Towards Obtaining a 3-Decomposition From a Perfect Matching

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Abstract

A decomposition of a graph is a set of subgraphs whose edges partition those of G. The 3-decomposition conjecture posed by Hoffmann-Ostenhof in 2011 states that every connected cubic graph can be decomposed into a spanning tree, a 2-regular subgraph, and a matching. It has been settled for special classes of graphs, one of the first results being for Hamiltonian graphs. In the past two years several new results have been obtained, adding the classes of plane, claw-free, and 3-connected tree-width 3 graphs to the list.

In this paper, we regard a natural extension of Hamiltonian graphs: removing a Hamiltonian cycle from a cubic graph leaves a perfect matching. Conversely, removing a perfect matching M from a cubic graph G leaves a disjoint union of cycles. Contracting these cycles yields a new graph G_M. The graph G is star-like if G_M is a star for some perfect matching M, making Hamiltonian graphs star-like. We extend the technique used to prove that Hamiltonian graphs satisfy the 3-decomposition conjecture to show that 3-connected star-like graphs satisfy it as well.

Mathematics Subject Classifications: 05C05,05C70

1 Introduction

A decomposition of a graph G is a set of subgraphs such that any edge of G is contained in exactly one of them. The 3-decomposition conjecture was posed by Hoffmann-Ostenhof in [1] and also appears in BCC22 [2] as Problem 516:

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Conjecture 1. Every connected cubic graph has a decomposition consisting of a spanning tree, a 2-regular subgraph, and a matching.

Any such decomposition is called a 3-decomposition. Note that formally the last component is a subgraph whose edge set is a matching, but, to adhere to the literature, we ignore this distinction. Moreover, the matching may be empty (and a simple counting argument shows that the cycle component is not).

The conjecture was proved to be true for connected cubic graphs that are Hamiltonian by Akbari, Jensen, and Siggers [3]. In 2016, Abdolhosseini et al. [4] showed that traceable is a sufficient requirement already, a result which was also obtained by Li and Liu [5]. Ozeki and Ye [6] proved that 3-connected cubic graphs satisfy the conjecture if they are planar or on the projective plane while Bachstein [7] deals with 3-connected cubic graphs on the Torus and Klein Bottle. The former of these results was extended to all connected plane cubic graphs by Hoffmann-Ostenhof, Kaiser, and Ozeki [8] in 2018. In the same year it was also proved to hold for claw-free (sub)cubic graphs by Aboomahigir, Ahanjideh, and Akbari [9] as well as by Hong et al. [10]. More recently, Lyngsie and Merker [11] showed that weakening the matching requirement to allow for paths of length 2 suffices to make the conjecture true and Heinrich [12] verified it for 3-connected cubic graphs of tree-width 3. Earlier this year, Xie, Zhou, and Zhou [13] proved the conjecture for graphs with a two-factor consisting of three cycles.

In this paper we look at graphs that are a natural extension of Hamiltonian cubic graphs in this context. Notice that a cubic graph G with a Hamiltonian cycle C has a perfect matching, namely the edges of $G - E(C)$ where $E(C)$ denotes the edges of C. In general, for a cubic graph G with a perfect matching M, $G - M$ is the disjoint union of cycles, leading us to the following definition.

Definition 2. Let G be a connected cubic graph with a perfect matching M. Then $G - M$ consists of disjoint cycles and contracting these in G to single vertices yields a new graph G_M, the *contraction graph*, that has a vertex for every cycle in $G - M$ and an edge between two vertices if the corresponding cycles are connected by an edge of M. If G has a perfect matching M such that G_M is a star (a tree with diameter at most 2), then G is *star-like*.

We wish to make a few remarks on this definition. First note that all Hamiltonian cubic graphs are star-like and, by Petersen's theorem [14], all bridgeless cubic graphs have a perfect matching. Since many conjectures in graph theory consider or can be reduced to bridgeless cubic graphs, obtaining structural information about these is of interest. A prominent example is the cycle double cover conjecture [15], two others are the shortest cycle cover [16] and the Fan-Raspaud conjecture [17]. Also, using this definition, the main theorem in [13] now reads that the conjecture is satisfied for any connected cubic graph with a perfect matching such that its contraction graph has order 3. This extends the previous proofs for Hamiltonian and traceable graphs, which handle contraction graphs of orders 1 and 2.

Furthermore, decompositions into trees and graphs of small maximum degree are of general interest and find applications, for example, in determining upper bounds for the game chromatic number [18]. Several such decompositions are known: it is possible to decompose a planar graph into three forests such that one of them has degree at most 8 [19] and into two forests and a graph of maximum degree at most 4 [20]. By requiring the girth to be at least 8, a decomposition into a forest and a matching can be found [21].

Our contribution We prove that:

Theorem 3. *Every 3-connected star-like cubic graph has a 3-decomposition.*

The idea of the proof is to construct a tree on the vertices of the centre cycle and to iteratively extend it to the tips of the star. Once extended to all cycles it yields a 3-decomposition. To make this precise, we introduce two types of decompositions. One describes this tree and the other formalises the properties that we need to extend it to further cycles. To achieve this extension, we show that certain decompositions always exist and that these are actually sufficient. We also exhibit an infinite family of graphs that are 3-connected and star-like, and therefore covered by Theorem 3, but for which the conjecture has not yet been proved.

We note that the result in [13] is similar to ours, both in their claim as in the approach, though the final result differs. However, our approach has two main advantages: due to the definition of the decompositions we provide, it is easier to obtain reusable components (whereas the proof in [13] consists of a series of very long case distinctions in which a lot of details have been omitted). In contrast to this, our final proof is very short and a direct application of the components we proved before.

Additionally, these decompositions have straight-forward extensions that allow them to deal with more general contraction graphs and our results carry over to this more general form. This leaves hope that the decompositions we introduce here could be a helpful tool for proving the conjecture for larger classes of graphs, tree-like being a natural candidate.

Outline We introduce our two decomposition types in the next section and show how they can be combined in a consistent fashion. In Section 3 we prove that certain decompositions always exist and these are shown to be sufficient to find a 3-decomposition in 3-connected star-like graphs in Section 4. Finally, we construct the promised star-like graphs which are now covered by our result in Section 5.

2 Decompositions and their extension

The basic notation for this paper is mainly based on [22], but we briefly summarise what we need here. All graphs are finite and contain neither self-loops nor parallel edges. The vertex and edge set of a graph G are denoted by $V(G)$ and $E(G)$. For sets $X, Y \subseteq V(G)$ we write $N(X)$ for the set of neighbours of X and $E(X, Y)$ for the edges of G with one end in X and the other in Y. If $Y = V(G) \setminus X$, we shorten this to $E(X)$. We write uv for an edge with ends u and v. A path P is a sequence of distinct vertices $v_0 v_1 \dots v_k$ such that $v_{i-1}v_i \in E(G)$ for $i \in \{1, \dots, k\}$. By $v_i P v_j$ with $i \leqslant j$ we denote the subpath $v_i \dots v_j$. The notation $\mathring{v}_i P v_j$, for $i < j$, describes the subpath $v_{i+1} \dots v_j$, $v_i P \mathring{v}_j$ and $\mathring{v}_i P \mathring{v}_j$ are defined analogously, and $\mathring{P} = \mathring{v}_0 P \mathring{v}_k$.

From now on, until the end of Section 5, let

- G be a star-like cubic graph
- with perfect matching M
- and cycles $C_1, \ldots, C_l$ in $G - M$, where C_1 is the centre cycle.

For $\emptyset \neq I \subseteq \{1, \ldots, l\}$, we denote $\bigcup_{i \in I} V(C_i)$ by V_I and $G[V_I]$ by G_I, writing G_i for $G_{\{i\}}$. We write $\partial(G_I)$ for the set of degree 2 vertices in G_I and call these vertices the *boundary of* G_I. Recall that a decomposition of a graph is a set of subgraphs such that any edge is contained in exactly one of them.

