Doubly stochastic matrices and
Schur–Weyl duality for partition algebras

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Abstract

We prove that the permutations of $\{1, \ldots, n\}$ having an increasing (resp., decreasing) subsequence of length $n-r$ index a subset of the set of all rth Kronecker powers of $n \times n$ permutation matrices which is a basis for the linear span of that set. Thanks to a known Schur–Weyl duality, this gives a new basis for the centralizer algebra of the partition algebra acting on the rth tensor power of a vector space. We give some related results on the set of doubly stochastic matrices in that algebra.

Mathematics Subject Classifications: 05A10, 05E10, 20C30, 20C08

Introduction

Let $\mathbf{V}$ be a free $\Bbbk$-module with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ over a unital commutative ring $\Bbbk$; identify $m \in \mathbb{Z}$ with its image under the natural ring morphism $\mathbb{Z} \to \Bbbk$. The symmetric group W_n on n letters acts on $\mathbf{V}$ by permuting the basis, via $w \cdot \mathbf{v}_j = \mathbf{v}_{w(j)}$ extended linearly. This action extends to a "diagonal" action on the rth tensor power $\mathbf{V}^{\otimes r} = \mathbf{V} \otimes \cdots \otimes \mathbf{V}$, by

$$w \cdot (\mathbf{v}_{j_1} \otimes \mathbf{v}_{j_2} \otimes \cdots \otimes \mathbf{v}_{j_r}) = \mathbf{v}_{w(j_1)} \otimes \mathbf{v}_{w(j_2)} \otimes \cdots \otimes \mathbf{v}_{w(j_r)} \qquad (1)$$

for any $j_1, j_2, \ldots, j_r \in [n] := \{1, \ldots, n\}$. In other words, the matrix of the action of w, taken with respect to the basis

$$\mathbf{v}_{i_1} \otimes \mathbf{v}_{i_2} \otimes \cdots \otimes \mathbf{v}_{i_r} \quad (i_1, \ldots, i_r \in [n]), \qquad (2)$$

is the rth Kronecker power $P(w)^{\otimes r}$ of the $n \times n$ permutation matrix $P(w) := [\delta_{i,w(j)}]$. Extending the action linearly to the group algebra $\Bbbk[W_n]$ makes $\mathbf{V}^{\otimes r}$ into a $\Bbbk[W_n]$-module. Identify $\operatorname{End}_\Bbbk(\mathbf{V}^{\otimes r})$ with the algebra $\mathsf{M}_{n^r}(\Bbbk)$ of $n^r \times n^r$ matrices, via the basis, and let

$$\Phi : \Bbbk[W_n] \to \operatorname{End}_\Bbbk(\mathbf{V}^{\otimes r}), \quad \sum_{w \in W_n} a_w w \mapsto \sum_{w \in W_n} a_w P(w)^{\otimes r} \qquad (3)$$

be the corresponding matrix representation. The image $\text{im}(\Phi)$ is the $\Bbbk$-linear span of the set Γ of rth Kronecker powers of $n \times n$ permutation matrices. Our main result is the following.

Theorem 1. *The set of all $P(w)^{\otimes r}$ such that w is a permutation in W_n and the sequence $(w(1), w(2), \ldots, w(n))$ has an increasing (resp., decreasing) subsequence of length $n - r$ is a $\Bbbk$-basis for $\text{im}(\Phi)$.*

The result is of interest only for $r < n - 1$, as the subsequence condition is vacuous for $r \geqslant n - 1$ (in which case Φ is faithful). In [5], it is shown that the problem of expressing an arbitrary element of $\text{im}(\Phi)$ as a linear combination of the basis in Theorem 1 reduces to inverting a $(0, 1)$-unitriangular matrix.

In Section 1 we assume that $\Bbbk = \mathbb{R}$, the field of real numbers, and consider nonnegative matrices in $\text{im}(\Phi)$ in the spirit of $[1, 6, 7, 28, 33, 34, 39]$. Of particular interest is Birkhoff's theorem [2] (see also von Neumann [42]), that the set of $n \times n$ doubly stochastic matrices is the convex hull of the set of $n \times n$ permutation matrices. We wondered whether Birkhoff's theorem extends to the set Ω of $n^r \times n^r$ doubly stochastic matrices in $\text{im}(\Phi)$; indeed, we conjectured that it does so extend in an earlier version of this paper. The conjecture is false, by a recent counterexample (Example 1) due to Roberson and Schmidt. In Section 1, we prove:

(i) Ω is convex and the points of Γ are extremal points of Ω.

(ii) Birkhoff's theorem extends to Ω if and only if a theorem of König extends to Ω, and it does so extend if $r \geqslant n - 1$.

