbb{F}_q}(XY, \tilde{g}(X)Y)$, where $\tilde{g} = g \circ f^{-1}$. Finally, the conclusion that $\tilde{g}$ is not injective follows from Theorem 13. $\square$

4 Applications to families of algebraically defined graphs over $\mathbb{R}$

As an application of Theorem 5, we consider a family of algebraically defined graphs over $\mathbb{R}$ whose adjacency conditions involve exponential functions. We define $x^{\frac{m}{n}}$ here as $\sqrt[n]{x^m}$ when m and n are coprime and n is odd, so $x^{\frac{m}{n}}$ exists for all $x \in \mathbb{R}$.

Theorem 15. *Let $h\colon \mathbb{R} \to \mathbb{R}$ be an injective function whose image contains a nonempty open interval. Let m and n be positive integers such that n is odd and $\gcd(m, n) = 1$, and let $a \in \mathbb{R}$ such that $a > 0$ and $a \neq 1$. Then $\Gamma = \Gamma_{\mathbb{R}}\left(X^{\frac{m}{n}}h(Y), a^X h(Y)\right)$ has girth eight if and only if $m = n = 1$. Otherwise, Γ has girth six.*

Proof. By Theorem 5, since $g(X) = a^X$ and h are injective, it follows that Γ has girth greater than four. We now show that if $m \neq n$, then Γ has a 6-cycle, and we proceed by considering cases. Our main tool is Theorem 5 statement 2b, i.e., we will determine in each case the values of $k, \ell \in \mathbb{R}$ for which $R(X) := a^X - kX^{\frac{m}{n}} - \ell$ has at least three real roots.

Case 1: m is odd and $m > n$. If $a > 1$, take $k = a$ and $\ell = 1$. Then $R(0) = 0$. Also, $R'(0) = \ln a > 0$, $R(1) < 0$, and $R(x) \to \infty$ as $x \to \infty$, so there is a root in each interval $(0, 1)$ and $(1, \infty)$. If $0 < a < 1$, take $k = -a^{-1}$ and $\ell = 1$. Then $R(0) = 0$. Also, $R(-1) < 0$, $R'(0) = \ln a < 0$, and $R(x) \to \infty$ as $x \to \infty$, so there is a root in each interval $(-1, 0)$ and $(0, \infty)$.

Case 2: m is odd and $m < n$. If $a > 1$, take $k = 1 - a^{-1}$ and $\ell = 1$. Then $R(0) = R(-1) = 0$. Also, $R'(x) \to -\infty$ as $x \to 0^+$ and $R(x) \to \infty$ as $x \to \infty$, so there exists another root in $(0, \infty)$. If $0 < a < 1$, take $k = a - 1$ and $\ell = 1$. Then $R(0) = R(1) = 0$. Also, $R(x) \to \infty$ as $x \to -\infty$ and $R'(x) \to \infty$ as $x \to 0^-$, so there exists another root in $(-\infty, 0)$.

Case 3: m is even and $m \neq n$. If $a > 1$, take $k = a + 1$ and $\ell = 0$. Then $R(x) \to -\infty$ as $x \to -\infty$, $R(0) > 0$, $R(1) < 0$, and $R(x) \to \infty$ as $x \to \infty$, so we have a root in each interval $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$. If $0 < a < 1$, take $k = a^{-1} + 1$ and $\ell = 0$. Then $R(x) \to \infty$ as $x \to -\infty$, $R(-1) < 0$, $R(0) > 0$, and $R(x) \to -\infty$ as $x \to \infty$, so we have a root in each interval $(-\infty, -1)$, $(-1, 0)$, and $(0, \infty)$.

Finally, if $m = n = 1$, then $R(x) = a^X - kX - \ell$, which has at most two distinct real roots. Therefore, Γ has girth eight by Theorem 5. □

We also consider algebraically defined graphs whose adjacency conditions involve a trigonometric function.

Theorem 16. *Let $f, h : \mathbb{R} \to \mathbb{R}$, and assume that h is an injective function whose image contains a nonempty open interval. Then $\Gamma = \Gamma_{\mathbb{R}}(f(X)h(Y), (\sin X)h(Y))$ has girth four if and only if there exist distinct $a, b \in \mathbb{R}$ satisfying $a-b = 2n\pi$ or $a+b = (2n+1)\pi$ for some $n \in \mathbb{Z}$ such that $f(a) = f(b)$. Otherwise, Γ has girth six. In particular, $\Gamma_{\mathbb{R}}(XY, (\sin X)Y)$ has girth six.*

