Level-Planarity: Transitivity vs. Even Crossings*

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Abstract

The strong Hanani-Tutte theorem states that a graph is planar if and only if it can be drawn such that any two edges that do not share an end cross an even number of times. Fulek et al. (2013, 2016, 2017) have presented Hanani-Tutte results for (radial) level-planarity where the y-coordinates (distances to the origin) of the vertices are prescribed. We show that the 2-Sat formulation of level-planarity testing due to Randerath et al. (2001) is equivalent to the strong Hanani-Tutte theorem for level-planarity (2013). By elevating this relationship to radial level planarity, we obtain a novel polynomial-time algorithm for testing radial level-planarity in the spirit of Randerath et al.

Mathematics Subject Classifications: 05C10, 68R10

1 Introduction

Planarity of graphs is a fundamental concept for graph theory as a whole, and for graph drawing in particular. Naturally, variants of planarity tailored specifically to directed graphs have been explored. A planar drawing is *upward planar* if all edges are drawn as monotone curves in the upward direction. A special case are *level-planar drawings* of level graphs, where the input graph $G = (V, E)$ comes with a level assignment $\ell\colon V \to \{1, 2, \ldots, k\}$ for some $k \in \mathbb{N}$ that satisfies $\ell(u) < \ell(v)$ for all $(u, v) \in E$. One then asks whether there is an upward planar drawing such that each vertex v is mapped to a point on the horizontal line $y = \ell(v)$ representing the level of v. Level-planar drawings have also been considered on the standing and rolling cylinder and on the torus [5, 3, 4, 1]. Level-planarity on the standing torus is also considered as *radial level-planarity*, where edges are drawn as curves that are monotone in the outward direction in the sense that a

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curve and any circle centered at the origin intersect in at most one point. Here the levels are represented as concentric circles around the origin.

Despite the similarity, the variants with and without levels differ significantly in their complexity. Whereas testing upward planarity and radial planarity are NP-complete [14], level-planarity and radial level-planarity can be tested in polynomial time. In fact, linear-time algorithms are known for both problems [17, 16, 2]. However, both algorithms are quite complicated, and subsequent research has led to slower but simpler algorithms for these problems [15, 24]. Recently also constrained variants of the level-planarity problem have been considered [6, 18].

The simpler algorithm by Randerath et al. [24] only considers *proper* level graphs, where each edge connects vertices on adjacent levels. This is not a restriction, because each level graph can be subdivided to make it proper, potentially at the cost of increasing its size by a factor of k. It is not hard to see that in this case a drawing is fully specified by the vertex ordering on each level. To represent this ordering, define a set of Boolean variables $\mathcal{V} = \{uw \mid u, w \in V, u \neq w, \ell(u) = \ell(w)\}$ where uw being true means u is left of w on level $\ell(w)$. Randerath et al. observe that there is a simple way of specifying the existence of a level-planar drawing by the following consistency (1), transitivity (2) and planarity constraints (3):

$$\forall uw \in \mathcal{V} \quad : \quad uw \Leftrightarrow \neg wu \tag{1}$$

$$\forall uw, wy \in \mathcal{V} \quad : \quad uw \wedge wy \Rightarrow uy \tag{2}$$

$$\forall uw, vx \in \mathcal{V} \text{ with } (u,v),(w,x) \in E \text{ independent} : \quad uw \Leftrightarrow vx \tag{3}$$

The surprising result due to Randerath et al. [24] is that the satisfiability of this system of constraints (and thus the existence of a level-planar drawing) is equivalent to the satisfiability of a *reduced constraint system* obtained by omitting the transitivity constraints (2). That is, transitivity is irrelevant for the satisfiability. Note that a satisfying assignment of the reduced system is not necessarily transitive. Rather Randerath et al. prove that a solution can be made transitive without invalidating the other constraints. Since the remaining conditions 1 and 3 can be easily expressed in terms of 2-Sat, which can be solved efficiently [19], this yields a quadratic-time algorithm for level-planarity.

A recent trend in planarity research are Hanani-Tutte style results. The (strong) Hanani-Tutte theorem [7, 25] states that a graph is planar if and only if it can be drawn so that any two independent edges (i.e., not sharing an endpoint) cross an even number of times. One may wonder for which other drawing styles such a statement is true. Pach and Tóth [20, 21] showed that the weak Hanani-Tutte theorem (which requires even crossings for all pairs of edges) holds for a special case of level-planarity and asked whether the result holds in general. This was shown in the affirmative by Fulek et al. [13], who also established the strong version for level-planarity. Both the weak and the strong Hanani-Tutte theorem have been established for radial level-planarity [11, 10]. Further, the weak version has been shown for all surfaces [23] and the strong version on the Torus [12] and the projective plane [22]. A counterexample for the strong version is known on the orientable surface of genus 4 [9].

Contribution. We show that the result of Randerath et al. [24] from 2001 is equivalent to the strong Hanani-Tutte theorem for level-planarity by Fulek et al. [13]. A key difference is that Randerath et al. consider proper level graphs, whereas Fulek et al. [13] work with graphs with only one vertex per level. For a graph G we define two graphs $G^\star$ and G^+ that are equivalent to G with respect to level-planarity. Graph $G^\star$ has only one vertex per level and graph G^+ is the proper subdivision of $G^\star$. We show how to transform a Hanani-Tutte drawing of graph $G^\star$ into a satisfying assignment for the constraint system of G^+ and vice versa. Since this transformation does not make use of the Hanani-Tutte theorem nor of the result by Randerath et al., this establishes the equivalence of the two results. Fulek et al. [13] indicate that the proof of Randerath et al. may contain a gap. In light of this, one can consider our result as an alternative proof of that result.

Moreover, we show that the transformation can be adapted to the case of radial level-planarity. This results in a novel polynomial-time algorithm for testing radial level-planarity by testing satisfiability of a system of constraints that, much like the work of Randerath et al., is obtained from omitting all transitivity constraints from a constraint system that trivially models radial level-planarity. Currently, we deduce the correctness of the new algorithm from the strong Hanani-Tutte theorem for radial level planarity [10]. However, also this transformation works both ways, and a new correctness proof of our algorithm in the style of the work of Randerath et al. [24] may pave the way for a simpler proof of the Hanani-Tutte theorem for radial level-planarity. We leave this as future work.

2 Preliminaries

A *level graph* is a directed graph $G = (V, E)$ together with a *level assignment* $\ell : V \to \{1, 2, \dots, k\}$ for some $k \in \mathbb{N}$ that satisfies $\ell(u) < \ell(v)$ for all $(u, v) \in E$. If $\ell(u)+1 = \ell(v)$ for all $(u, v) \in E$, the level graph G is *proper*. For any level graph G let $\overline{G}$ be the graph obtained by subdividing the edges of G so that the graph becomes proper; see G^+ in Figure 1.

Two independent edges $(u, v), (w, x)$ are *critical* if $\ell(u) \leqslant \ell(x)$ and $\ell(v) \geqslant \ell(w)$. Note that only pairs of critical edges can cross. Throughout this paper, we consider drawings that may be non-planar, but we assume at all times that no two distinct vertices are drawn at the exact same point, no edge passes through a vertex, no three (or more) edges cross in a single point, and any two edges have only finitely many points in common. If any two independent edges cross an even number of times in a drawing Γ of G, it is called a *Hanani-Tutte drawing* of G [8].

We now describe graphs G^* and G^+ for a given level graph G that are level-planar if and only if G is level-planar. We will show that one can translate between Hanani-Tutte drawings and satisfying assignments of the constraint system by switching between $G^\star$ and G^+. Note that a level graph with only one vertex per level can be considered as an ordered graph (where the vertices are ordered as their levels are ordered), and vice versa. For any k-level graph G we define a *narrow form* $G^\star$ that is an ordered graph. The construction is similar to the one used by Fulek et al. [13]. Let $n(i)$ denote the number of vertices on level i for $1 \leqslant i \leqslant k$. Further, let $v_1, v_2, \dots, v_{n(i)}$ denote the vertices on level i.

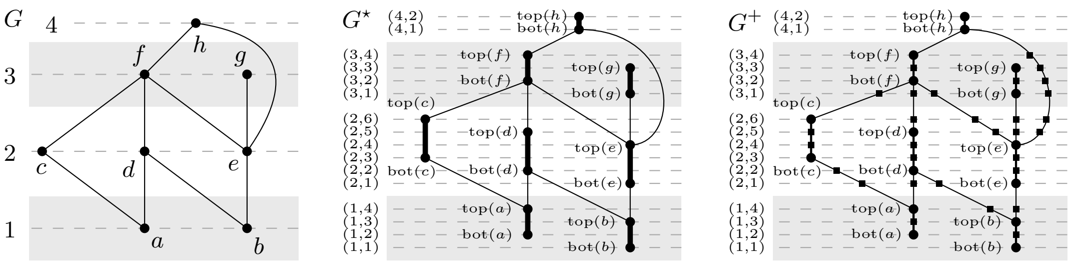
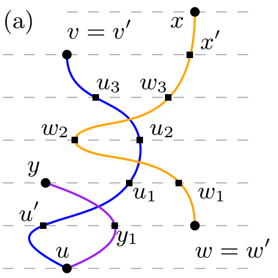


Figure 1: A level graph G (a) modified to graph $G^\star$ so as to have only one vertex per level; stretch edges are bold (b) and its proper subdivision $G^+ = \overline{G^\star}$ (c).



(b)

$\psi(v'x') = $ true $\quad \psi(yu_1) = $ true

$\psi(u_3w_3) = $ true $\quad \psi(yw_1) = $ true

$\psi(u_2w_2) = $ false $\quad \psi(y_1u') = $ false

$\psi(u_1w_1) = $ true $\quad \psi(y_1w') = $ true

$\psi(u'w') = $ true

$\varphi(v'x') = $ true $\quad \varphi(yu_1) = $ true

$\varphi(u_3w_3) = $ true $\quad \varphi(yw_1) = $ true

$\varphi(u_2w_2) = $ true $\quad \varphi(y_1u') = $ true

$\varphi(u_1w_1) = $ true $\quad \varphi(y_1w') = $ true

$\varphi(u'w') = $ true

Figure 2: (a) A Hanani-Tutte drawing and the induced truth assignment ψ; (b) the deduced truth assignment φ. The value where φ differs from ψ is highlighted in red.

Subdivide every level i into $2n(i)$ intermediate levels $(i,1),(i,2),\ldots,(i,2n(i))$ and order them all lexicographically. For $1 \leqslant j \leqslant n(i)$, replace vertex v_j by two vertices $\mathrm{bot}(v_j)$, $\mathrm{top}(v_j)$ with $\ell(\mathrm{bot}(v_j)) = (i,j)$ and $\ell(\mathrm{top}(v_j)) = (i, j+n(i))$ and connect them by an edge $(\mathrm{bot}(v_j), \mathrm{top}(v_j))$, referred to as the *stretch edge* $e(v_j)$. Connect all incoming edges of v_j to $\mathrm{bot}(v_j)$ and connect all outgoing edges of v_j to $\mathrm{top}(v_j)$. Let $e = (u,v)$ be an edge of G. Then let e' denote the edge of $G^\star$ that connects the endpoint of $e(u)$ with the starting point of $e(v)$; see Figure 1. Note that a Hanani-Tutte drawing of G induces a Hanani-Tutte drawing of $G^\star$ (where stretch edges do not cross). Define $G^+ = \overline{G^\star}$ as the graph obtained by subdividing the edges of $G^\star$ so that the graph becomes proper; see Figure 1(c). For any vertex u in G we say the vertices $\mathrm{bot}(u), \mathrm{top}(u)$ and all subdivision vertices of $(\mathrm{bot}(u), \mathrm{top}(u))$ *originate* from u. Let $(u,v),(w,x)$ be critical edges in $G^\star$. Define their *limits* in G^+ as $(u',v'),(w',x')$ where u',v' are endpoints or subdivision vertices of (u,v), the vertices w',x' are endpoints or subdivision vertices of (w,x) and it is $\ell(u') = \ell(w') = \max(\ell(u),\ell(w))$ and $\ell(v') = \ell(x') = \min(\ell(v),\ell(x))$; see Figure 2(a). Finally, we define the function L that maps each level (i,j) of $G^\star$ or G^+ to the level i of G.