As promised, we begin with the two types of decompositions we need, starting with the one describing the tree we wish to extend. Intuitively, it describes a tree T in G_I, for $1 \in I \subseteq \{1, \ldots, l\}$, that could be part of a 3-decomposition of the entire graph. The definition does this by ensuring a few necessary conditions: it requires that all vertices of degree 3 in G_I are part of the tree and that edges not in T should either be matching edges or in the set of cycles and paths one gets by restricting a collection of cycles to a subgraph. These paths need to be extended to cycles in a later step, so they must end at the boundary $\partial(G_I)$.

Definition 4. Let $1 \in I \subseteq \{1, \ldots, l\}$ and $\mathcal{D}_I = \{T_I, \mathcal{C}_I, M_I\}$ be a decomposition of G_I such that

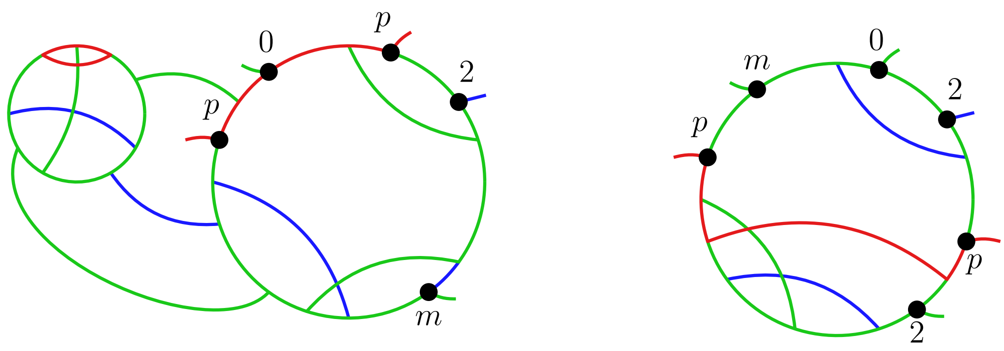
- T_I is a tree spanning all degree 3 vertices of G_I,
- $\mathcal{C}_I$ is a disjoint union of (positive length) paths and cycles in G_I, and
- M_I is a matching.

If all path components (components that are paths) of $\mathcal{C}_I$ end at vertices in the boundary $\partial(G_I)$, then $\mathcal{D}_I$ is an *I-decomposition*.

Note that for $I = \{1, \ldots, l\}$ this is just a 3-decomposition since $\mathcal{C}_I$ is no longer allowed to contain path components. We also remark that this does not describe all possible restrictions of a 3-decomposition of G to the graph G_I, we would have to allow forests for that to be true, but for the upcoming proof trees suffice.

An example of an I-decomposition is shown in Figure 1a. There the edges in T_I are coloured in green, those in $\mathcal{C}_I$ are red, and the ones in M_I are blue. This is also our colour scheme for figures throughout this paper. The edges on the boundary are not actually part of the decomposition, but they exist and their colours describe which component they should eventually end up in.

A bit of additional notation is useful at this point. We write $A_i(\mathcal{D}_I)$ for the set of vertices in $\partial(G_I)$ that have degree i in T_I, where $A_0(\mathcal{D}_I)$ denotes those that are not in T_I at all. Moreover, we split the set $A_1(\mathcal{D}_I)$ into $A_p(\mathcal{D}_I)$ and $A_m(\mathcal{D}_I)$, where the former contains those degree 1 vertices of T_I that are ends of path components of $\mathcal{C}_I$, whereas the latter contains the ends of matching edges. As the vertices in $A_1(\mathcal{D}_I)$ have degree 2 in G_I



(a) An I-decomposition with two cycles. (b) An (A_0, A_p, A_m, A_2)-decomposition.

Figure 1: Sketches of the two types of decompositions introduced in Definitions 4 and 5. Green edges should be part of the tree in the final decomposition, red ones are on cycles, and the blue ones form a matching.

and degree 1 in T_I, these are the only two possibilities. As a result, we have that $\partial(G_I)$ is the disjoint union of the four sets $A_x(\mathcal{D}_I)$ for $x \in \{0, p, m, 2\}$. The drawn vertices in Figure 1a represent the boundary and those in set $A_x(\mathcal{D}_I)$ are labelled by x.

We can now move on to the second type of decomposition we need. The next definition might seem cryptic at first glance, but it, like the previous one, is essentially just a collection of necessary conditions. Our goal is to formalise the extension of an I-decomposition $\mathcal{D}_I$ to another cycle C_i by describing a spanning forest F_i of G_i that satisfies conditions analogous to those of the tree T_I above. Once again, the remaining edges of G_i should either be part of a matching M_i or of cycles or paths in $\mathcal{C}_i$.

But it also needs to "fit together" with the I-decomposition. We notice that we do not actually need the details of $\mathcal{D}_I$, but it suffices to know the behaviour of the vertices on the boundary of G_I with an edge to C_i. Let u be a vertex in $\partial(G_I)$ with unique neighbour v in C_i, more precisely in $\partial(G_i)$. We now classify the vertices v based on the behaviour of u, placing them into the four sets $A_x = V(C_i) \cap N(A_x(\mathcal{D}_I))$ for $x \in \{0, p, m, 2\}$. Let us see what we want v to satisfy in each of these cases.

If $u \in A_2(\mathcal{D}_I)$, then uv can either be in the tree or matching part, depending on whether we need it to connect to C_i or not.

When u is in $A_p(\mathcal{D}_I)$, we need to extend the path ending at this vertex to a cycle, meaning it must continue in C_i. Hence, we require a path at v to another vertex in $\partial(G_i)$ of this type. Thus, we require that the vertices in A_p are exactly the ends of path components of $\mathcal{C}_i$. This is a necessary condition as paths must end at the boundary and all other types of vertices have conflicting behaviour.

If u is in $A_m(\mathcal{D}_I)$, then the edge uv must be part of the tree and we have to ensure that it does not create cycles. This is achieved by requiring that any component of F_i contains at most one vertex that is either in A_m or is both a leaf of F_i and in A_2. Such vertices need a tree edge to T_I, which we have already seen for those in A_m and it holds for the leaves as well: the missing edge in G_i at such a vertex must be in M_i as the ends of paths are in A_p.

Finally, u can be in $A_0(\mathcal{D}_I)$, where it needs the edge uv to be part of the tree and v must be connected to T_I in C_i. We ensure that v ends up in a component of F_i that can be connected to T_I. For this we require every component of F_i to contain an element of $A_2 \cup A_m$, which are vertices that can or need to connect to T_I. With these ideas at hand, let us give the definition.

Definition 5. Let A_0, A_p, A_m, and A_2 be disjoint subsets of $\partial(G_i)$ for some $i \in \{1, \dots, l\}$ whose union is $\partial(G_i)$. Also let $\mathcal{D}_i = \{F_i, \mathcal{C}_i, M_i\}$ be a decomposition of G_i such that

- F_i is a spanning forest in G_i,
- $\mathcal{C}_i$ is a disjoint union of (positive length) paths and cycles in G_i, and
- M_i is a matching.

The decomposition $\mathcal{D}_i$ is an (A_0, A_p, A_m, A_2)-*decomposition of* C_i if it satisfies the following two conditions.

(i) Every component of F_i contains a vertex in $A_2 \cup A_m$ and it contains at most one vertex v such that $v \in A_m$ or v is both in A_2 and a leaf of F_i.

(ii) The set of ends of path components of C_i is exactly A_p.

Figure 1b visualises such a decomposition, where vertices in A_x are labelled by x for $x \in \{0, p, m, 2\}$ and the colour scheme is analogous to before: the edges in F_i are coloured in green, those in $\mathcal{C}_i$ are red, and the ones in M_i are blue.