Proof. Since h is injective, by Theorem 5 statement 1, Γ has girth four if and only if there exist distinct $a, b \in \mathbb{R}$ such that $f(a) = f(b)$ and $\sin a = \sin b$. If Γ has girth greater than four, then Γ has girth six by Theorem 5 statement 2 since $\sin 0 = \sin \pi = \sin 2\pi$. Finally, since $X - \sin X$ has a unique root in $\mathbb{R}$, $\Gamma_{\mathbb{R}}(XY, (\sin X)Y)$ has girth six. □

Another interesting family of algebraically defined graphs is $\Gamma_{\mathbb{R}}(X^m Y^n, a^{X+Y})$ with $m, n \in \mathbb{N}$, $a > 0$, and $a \neq 1$. Since $g(X) = a^X$ and $j(Y) = a^Y$ are both injective, the conditions given by Theorem 8 statement 2 do not apply. Nevertheless, all graphs in this family have girth six, showing that the conditions in Theorem 8 statement 2 are only sufficient but not necessary.

Theorem 17. *Let m and n be positive integers, and let $a \in \mathbb{R}$ be such that $a > 0$ and $a \neq 1$. Then $\Gamma = \Gamma_{\mathbb{R}}(X^m Y^n, a^{X+Y})$ has girth six.*

Proof. Since $g(X) = a^X$ and $j(Y) = a^Y$ are both injective, by Theorem 8 statement 1, Γ has girth greater than four. If m and n are both even, then it is a straightforward verification that Γ has a 6-cycle of type $(1, -1, 0; 0, 1, -1)$. If m and n are not both even, then without loss of generality due to $(\mathcal{I}_1)$, we may assume that n is odd. It is a straightforward verification that Γ has a 6-cycle of type

$$\left(\sqrt[m]{2}, 1, 0; -\log_a \frac{a^{\sqrt[m]{2}} - a}{a-1}, \log_a \frac{a^{\sqrt[m]{2}} - a}{a-1}, 0\right). \qquad \square$$

5 Concluding Remarks

We finish this paper with a few brief comments and open questions. First, we note that Theorem 8 does not provide a complete classification, which leads to the following open problem.

Open Problem 1. Complete the classification of graphs $\Gamma = \Gamma_{\mathbb{R}}(f(X)h(Y), g(X)j(Y))$ from Theorem 8. For instance, determine necessary and sufficient conditions for Γ to have girth six. Also, is it possible for Γ to have girth greater than eight?

Of course, this problem only accounts for graphs using separable adjacency conditions. Removing this condition would make for a more comprehensive classification.

Open Problem 2. Classify $\Gamma_{\mathbb{R}}(f_2(X,Y), f_3(X,Y))$ by girth for all functions $f_2, f_3 : \mathbb{R}^2 \to \mathbb{R}$.

On top of classification based on girth, we are also interested in proving or disproving graph isomorphism between algebraically defined graphs with the same girth. As proved in Theorem 15, $\Gamma_{\mathbb{R}}(XY, a^X Y)$ has girth eight. Also, certain monomial graphs have girth eight [12], and we are curious whether they could possibly be isomorphic.

Open Problem 3. Do there exist $a \in \mathbb{R}$ and $m, n \in \mathbb{Z}$, where $a > 0$, $a \neq 1$, and $0 \leqslant m < n$, such that the girth eight graphs $\Gamma_{\mathbb{R}}(XY, a^X Y)$ and $\Gamma_{\mathbb{R}}(X^{2m+1}Y, X^{2n}Y)$ are isomorphic?

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Appendix

We refer back to Remark 6 to comment on some properties of the set H. Note first of all that H can be constructed as a Cantor-like set: $H = \bigcap_{n=1}^{\infty} H_n$, where $H_1 = (0.1, 0.2) \cup (0.6, 0.7)$, $H_2 = (0.11, 0.12) \cup (0.16, 0.17) \cup (0.61, 0.62) \cup (0.66, 0.67)$, and in general H_{n+1} is constructed by dividing each of the 2^n intervals of H_n into ten subintervals, each of length 10^{-n-1}, selecting the second and seventh subintervals from each interval (i.e., selecting those 2^{n+1} subintervals in which all elements have all of their first $n+1$ decimal digits either 1 or 6), and taking their union. Clearly, the same set could be obtained by taking the intersection of the unions of these intervals when their endpoints are included. We note the following properties of H, which it shares with the "classical" Cantor set:

(i) It is an uncountable set, which is easily established by the diagonal argument.