Lemma 1. *Let G be a level graph. Then*

$$G \text{ is level-planar} \Leftrightarrow G^\star \text{ is level-planar} \Leftrightarrow G^+ \text{ is level-planar.}$$

Proof. The first equivalence is due to Fulek et al. [13]. The forward direction holds trivially. For the reverse direction, the key insight is that, for every level i of G, there is a level i' of $G^\star$ so that for each vertex v with $\ell(v) = i$ its stretch edge $e(v)$ crosses level i' in $G^\star$. The order in which the stretch edges cross level i' then determines a vertex ordering for level i. The second equivalence is obvious, because G^+ is the proper subdivision of $G^\star$. □

3 Level-Planarity

Recall from the introduction that Randerath et al. formulated level planarity of a proper level graph G as a Boolean satisfiability problem $\mathcal{S}'(G)$ on the variables $\mathcal{V} = \{uw \mid u \neq w, \ell(u) = \ell(w)\}$ and the clauses given by Eq. (1)–(3).

It is readily observed that G is level-planar if and only if $\mathcal{S}'(G)$ is satisfiable. Now let $\mathcal{S}(G)$ denote the SAT instance obtained by removing the transitivity clauses (2) from $\mathcal{S}'(G)$. Note that $\mathcal{S}(G)$ is an instance of 2-SAT, which can be solved efficiently [19]. The key claim of Randerath et al. is that $\mathcal{S}'(G)$ is satisfiable if and only if $\mathcal{S}(G)$ is satisfiable, i.e., dropping the transitivity clauses does not change the satisfiability of $\mathcal{S}'(G)$. In this section, we show that $\mathcal{S}(\overline{G})$ is satisfiable if and only if $\overline{G}^\star$ has a Hanani-Tutte level drawing (Theorem 6). Of course, we do not use the equivalence of both statements to level-planarity of G. Instead, we show that $\mathcal{S}(\overline{G})$ and $\mathcal{S}(\overline{G}^+)$ are equivalent (Lemma 4) and construct a satisfying truth assignment of $\mathcal{S}(G^+)$ from a given Hanani-Tutte level drawing of $G^\star$ (Lemma 3, with $\overline{G^\star} = G^+$), and vice versa (Lemma 5). This implies the equivalence of the results of Randerath et al. and Fulek et al. (Corollary 7).

The common ground for our constructions is the constraint system $\mathcal{S}'(G^+)$. Here a Hanani-Tutte drawing implies a variable assignment that does not necessarily satisfy the planarity constraints (3), though in a controlled way. On the other hand, a satisfying assignment of $\mathcal{S}(G)$ induces an assignment for $\mathcal{S}'(G^+)$ that satisfies the planarity constraints but not necessarily the transitivity constraints (2). Thus, in a sense, our transformation trades planarity for transitivity and vice versa.

A (not necessarily planar) drawing Γ of G *induces* a truth assignment φ of $\mathcal{V}$ by defining for all $uw \in \mathcal{V}$ that $\varphi(uw)$ is true if and only if u lies to the left of w in Γ. Note that this truth assignment must satisfy the consistency and transitivity clauses, but does not necessarily satisfy the planarity constraints. The following lemma describes a relationship between certain truth assignments of $\mathcal{S}(G)$ and crossings in Γ that we use to prove Lemmas 3 and 5.

Lemma 2. *Let G be a level graph. Let $(u, v), (w, x)$ be two critical edges of G and let $(u', v'), (w', x')$ be their limits in $\overline{G}$. Further, let Γ be a drawing of G, let $\overline{\Gamma}$ be the drawing of $\overline{G}$ induced by Γ and let φ be the truth assignment of $\mathcal{S}\left(\overline{G}\right)$ induced by $\overline{\Gamma}$. Then (u, v) and (w, x) intersect an even number of times in Γ if and only if $\varphi(u'w') = \varphi(v'x')$.*

Proof. We may assume without loss of generality that any two edges cross at most once between consecutive levels by introducing intermediate levels if necessary. Let X be a crossing between (u, v) and (w, x) in G; see Figure 2 (a). Further, let u_1, w_1 and u_2, w_2 be the subdivision vertices of (u, v) and (w, x) on the levels directly below and above X

in G, respectively. It is $\varphi(u_1w_1) = \neg\varphi(u_2w_2)$. Conversely, $\varphi(u_1w_1) = \neg\varphi(u_2w_2)$ implies that (u, v) and (w, x) cross between the levels $\ell(u_1)$ and $\ell(u_2)$. Due to the definition of limits, any crossing between (u, v) and (w, x) in G must occur between the levels $\ell(u') = \ell(w')$ and $\ell(v') = \ell(x')$. Therefore, it is $\varphi(u'w') = \varphi(v'x')$ if and only if (u, v) and (w, x) cross an even number of times. □

We obtain that, if $\Gamma^\star$ is a Hanani-Tutte drawing, then for any two critical edges $(u, v), (w, x)$ the limits u', w' are ordered the same way as the limits v', x'. To obtain a satisfying assignment for $S(G)$, we then choose on each level the same ordering for the corresponding subdivision vertices. For any vertex v of $G^\star$ we use for every subdivision vertex on level $\ell(v)$ the ordering induced by $\Gamma^\star$ with regards to v. Note that these orderings are only assigned for pairs of vertices and may be non-transitive on some layers.

Lemma 3. *Let G be a level graph and let Γ be a Hanani-Tutte drawing of G. Then $\mathcal{S}\left(\overline{G}\right)$ is satisfiable.*

Proof. We use Γ to define a truth assignment φ that satisfies all clauses of $\mathcal{S}(\overline{G})$. Let $\overline{\Gamma}$ be the drawing of $\overline{G}$ induced by Γ and let ψ denote the truth assignment induced by $\overline{\Gamma}$; see Figure 2(a). Note that ψ does not necessarily satisfy the planarity constraints. Define φ so that it satisfies all clauses of $\mathcal{S}(\overline{G})$ as follows; see Figure 2(b). Let u'', w'' be two vertices of $\overline{G}$ with $\ell(u'') = \ell(w'')$. If one of them is a vertex in G, then set $\varphi(u'', w'') = \psi(u'', w'')$. Otherwise u'', w'' are subdivision vertices of two edges $(u, v), (w, x) \in E(G)$. If they are independent, then they are critical. In that case their limits $(u', v'), (w', x')$ are already assigned consistently by Lemma 2 since Γ is a Hanani-Tutte drawing. Then set $\varphi(u''w'') = \psi(u'w')$. If (u, v) and (w, x) are adjacent, then $u = w$ or $v = x$. In the first case, we set $\varphi(u''w'') = \psi(v'x')$. In the second case, we set $\varphi(u''w'') = \psi(u'w')$.

Thereby, we have for any critical pair of edges $(u'', v''), (w'', x'') \in E(\overline{G})$ that $\varphi(u''w'') = \varphi(v''x'')$ and clearly $\varphi(u''w'') = \neg\varphi(w''u'')$. Hence, assignment φ satisfies $\mathcal{S}(\overline{G})$. □

Lemma 4. *Let G be a level graph. Then $\mathcal{S}(\overline{G})$ is satisfiable if and only if $\mathcal{S}(\overline{G}^+)$ is satisfiable.*

Proof. First assume there is a satisfying assignment φ for $\mathcal{S}(\overline{G})$. We define a mapping $O\colon V(\overline{G}^+) \to V(\overline{G})$ that maps each vertex of $V(\overline{G}^+)$ to the vertex in $\overline{G}$ that it originates from. This will allow us to describe φ^+ in relation to φ. For each vertex $v \in V(\overline{G}^+)$ that is part of a stretch edge $e(w)$ of $\overline{G}^\star$ for some vertex w of $\overline{G}$, we set $O(v) = w$. Each other vertex $v \in V(\overline{G}^+)$ is a subdivision vertex of an edge (x, x') of $\overline{G}^\star$ that is not a stretch edge. We then map v to vertex $w \in \{x, x'\}$ with $L(\ell(v)) = L(\ell(w))$. Now, for distinct $x, y \in V(\overline{G}^+)$ with $O(x) \neq O(y)$, we set $\varphi^+(xy) = \varphi(O(x)O(y))$. If $O(x) = O(y)$, then x, y are subdivision vertices of non-stretch edges e_x, e_y in $\overline{G}^\star$. Then let v_x, v_y be the disjoint ends of e_x, e_y and we set $\varphi^+(xy) = \varphi(O(v_x), O(v_y))$.

We obtain directly that φ^+ satisfies the consistency constraints, since φ satisfies the consistency constraints. Next, consider the planarity constraint. Let $(u, v), (w, x) \in E(\overline{G}^+)$ with $\ell(u) = \ell(w)$ be independent edges. If $L(\ell(u)) = L(\ell(v))$, then $O(u) = O(v)$

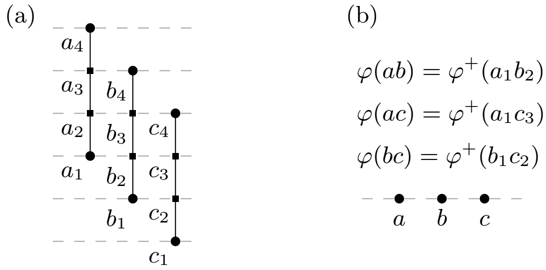


Figure 3: Using the subdivided stretch edges of $\overline{G}^\star$ (a), translate φ^+ to a satisfying assignment φ of $\mathcal{S}(\overline{G})$ (b).

and $O(w) = O(x)$ and thus $\varphi^+(uw) = \varphi(O(u)O(w)) = \varphi(O(v)O(x)) = \varphi^+(vx)$. Otherwise, $L(\ell(u)) \neq L(\ell(v))$. Then (u, v), (w, x) are subdivision edges of non-stretch edges e_u, e_w in $\overline{G}^\star$. If e_u and e_w are not independent, then $\varphi^+(uw) = \varphi(yz) = \varphi^+(vx)$ where y, z are the distinct ends of e_u, e_w. Otherwise, e_u and e_w are independent. Then $\varphi^+(uw) = \varphi(O(u)O(w)) = \varphi(O(v)O(x)) = \varphi^+(vx)$ since φ satisfies the planarity constraints.

For the reverse direction, assume there is a satisfying assignment φ^+ of $\mathcal{S}(\overline{G}^+)$. Let u, w be two vertices of $\overline{G}$ with $\ell(u) = \ell(w)$. Then the stretch edges $e(u), e(w)$ in $\overline{G}^\star$ are critical by construction. Let $(u', u''), (w', w'')$ be their limits in $\overline{G}^+$. Set $\varphi(uw) = \varphi^+(u'w')$; see Figure 3. Because φ^+ is a satisfying assignment, all planarity constraints of $\mathcal{S}(\overline{G}^+)$ are satisfied, which implies $\varphi^+(u'w') = \varphi^+(u''w'')$. The same is true for all subdivision vertices of $e(u), e(w)$ in $\overline{G}^+$. Because φ^+ also satisfies the consistency clauses of $\mathcal{S}(\overline{G}^+)$, this means that φ satisfies the consistency clauses of $\mathcal{S}(\overline{G})$. Note that the resulting assignment is not necessarily transitive, e.g., it could be $\varphi(uv) = \varphi(vw) = \neg\varphi(uw)$.