We can now prove that compatible decompositions can be combined.

Lemma 6. *Let* $1 \in I \subseteq \{1, \dots, l\}$, $i \notin I$, $J = I \cup \{i\}$, *and* $\mathcal{D}_I = (T_I, \mathcal{C}_I, M_I)$ *be an* I*-decomposition of* G. *If there exists an* (A_0, A_p, A_m, A_2)*-decomposition* $\mathcal{D}_i = (F_i, \mathcal{C}_i, M_i)$ *of* C_i *where*

$$A_x = N(A_x(\mathcal{D}_I)) \cap V(C_i) \text{ for } x \in \{0, p, m, 2\},$$

then G *has a* J*-decomposition* $(T_J, \mathcal{C}_J, M_J)$ *satisfying* $T_I \cup F_i \subseteq T_J$, $\mathcal{C}_I \cup \mathcal{C}_i \subseteq \mathcal{C}_J$, *and* $M_I \cup M_i \subseteq M_J$.

Proof. Let $\mathcal{D}_I$ and $\mathcal{D}_i$ be decompositions as in the claim. In order to get a decomposition $\mathcal{D}_J$ as desired, we need to assign the edges in $E(V(G_I), V(C_i))$ to the graphs $T_I \cup F_i$, $\mathcal{C}_I \cup \mathcal{C}_i$, and the set $M_I \cup M_i$. Note that, by definition of the sets A_x,

$$E(V(G_I), V(C_i)) = E(\partial(G_I), V(C_i)) = \bigcup_x E(A_x(\mathcal{D}_I), A_x) \text{ where } x \in \{0, p, m, 2\}.$$

We add the set $E(A_p(\mathcal{D}_I), A_p)$ to $\mathcal{C}_I \cup \mathcal{C}_i$ to get $\mathcal{C}_J$. The sets $E(A_m(\mathcal{D}_I), A_m)$ and $E(A_0(\mathcal{D}_I), A_0)$ are both added to $T_I \cup F_i$. Additionally, for any component K of F_i that contains a vertex of A_2 but none of A_m, we pick a vertex in $A_2 \cap V(K)$ of least degree in K and add the edge incident to it with an end in $A_2(\mathcal{D}_I)$ to the tree part as well. (Such vertices exist by Condition (i) of Definition 5 and the minimality just means that we choose a leaf in case one is present.) This yields T_J. The remaining edges of $E(A_2(\mathcal{D}_I), A_2)$ are added to $M_I \cup M_i$ to get M_J.

We claim that $\mathcal{D}_J = (T_J, \mathcal{C}_J, M_J)$ is a desired J-decomposition of G. The set $\mathcal{C}_J$ is the union of two disjoint graphs $\mathcal{C}_I$ and $\mathcal{C}_i$, both of which consist of paths and cycles, together with edges $E(A_p(\mathcal{D}_I), A_p)$ connecting degree-1 vertices of these subgraphs. Hence it, too, is a disjoint union of paths and cycles as required. Furthermore, a degree-1 vertex in $\mathcal{C}_J$ must have degree 1 in $\mathcal{C}_I$ or $\mathcal{C}_i$. In the first case it is an element of $\partial(G_I)$ by definition of an I-decomposition and it cannot be part of $N(C_i)$ without increasing its degree when we add the edges in $E(A_p(\mathcal{D}_I), A_p)$. So it is in $\partial(G_J)$ as desired. The second case does not occur as vertices of degree 1 in $\mathcal{C}_i$ are in A_p and have degree 2 in $\mathcal{C}_J$.

The set M_J is also a matching as it is the union of two matchings M_I, M_i in disjoint subgraphs and the additional edges are part of $E(A_2(\mathcal{D}_I), A_2)$, meaning their ends in G_I have degree 2 in $T_I \subseteq T_J$. Their ends in C_i also have degree 2 in $F_i \subseteq T_J$ as a lower degree makes them a leaf or an isolated vertex of F_i. In the first case the component containing that vertex cannot contain a vertex in A_m and the leaf is unique, meaning the edge is added to T_J by our construction. The second case is faced with a component having a unique edge to G_I, which is also added to T_J.

This just leaves T_J. Let $\mathcal{K}$ be the set of components of F_i and let F be the union of T_I with the components in $\mathcal{K}$. By adding the edges of $E(A_m(\mathcal{D}), A_m)$ to F we have connected all components $K \in \mathcal{K}$ that contain a vertex of A_m to T_I by exactly one edge each. The result is a new forest F' consisting of a tree $T' \supseteq T_I$ and remaining components $\mathcal{K}' \subseteq \mathcal{K}$ that have no vertex in A_m. By adding our chosen elements of $E(A_2(\mathcal{D}_I), A_2)$ we connect the components of $\mathcal{K}'$ (as these contain an element of A_2) to T' by exactly one edge. This results in a tree T''. Finally, the edges in $E(A_0(\mathcal{D}_I), A_0)$ connect vertices of G_I that are not in T'' to it by a single edge, creating the tree T_J.

Now we only need to check that T_J spans all vertices of degree 3 in G_J. To this end regard a vertex of G_J that is not part of T_J. It cannot be in C_i as all the components of F_i are part of T_J and F_i was spanning. A vertex in G_I that is not part of T_J is also not in T_I, putting it in $A_0(\mathcal{D}_I)$. But such vertices still have degree 2 in G_J as they cannot be in $A_0(\mathcal{D}_I) \cap N(C_i)$ because the degree of such a vertex is 1 now. □

3 Finding decompositions in cycles

In this section, let C_i be some cycle in $G - M$ (as defined in Section 2) and A_0, A_p, A_m, and A_2 be disjoint subsets of $\partial(G_i)$ for which we want to find an (A_0, A_p, A_m, A_2)-decomposition. We need four different types of decompositions in order to handle all cases that occur when piecing them together.

Before we start, a bit more notation will come in handy, that we now introduce. For a chord $e \in G_i$ of C_i we obtain two paths in C_i between its ends which, together with the chord, yield two cycles, say C_i' and C_i''. We call $C \in \{C_i', C_i''\}$ *minimal* if it is a chordless cycle in G. The unique edge in $M \cap E(C)$ of a minimal cycle C is denoted by e_C and we write P_C for the path $C - e_C$. Note that C_i has a minimal cycle avoiding any specific vertex in $\partial(G_i)$ if it has a chord: take a chord e with cycles $P_1 + e$ and $P_2 + e$.

By choosing v_j, w_j as ends of an edge in M of minimal distance in P_j, for $j \in \{1, 2\}$, we find two minimal cycles $v_j P_j w_j v_j$ that meet disjoint sets of vertices of $\partial(G_i)$. Note that the vertices v_j, w_j always exist as the ends of e are candidates. The cycles are minimal as a chord of $v_j P_j w_j v_j$ must be an edge of M whose ends have smaller distance.

A useful construction that we apply regularly is the following. Let C be a minimal cycle of C_i that does not contain some vertex $x \in A_2 \cup A_m$ and where $V(C) \cap \partial(G_i)$ contains only vertices of A_2. We assign the edges of $E(G_i)$ to our three components by setting $\mathcal{C}_i = C$ and $M' = M \cap E(G_i) \setminus E(C)$, $F' = C_i - E(C)$. In this assignment $\mathcal{C}_i$ is a cycle and thus contains no path components, M' is a matching, and F' consists of a path P together with a set of isolated vertices. As $x \in P$, this path is not disjoint from $A_2 \cup A_m$ and has no leaf in $\partial(G_i)$ as its ends are incident to a chord, so it satisfies Condition (i) of Definition 5. Let v be an isolated vertex of F'. If v has degree 3 in G_i, then it is incident to an edge $vu \notin C$ whose other end is in P. We remove this edge from M' and add it to F', leaving F' acyclic by adding a new leaf to the tree P. As this leaf has degree 3 in G_i, the larger component continues to satisfy Condition (i). In the case where v has degree 2 in G_i, we assumed that $v \in A_2$ and this component also satisfies Condition (i). The resulting spanning forest F_i and matching M_i therefore form an (A_0, A_p, A_m, A_2)-decomposition of C_i if $A_p = \emptyset$ and $|A_m| \leqslant 1$. We call this the *decomposition given by* C.