(ii) It is a perfect set (and it particular, it is dense-in-itself), that is, every point of H is a limit point. Given $x = 0.a_1a_2a_3\ldots \in H$, let, for every $n \in \mathbb{N}$, $x_n \in H$ be a decimal fraction whose n^{th} digit is 6 if $a_n = 1$, and 1 if $a_n = 6$, with all other digits equal to those of x. Then for all $n \in \mathbb{N}$, $|x - x_n| = 5 \cdot 10^{-n}$, so $x_n \to x$ as $n \to \infty$.

(iii) It is a nowhere dense set in $\mathbb{R}$ since it is closed, and any neighborhood of any point contains a point not from H, so the interior of the closure of H is empty.

(iv) It is a set of Lebesgue measure zero. As $\{H_n\}_{n=1}^{\infty}$ is a descending chain of measurable sets, their intersection H is measurable, and for the measure m of H we have (by continuity of measure)

$$m(H) = m\left(\bigcap_{n=1}^{\infty} H_n\right) = \lim_{n\to\infty} m(H_n) = \lim_{n\to\infty} (0.2)^n = 0.$$

(v) Finally we note that each of the intervals in the definition of H_n contains a maximal and minimal element that is an element of H, e.g., $\frac{1}{6} = 0.1\overline{6}$ is the maximal element of $(0.1, 0.2) \cap H$, and $\frac{11}{18} = 0.6\overline{1}$ is the minimal element of $(0.6, 0.7) \cap H$. It is therefore clear that there are points in H between which there is no other point of H. Also, even for a pair of points x and y of H between which there is a point of H, there exists a point between x and y that is not in H, so H is totally disconnected. It is remarkable that for any $x, y \in H$, the point that is one-fifth of the way from y to x is never in H.

We now give another example demonstrating that the condition of Theorem 5 statement 2, which states that the image H of h contains a nonempty open interval, is not overly strong. This time the set H will be uncountable, dense in $\mathbb{R}$ (and so dense-in-itself), and will have the "betweenness property" (the property that strictly between any two points of the set there is another point of the set.) Unlike the situation of Remark 6, there will be uncountably many values of the weight t for which (8) fails for *all* $x, y, z \in \mathbb{R}$. This means that statement 2 of Theorem 5 does not imply that Γ has girth six.

An easy argument utilizing Zorn's lemma shows that if all proper subfields of a given field are countable, then the field itself is countable (see Butcher, Hamilton, and Milcetich [1]). The field $\mathbb{R}$ of real numbers therefore contains an uncountable proper subfield $\mathbb{K}$. A somewhat more constructive approach would be to consider subfields of $\mathbb{R}$ generated over $\mathbb{Q}$ by a transcendency basis, a maximal subset S of $\mathbb{R}$ that is algebraically independent over $\mathbb{Q}$ with the property that $\mathbb{R}/\mathbb{Q}(S)$ is an algebraic extension. In fact, for any maximal algebraically independent set $S \subseteq \mathbb{R}$, the extension $\mathbb{R}/\mathbb{Q}(S)$ is algebraic. For a proof of the existence of transcendency bases (which also uses Zorn's lemma) and uniqueness of their cardinality, see Hähl, Löwen, Grundhöfer, and Salzmann [8, p. 355], and references therein; see also Milne [23, Chapter 9] and the Stacks Project [25, Section 9.26]. Clearly a transcendency basis S over $\mathbb{Q}$ is uncountable, for otherwise $\mathbb{Q}(S)$ is countable, which would violate the condition that $\mathbb{R}/\mathbb{Q}(S)$ is algebraic since the set of elements algebraic over a countable field is itself countable. Thus if S is a transcendency basis of $\mathbb{R}$ over $\mathbb{Q}$, then $\mathbb{Q}(S)$ is easily seen to be a proper uncountable subfield of $\mathbb{R}$ since for any $x \in S$, $\mathbb{Q}(S)$ contains neither the square root of x, nor the square root of $-x$. Now let h be such that $h(\mathbb{R}) = H = \mathbb{K}$, a proper uncountable subfield of $\mathbb{R}$, and let $t \in \mathbb{R} \setminus \mathbb{K}$, where $\mathbb{R} \setminus \mathbb{K}$ is clearly uncountable. Then for any distinct $x, y \in \mathbb{K}$, $c := t \cdot h(x) + (1-t)h(y) \notin h(\mathbb{R})$, for otherwise, by the injectivity of h, $t = (c - h(y))/(h(x) - h(y)) \in h(\mathbb{R})$. Clearly $H \supseteq \mathbb{Q}$ is dense in $\mathbb{R}$ and has the property that strictly between any distinct $x, y \in H$, there is another point from H. We note that there may exist uncountable subfields of $\mathbb{R}$ that are non-measurable or of measure zero (see Hamkins [9] and Goldstern [7]).