Consider two edges (u, v), (w, x) in $\overline{G}$ with $\ell(u) = \ell(w)$. Because $\overline{G}$ is proper, we do not have to consider other pairs of edges. Let $(u', u''), (w', w'')$ be the limits of $e(u), e(w)$ in G^+. Further, let $(v', v''), (x', x'')$ be the limits of $e(v), e(x)$ in G^+. Because there are disjoint directed paths from u', w' to v', x' and φ^+ is a satisfying assignment, it is $\varphi^+(u'w') = \varphi^+(v'x')$. Due to the construction of φ described in the previous paragraph, this means that it is $\varphi(uw) = \varphi(vx)$. Therefore, φ is a satisfying assignment of $\mathcal{S}(\overline{G})$. ☐

Lemma 5. *Let G be a level graph together with a satisfying truth assignment φ^+ of $\mathcal{S}(G^+)$. Then there exists a Hanani-Tutte drawing $\Gamma^\star$ of $G^\star$.*

Proof. We construct a drawing Γ^+ of G^+ from φ^+ as follows; see Figure 4. Recall that by construction, every level of G^+ contains exactly one non-subdivision vertex. Let u denote the non-subdivision vertex of level i. Draw a subdivision vertex w on level i to the right of u if $\varphi^+(uw)$ is true and to the left of u otherwise. The relative ordering of subdivision vertices on either side of u can be chosen arbitrarily. Let $\Gamma^\star$ be the drawing of $G^\star$ induced by Γ^+. To see that $\Gamma^\star$ is a Hanani-Tutte drawing, consider two critical edges $(u, v), (w, x)$ of $G^\star$. Let $(u', v'), (w', x')$ denote their limits in G^+. One vertex of u' and v' (w' and x') is a subdivision vertex and the other one is not. The planarity constraint gives $\varphi^+(u'w') = \varphi^+(v'x')$ and by construction u', w' and v', x' are placed consistently on

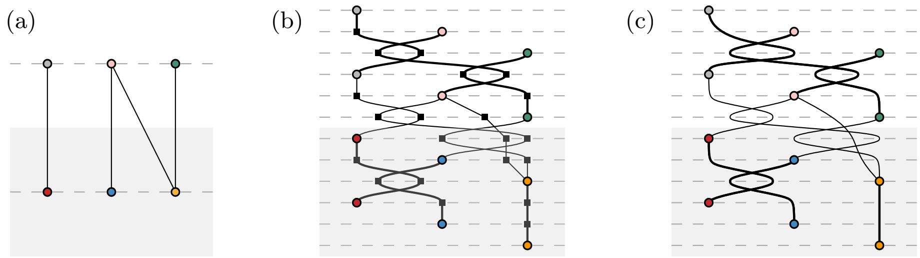


Figure 4: (a) A proper level graph G. Let φ be the truth assignment induced by the drawing (for easy readability). (b) A drawing Γ^+ of G^+ corresponding to φ, (c) the Hanani-Tutte drawing of $G^\star$ induced by Γ^+.

their respective levels. Moreover, Lemma 2 yields that (u, v) and (w, x) cross an even number of times in $\Gamma^\star$. □

Theorem 6. *Let G be a level graph. Then*

1. $\mathcal{S}(\overline{G}^+)$ *is satisfiable* $\Leftrightarrow \overline{G}^\star$ *has a Hanani-Tutte level drawing,*
2. G *has a Hanani-Tutte level drawing* $\Rightarrow \mathcal{S}(\overline{G})$ *is satisfiable,*
3. $\mathcal{S}(\overline{G})$ *is satisfiable* $\Leftrightarrow \overline{G}^\star$ *has a Hanani-Tutte level drawing.*

Proof. The first statement follows from Lemmas 5 and 3. The second statement is immediate from Lemma 3. The third statement follows by applying Lemmas 4 and 5 in one direction and Lemmas 3 and 4 in the other. □

With Lemma 1 we obtain the claimed equivalence of the result of Randerath et al. [24] and the strong Hanani-Tutte Theorem. Namely, on the one hand, if every level graph with a Hanani-Tutte level drawing is level-planar, then we obtain for every level graph G where $\mathcal{S}(\overline{G})$ is satisfied that $\overline{G}^\star$ has a Hanani-Tutte level drawing and is thus level-planar. This implies that G is level-planar. On the other hand, for every level-planar graph G with a Hanani-Tutte level drawing, $\mathcal{S}(\overline{G})$ is satisfiable. If every level graph H where $\mathcal{S}(H)$ is satisfiable is level-planar, this implies $\overline{G}$ is level-planar and thus G is also level-planar.

Corollary 7. *The level-planar graphs are exactly the level graphs with a Hanani-Tutte level drawing if and only if they are exactly the level graphs G where $\mathcal{S}(G)$ is satisfiable.*

4 Radial Level-Planarity

In this section we present an analogous construction for radial level planarity. In contrast to level-planarity, we now have to consider cyclic orderings on the levels, and even those may still leave some freedom for drawing the edges between adjacent levels. In the following we first construct a constraint system of radial level planarity for a proper level graph G, which is inspired by the one of Randerath et al. (Section 4.1). Afterwards, we slightly modify the construction of $G^\star$ (Section 4.2). Finally, in analogy to the level-planar case, we show that a satisfying assignment of our constraint system defines a satisfying assignment of the constraint system of G^+ (Section 4.3), and that this in turn corresponds to a Hanani-Tutte radial level drawing of $G^\star$ (Section 4.4) and vice versa (Section 4.5).

4.1 A Constraint System for Radial Level-Planarity

To deal with the increased complexity in the radial case, we state the constraints aiming for a linear equation system over $\mathbb{F}_2$, the field over $\{0,1\}$. First, observe that we can formulate the constraints for level-planarity as follows. We again use the set of variables $\mathcal{V} = \{uw \mid u, w \in V, u \neq w, \ell(u) = \ell(w)\}$ with possible values in $\{0,1\}$ where now $uw \equiv 0$ means u is left of w on level $\ell(w)$.

$$\begin{array}{lll}
\forall uw \in \mathcal{V} & : & uw \equiv wu + 1 \\
\forall uw, wy \in \mathcal{V} & : & uw \equiv 0 \wedge wy \equiv 0 \Rightarrow uy \equiv 0 \\
\forall uw, vx \in \mathcal{V} \text{ with } (u,v),(w,x) \in E \text{ independent} & : & uw \equiv vx
\end{array}$$

To obtain a constraint system for the radial case, we start with a special case that bears a strong similarity with the level-planar case. Namely, assume that G is a proper level graph that contains a directed path $P = \alpha_1, \dots, \alpha_k$ that has exactly one vertex α_i on each level i. We now express the cyclic ordering on each level as linear orderings whose first vertex is α_i. To this end, we introduce for each level the variables $\mathcal{V}_i = \{\alpha_i uv \mid u, v \in V_i \setminus \{\alpha_i\}\}$, where $\alpha_i uv \equiv 0$ means α_i, u, v are arranged clockwise in this oder on the circle representing level i. We further impose the following necessary and sufficient *linear ordering constraints* $\mathcal{L}_G(\alpha_i)$.

$$\forall \text{ distinct } u, v \in V \setminus \{\alpha_i\} : \quad \alpha_i uv \equiv \alpha_i vu + 1 \tag{4}$$

$$\forall \text{ pairwise distinct } u, v, w \in V \setminus \{\alpha_i\} : \quad \alpha_i uv \equiv 0 \wedge \alpha_i vw \equiv 0 \Rightarrow \alpha_i uw \equiv 0 \tag{5}$$

It remains to constrain the cyclic orderings of vertices on adjacent levels so that the edges between them can be drawn without crossings. For two adjacent levels i and $i+1$, let $\varepsilon_i = (\alpha_i, \alpha_{i+1})$ be the *reference edge*. Let E_i be the set of edges (u,v) of G with $\ell(u) = i$ that do not share an endpoint of ε_i. Further let $E_i^+ = \{(\alpha_i, v) \in E \setminus \{\varepsilon_i\}\}$ and $E_i^- = \{(u, \alpha_{i+1}) \in E \setminus \{\varepsilon_i\}\}$ denote the edges between levels i and $i+1$ adjacent to the reference edge ε_i.

In the context of the constraint formulation, we only consider drawings of the edges between levels i and $i+1$ where any pair of edges crosses at most once and, moreover, ε_i is not crossed. Note that this can always be achieved, independently of the orderings chosen for levels i and $i+1$. Then, the cyclic orderings of the vertices on the levels i and $i+1$ determine the drawings of all edges in E_i. In particular, two edges (u,v), $(u',v') \in E_i$ do not intersect if and only if $\alpha_i uu' \equiv \alpha_{i+1} vv'$; see Figure 5 (a). Therefore, we introduce constraint (6) below. For each edge $e \in E_i^+ \cup E_i^-$ it remains to decide whether it is embedded locally to the left or to the right of ε_i (i.e., whether in some small

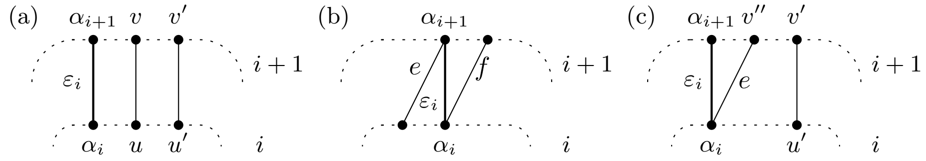


Figure 5: Illustration of the planarity constraints for radial planarity for the case of two edges in E_i (a), constraint (6); the case of an edge in $e \in E_i^-$ and an edge $f \in E_i^+$ with $l(e) \equiv 0$ and $l(f) \equiv 1$ (b), constraint (7); and the case of an edge in E_i and an edge $e \in E_i^+$ with $l(e) \equiv 1$ (c), constraint (8).

environment of α_i^+ or α_i^-, e precedes ε_i in clockwise order or not). For each such edge e, we introduce a variable $l(e)$ where $l(e) \equiv 0$ represents the case where e is to the left of ε_i. Two edges $e \in E_i^-$, $f \in E_i^+$ do not cross if and only if $l(e) \equiv l(f) + 1$; see Figure 5 (b). This gives us constraint (7) below. It remains to forbid crossings between edges in E_i and edges in $E_i^+ \cup E_i^-$. An edge $e = (\alpha_i, v'') \in E_i^+$ and an edge $(u', v') \in E_i$ do not cross if and only if $l(e) \equiv \alpha_{i+1}v'v''$; see Figure 5 (c). Crossings with edges $(v, \alpha_{i+1}) \in E_i^-$ can be treated analogously. This yields constraints (8) and (9). We denote the planarity constraints (6)–(9) by $\mathcal{P}_G(\varepsilon_i)$, where $\varepsilon_i = (\alpha_i, \alpha_{i+1})$.

$$\begin{aligned}
&\forall \text{ independent } (u,v),(u',v') \in E_i &&: \quad \alpha_i uu' &&\equiv \alpha_{i+1}vv' &&(6)\\
&\forall e \in E_i^+, f \in E_i^- &&: \quad l(e) &&\equiv l(f)+1 &&(7)\\
&\forall \text{ independent } (\alpha_i,v'') \in E_i^+, (u,v) \in E_i &&: \quad l(\alpha_i,v'') &&\equiv \alpha_{i+1}vv'' &&(8)\\
&\forall \text{ independent } (u'',\alpha_{i+1}) \in E_i^-, (u,v) \in E_i &&: \quad l(u'',\alpha_{i+1}) &&\equiv \alpha_i uu'' &&(9)
\end{aligned}$$

It is not difficult to see that the transformation between Hanani-Tutte drawings and solutions of the constraint system without the transitivity constraints (5) can be performed as in the previous section. The only difference is that one has to deal with edges that share an endpoint with a reference ε_i.

In general, however, such a path P from level 1 to level k does not necessarily exist. Instead, we use an arbitrary reference edge between any two consecutive levels. If there is no edge between two consecutive levels with vertices u, v, we can insert the edge (u, v) into every radial drawing of G without creating new crossings. We therefore assume from now on that between any two consecutive levels there is at least one edge. Formally, we call a pair of sets $A^+ = \{\alpha_1^+, \dots, \alpha_k^+\}$, $A^- = \{\alpha_1^-, \dots, \alpha_k^-\}$ *reference sets* for G if we have: (i) $\alpha_1^- = \alpha_1^+$ and $\alpha_k^+ = \alpha_k^-$, (ii) for $1 \leqslant i \leqslant k$ the *reference vertices* α_i^+, α_i^- lie on level i and (iii) for $1 \leqslant i < k$ graph G contains the *reference edge* $\varepsilon_i = (\alpha_i^+, \alpha_{i+1}^-)$.