We now show that certain (A_0, A_p, A_m, A_2)-decompositions exist, starting with $A_0 = A_p = \emptyset$, $A_m = \{x\}$ for some $x \in \partial(G_i)$, and $A_2 = \partial(G_i) \setminus A_m$.

Lemma 7. *There exists an $(\emptyset, \emptyset, \{x\}, A_2)$-decomposition of C_i.*

Proof. If the cycle C_i is chordless, $V(C_i) = \partial(G_i)$ and $E(G_i) = E(C_i)$. Here, setting $F_i = (V(C_i), \emptyset)$, $\mathcal{C}_i = C_i$, and $M_i = \emptyset$ does the trick.

In the case where the cycle C_i has a chord, it contains a minimal cycle C that avoids x and we can use the decomposition given by C. □

We also find decompositions this way when all elements of $\partial(G_i)$ are in A_2.

Corollary 8. *If $\partial(G_i) \neq \emptyset$, then C_i has an $(\emptyset, \emptyset, \emptyset, \partial(G_i))$-decomposition.*

Proof. By Lemma 7 there exists an $(\emptyset, \emptyset, \{x\}, A_2 \setminus \{x\})$-decomposition of C_i for some $x \in A_2$. This is an $(\emptyset, \emptyset, \emptyset, A_2)$-decomposition by definition. □

Next, let $A_0 = \{x\}$ for some $x \in \partial(G_i)$, $A_2 = \partial(G_i) \setminus A_0$, and $A_m = A_p = \emptyset$.

Lemma 9. *If $A_2 \neq \emptyset$, then there exists an $(\{x\}, \emptyset, \emptyset, A_2)$-decomposition of C_i.*

Proof. We begin by looking at the case where C_i is chordless. Let y be a neighbour of x in C_i and regard the spanning tree $F_i = C_i - xy$. This contains an element of A_2 and it has only one leaf in A_2, namely y. Thus, F_i satisfies Condition (i). The last missing edge xy of $E(G_i)$ is assigned to M_i, making this a matching and leaving $\mathcal{C}_i$ with no edges and thus no path component. This gives us an $(\{x\}, \emptyset, \emptyset, A_2)$-decomposition.

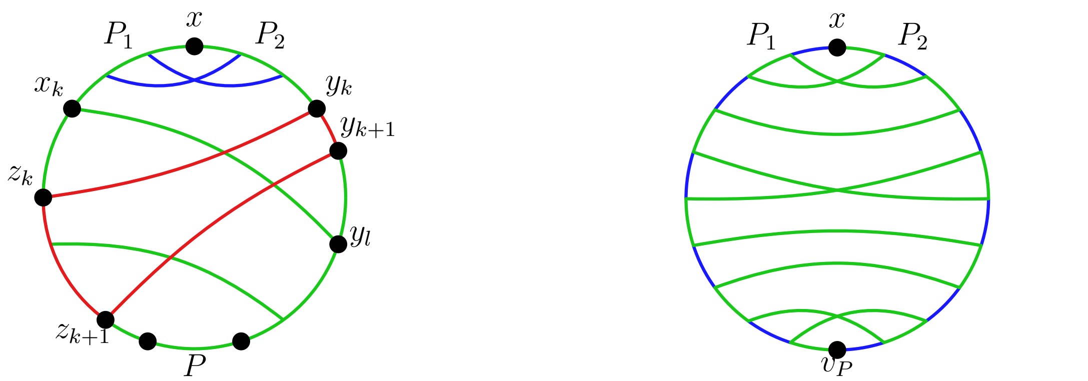
Now we assume that C_i has a chord. The existence of a minimal cycle C that neither contains x nor all elements of A_2 is another good case as it satisfies both requirements necessary for us to obtain a decomposition given by C.

In the final and most complicated case, we may assume that C_i has chords but none of them yield a cycle as described above. We have already seen that any chord naturally gives rise to two minimal cycles C, C' for which P_C, $P_{C'}$ have no inner vertex in common. Consequently one of them must contain x while the other contains all vertices of A_2. Hence, all vertices of A_2 must form a path P: a vertex of degree 3 between them is incident to a chord and this would yield a minimal cycle as above. Let P_1 and P_2 be the two paths in $C_i - E(P)$ between x and the ends of P. Then any chord uv of C_i must connect an inner vertex of P_1 to one of P_2: if both were on the same path, then one of the two minimal cycles that uv yields would contain neither x nor any element of A_2, contradicting our assumption. Hence, P_1 and P_2 have the same length.

Let $x_1, \ldots, x_r$ and $y_1, \ldots, y_r$ be the inner vertices of P_1 and P_2 respectively, ordered by increasing distance to x. We call a chord $x_k y_l$ of C_i *short* if $|k-l| \leqslant 1$ and *long* otherwise. As an illustration, the two blue chords in Figure 2a are short, whereas the edge $x_k y_l$ is a long chord. It turns out that long chords are helpful and the presence of only short ones is structurally very restrictive, which we shall see shortly. We first prove inductively that if all chords $x_k y_l$ in G_i with $k,\, l \leqslant d$ are short, then it holds that $x_k y_l$ is in G if and only if $x_l y_k$ is, for all $k,\, l \leqslant d$. The case that $d \leqslant 1$ is clear as the only candidate is $x_1 y_1$. So let it hold up to $d-1$ for $d \geqslant 2$ and let $x_k y_l$ satisfy $k,\, l \leqslant d$. If $k,\, l < d$ or $k,\, l = d$ we are done, so we may assume, by symmetry, that $k = d$ and $l = d-1$. But x_{d-1} is matched to a vertex in $\{y_{d-2}, y_{d-1}, y_d\}$ by M. Of these three, only y_d is an option as y_{d-1} is taken by x_d and $x_{d-1} y_{d-2}$ cannot be in M by induction since $x_{d-2} y_{d-1}$ is not.

With this preparation, we can now look at the case in which a long chord $x_k y_l$ exists. We choose one minimising $\min\{k, l\}$ and may, by symmetry, assume that $k < l$ and all chords with an end of index at most $k-1$ are short. As a result, the vertices y_k, y_{k+1} are matched to vertices in $x_{k+1} P_1$: this holds as neither is matched to x_k whose neighbour in $G[M]$ is y_l with $l \geqslant k+2$. All vertices in $P_1 x_{k-2}$ have neighbours in $P_2 y_{k-1}$, so none of these are possible either. This just leaves the vertex x_{k-1}. Since $x_k y_{k-1} \notin M$, y_{k-1} is matched to x_{k-1} or x_{k-2}, giving us $x_{k-1} y_{k-1} \in M$ or $x_{k-1} y_{k-2} \in M$, eliminating x_{k-1} as well.

Now, let z_k, z_{k+1} be the neighbours of y_k, y_{k+1} in $G[M]$ and take a look at the cycle $C = y_k y_{k+1} z_{k+1} P_1 z_k y_k$ shown in Figure 2a. We now apply a construction similar to the one for minimal cycles: assign the edges of $E(G_i)$ to the three components by setting $\mathcal{C}_i = C$ and $M' = M \cap E(G_i) \setminus E(C)$, $F' = C_i - E(C)$. Then $\mathcal{C}_i$ is a cycle, M' is a matching, and F' consists of two paths together with isolated vertices. The ends of these paths are part of C, so they have degree 3 in G_i and we can connect the two paths using the long chord $x_k y_l$. This replaces the two paths by a tree T' and the isolated vertices have degree 3 in G_i with a neighbour in P_2. As their neighbours are not on C, they are part of T' and we can connect them by adding such edges to T'. Give F_i the edges of T' and put the remaining edges into M_i, then $M_i \subseteq M'$ is still a matching and F_i is a spanning tree. Since F_i contains P and thus (all) vertices of A_2, the conditions of an



(a) The cycle obtained from a long chord $x_k y_l$.