The interesting question of determining the convex structure of Ω, and in particular finding its set of vertices, is highlighted.

The rest of the paper takes place over a general unital commutative ring $\Bbbk$, unless indicated otherwise. Section 2 looks at $r = 1$ in detail; the bases in Theorem 1 appear to be new even in that case, and we show that they are indexed by the set of "consecutive" cycles in W_n. Section 3 contains the proof of Theorem 1. Although the proof is straightforward it is heavy on technical notation; in particular we need to work in the Iwahori–Hecke algebra of W_n in most of that section. Note that Schur–Weyl duality is not needed to prove Theorem 1. Theorem 2 in Section 3 obtains a new "Kazhdan–Lusztig" basis for the annihilator of a certain key permutation module, which may be of independent interest. Section 4 explains the connections to integral Schur–Weyl duality for partition algebras, proved in $[3, 10]$, and shows for instance that $\text{im}(\Phi) = \text{End}_{\mathcal{P}(r,n)}(\mathbf{V}^{\otimes r})$, the centralizer algebra for the usual action of the partition algebra on tensor space. Section 5 applies Schur–Weyl duality to describe $\text{im}(\Phi)$ by an explicit linear system, which relates back to doubly stochastic matrices in case $\Bbbk = \mathbb{R}$.

1 Convexity and doubly stochastic matrices

Assume that $\Bbbk = \mathbb{R}$ in this section. Recall that a square matrix is *doubly stochastic* if its entries are nonnegative real numbers and all its rows and columns sum to 1.

Proposition 1. *If $\Bbbk = \mathbb{R}$, the set Ω of doubly stochastic matrices in $\mathrm{im}(\Phi)$ is convex, and every $P(w)^{\otimes r}$ is an extremal point (vertex) of Ω.*

Proof. Let M, M' be in Ω, the set of doubly stochastic elements of $\mathrm{im}(\Phi)$, the $\mathbb{R}$-span $\Gamma = \{P(w)^{\otimes r} \mid w \in W_n\}$. Then for any $0 \leqslant t \leqslant 1$, $tM + (1-t)M'$ is again doubly stochastic, and it is also in the $\mathbb{R}$-span of Γ, so it is in Ω. Thus Ω is convex.

Furthermore, if $M = P(w)^{\otimes r}$ is not an extremal point of Ω, then it must be the midpoint $M = \frac{1}{2}(A + B)$ of the line segment between two *distinct* points A, B of Ω. Hence

$$m_{i_1 \cdots i_r, \, j_1 \cdots j_r} = \tfrac{1}{2}(a_{i_1 \cdots i_r, \, j_1 \cdots j_r} + b_{i_1 \cdots i_r, \, j_1 \cdots j_r})$$

for all $i_1 \cdots i_r$, $j_1 \cdots j_r$ in $[n]^r$, where each

$$0 \leqslant a_{i_1 \cdots i_r, \, j_1 \cdots j_r}, b_{i_1 \cdots i_r, \, j_1 \cdots j_r} \leqslant 1.$$

If $m_{i_1 \cdots i_r, \, j_1 \cdots j_r} = 0$ then $a_{i_1 \cdots i_r, \, j_1 \cdots j_r} = b_{i_1 \cdots i_r, \, j_1 \cdots j_r} = 0$. If $m_{i_1 \cdots i_r, \, j_1 \cdots j_r} = 1$ then $a_{i_1 \cdots i_r, \, j_1 \cdots j_r} = b_{i_1 \cdots i_r, \, j_1 \cdots j_r} = 1$. Since $0, 1$ are the only possible values of the entries of M, we see that $A = B$, which is the desired contradiction. $\qquad\square$

In case $r = 1$, Birkhoff's theorem [2] characterizes the set of $n \times n$ doubly stochastic matrices as the convex hull of the set of $n \times n$ permutation matrices. In light of Proposition 1, it is natural to ask the following question.

Question 1. Does Birkhoff's result extend to the set Ω, for all n, r? In other words, is Ω the convex hull of Γ, for all n, r?

Proposition 3 shows that the answer is yes for all r sufficiently large, but it is no in general, as we will see. Recall that Birkhoff's theorem is implied by a theorem of König [24], which states that any $n \times n$ doubly stochastic matrix $M = [m_{ij}]$ has a positive diagonal, where a *diagonal* is defined to be $\{m_{w(j),j}\}_{j \in [n]}$, for some $w \in W_n$ (the entries corresponding to the nonzero entries in $P(w)$). Equivalently, König's result is that the permanent of any doubly stochastic matrix is positive; recall that the permanent is the sum of all diagonal products.