To express radial level-planarity, we express the cyclic orderings on each level twice, once with respect to the reference vertex α_i^+ and once with respect to the reference vertex α_i^-. To express planarity between adjacent levels i and $i+1$, we use the planarity constraints with respect to the reference edge ε_i. It only remains to specify that, if $\alpha_i^+ \neq \alpha_i^-$, the linear ordering with respect to these reference vertices must be linearizations of the same cyclic ordering. This is expressed by the following *cyclic ordering constraints* $\mathcal{C}_G(\alpha_i^+, \alpha_i^-)$; see Figure 6. Here constraint (10) means that each vertex lies in

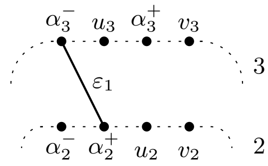


Figure 6: Illustration of the cyclic ordering constraints with $\alpha_2^- u_2 v_2 \equiv \alpha_2^+ u_2 v_2$ on level 2 and with $\alpha_3^- u_3 v_3 \not\equiv \alpha_3^+ u_3 v_3$ on level 3. Regarding constraint (10), we have for example $\alpha_3^- u_3 \alpha_3^+$ and $\alpha_3^+ \alpha_3^- u_3$.

both linar orderings on the same side of α_i^-, α_i^+. Constraint (11) means that the ordering of two vertices is different in the two linear orderings if and only if they are not on the same side of α_i^-, α_i^+.

$$\forall \qquad v \in V_i \setminus \{\alpha_i^-, \alpha_i^+\} : \qquad \alpha_i^- v \alpha_i^+ \; + \; \alpha_i^+ \alpha_i^- v \equiv 0 \tag{10}$$

$$\forall \text{ distinct } u, v \in V_i \setminus \{\alpha_i^-, \alpha_i^+\} : \; \alpha_i^- uv \; + \; \alpha_i^+ uv \; + \; \alpha_i^- u \alpha_i^+ \; + \; \alpha_i^- v \alpha_i^+ \equiv 0 \tag{11}$$

The constraint set $\mathcal{S}'(G, A^+, A^-)$ consists of the linear ordering constraints $\mathcal{L}_G(\alpha_i^+)$ and $\mathcal{L}_G(\alpha_i^-)$ and the cyclic ordering constraints $\mathcal{C}_G(\alpha_i^+, \alpha_i^-)$ for $i = 1, 2, \ldots, k$ if $\alpha_i^+ \neq \alpha_i^-$, plus the planarity constraints $\mathcal{P}_G(\varepsilon_i)$ for $i = 1, 2, \ldots, k-1$. This completes the definition of our constraint system.

Theorem 8. *Let G be a proper level graph with reference sets A^+ and A^-. Then the constraint system $\mathcal{S}'(G, A^+, A^-)$ is satisfiable if and only if G is radial level-planar. Moreover, the radial level planar drawings of G correspond bijectively to the satisfying assignments of $\mathcal{S}'(G, A^+, A^-)$.*

Proof. Clearly, the radial level-planar drawings of G correspond injectively to the satisfying assignments of $\mathcal{S}'(G, A^+, A^-)$. For the reverse direction, consider a satisfying assignment φ of $\mathcal{S}'(G, A^+, A^-)$. Start by observing that the constraints (4) and (5) ensure that the orderings defined for all $u, v \in V_i$ by the variables $\alpha_i^- uv$ and $\alpha_i^+ uv$ (with $\alpha_i^-, \alpha_i^+ \notin \{u, v\}$, respectively) are linear. Let σ_i^-, σ_i^+ denote the cyclic orderings of the vertices on level i induced by the assignments for variables starting with a reference vertex in A^- and A^+, respectively. We next show $\sigma_i^- = \sigma_i^+$. Let w_-, w_+, w'_-, w'_+ be words over V such that the words $\alpha_i^- w_- \alpha_i^+ w_+$ and $\alpha_i^+ w'_+ \alpha_i^- w'_-$ correspond to σ_i^- and σ_i^+. First observe that constraint (10) ensures that w_- and w'_- contain the same vertices (and likewise, w_+ and w'_+ contain the same vertices). Then constraint (11) ensures that the ordering of any two vertices u, v in w_- is the same in w'_- and thus $w_- = w'_-$. Similarly, we get $w_+ = w'_+$ and thus $\sigma_i^- = \sigma_i^+$.
Therefore, φ induces well-defined cyclic orderings of the vertices on all levels. It remains to show that no two edges cross. Recalling that G is proper, it is sufficient to show that no two edges $e, f \in (E_i \cup E_i^- \cup E_i^+)$ cross for $i = 1, 2, \ldots, k$. We distinguish five cases based on how the edges e, f are distributed across the sets E_i, E_i^- and E_i^+.

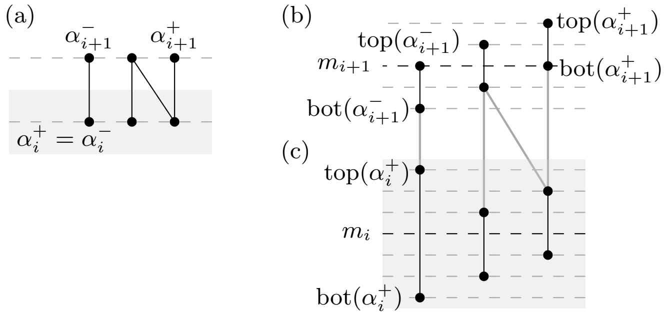


Figure 7: Illustration of the modified construction of the stretch edges for $G^\star$ for graph G in (a). The stretch edges for level $i+1$ where $\alpha_{i+1}^+ \neq \alpha_{i+1}^-$ (b) and for level i where $\alpha_i^+ = \alpha_i^-$ (c).

1. $e, f \in E_i$. Then constraint (6) together with the fact that no edge may cross ε_i implies that e and f do not cross.
2. $e \in E_i, f \in E_i^-$. Let $e = (u, v)$ and $f = (u'', \alpha_{i+1}^-)$ and suppose $l(f) \equiv 0$ (the case $l(f) \equiv 1$ works symmetrically). Then constraint (9) ensures that e and f do not cross.
3. $e \in E_i, f \in E_i^+$; works symmetrically to case 2 with constraint (8).
4. $e \in E_i^-, f \in E_i^+$. Then constraint (7) ensures that e and f are embedded locally to the left and right of ε_i, respectively, or vice versa. Together with the fact that no edge may cross ε_i, this means that e and f do not cross.
5. $e, f \in E_i^-$ or $e, f \in E_i^+$. Because e and f share an endpoint they do not cross.

Thus, no two edges cross, which means that φ induces a radial level-planar drawing. $\square$

Similar to Section 3, we define a reduced constraints system $\mathcal{S}(G, A^+, A^-)$ obtained from $\mathcal{S}'(G, A^+, A^-)$ by dropping constraint (5). Our main result is that $S(G, A^+, A^-)$ is satisfiable if and only if G is radial level-planar. The proof works by showing equivalence to the existence of a Hanani-Tutte drawing of $G^\star$.

4.2 Modified Narrow Form

We slightly modify the construction of the narrow form $G^\star$ of G. This is necessary since we need a special treatment of the reference vertices α_i^+, α_i^- on each level i. Namely, we want that the endpoint of the stretch edge of α_i^+ lies on the highest level and that the starting point of the stretch edge of α_i^- lies at the lowest level replacing level i. Further, for $\alpha_i^- \neq \alpha_i^+$ we want the starting point of the stretch edge of α_i^+ and the endpoint of the stretch edge of α_i^- to lie on the middle level m_i. In each case, we want level m_i to intersect all stretch edges of level i; see Figure 7.

Consider the level i containing the n_i vertices $v_1, \ldots, v_{n_i}$. If $\alpha_i^+ \neq \alpha_i^-$, then we choose the numbering of the vertices such that $v_1 = \alpha_i^-$ and $v_{n_i} = \alpha_i^+$ and arbitrary for the other vertices. We replace i by $2n_i - 1$ levels $(i, 1), (i, 2), \dots, (i, 2n_i - 1)$, which is one level less than previously. Similar to before, we replace each vertex v_j by two vertices $\mathrm{bot}(v_j)$ and $\mathrm{top}(v_j)$ with $\ell(\mathrm{bot}(v_j)) = (i, j)$ and $\ell(\mathrm{top}(v_j)) = (i, n_i - 1 + j)$ and the corresponding stretch edge $(\mathrm{bot}(v_j), \mathrm{top}(v_j))$; see Figure 7 (b). This ensures that the construction works as before, except that the middle level $m_i = (i, n_i)$ contains two vertices, namely $\mathrm{bot}(\alpha_i^+)$ and $\mathrm{top}(\alpha_i^-)$.

If, on the other hand, $\alpha_i^+ = \alpha_i^-$, then we choose $v_1 = \alpha_i^+$. But now we replace level i by $2n_i + 1$ levels $(i, 1), \dots, (i, 2n_i + 1)$. Replace v_1 by vertices $\mathrm{bot}(v_1), \mathrm{top}(v_1)$ with $\ell(\mathrm{bot}(v_1)) = (i, 1)$ and $\ell(\mathrm{top}(v_1)) = (i, 2n_i + 1)$. Replace all other v_j by vertices $\mathrm{bot}(v_j), \mathrm{top}(v_j)$ with $\ell(\mathrm{bot}(v_j)) = (i, j)$ and $\ell(\mathrm{top}(v_j)) = (i, n_i + 1 + j)$. For all j, we add the stretch edge $(\mathrm{bot}(v_j), \mathrm{top}(v_j))$ as before; see Figure 7 (c). This construction ensures that the stretch edge of $\alpha_i^+ = \alpha_i^-$ starts in the first new level $(i, 1)$ and ends in the last new level $(i, 2n_i + 1)$, and the middle level $m_i = (i, n_i + 1)$ contains no vertex.

As before, we replace each original edge (u, v) of the input graph G by the edge $(\mathrm{top}(u), \mathrm{bot}(v))$ connecting the upper endpoint of the stretch edge of u to the lower endpoint of the stretch edge of v. Observe that the construction preserves the property that for each level i the stretch edges of all its vertices are intersected by the middle level m_i and no other edge crosses m_i if G is proper. Therefore, we obtain the following result analogously to Lemma 1.

Lemma 9. *Let G be a level graph. Then*

G is radial level-planar $\Leftrightarrow$ $G^\star$ is radial level-planar $\Leftrightarrow$ G^+ is radial level-planar.

For each vertex v of G we use $e(v) = (\mathrm{bot}(v), \mathrm{top}(v))$ to denote its stretch edge. Again, we use the function L that maps each level (i, j) of $G^\star$ or G^+ to the level i of G. We order the tuples of levels lexicographically and consider them as natural numbers as suitable. For an edge e of $G^\star$ and a level i that intersects e, we denote by e_i the subdivision vertex of e at level i in G^+. For two levels i and j that both intersect an edge e of $G^\star$, we denote by e_i^j the path from e_i to e_j in G^+.

4.3 Constraint System and Assignment for G^+

Let G be a proper level graph with reference sets A^+, A^-. We now choose reference sets B^+, B^- for G^+ that are based on the reference sets A^+, A^-. Let j be any level of $G^\star$ and let $L(j) = i$ be the corresponding level of G. Define two vertices β_j^+, β_j^- as follows. If $\alpha_i^- = \alpha_i^+$, set $\beta_j^- = \beta_j^+ = e(\alpha_i^-)_j$; see Figure 8 (c). Otherwise, the choice is based on whether j is the middle level $m = m_i$ of the levels $L^{-1}(i)$ that replace level i of G, or whether j lies above or below m. Choose $\beta_m^- = \mathrm{top}(\alpha_i^-)$ and $\beta_m^+ = \mathrm{bot}(\alpha_i^+)$.For $j < m$, choose $\beta_j^- = \beta_j^+ = e(\alpha_i^-)_j$ and for $j > m$, choose $\beta_j^- = \beta_j^+ = e(\alpha_i^+)_j$; see Figure 8 (b).

We set B^+ to be the set containing all β_j^+ and likewise for B^-. Our next step is to construct from a satisfying assignment φ of $\mathcal{S}(G, A^+, A^-)$ a corresponding satisfying assignment φ^+ of $\mathcal{S}(G^+, B^+, B^-)$. The construction follows the approach from Lemma 5 and makes use of the fact that G^+ is essentially a stretched and perturbed version of G.