(b) The case when all chords are short.

Figure 2: Illustrations of the decompositions used in the proof Lemma 9, using the presence or non-existence of long chords in the cycle C_i.

$(\{x\}, \emptyset, \emptyset, A_2)$-decomposition are satisfied.

This just leaves the case that all chords are short, which makes use of the very potent knowledge of the way these behave. Here we give $\mathcal{C}_i$ no edges of $E(G_i)$, instead dividing them up amongst F_i and M_i. Note that the degree 2 vertices of G_i are those in the set $\{x\} \cup A_2$ where the vertices of A_2 are all on P. We suppress all vertices of degree 2 except x and one element of A_2. Here, suppressing a degree 2 vertex v means removing it and the edges uv, vw to its neighbours and adding the direct edge uw between these. The resulting graph G' is left with just two vertices of degree 2, namely x and a vertex v_P that has replaced the entire path P. The paths P_1 and P_2 are now xv_P-paths, so we have $P_1 = xx_1 \dots x_r v_P$ and $P_2 = xy_1 \dots y_r v_P$. An illustration of this and the decomposition we now choose can be found in Figure 2b.

Since P_1 and P_2 have the same length, $P_1 \cup P_2$ is a Hamiltonian cycle of even length. Its edges thus decompose into two perfect matchings M_1 and M_2, and we claim that the graph $Q = G' - M_1$ is a Hamiltonian xv_P-path in G'. To see that this is indeed the case, notice that all vertices in $V(G') \setminus \{x, v_P\}$ have degree 2 in Q, where the two excluded ones have degree 1. Hence, Q consists of an xv_P-path and possibly additional cycles. Suppose it contains a cycle C and choose a vertex of minimal index in C. By symmetry, we assume this vertex is $x_k \in P_1$. Then the edge $x_k x_{k-1}$ is part of M_1 as $x_{k-1} \notin C$ (where $x_0, y_0 = x$ and $x_{r+1}, y_{r+1} = v_P$). Thus, $x_k x_{k+1}$ and $y_k y_{k-1}$ are in M_2 and Q. The edge $x_k y_{k-1}$ cannot be in G' either since y_{k-1} is not part of C. But now $x_k y_k \in Q$ or $x_k y_{k+1}, x_{k+1} y_k \in Q$. Both yield an $x_k y_{k-1}$-path in C, $x_k y_k y_{k-1}$ or $x_k x_{k+1} y_k y_{k-1}$ respectively, a contradiction.

Hence, Q is a Hamiltonian path and we can obtain the desired decomposition by replacing v_P by the path P again and putting all edges of $P \cup Q$ into F_i, which is a Hamiltonian path in G_i ending at x and at an end of P. The graph $\mathcal{C}_i$ receives no edges and $M_i = M_1$. This is an $(\{x\}, \emptyset, \emptyset, A_2)$-decomposition of C_i since the only component of F_i contains an element of A_2 and it only has one leaf in A_2. □

Here we remark that this lemma was also obtained by Xie, Zhou, and Zhou and can be found in [13, Lemma 2.3]. Their formulation basically describes the two cases in the proof, as they claim to either get a decomposition containing a cycle with two chords or a Hamiltonian path. We repeated the statement to make it fit into our notation. The reason we also presented our proof is that it is different and we believe that it reveals more structure. Our case distinction is based on the existence of long chords and we obtained that either the graph has one or all chords are short, in which case it has a Hamiltonian path. But we also know that all chords in this case are either of the form x_iy_i or they come in pairs $x_iy_{i+1}, x_{i+1}y_i$. Xie, Zhou, and Zhou prove this by distinguishing whether or not C_i has a non-separating two-chord cycle. This turns out to be exactly our distinction, as we obtain such cycles in the case that there is a long chord and they do not exist when all chords are short, but it obscures the structure of the chords. They also construct a different Hamiltonian path as a result.

Lastly, we let $A_p = \{x, y\}$ for $x, y \in \partial(G_i)$ and $A_2 = \partial(G_i) \setminus A_p$. Due to the abundance of indices needed in the proof and the lack of cycles therein, we denote the regarded cycle by C instead of C_i and write G_C for $G[V(C)]$.

Lemma 10. *If G is 3-connected, then C has an $(\emptyset, \{x, y\}, \emptyset, A_2)$-decomposition.*

Proof. Let P, P' be the two xy-paths in C. Since $G - \{x, y\}$ is connected, A_2 is non-empty and we may assume that, without loss of generality, $V(P') \cap A_2 \neq \emptyset$. Next, let $u_1v_1, u_2v_2, \ldots, u_sv_s$ be a maximal sequence of edges of M with ends in P satisfying, for all $i \in \{1, \ldots, s\}$, that

(1) the path $P_i = u_iPv_i$ is disjoint from all previous ones, meaning that $P_i \subseteq P - \bigcup_{k<i} P_k$,
(2) P_i either contains an element of A_2 or there is an edge $e_i \in E(V(P_i), X_i)$ where $X_i = V(P') \cup \bigcup_{k<i} V(P_k)$, and
(3) P has no vertices u', v' with $u'v' \in M$ and $P_i \subsetneq u'Pv' \subseteq P - \bigcup_{k<i} P_k$.

We remark that these paths P_i end up as part of the tree and Property (2) just ensures that they either contain an element of A_2, and can form a component, or can connect to a prior path or P', which will have such a vertex by induction or our assumption. Also notice that Property (3) can be read as: "P_i is chosen maximally", in the sense that it forbids the existence of candidates u', v' for u_i, v_i that would yield a longer path when picked instead.

Let $X_u = \{\, u_i : i = 1, \ldots, s \,\}$, $X_v = \{\, v_i : i = 1, \ldots, s \,\}$. We assume that u_i occurs before v_i in P for all i. By Property (1), no P_i contains an element of $X_u \cup X_v$ as an inner vertex, meaning vertices of X_u and X_v alternate. Since we shall need access to the vertices u_i and v_i in the order they appear in P momentarily, we introduce new labels that enable this: let y_i (x_i) be the ith occurrence of a vertex of X_u (X_v) in P, for $i \in \{1, \ldots, s\}$. We refer to Figure 3 to keep track of the notation.

Note that for $\mathring{Q}_i = x_iPy_{i+1}$, where $i \in \{0, \ldots, s\}$ and $x_0 = x, y_{s+1} = y$, there are no edges $u'v'$ in $E(V(\mathring{Q}_i), V(\mathring{Q}_j)) \cap M$, for $i < j$. To see this let m be the minimal index in $\{1, \ldots, s\}$ such that P_m is part of $y_{i+1}Px_j$ where m is well-defined as this contains at

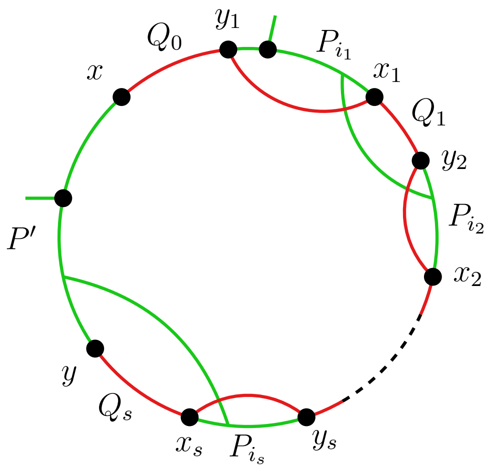


Figure 3: An illustration of the notation and the decomposition used in the proof of Lemma 10. The visible part of Q is red, that of F is green, and matching edges are omitted in favour of clarity.

least the path $y_{i+1}Px_{i+1}$. (Be aware that this is not necessarily P_m, as we choose the path observed first by the construction, which might appear later in the ordering.) The vertices $u' \in \mathring{Q}_i, v' \in \mathring{Q}_j$ satisfy $P_m \subsetneq u'Pv' \subseteq P - \bigcup_{k<m} P_k$, where the last subset relation uses the choice of m: before it, no paths in $x_iPy_{j+1} \supseteq u'Pv'$ were chosen. Hence, Property (3) implies that $u'v' \notin M$.

As exemplified in Figure 3, we now set

$$\begin{aligned} Q &= \bigcup_{i=0}^{s} Q_i + \{\, x_iy_i : i \in \{1, \ldots, s\} \,\} = Q_0y_1x_1Q_1y_2 \ldots x_sQ_s, \\ F' &= P' \cup \bigcup_{i=1}^{s} P_i + \{\, e_i : V(P_i) \cap A_2 = \emptyset \,\}, \text{ and} \\ F &= (V(C), E(F') \cup (E(S, V(F')) \cap M)) \text{ where } S = \bigcup_{i=0}^{s} V(\mathring{Q}_i). \end{aligned}$$

Notice that F' is well-defined: the edges e_i exist in the specified case by (2) and we have just fixed one of them arbitrarily. We now show that $F_C = F$, $\mathcal{C}_C = Q$, and $M_C = M \setminus (E(F) \cup E(Q))$ is an $(\emptyset, \{x, y\}, \emptyset, A_2)$-decomposition of C. The edges of G_C are partitioned completely as $F \cup Q$ contains all edges of C. Additionally, M_C is a matching and $\mathcal{C}_C$ consists of a path from x to y. We thus need to show that F is a spanning forest in G_C whose components each contain an element of A_2, at most one of which is a leaf.

We first show that every vertex of $\mathring{Q}_i$, for $i \in \{0, \ldots, s\}$, is either in A_2 or adjacent to a vertex of F' in $G[M]$. To see this, let W be the set of vertices in $\mathring{Q}_i$ that satisfy one of these two conditions and assume $W \neq V(\mathring{Q}_i)$. Then take two elements w, z of $W \cup \{x_i, y_{i+1}\}$ of minimal distance in Q_i such that wPz contains an element of $V(\mathring{Q}_i) \setminus W$. As $G - \{w, z\}$ is connected, there exists an edge e incident to a vertex u of $\mathring{w}P\mathring{z}$ with other end in $G - wPz$. We show that this edge could be used to extend the sequence $u_1v_1, u_2v_2, \ldots, u_sv_s$, contradicting maximality.

Since $A_2 \cup \{x, y\}$ is disjoint from $\mathring{w}P\mathring{z}$, the other end v of e is also in C. By assumption, it is a matching edge that does not end at a vertex in F'. But we already know that e cannot end in another $\mathring{Q}_j$, so it must have both ends in $\mathring{Q}_i$. Hence it satisfies Properties (1) and (2), where the latter holds as either w or z is part of the resulting path and this vertex is in W. (Note that if w is part of the path, then it cannot be x_i or y_{i+1} as it is an inner vertex of Q_i since both ends of e are and it lies between them. The same holds for z.) Consequently, e either satisfies Property (3) or there exist vertices u', v' as stated there. Choosing them to have maximal distance (in $\mathring{Q}_i$) yields an edge $u'v'$ that satisfies all three properties. In either case we get a contradiction to the maximality of the sequence $u_1v_1, u_2v_2, \ldots, u_sv_s$.

To see that F is a forest, note that F' is the disjoint union of paths with solitary edges connecting those with $V(P_i) \cap A_2 = \emptyset$ to ones prior in the ordering. This ensures that F' is acyclic and its leaves are precisely the ends of the paths P_i and P', none of which are in A_2. The transition to F now only adds edges of M between a vertex of F' and one not in F', which becomes a leaf of the component and is not in A_2. So F is a spanning forest of G_C without any leaves in A_2. Now take a component K of F'. If it contains one of the paths P_i then it has a vertex of A_2. This just follows from Property (2), for P_1 directly and for the others inductively. Any component of F that does not contain such a path must be an isolated vertex in $S = \bigcup_{i=0}^{s} V(\mathring{Q}_i)$ without an edge of M to a vertex in F'. But as vertices of S that do not have such an edge are in A_2 by the previous two paragraphs, we get that these components have a vertex of A_2, too. □

This lemma also requires us to take a look at [13] again. Our construction is similar to the one found in Lemma 2.1 there, though our assumption and obtained decomposition differ. If we formulate their lemma in our notation, we obtain the following.

Lemma 11. *Let $x, y \in \partial(G_i)$ and $A_2 = \partial(G_i) \setminus \{x, y\} \neq \emptyset$. The cycle C_i has an $(\emptyset, \{x, y\}, \emptyset, A_2)$-decomposition or an $(A'_0, \emptyset, \emptyset, A'_2)$-decomposition exists for any choice of A'_0, A'_2 with $A'_0 \cup A'_2 = \partial(G_i)$, $A'_0 \cap A'_2 = \emptyset$, and $A'_2 \neq \emptyset$.*

We do, however, need the first decomposition to exist in our proof and cannot use this to eliminate the 3-connectivity requirement from Lemma 10.

4 Proof of the main theorem

To shorten the proof of Theorem 3, we define *good* I-decompositions and show that they exist. These are basically just decompositions of the centre cycle where the required behaviour of the remaining cycles corresponds to one of our previous lemmas.

Definition 12. Let $I = \{1\}$. We call an I-decomposition $\mathcal{D}_I$ of G *good* if every cycle C_j with $j > 1$ satisfies that

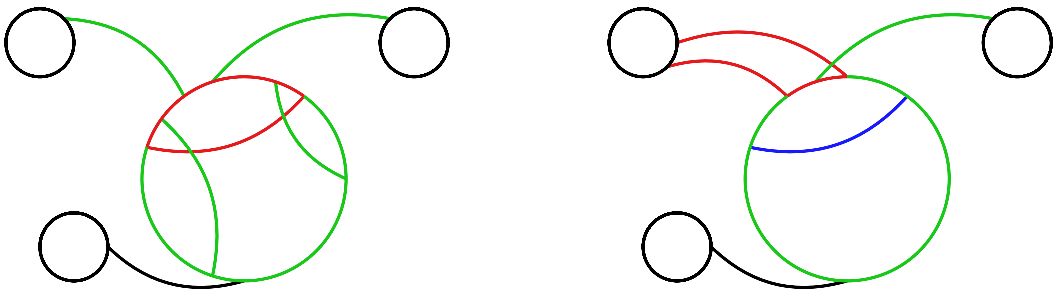
- $|A_0(\mathcal{D}_I) \cap N(C_j)| \leqslant 1$ and $|A_m(\mathcal{D}_I) \cap N(C_j)| \leqslant 1$,

- $|A_p(\mathcal{D}_I) \cap N(C_j)| \in \{0, 2\}$, and
- at most one of the sets $A_x(\mathcal{D}_I) \cap N(C_j)$ for $x \in \{0, m, p\}$ is non-empty.

We begin by showing that good decompositions exist.