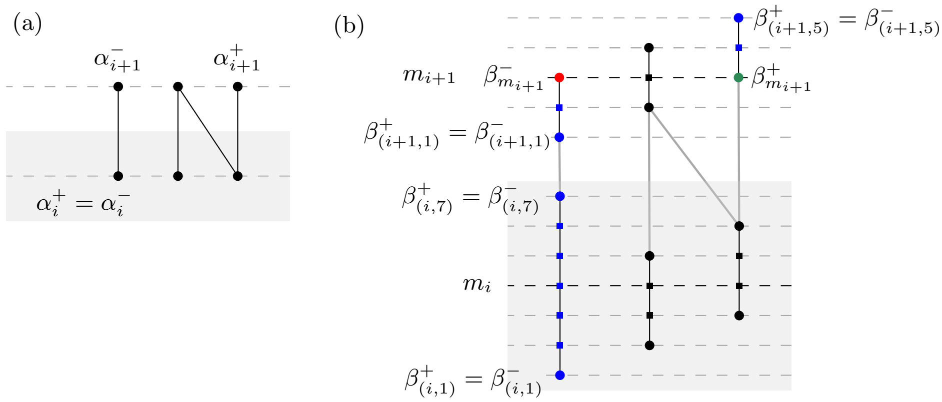


Figure 8: Definition of β^+, β^- in the assignment for G^+ for the same graph as in Figure 7 (a). Vertices β^+ (β^-) are drawn in green (red), or in blue if they coincide.

Lemma 10. *If $\mathcal{S}(G, A^+, A^-)$ is satisfiable, then $\mathcal{S}(G^+, B^+, B^-)$ is satisfiable.*

For the proof of this result, we consider a satisfying assignment φ for $\mathcal{S}(G, A^+, A^-)$. We now derive an assignment φ^+ for $\mathcal{S}(G^+, B^+, B^-)$ from φ. Afterwards, we show that φ^+ satisfies $\mathcal{S}(G^+, B^+, B^-)$.

Construction of $\boldsymbol{\varphi^+}$. Let $e = (u, u')$ and $f = (v, v')$ be two edges between level i and level $i+1$ of G and let $\varepsilon_i = (\alpha^+_i, \alpha^-_{i+1})$ denote the reference edge between these levels. We introduce a function $\psi(\varepsilon_i, e, f)$ to deduce the ordering of the edges ε_i, e, f in a drawing that corresponds to φ by setting:

$$\psi(\varepsilon_i, e, f) \equiv \begin{cases} \varphi(\alpha^+_i uv) & \text{if } \alpha^+_i, u, v \text{ are pairwise distinct,} \\ \varphi(\alpha^-_{i+1} u'v') & \text{if } \alpha^-_{i+1}, u', v' \text{ are pairwise distinct,} \\ \varphi(l(f)) & \text{if } \alpha^+_i = v \neq u \text{ or } \alpha^-_{i+1} = v' \neq u', \\ \varphi(l(e)) + 1 & \text{if } \alpha^+_i = u \neq v \text{ or } \alpha^-_{i+1} = u' \neq v'. \end{cases}$$

Note that if α^+_i, u, v and α^-_{i+1}, u', v' are pairwise distinct, then by Eq. (6) it follows $\varphi(\alpha^+_i uv) \equiv \varphi(\alpha^-_{i+1} u'v')$. Similarly, if $\alpha^+_i = v$ and $\alpha^-_{i+1} = u'$ (or $\alpha^+_i = u$ and $\alpha^-_{i+1} = v'$), then by Eq. (7) it is $\varphi(l(f)) \equiv \varphi(l(e)) + 1$. Therefore ψ is well-defined.

Based on this, we can now define the orderings of triples of subdivision vertices of G^+, which leads to an assignment φ^+ for $\mathcal{S}(G^+, B^+, B^-)$. To this end, we define a mapping $O\colon V(G^+) \to V$ that maps each vertex of $V(G^+)$ to a vertex in G that it originates from. For each vertex v of G^+ that is part of a stretch edge $e(w)$ of $G^\star$ for some vertex w of G, we set $O(v) = w$. For an example, see the orange vertex w of G in Figure 9. The encircled orange vertices in G^+ are mapped to w. It remains to define $O(v)$ for vertices $v \in V(G^+)$ that are subdivision vertices of an edge (x, x') of $G^\star$ that its not a stretch edge. Then the original levels satisfy $L(\ell(x')) = L(\ell(x)) + 1$, and we map v to the vertex w with $x = \mathrm{top}(w)$ if $L(\ell(v)) = L(\ell(x))$ and to the vertex w' with $x' = \mathrm{bot}(w')$, otherwise. For an example, see the edge (x, x') of G in Figure 9. The encircled purple (green) subdivision vertices in G^+ are mapped to x (to x').

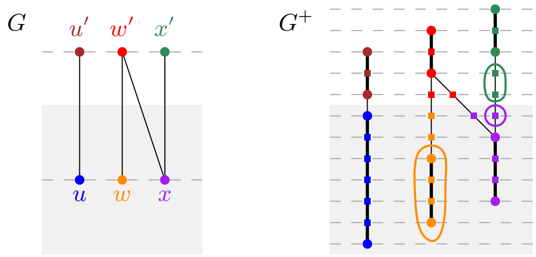


Figure 9: Definition of the mapping $O\colon V(G^+) \to V$. Vertices of G^+ are mapped to the vertex of G with the matching color.

We are now ready to define the assignment φ^+ for $\mathcal{S}(G^+, B^+, B^-)$ for a satisfying assignment φ of $\mathcal{S}(G, A^+, A^-)$. Let $x, y, z \in V_j$ be three distinct vertices on level (i, j) and such that $x \in \{\beta_j^-, \beta_j^+\}$. Observe that $O(x) \in \{\alpha_i^+, \alpha_{i+1}^-\}$. If the vertices $O(x)$, $O(y)$, $O(z)$ are pairwise distinct, we set

$$\varphi^+(xyz) = \varphi\left(O(x)O(y)O(z)\right).$$

Otherwise, two of the corresponding vertices are the same. Hence, two vertices of x, y, z are subdivision vertices of original edges in G. Note that x can by definition of $G^\star$ not be such a subdivision vertex. Hence, vertex y is a subdivision vertex of an edge $e \in G$ and vertex z is a subdivision vertex of an edge $f \in G$, both of which connect a vertex of level i to a vertex on level $i+1$. Then we set

$$\varphi^+(xyz) \equiv \psi(\varepsilon_i, e, f),$$

where $\varepsilon_i = (\alpha_i^+, \alpha_{i+1}^-)$ is the reference edge between these levels. Moreover, we set $\varphi^+(l(e)) \equiv \varphi(l(O(u), O(v)))$ for each edge $e = (u, v)$ of G^+ with $u = \beta_j^+$ or $v = \beta_{j+1}^-$ but not both for some level j of G^+.

Lemma 11. *The assignment φ^+ satisfies $\mathcal{S}(G^+, B^+, B^-)$.*

Proof. We check that φ^+ satisfies each part of the constraint system. We denote by V_j the vertices of G^+ on level j.

$\boldsymbol{\varphi^+}$ satisfies $\boldsymbol{\mathcal{L}'_{G^+}(\beta_j^+)}$. Let $\beta_j^+ \in B^+$. We aim to show that φ^+ satisfies $\mathcal{L}'_{G^+}(\beta_j^+)$. Let $y, z \in V_j \setminus \{\beta_j^+\}$ be distinct. If $O(\beta_j^+), O(y), O(z)$ are distinct, then $\varphi^+(\beta_j^+ yz) \equiv \varphi(O(\beta_j^+)O(y)O(z))$ as well as $\varphi^+(\beta_j^+ zy) \equiv \varphi(O(\beta_j^+)O(z)O(y))$. Note that $O(\beta_j^+) \in \{\alpha_{L(j)}^+, \alpha_{L(j)}^-\}$. Since φ satisfies $\mathcal{L}'_G(\alpha_{L(j)}^+)$ and $\mathcal{L}'_G(\alpha_{L(j)}^-)$, it is $\varphi^+(\beta_j^+ yz) \equiv \varphi^+(\beta_j^+ zy) + 1$.

If $O(\beta_j^+), O(y), O(z)$ are not distinct, then $O(y) = O(z)$, since β_j^+ cannot be a subdivision vertex of an edge that is not a stretch edge. Hence, the vertices y, z are subdivision vertices of edges e, f of $G^\star$ connecting vertices on levels i and $i+1$. Note that this also implies $O(\beta_j^+) \neq O(y)$. We obtain $\varphi^+(\beta_j^+, y, z) \equiv \psi(\varepsilon_i, e, f)$ and $\varphi^+(\beta_j^+, z, y) \equiv \psi(\varepsilon_i, f, e)$. Since $O(y) = O(z) \neq O(\beta_j^+)$, we have with the definition of ψ that $\psi(\varepsilon_i, e, f) \equiv \psi(\varepsilon_i, f, e) + 1$. This yields $\varphi^+(\beta_j^+, y, z) \equiv \varphi^+(\beta_j^+, z, y) + 1$.

φ^+ satisfies $\mathcal{L}'_{G^+}(\beta_j^-)$. This can be argued analogously to $\mathcal{L}'_{G^+}(\beta_j^+)$.

φ^+ satisfies $\mathcal{C}_{G^+}(\beta_j^+, \beta_j^-)$. We next show that φ^+ satisfies the constraints $\mathcal{C}_{G^+}(\beta_j^+, \beta_j^-)$ for all j with $\beta_j^- \neq \beta_j^+$. By definition of $G^\star$ and B^+, B^-, these constraints are non-trivial only for levels $j = m_i$ for a level i of G with $\alpha_i^+ \neq \alpha_i^-$.

Note that due to the construction of $G^\star$, level m_i is only crossed by stretch edges, which implies that the restriction of O to the vertices of G^+ on level m_i is injective. Moreover, $O(\beta_j^+) = \alpha_i^+$ and $O(\beta_j^-) = \alpha_i^-$. Hence, the triple $(x, y, z) \in V_j$ of distinct vertices maps injectively to triples of distinct vertices of G on level i. Since the value of $\varphi^+(xyz)$ is defined in terms of $\varphi(O(x)O(y)O(z))$, it follows that φ^+ satisfies Eq. (11) and (10) if φ does.

φ^+ satisfies $\mathcal{P}_{G^+}(\beta_j^+, \beta_{j+1}^-)$. We finally show that φ^+ satisfies $\mathcal{P}_{G^+}(\beta_j^+, \beta_{j+1}^-)$ for any two consecutive levels j and $j+1$ of G^+. We distinguish two cases, based on whether $L(j) = L(j+1)$ or $L(j+1) = L(j) + 1$.

1. $L(j) = L(j+1) = i$. First observe that, except for $(\beta_j^+, \beta_{j+1}^-)$, the reference vertex β_j^+ has no outgoing edges and β_{j+1}^- has no incoming edges. Namely, vertex β_j^+ can only have more outgoing edges if $\beta_j^+ = \mathrm{top}(\alpha_i^+)$, and vertex β_{j+1}^- can only have more incoming edges if $\beta_{j+1}^- = \mathrm{bot}(\alpha_i^-)$. But $\mathrm{bot}(\alpha_i^-)$ and $\mathrm{top}(\alpha_i^+)$ occupy the first and the last level of G^+ corresponding to level i of G. Hence this is not possible since $L(j) = L(j+1)$. Therefore the equations (7)–(9) are trivially satisfied.

   Now consider two independent edges $(y, y'), (z, z')$ between levels j and $j+1$ that are different from $(\beta_j^+, \beta_{j+1}^-)$.

   Since the vertices y, y' are adjacent in G^+, they either both subdivide the same edge of $G^\star$, or one of them is a vertex of $G^\star$ and the other one subdivides an incident edge. As $L(j) = L(j+1)$, we obtain $O(y) = O(y')$ in both cases. Similarly, we obtain $O(z) = O(z')$. If $j \geqslant m_i$, then $O(\beta_j^+) = O(\beta_{j+1}^-) = \alpha_i^+$. If $j < m_i$, then $O(\beta_j^+) = O(\beta_{j+1}^-) = \alpha_i^-$. In both cases $O(\beta_j^+) = O(\beta_{j+1}^-)$.