Lemma 13. *In the situation of Definition 12, G has a good I-decomposition.*

Proof. We may assume that $l > 1$ as otherwise G is Hamiltonian and has a 3-decomposition. We begin by taking a look at the case where C_1 has a chord. This lets us basically repeat the construction from before Lemma 7 for decompositions given by cycles. We choose some minimal cycle C of G_1 and regard the number of elements in $N(C_j)$ that are on C for cycles C_j with $j > 1$. If these sets contain at most one element for all cycles, we set $\mathcal{C}_I = C$ and let T be the path in $C_1 - E(C)$. The path T is a tree spanning all degree 3 vertices of G_1 except those on C. As C is minimal, any such vertex is connected to T by an edge of M and we add this to T to obtain T_I. The remaining edges, which are a subset of M, form M_I and this is an I-decomposition $\mathcal{D}_I$. For this decomposition the sets $A_m(\mathcal{D}_I)$ and $A_p(\mathcal{D}_I)$ are both empty. Furthermore $A_0(\mathcal{D}_I) \cap N(C_j)$ contains at most one element of C and all vertices of $\partial(G_1)$ that are not in C are in $A_2(\mathcal{D}_I)$, so the conditions of a good decomposition are satisfied. This case is illustrated in Figure 4a.



(a) T_I in case no cycle has multiple edges to C. (b) And T_I in case the cycle $\mathcal{C}_k$ does.

Figure 4: The decompositions used in the proof of Lemma 13 when the centre has a chord.

Next we assume that C contains multiple vertices of some $N(C_k)$ for some k. Choosing two vertices in $P_C \cap N(C_k)$ for some k so as to minimise the distance between them, let P by the path between them in P_C. This path contains at most one vertex from sets $N(C_j)$ for $j \neq k, j > 1$. We apply a similar construction, where we set $\mathcal{C}_I = P$ and let T be the path $C_i - \mathring{P}$. Again we connect vertices of degree 3 in G_I that are not part of T yet and obtain T_I and an I-decomposition. This, too, is good as $A_m(\mathcal{D}_I)$ is still empty, $A_p(\mathcal{D}_I)$ contains exactly the two ends of P, which are in $N(C_k)$, and $A_0(\mathcal{D}_I) \cap N(C_j)$ is empty for $j = k$ and has at most one element otherwise. Figure 4b shows the decomposition constructed in this case.

Finally, in the case that C_1 is chordless, we take two adjacent vertices u, v on C_1. If they have neighbours in different cycles, we set $T_I = C_i - uv$ and $M_I = \{uv\}$, leaving $\mathcal{C}_I$

empty. This gives us a spanning tree and a matching that form a good I-decomposition $\mathcal{D}_I$ as all vertices are in $A_2(\mathcal{D}_I)$ except for u, v which are part of $A_m(\mathcal{D}_I)$ and in different sets $N(C_j)$. Should u, v be in the same set $N(C_j)$, then we add uv to $\mathcal{C}_I$ instead and obtain another good I-decomposition, this time with $A_p(\mathcal{D}_I) = \{u, v\}$ and these being part of the same set $N(C_j)$. □

Now we can finally finish up the proof of Theorem 3.

Proof. Let $I = \{1\}$. By Lemma 13 there exists a good I-decomposition of G. We now iteratively extend this decomposition to more cycles by checking the conditions of Lemma 6 and verifying that we can satisfy them with the help of Corollary 8 and Lemmas 7, 9, and 10.

As long as $I \neq \{1, \ldots, l\}$ take an element $i \notin I$ and set $J = I \cup \{i\}$. Then we can apply Lemma 6 which gives us a J-decomposition if we can exhibit an (A_0, A_p, A_m, A_2)-decomposition where $A_x = N(A_x(\mathcal{D}_I)) \cap V(C_i)$, $x \in \{0, p, m, 2\}$. As G is star-like, we know that $\partial(G_I) \cap N(C_i) = \partial(G_1) \cap N(C_i)$. Using that $\mathcal{D}_{\{1\}}$ is good we can conclude that all vertices in $\partial(G_I) \cap N(C_i)$ are in $A_2(\mathcal{D}_I)$, with the possible exception of either one element in $A_0(\mathcal{D}_I)$, one in $A_m(\mathcal{D}_I)$, or two in $A_p(\mathcal{D}_I)$. Consequently, we have that all vertices in $\partial(G_i)$ are in A_2 aside from a single one in either A_0 or A_m, or two in A_p. (Note that this is true initially and remains true in later steps as Lemma 6 ensures that edges in the centre are never reassigned.) The set $\partial(G_i)$ contains at least three elements as it separates C_i from C_1 in G and G is 3-connected. Hence, $A_2 \neq \emptyset$. We have thus fulfilled the premise of Corollary 8 and Lemmas 7, 9, and 10, giving us an (A_0, A_p, A_m, A_2)-decomposition and completing the proof. □

5 Constructing 3-connected star-like graphs for which the conjecture was not already known

In this section we construct 3-connected star-like graphs, for which Theorem 3 shows there is a 3-decomposition. As the conjecture is already proved for graphs that are traceable, planar, claw-free, 3-connected and of tree-width at most 3, embeddable in the Torus or Klein bottle, or have a matching with a contraction graph of order at most 3, our goal is to find examples that have none of these properties. We present a construction closely based on [23], which we have modified in order to obtain graphs that are actually star-like.

Definition 14. A graph H is hypohamiltonian if it is not Hamiltonian, but $H - v$ is for all $v \in V$.

Observation 15 ([23, Lemma 3.1 (b)]). *For a hypohamiltonian graph H and $z \in H$, $H - z$ has no Hamiltonian path between two neighbours of z, as this could be extended to a Hamiltonian cycle of H.*

For the construction, let H_i, $i \in I = \{1, 2, 3\}$, be graphs that are 3-connected, cubic, non-planar, and hypohamiltonian. In order to see that such H_i exist and to obtain infinitely many examples, we exhibit an infinite family of candidates for the H_i. First, note that we can drop the 3-connectivity requirement.

Observation 16. *Any hypohamiltonian graph is 3-connected.*

Proof. Let G be a hypohamiltonian graph and $v \in V(G)$. Since $G - v$ is Hamiltonian, it has at least three vertices, G has order at least 4, and $G - \{u, v\}$ is connected for any other vertex $u \in V(G)$. □

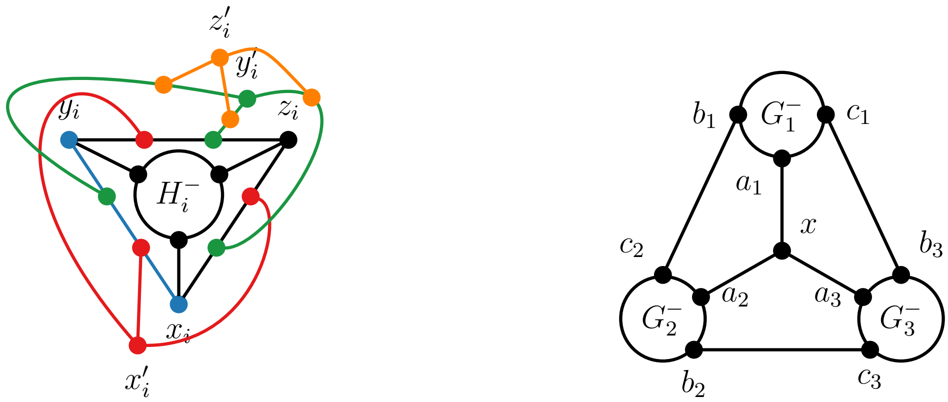
So we only need to find a family of cubic graphs that are hypohamiltonian and non-planar. The Petersen graph and the family of flower snarks satisfies both these properties. We start with the former.

Lemma 17. *The Petersen graph is cubic and hypohamiltonian. It also has a K_5 as a minor and is thus non-planar.*

Proof. All these properties can be found in [24]. □

Lemma 18. *There exists an infinite family of cubic, non-planar, and hypohamiltonian graphs, the flower snarks.*

Proof. We refer to [25] for the definition of these cubic graphs. The same paper shows that they are hypohamiltonian and [26] proves non-planarity. □



(a) The extension G_i of the graph H_i.

(b) The graph $G = K_4[G_1, G_2, G_3]$.

Figure 5: Illustrations describing the construction of the star-like graph G that has none of the properties for which the conjecture is known.

Now on to the construction. For each $i \in I$ we pick some vertex $z_i \in H_i$ and set H_i^- to $H_i - z_i$. Our next goal is to expand H_i to the slightly larger 3-connected cubic graph G_i shown in Figure 5a. To ensure 3-connectivity we may iteratively subdivide two distinct edges and connect the resulting degree two vertices [27]. We apply this first to two of the edges incident to z_i and call the subdivision vertices x_i and y_i. This step is drawn in blue in the figure. Next note that we can also subdivide three edges and connect the subdivision vertices to a single new vertex since this is just a sequence of two subdivision steps. We apply this to the edges of the triangle $x_iy_iz_ix_i$ and call the new vertex x_i', which is drawn in red. Next, we subdivide three of the newly split edges of the triangle that form a matching and connect them to the new vertex y_i', shown in green. Finally, we subdivide all edges incident to y_i' and connect them to z_i', which are the orange vertices and edges. Let the resulting graph be called G_i, denote $G_i - z_i'$ by G_i^-, and set $N_{G_i}(z_i') = \{a_i, b_i, c_i\}$.

Before we go on, we show some properties that the graphs G_i and G_i^- possess.

Lemma 19. *The following properties hold:*

(i) *The graph G_i is 3-connected and cubic.*
(ii) *The graph G_i^- has H_i as a minor.*
(iii) *The graph G_i^- has no Hamiltonian path with both ends in $N_{G_i}(z_i')$.*
(iv) *For any $u \in G_i^-$, three $uN_{G_i}(z_i')$-paths exist that are disjoint aside from u.*
(v) *For any pair of distinct vertices $u, v \in N_{G_i}(z_i')$, $G_i^- - H_i^-$ contains a Hamiltonian uv-path $P_i(u, v)$.*
(vi) *The graph H_i^- is Hamiltonian.*

Proof. Part (i) holds by the assumption on H_i and the construction, while the minor for Part (ii) is obtained by taking the subgraph consisting only of the black edges in Figure 5a and removing the subdivision vertices. For (iii) we notice that a Hamiltonian path in G_i^- that does not end in H_i^- must use exactly two of the three edges between $\{x_i, y_i, z_i\}$ and $N_{H_i}(z_i)$ (as they form a cut). Thus it would induce a Hamiltonian path in H_i^- with both ends in $N_{H_i}(z_i)$, which does not exist by Observation 15. The fourth part follows from the 3-connectivity of G, the paths for part (v) are shown in Figure 6 (and are symmetric), and the last part holds because H_i is hypohamiltonian. □

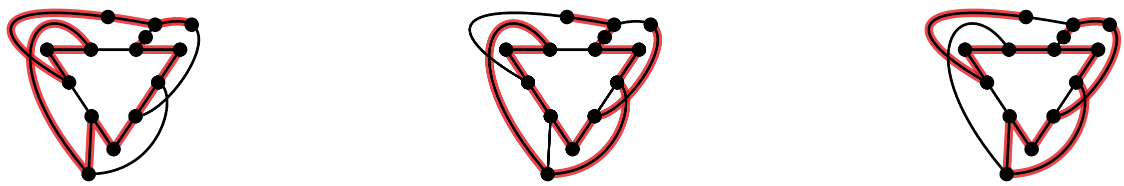


Figure 6: The three Hamiltonian paths between pairs of neighbours of z_i in $G_i^- - H_i^-$.

With this out of the way we now construct our desired graph $G = K_4[G_1, G_2, G_3]$ as follows: let $G_4 = (\{x\}, \emptyset)$ be a further vertex. Take $G_1^- \cup G_2^- \cup G_3^- \cup G_4$ and add the following six edges: xa_1, xa_2, xa_3, b_1c_2, b_2c_3, and b_3c_1. This graph is visualised in Figure 5b.

We now prove that this does the trick.

Theorem 20. *The graph G is cubic, non-traceable, star-like, and 3-connected.*

Proof. The graph is cubic by construction. To see that G is not traceable, we simply need to realise that a Hamiltonian path P would yield a graph G_i^- in which no end of the path resides. This means that P restricted to G_i^- would necessarily be a Hamiltonian path with ends in $N_{G_i}(z_i')$, which does not exist by Lemma 19 (iii).

In order to prove that G is star-like, we just need to specify the cycles. We take a Hamiltonian cycle in each H_i^- for $i \in I$, which exist by Lemma 19 (vi). In addition, we use the paths given by Lemma 19 (v) to obtain a final Hamiltonian cycle C in $G - \bigcup_{i\in I} H_i^-$, namely

$$C = xa_1P_1(a_1, b_1)b_1c_2P_2(c_2, b_2)b_2c_3P_3(c_3, a_3)a_3x.$$

This yields a decomposition where the contraction graph is a star in which the centre corresponds to C and the three tips to the cycles in the H_i^-.

Finally, we are only left with the proof that G is 3-connected. To this end we prove the existence of three internally vertex-disjoint paths between any pair u, v of distinct vertices in G. Let $u \in G_1^-$ without loss of generality. If $v \in G_1^-$, then $G/(V(G) \setminus V(G_1^-)) \cong G_1$ contains three uv-paths by the 3-connectivity of G_1. If one such path uses the super node, we replace it by a path through the subgraph we contracted. So assume that $v \notin G_1^-$. If $v \in G_2^-$ (G_3^- is analogous), we use Lemma 19 (iv) to reduce the problem to finding three disjoint $N_{G_1}(z_1')N_{G_2}(z_2')$-paths. But these exist, just take

$$P_1 = a_1xa_2, \quad P_2 = b_1c_2, \quad P_3 = c_1b_3P_3(b_3, c_3)c_3b_2.$$

Similarly we deal with the case that $v = x$, where it suffices to find three $xN(z_1')$-paths that are disjoint aside from x. This time we use

$$P_1 = xa_1, \quad P_2 = xP_2a_2(a_2, c_2)c_2b_1, \quad P_3 = xa_3P_3(a_3, b_3)b_3c_1.$$

This completes the proof. □

The theorem below follows from [28] and [29, Theorem 1] by noticing that G contains a minor H with three blocks, each of which is a K_5 or a $K_{3,3}$.

Theorem 21. *The graph G has genus and non-orientable genus at least 3.*

We have now seen that the star-like graphs constructed this way fulfil none of the requirements necessary to apply one of the previously existing results, except for the tree-width 3 and order 3 contraction graph results. But as soon as one of the non-planar graphs contains a K_5-minor, then the tree-width is at least 4. So, by assuming that H_1 is the Petersen graph for example, we can assure that the graphs constructed here do not have the necessary tree-width.

To see that they do not admit a matching with a contraction graph of order 3, we show that G has no 2-factor consisting of 3 cycles. Let F by any 2-factor of G, then F contains a cycle C_i completely contained in G_i^- for all $i \in \{1, 2, 3\}$. This is true as any cycle in F is either completely contained in G_i^-, disjoint from it, or consists of a path in it. But there can be at most one cycle that restricts to a path because $E(V(G_i^-))$ contains only three edges. Since this path is not Hamiltonian by Lemma 19 (iii), G_i^- contains at least one cycle of F. Thus F is made up of at least four cycles, one in each G_i^- and one containing x. This completes the proof that our examples fall in none of the classes covered previously.

6 Conclusion

In this paper we have extended the idea of the proof that Hamiltonian graphs have a 3-decomposition to the class of 3-connected star-like graphs. There is hope that this result could be extended to the class of graphs where G_M is a tree, with just a slight generalisation of the definitions and techniques used here. The only construction that does not generalise is the one in Lemma 10, where 3-connectedness is no longer strong enough to ensure the existence of edges connecting our path segments to the tree T_I.

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