   Observe that $O(y) = O(\beta_j^+) = \alpha_i^-$ implies that $j < m_i$, and hence y is a subdivision vertex of an outgoing edge of $\mathrm{bot}(\alpha_i^-)$ in $G^\star$. This is, however, impossible since the only outgoing edge of $\mathrm{bot}(\alpha_i^-)$ is the stretch edge of α_i^-. Symmetrically, $O(y) = O(\beta_j^+) = \alpha_i^+$ implies $j \geqslant m_i$, and hence that y is a subdivision vertex of an incoming edge of $\mathrm{top}(\alpha_i^+)$, which is again impossible. Thus we find that $O(\beta_j^+) \neq O(y)$. Likewise, it is $O(\beta_j^+) \neq O(z)$.

   If $O(y) \neq O(z)$, then the vertices $O(\beta_j^+) = O(\beta_j^-)$, $O(y) = O(y')$ and $O(z) = O(z')$ are pairwise distinct. Thus O maps the triples $t_1 = \beta_j^+ yz$ and $t_2 = \beta_{j+1}^- y'z'$ to the same triple t. Since $\varphi^+(t_i) \equiv \varphi(t)$ for $i = 1, 2$, it follows that $\varphi^+(t_1) \equiv \varphi^+(t_2)$, i.e., φ^+ satisfies Eq. (6).

   If $O(y) = O(z)$, then $O(y') = O(z')$, and $(y, y'), (z, z')$ originate from edges $e, f \in G$. Now, if $j < m_i$, then e, f connect vertices on level $i-1$ to vertices on level i. In this case, $\varphi^+(\beta_j^+ ef)$ and $\varphi^+(\beta_j^- ef)$ are both defined in terms of $\psi(\varepsilon, e, f)$, where $\varepsilon =$

$(\alpha_{i-1}^+, \alpha_i^-)$. If $j \geqslant m_i$, then e, f connect vertices on levels i and $i+1$. Then φ^+ of both triples is defined as $\psi(\varepsilon, e, f)$ for $\varepsilon = (\alpha_i^+, \alpha_{i+1}^-)$. In both case Eq. (6) is satisfied.

2. $i = L(j) < L(j+1) = i+1$. In this case $\beta_j^+ = \mathrm{top}(\alpha_i^+)$ and $\beta_{j+1}^- = \mathrm{bot}(\alpha_{i+1}^-)$. Let $(x, x') \in G^+$ be any edge between level j and level $j+1$. Since $L(j) \neq L(j+1)$, $e = (O(x), O(x'))$ is an edge of G. Further, we obtain

$$\begin{aligned}
(x, x') \in E_j(G^+) &\Leftrightarrow e \in E_i \\
(x, x') \in E_j^+(G^+) &\Leftrightarrow x = \beta_j^+ \quad \Leftrightarrow O(x) = \alpha_i^+ \quad \Leftrightarrow e \in E_i^+ \\
(x, x') \in E_j^-(G^+) &\Leftrightarrow x' = \beta_{j+1}^- \Leftrightarrow O(x') = \alpha_{i+1}^- \Leftrightarrow e \in E_i^- .
\end{aligned}$$

Let $(y, y'), (z, z')$ be distinct edges between levels j and $j+1$ in G^+ that are different from $(\beta_j^+, \beta_{j+1}^-)$. We have that $e = (O(y), O(y'))$ and $f = (O(z), O(z'))$ are edges of G. We further distinguish cases based on whether e and f are independent.

(a) e and f are independent. If $e, f \in E_i$, then $(y, y'), (z, z') \in E_i(G^+)$ and, by the definition of φ^+, it is $\varphi^+(\beta_j^+ yz) \equiv \varphi(\alpha_i^+ O(y) O(z))$ and $\varphi^+(\beta_{j+1}^- y'z') \equiv \varphi(\alpha_{i+1}^- O(y') O(z'))$. Since φ satisfies $\mathcal{P}_G(\alpha_i^+, \alpha_{i+1}^-)$, it follows that φ^+ satisfies Eq. (6) for G^+ and edges $(y, y'), (z, z')$.

If $e \in E_i^+$ and $f \in E_i^-$, then $(y, y') \in E_j^+(G^+)$ and $(z, z') \in E_i^-(G^+)$. By the definition of φ^+, it is $\varphi^+(l(y, y')) \equiv \varphi(l(\alpha_i^+ O(y')))$ and $\varphi^+(l(z, z')) \equiv \varphi(l(O(z)\alpha_{i+1}^-))$. Since φ satisfies $\mathcal{P}_G(\varepsilon_i)$, it follows that φ^+ satisfies Eq. (7) for G^+ and edges $(y, y'), (z, z')$.

If $e \in E_i^+$ and $f \in E_i$, then $(y, y') \in E_i^+(G^+)$ and $(z, z') \in E_i(G^+)$. By the definition of φ^+ it is $\varphi^+(l(\beta_j^+, y')) \equiv \varphi(l(\alpha_i^+, O(y')))$ and $\varphi^+(\beta_{j+1}^- z'y') \equiv \varphi(\alpha_{i+1}^- O(z') O(y'))$. Since φ satisfies $\mathcal{P}_G(\varepsilon_i)$, it follows that φ^+ satisfies Eq. (8) for G^+ and edges $(y, y'), (z, z')$. The case $e \in E_i^-$ and $f \in E_i$ can be argued analogously.

(b) e, f are dependent. If $O(y) = O(z)$, then y, z are subdivision vertices of two edges e', f' in $G^\star$ that share their source, which hence lies on a level strictly below j. In particular, it is $y, z \neq \mathrm{top}(\alpha_i^+) = \beta_j^+$. Moreover, it is $O(y') \neq O(z')$. Thus, by definition, $\varphi^+(\beta_j^+ yz) \equiv \varphi^+(\beta_{j+1}^- y'z') \equiv \psi(\varepsilon_i, e, f)$, where $\varepsilon_i = (\alpha_i^+, \alpha_{i+1}^-)$.

If the vertices $O(\beta_j^+) = \alpha_{i+1}^-$, $O(y')$, $O(z')$ are pairwise distinct, then we have $(y, y'), (z, z') \in E_i(G^+)$. This implies $\varphi^+(\beta_j^+ yz) \equiv \varphi(\alpha_{i+1}^- O(y') O(z'))$ and $\varphi^+(\beta_{j+1}^- y'z') \equiv \varphi(\alpha_{i+1}^- O(y') O(z'))$. Therefore φ^+ satisfies Eq. (6) for G^+ and edges $(y, y'), (z, z')$.

If $\alpha_{i+1}^- = O(y') \neq O(z')$, then $(y, y') \in E_i^-(G^+)$ and $(z, z') \in E_i(G^+)$. Then $\varphi^+(\beta_j^+ yz) \equiv \varphi(l(e))$ and $\varphi^+(\beta_{j+1}^- y'z') \equiv \varphi(l(e))$. Therefore φ^+ satisfies Eq. (8) for G^+ and edges $(y, y)', (z, z')$.

Analogously, we obtain for $\alpha_{i+1}^- = O(z') \neq O(y')$ that φ^+ satisfies Eq. (8) for G^+ and edges (y, y') and (z, z').

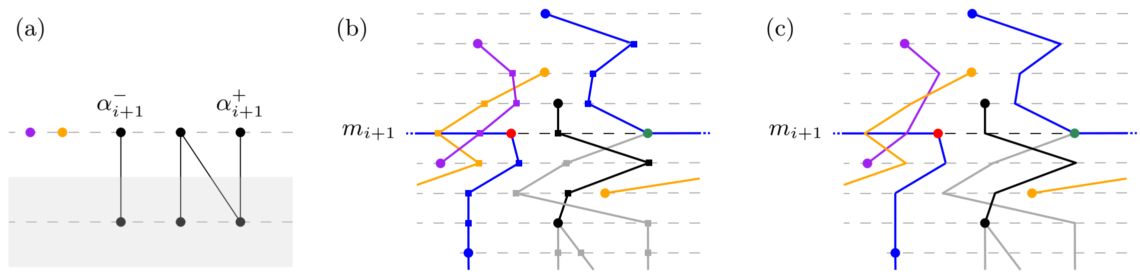


Figure 10: Let φ^+ be the truth assignment induced by the drawing in (a) for $\mathcal{S}(G^+, B^+, B^-)$. We only consider level i+1. (b) A drawing Γ^+ of G^+ for level $i+1$ corresponding to φ^+ and (c) the induced Hanani-Tutte drawing of $G^\star$. Curve γ is drawn in blue.

Finally, the case $O(y') = O(z')$ can be handled analogously to the case $O(y) = O(z)$.

This concludes the proof that φ^+ satisfies $\mathcal{S}(G^+, B^+, B^-)$. $\square$

4.4 From a Satisfying Assignment to a Hanani-Tutte Drawing

Let G be a proper level graph with reference sets A^+, A^- and let B^+, B^- be corresponding reference sets for G^+. Finally, let φ^+ be a satisfying assignment for $\mathcal{S}(G^+, B^+, B^-)$. We construct a radial drawing Γ^+ of G^+, from which we obtain the drawing $\Gamma^\star$ of $G^\star$ by smoothing the subdivision vertices. Afterwards we show that $\Gamma^\star$ is a Hanani-Tutte drawing.

We construct Γ^+ as follows. Consider a level j of G^+ and let $i = L(j)$ be the original level of G. First assume $j = m_i$. If $\beta^-_j = \beta^+_j$, then we place all vertices of $V_j(G^+)$ in arbitrary order. Otherwise, we place β^-_j, β^+_j arbitrarily on the circle representing the level m_i. We then place each vertex $v \in V_j(G^+) \setminus \{\beta^-_j, \beta^+_j\}$ such that β^-_j, v, β^+_j are ordered clockwise if and only if $\varphi(\beta^-_j v \beta^+_j) \equiv 0$, i.e., we place v on the correct side of β^-_j and β^+_j and arrange the vertices on both sides of β^-_j and β^+_j arbitrarily; see Figure 10.

Next assume $j \neq m_i$. Then there is exactly one vertex $\xi \in V_j(G^+) \cap V(G^\star)$. If $\xi \in B^-$, then we place all vertices of $V_j(G^+)$ in arbitrary order on the circle representing the level j. Otherwise, we place β^-_j and ξ arbitrarily. We then place any vertex $v \in V_j(G^+) \setminus \{\beta^-_j, \xi\}$ such that β^-_j, ξ, v are ordered clockwise if and only if $\varphi^+(\beta^-_j \xi v) \equiv 0$. Again, we arrange the vertices on either side of β^-_j and ξ arbitrarily. We have now fixed the positions of all vertices and it remains to draw the edges.

Consider two consecutive levels j and $j+1$ of G^+. We draw the edges in $E_j(G^+)$ such that they do not cross the reference edges in $E(G^+) \cap (B^+ \times B^-)$. We draw an edge $e = (\beta^+_j, x') \in E^+_j(G^+)$ such that it is locally left of (β^+_j, β^-_j) if and only if $\varphi^+(l(e)) \equiv 0$. By smoothing the subdivision vertices of the edges in G^+ we obtain $G^\star$ and along with that we obtain a drawing $\Gamma^\star$ of $G^\star$ from Γ^+.

Let a, b, c be curves or edges in a drawing. Then we write $\mathrm{cr}(a, b)$ for the number of crossings between a, b and set $\mathrm{cr}(a, b, c) = \mathrm{cr}(a, b) + \mathrm{cr}(a, c) + \mathrm{cr}(b, c)$. We consider any number of crossings modulo 2. The following lemma is the radial equivalent to Lemma 2 and constitutes our main tool for showing that edges in our drawing cross evenly. Note that cyclic orderings consider triples of points instead of pairs and that one can have e.g. $\operatorname{cr}(a, b) \equiv 1$ while $\operatorname{cr}(a, b, c) \equiv 0$.

Lemma 12. *Let C_1, C_2 be distinct concentric circles and let a, b, c be radially monotone curves from C_1 to C_2 with pairwise distinct start- and endpoints that only intersect at a finite number of points. Then the start- and endpoints of a, b, c have the same ordering on C_1 and C_2 if and only if $\operatorname{cr}(a, b, c) \equiv 0$.*

Proof. If there are no crossings, then both sides hold. Hence, assume there is at least one crossing. By perturbations, we achieve that every crossing has a distinct distance to the center m of the circles. We order the crossings by distance to m. We add a concentric circle C_X^Y between any two consecutive crossings X, Y such that C_X^Y intersects each of a, b, c once.

Then, the ordering of the intersection points of a, b, c must change between every two consecutive circles. Thus, that ordering is the same in C_1 and C_2, if and only if we added an odd number of circles. This in turn holds if and only if we have an even number of crossings. □

Lemma 13. *The drawing $\Gamma^\star$ is a Hanani-Tutte drawing of $G^\star$.*

Proof. We show that each pair of independent edges of $G^\star$ crosses evenly in $\Gamma^\star$. Of course it suffices to consider critical pairs of edges, since our drawing is radial by construction, and therefore non-critical independent edge pairs cannot cross. Every edge $(\alpha_i^+, \alpha_{i+1}^-)$ is subdivided into edges of the form $(\beta_j^+, \beta_{j+1}^-)$ and therefore it is not crossed.

Let e, f be two independent edges in $E(G^\star) \setminus (A^+ \times A^-)$ that are critical. Let a and b be the innermost and outermost level shared by e and f. We seek to use Lemma 12 to analyze the parity of the crossings between e and f. To this end, we construct a curve γ along the edges of the form $(\beta_j^+, \beta_{j+1}^-)$ as follows. For every level j we add a curve c_j going counterclockwise from β_j^- to β_j^+ on the circle representing the level j (a point for $\beta_j^- = \beta_j^+$; chosen arbitrarily otherwise). The curve γ is the union of these curves c_j and the curves for the edges of the form $(\beta_j^+, \beta_{j+1}^-)$; see Figure 10. Note that γ spans from the innermost level 1 to the outermost level $(k, 2n_k + 1)$ with endpoints $\operatorname{bot}(\alpha_1^+)$ and $\operatorname{top}(\alpha_k^-)$.

For any edge $g \in G^\star$, we denote its curve in $\Gamma^\star$ by $c(g)$. For any radial monotone curve c we denote its subcurve between level i and level j by c_i^j (using only one point on circle i and circle j each). We consider the three curves $g' = \gamma_a^b, e' = c(e)_a^b, f' = c(f)_a^b$. We now distinguish cases based on whether one of the edges e, f starts at the bottom end or ends at the top end of the reference edges on level a or b.

1. $e_a, f_a \neq \beta_a^+$ and $e_b, f_b \neq \beta_b^-$. Then for $a \leqslant j \leqslant b-1$:

$$
\begin{aligned}
\varphi^+(\beta_{j+1}^+ e_{j+1} f_{j+1}) &\overset{(11)}{\equiv} \varphi^+(\beta_{j+1}^- e_{j+1} f_{j+1}) + \varphi^+(\beta_{j+1}^- e_{j+1} \beta_{j+1}^+) + \varphi^+(\beta_{j+1}^- f_j \beta_{j+1}^+) \\
&\equiv \varphi^+(\beta_{j+1}^- e_{j+1} f_{j+1}) \quad + \operatorname{cr}(e, c_{j+1}) + \operatorname{cr}(f, c_{j+1}) \\
&\overset{(6)}{\equiv} \varphi^+(\beta_j^+ e_j f_j) \qquad\quad + \operatorname{cr}(e, c_{j+1}) + \operatorname{cr}(f, c_{j+1})
\end{aligned}
$$

Note that the second equivalence follows from the definition of c_{j+1}. This implies

$$\varphi^+(\beta_a^+ e_a f_a) + \varphi^+(\beta_b^- e_b f_b) \equiv \sum_{j=a}^{b-1} \mathrm{cr}(c_{j+1}, e) + \sum_{j=a}^{b-1} \mathrm{cr}(c_{j+1}, f) \equiv \mathrm{cr}(e, \gamma) + \mathrm{cr}(f, \gamma)$$

where the third equation holds since edges of the form $(\beta_j^+, \beta_{j+1}^-)$ are not crossed. On the other hand, by Lemma 12 we have $\varphi^+(\beta_a^+ e_a f_a) + \varphi^+(\beta_b^- e_b f_b) \equiv \mathrm{cr}(e, f, \gamma)$ since β_a^+, e_a, f_a and β_b^-, e_b, f_b are the endpoints of g', e', f'. This yields $\mathrm{cr}(e, f) \equiv \mathrm{cr}(e, f, \gamma) + \mathrm{cr}(e, \gamma) + \mathrm{cr}(f, \gamma) \equiv 0$.

2. We do not have $e_a, f_a \neq \beta_a^+$ and $e_b, f_b \neq \beta_b^-$. For example, assume $e_a = \beta_a^+$; the other cases work analogously. Then $\beta_a^+ = \mathrm{top}(\alpha_i^+)$. This means e originates from an edge in G. Since such edges do not cross middle levels, g' is a subcurve of an original edge ε_i. In particular, there are only three vertices per level between a and b that correspond to γ, e, f.

   Let $H \subseteq G^+$ be the subgraph induced by the vertices of $(\varepsilon_i)_a^b$, e_a^b, f_a^b. Then φ^+ satisfies all the constraints of $\mathcal{S}(H, V((\varepsilon_i)_a^b), V((\varepsilon_i)_a^b))$. However, each level of H contains only three vertices, and therefore the transitivity constraints are trivially satisfied, i.e., φ^+ satisfies all the constraints of $\mathcal{S}'(H, V((\varepsilon_i)_a^b), V((\varepsilon_i)_a^b))$. Thus, by Theorem 8, a drawing Γ_H of H according to φ^+ is planar. I.e., $\mathrm{cr}((\varepsilon_i)_a^b, e_a^b, f_a^b) = 0$ with regards to Γ_H. Let C_a, C_b be ε-close circles to levels a and b, respectively, that lie between levels a and b. With Lemma 12 we obtain that ε_i, e, f intersect C_a and C_b in the same order.

   Note that Γ^+ is drawn according to φ^+ on level a and on level b. We obtain that the curves for ε_i, e, f intersect C_a in the same order in Γ^+ and in Γ_H. The same holds for C_b. Hence, the curves intersect C_a and C_b in the same order in Γ^+. With Lemma 12 $\mathrm{cr}((\varepsilon_i)_a^b, e_a^b, f_a^b) \equiv 0$ with regards to Γ^+. Since γ is a subcurve of ε_i and thus not crossed in Γ^+, this yields $\mathrm{cr}(e_a^b, f_a^b) \equiv 0$ with regards to Γ^+.

Thus any two independent edges have an even number of crossings. □

4.5 From a Hanani-Tutte Drawing to a Satisfying Assignment

As in the level-planar case the converse also holds.

Lemma 14. *Let G be a level graph with reference sets A^+, A^- for $\overline{G}$. If G admits a Hanani-Tutte drawing, then there exists a satisfying assignment φ of $\mathcal{S}(\overline{G}, A^+, A^-)$.*

For the proof of the lemma we first construct the assignment φ and then show that it satisfies $\mathcal{S}(\overline{G}, A^+, A^-)$. Let $G = (V, E)$ have k levels and a radial Hanani-Tutte drawing Γ. For three distinct vertices x, y, z on level j with $x = \alpha_j^-$ or $x = \alpha_j^+$, we set $\psi(xyz) = 0$ if and only if x, y, z appear clockwise on the circle representing level j. If two edges e, f are adjacent in a vertex v with $\mathrm{cr}(e, f) \equiv 1$, then we say they have a *phantom crossing* at v. We denote by $\mathrm{cr}^\star$ the function counting crossings and additionally adding 1 if there is a phantom crossing. With the phantom crossings, any two edges in G cross an even number of times, even if they are not independent. For any edge $e = uv$ in $\overline{G}$ between level j and $j+1$ with $u = \alpha_j^+$ or $v = \alpha_{j+1}^-$ we set $\psi(l(e)) = 0$ if and only if e is locally left of ε_j. We further set $\varphi(l(e)) \equiv \psi(l(e)) + \mathrm{cr}(e, \varepsilon_j)$ to switch that value in case of a phantom crossing.

Let $v \in V(\overline{G})$ be a vertex. If v is a subdivision vertex of an edge e, then we set $e(v) = e$. Otherwise we set $e(v) = \emptyset$. We say $\emptyset$ has no crossings with anything but stretches over all levels. This helps to avoid case distinctions. For an edge $e = (u, v)$ of G we write e^j for the subdivision path of e that starts at u and ends on level j. We set $\emptyset^j = \emptyset$. Let $x, y, z \in V_j(\overline{G})$ be disjoint with $x = \alpha_j^-$ or $x = \alpha_j^+$. We set

$$\varphi(xyz) \equiv \psi(xyz) + \mathrm{cr}^\star(e(x)^j, e(y)^j, e(z)^j)\,.$$

We thereby switch the ordering of x, y, z if and only if at least two of them are subdivision vertices and the corresponding edges cross an odd number of times up to level j. This finishes the construction of φ.

Lemma 15. *The assignment φ satisfies $\mathcal{S}(\overline{G}, A^+, A^-)$.*

Proof. First note that ψ satisfies $\mathcal{L}(\alpha_j^+), \mathcal{L}(\alpha_j^-)$ and $\mathcal{C}(\alpha_j^+, \alpha_j^-)$ for $1 \leqslant j \leqslant k$.

$\boldsymbol{\varphi}$ satisfies $\boldsymbol{\mathcal{L}(\alpha_j^+), \mathcal{L}(\alpha_j^-)}$. For three distinct vertices $x, y, z \in V_j(\overline{G})$, by definition of φ, $\varphi(xyz) + \varphi(xzy) \equiv \psi(xyz) + \psi(xzy)$. Since ψ satisfies Eq. (4), so does φ.

$\boldsymbol{\varphi}$ satisfies $\boldsymbol{\mathcal{C}(\alpha_j^+, \alpha_j^-)}$. For Eq. (11) let $1 \leqslant j \leqslant k$ such that $\alpha_j^- \neq \alpha_j^+$. Let $u, v \in V_j(\overline{G}) \setminus \{\alpha_j^-, \alpha_j^+\}$ be distinct. By definition of φ:

$$\begin{aligned}
&(\varphi(\alpha_j^- uv) + \varphi(\alpha_j^+ uv)) + (\varphi(\alpha_j^- u\alpha_j^+) + \varphi(\alpha_j^- v\alpha_j^+))\\
\equiv\ &(\psi(\alpha_j^- uv) + \psi(\alpha_j^+ uv)) + (\psi(\alpha_j^- u\alpha_j^+) + \psi(\alpha_j^- v\alpha_j^+))\\
&+ 2\big(\mathrm{cr}^\star(e(\alpha_j^-), u) + \mathrm{cr}^\star(e(\alpha_j^-), v) + \mathrm{cr}^\star(e(\alpha_j^+), u) + \mathrm{cr}^\star(e(\alpha_j^+), v)\\
&+ \mathrm{cr}^\star(u, v) + \mathrm{cr}^\star(e(\alpha_j^-), e(\alpha_j^+))\big)\\
\equiv\ &(\psi(\alpha_j^- uv) + \psi(\alpha_j^+ uv)) + (\psi(\alpha_j^- u\alpha_j^+) + \psi(\alpha_j^- v\alpha_j^+))
\end{aligned}$$

Thereby Eq. (11) holds.

For Eq. (10) let $u \in V_j(\overline{G}) \setminus \{\alpha_j^+, \alpha_j^-\}$. With the definition of φ we obtain

$$\varphi(\alpha_j^- u\alpha_j^+) + \varphi(\alpha_j^+\alpha_j^- u) \equiv \psi(\alpha_j^- u\alpha_j^+) + \psi(\alpha_j^+\alpha_j^- u)\,.$$

Thereby Eq. (10) holds.
It remains to argue that φ satisfies $\mathcal{P}(\varepsilon_j)$ for all levels.

$\boldsymbol{\varphi}$ satisfies Eq. (6) **of $\boldsymbol{\mathcal{P}(\varepsilon_j)}$.** Let $1 \leqslant j \leqslant k-1$ and let $(u, u'), (v, v') \in E_j(\overline{G})$ be independent. Then we argue that

$$\begin{aligned}
&\varphi(\alpha_j^+ uv) + \varphi(\alpha_{j+1}^- u'v')\\
\equiv\ &\psi(\alpha_j^+ uv) + \psi(\alpha_{j+1}^- u'v') + \mathrm{cr}^\star((u, u'), (v, v'), \varepsilon_j)
\end{aligned} \tag{12}$$

This means we change the ordering of the ends of $(u,u'),(v,v'),\varepsilon_j$ on exactly one of the levels j and $j+1$ if and only if they cross an odd number of times. With Lemma 12 we then obtain that Eq. (6) holds for $(u,u'),(v,v')$.

With the definition of φ it suffices to show the following three equations

$$\operatorname{cr}^\star(e(\alpha^-_{j+1})^{j+1}, e(u')^{j+1}) + \operatorname{cr}^\star(e(\alpha^+_j)^j, e(u)^j) \equiv \operatorname{cr}^\star(\varepsilon_j,(u,u')) \tag{13}$$
$$\operatorname{cr}^\star(e(\alpha^-_{j+1})^{j+1}, e(v')^{j+1}) + \operatorname{cr}^\star(e(\alpha^+_j)^j, e(v)^j) \equiv \operatorname{cr}^\star(\varepsilon_j,(v,v')) \tag{14}$$
$$\operatorname{cr}^\star(e(u')^{j+1}, e(v')^{j+1}) \quad + \operatorname{cr}^\star(e(u)^j, e(v)^j) \quad \equiv \operatorname{cr}^\star((u,u'),(v,v')) \tag{15}$$

We show Eq. (15). Eq. (13) and Eq. (14) can be shown analogously noting that $\varepsilon_j = (\alpha^+_j, \alpha^-_{j+1})$ is independent from $(u,u'),(v,v')$. We distinguish cases where u, u', v, v' are vertices of different kinds.

1. u, u' or v, v' are original vertices. Then $e(u) = e(u') = \emptyset$ or $e(v) = e(v') = \emptyset$. Then the left side equals 0. The right side also equals 0, since (u,u') or (v,v') is an edge of $\overline{G}$.

2. u, u' or v, v' are subdivision vertices. Without loss of generality assume u, u' are subdivision vertices. Then we obtain $e(u) = e(u')$ and the left side reduces to $\operatorname{cr}^\star(e(u)^{j+1}, e(v')^{j+1}) + \operatorname{cr}^\star(e(u)^j, e(v)^j)$.

   (a) The case where v, v' are original vertices is already covered.

   (b) If v, v' are subdivision vertices, then $\operatorname{cr}^\star(e(u)^{j+1}, e(v')^{j+1})+ \operatorname{cr}^\star(e(u)^j, e(v)^j)$ $\equiv \operatorname{cr}^\star(e(u)^{j+1}, e(v)^{j+1}) + \operatorname{cr}^\star(e(u)^j, e(v)^j) \equiv \operatorname{cr}^\star(e(u)^{j+1}_j, e(v)^{j+1}_j)$ $\equiv \operatorname{cr}^\star((u,u'),(v,v'))$

   (c) If v is an original vertex and v' is a subdivision vertex, we get $\operatorname{cr}^\star(e(u)^{j+1}, e(v')^{j+1}) + \operatorname{cr}^\star(e(u)^j, e(v)^j) \equiv \operatorname{cr}^\star(e(u)^{j+1}, vv') + 0 \equiv \operatorname{cr}^\star(uu', vv')$

   (d) If v is a subdivision vertex and v' is an original vertex, then $\operatorname{cr}^\star(e(u')^{j+1}, e(v')^{j+1}) + \operatorname{cr}^\star(e(u)^j, e(v)^j) \equiv \operatorname{cr}^\star(e(u)^j, e(v)^j) \equiv \operatorname{cr}^\star(e(u), e(v)) + \operatorname{cr}^\star(e(u)^j, e(v)^j) \equiv \operatorname{cr}^\star((u,u'),(v,v'))$. Note that $e(u)$, $e(v)$ are edges of G.

3. u, v are original vertices and u', v' are subdivision vertices. Then $e(u) = e(v) = \emptyset$ and $e(u')^{j+1} = uu'$ and $e(v')^{j+1} = vv'$ and the equivalence holds.

4. u', v' are original vertices and u, v are subdivision vertices. Noting $\operatorname{cr}^\star(e(u)^j, e(v)^j)+ \operatorname{cr}^\star(e(u)^k_j, e(v)^k_j) = \operatorname{cr}^\star(e(u), e(v)) \equiv 0$ we can argue analogously.

5. u, v' are original vertices and u', v are subdivision vertices. Then $e(u) = e(v') = \emptyset$ and the left side equals 0. Further, $\operatorname{cr}^\star((u,u'),(v,v')) = \operatorname{cr}^\star(e(u'), e(v)) \equiv 0$. I.e., both sides are even.

6. u', v are original vertices and u, v' are subdivision vertices. Then we can argue analogously to the previous case.

Hence, φ satisfies Eq. (6).

φ satisfies Eq. (7) **of $\mathcal{P}(\varepsilon_j)$.** Let $e = (u, u') \in E_j^+(\overline{G})$ and let $f = (v, v') \in E_j^-(\overline{G})$. Then we adjust the drawing by perturbing possible phantom crossings of e, f with ε_j at α_j^+ and α_{j+1}^- to the space between circle j and circle $j + 1$. Thereby, the states of e, f being locally left of ε_j change if and only if they have a phantom crossing. This is the case if and only if ψ and φ differ for l of the corresponding edge. I.e., $\varphi(l(e))$, $\varphi(l(f))$ describe, whether e, f are locally left of ε_j. Consider the closed curve c that is the union of $c(e), c(\varepsilon_j)$ and the part d of circle $j + 1$ between u' and α_{j+1}^-, such that circle j is on the outside of c. Then the edge f is outside of c at circle j. Note that ε_j has to be an original edge. Since f crosses e, ε_j an even number of times each (as their corresponding original edges cross only between level j and $j+1$), and it does not cross d, edge f has to be outside of c at ε-close distance to circle $j+1$. We thereby obtain $\varphi(l(e)) \equiv \varphi(l(f)) + 1$.

φ satisfies Eq. (8) **of $\mathcal{P}(\varepsilon_j)$.** Let $(u, u') \in E_j^+$ and let $(v, v') \in E_j$ be independent. Note that $e(\alpha_{j+1}^-)^{j+1} = \varepsilon_j$ and that $e(u')^{j+1} = (u, u')$ since $\alpha_j^+ = u$ must be an original vertex. Let circle a be a circle that is ε-close outside of circle j. Then if v' is a subdivision vertex:

$$\begin{aligned}
&\varphi(\alpha_{j+1}^- v'u') - \psi(\alpha_{j+1}^- v'u') \\
&\equiv \mathrm{cr}^\star(e(v')^{j+1}, e(u')^{j+1}) + \mathrm{cr}^\star(e(\alpha_{j+1}^-)^{j+1}, e(v')) + \mathrm{cr}^\star(e(\alpha_{j+1}^-)^{j+1}, e(u')) \\
&\equiv \mathrm{cr}^\star((v, v'), (u, u')) + \mathrm{cr}^\star(\varepsilon_j, (v, v')) + \mathrm{cr}^\star(\varepsilon_j, (u, u')) \\
&\equiv \mathrm{cr}^\star(\varepsilon_j, (v, v'), (u, u')) \\
&\equiv \mathrm{cr}(\varepsilon_j, (v, v'), (u, u')) + \mathrm{cr}^\star((u, u')_a, (v, v')_a, (\varepsilon_j)_a) \\
&\equiv (\psi(l(u, u')) - \psi(\alpha_{j+1}^- v'u')) + \mathrm{cr}^\star((u, u')_a, (v, v')_a, (\varepsilon_j)_a) \\
&\equiv \varphi(l(u, u')) - \psi(\alpha_{j+1}^- v'u')\,.
\end{aligned}$$

If v' is an original vertex:

$$\begin{aligned}
&\varphi(\alpha_{j+1}^- v'u') - \psi(\alpha_{j+1}^- v'u') \\
&\equiv \mathrm{cr}^\star(e(v')^{j+1}, e(u')^{j+1}) + \mathrm{cr}^\star(e(\alpha_{j+1}^-)^{j+1}, e(v')) + \mathrm{cr}^\star(e(\alpha_{j+1}^-)^{j+1}, e(u')) \\
&\equiv \mathrm{cr}^\star(\varepsilon_j, (u, u')) \\
&\equiv \mathrm{cr}(\varepsilon_j, (u, u')) + \mathrm{cr}^\star((u, u')_a, (\varepsilon_j)_a) \\
&\equiv (\psi(l(u, u')) - \psi(\alpha_{j+1}^- v'u')) + \mathrm{cr}^\star((u, u')_a, (\varepsilon_j)_a) \\
&\equiv \varphi(l(u, u')) - \psi(\alpha_{j+1}^- v'u')\,.
\end{aligned}$$

We obtain $\varphi(l(u, u')) = \varphi(\alpha_{j+1}^- v'u')$.

φ satisfies Eq. (9) **of $\mathcal{P}(\varepsilon_j)$.** Can be argued similarly to the previous case. □

We obtain the following implications.

Theorem 16. *Let G be a level graph and let A^+, A^- be reference sets of $\overline{G}$. Let B^+, B^- be corresponding reference sets of $\overline{G}^+$. Then*

1. *$\mathcal{S}(\overline{G}^{+}, B^+, B^-)$ is satisfiable $\Leftrightarrow \overline{G}^{\star}$ has a Hanani-Tutte radial level drawing,*

2. *G has a Hanani-Tutte radial level drawing $\Rightarrow \mathcal{S}(\overline{G}, A^+, A^-)$ is satisfiable,*

3. *$\mathcal{S}(\overline{G}, A^+, A^-)$ is satisfiable $\Rightarrow \overline{G}^{\star}$ has a Hanani-Tutte radial level drawing.*

Proof. The first statement follows by Lemmas 13, 14. The second statement is immediate from Lemma 14. For the third statement apply Lemma 10 and use the first statement. □

Note that in contrast to Theorem 6, we do not provide the reverse for the third statement. This would require showing $\mathcal{S}(\overline{G}^{+}, B^+, B^-) \Rightarrow \mathcal{S}(\overline{G}, A^+, A^-)$. We still obtain the equivalence of the two radial level-planarity characterizations as follows. Assume all level graphs with a Hanani-Tutte radial level drawing are radial level-planar. Then for every level graph G with reference sets A^+, A^- for $\overline{G}$ where $\mathcal{S}(\overline{G}, A^+, A^-)$ is satisfiable, $\overline{G}^{\star}$ has a Hanani-Tutte radial level drawing by the third statement. Thus $\overline{G}^{\star}$ is radial level-planar. It follows that $\overline{G}$ and G are also radial level-planar. On the other hand, assume each level graph G with reference sets A^+, A^- for $\overline{G}$ where $\mathcal{S}(\overline{G}, A^+, A^-)$ is satisfiable is radial level-planar. Then for every level graph G with reference sets A^+, A^- for $\overline{G}$ and a Hanani-Tutte radial level drawing, $\mathcal{S}(\overline{G}, A^+, A^-)$ is satisfiable by the second statement and thus G is radial level-planar. Note that we did not actually use the Hanani-Tutte result for the radial case [10].

Corollary 17. *The radial level-planar graphs are exactly the level graphs with a Hanani-Tutte radial level drawing if and only if they are exactly the level graphs G for which $\mathcal{S}(\overline{G}, A^+, A^-)$ is satisfiable.*

With the known Hanani-Tutte result for the radial case [10], we conclude that radial level-planarity can be characterized in the spirit of Randerath et al. [24].

Corollary 18. *Let G be a level graph with reference sets A^+, A^-. Then $\mathcal{S}(\overline{G}, A^+, A^-)$ is satisfiable if and only if G is radial level-planar.*

5 Conclusion

We have established an equivalence of two results on level-planarity that have so far been considered as independent. The novel connection has further led us to a new testing algorithm for radial level planarity. Can similar results be achieved for level-planarity on a rolling cylinder or on a torus [1]? It would be interesting to establish the equivalence of the two results for single instances. That is, it would be interesting to prove the reverse of statement two of Theorem 6 or Theorem 16 without using planarity. This would be achieved by obtaining a Hanani-Tutte drawing of G from a Hanani-Tutte drawing of $\overline{G}^{\star}$.

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