Proposition 2. *Let $\Bbbk = \mathbb{R}$. For any n, r the following are equivalent:*

(a) *The set Ω of doubly stochastic matrices in $\mathrm{im}(\Phi)$ is equal to the convex hull of $\Gamma = \{P(w)^{\otimes r} \mid w \in W_n\}$.*

(b) *Every $M = [m_{i_1 \cdots i_r, \, j_1 \cdots j_r}]$ in Ω has a positive "Kronecker power" diagonal; that is, a diagonal, with all entries positive, of the form*

$$\{m_{w(j_1)w(j_2)\cdots w(j_n), \, j_1 j_2 \cdots j_r} \mid j_1, j_2, \ldots, j_r = 1, \ldots, n\}$$

corresponding to the nonzero entries in $P(w)^{\otimes r}$, for some $w \in W_n$.

Proof. This is an extension to higher Kronecker powers of standard arguments, e.g., [28, II.1.7], [1, Thm. 2.1.4], or [7, Thm. 1.2.1].

We first show that (b) implies (a). Assume that (b) holds. Any convex linear combination of the $P(w)^{\otimes r}$ clearly belongs to Ω, so we only need to show the reverse inclusion. Let M be in Ω. By (b), there is a positive Kronecker power diagonal in M, indexed by some $w \in W_n$. Let c_w be the minimum entry in that diagonal. If $c_w = 1$ then $M = P(w)^{\otimes r}$ and we are done. Otherwise $0 < c_w < 1$ and the matrix $M' := \frac{1}{1-c_w}(M - c_w P(w)^{\otimes r})$ is again in Ω. Note that M' has at least one more zero entry than M, and $M = c_w P(w)^{\otimes r} + (1-c_w)M'$. We then repeat the argument with M' in place of M. The process must terminate in finitely many steps, as the number of nonzero entries in the sequence of matrices forms a strictly decreasing sequence. Upon termination, we have found real scalars $c_w \geqslant 0$ such that

$$M = \sum_{w \in W_n} c_w P(w)^{\otimes r} \quad \text{and} \quad \sum_{w \in W_n} c_w = 1$$

which is a convex linear combination, thus proving the reverse inclusion and the desired equality.

Conversely, the fact that (a) implies (b) is immediate, as the diagonal corresponding to any nonzero summand in a convex linear combination of the $P(w)^{\otimes r}$ must be positive. $\square$

Proposition 3. *Assume that $\Bbbk = \mathbb{R}$. If $r \geqslant n-1$ then Ω is equal to the convex hull of the set $\Gamma = \{P(w)^{\otimes r} \mid w \in W_n\}$. In other words, the analogue of Birkhoff's theorem holds.*

Proof. By [4, Cor. 4.13], the representation Φ is injective for any $r \geqslant n-1$, hence induces an isomorphism $\mathbb{R}[W_n] \cong \mathrm{im}(\Phi)$. Thus there is always a unique solution to the equation

$$\Phi\big(\sum_{w \in W_n} c_w w\big) = \sum_{w \in W_n} c_w P(w)^{\otimes r} = M$$

for any given M in $\mathrm{im}(\Phi)$. We have

$$m_{i_1 \cdots i_r, \, j_1 \cdots j_r} = \sum_{w \in W_n : w(j_\alpha) = i_\alpha, \forall \alpha} c_w.$$

If $r \geqslant n$, only one $P(w)^{\otimes r}$ can contribute to any $m_{w(j_1) \cdots w(j_r), j_1 \cdots j_r}$, where there are exactly n distinct values (the maximum possible) in $\{j_1, \ldots, j_r\}$. So we must take $c_w = m_{w(j_1) \cdots w(j_r), j_1 \cdots j_r}$ equal to that entry of M, for each w. Thus, if M happens to be doubly stochastic, then each $c_w \geqslant 0$. At least one of the c_w is positive, and the corresponding diagonal is a positive Kronecker power diagonal in M, so by Proposition 2, Birkhoff's theorem holds in this case.

If $r = n-1$ then the same reasoning applies to any $m_{w(j_1) \cdots w(j_{n-1}), j_1 \cdots j_{n-1}}$, where the values in $\{j_1, \ldots, j_{n-1}\}$ are all distinct (we can take $j_k = k$ here, for instance). The point is that any permutation of n objects is determined by its values on $n-1$ of them. So the rest of the argument goes through as in the preceding paragraph. $\square$

However, in general the answer to Question 1 is no, as shown in the following simple counterexample, based on Roberson and Schmidt [38, Sect. 3]. To set the stage, we observe that $P(w)^{\otimes 2}$ is the block matrix $[\delta_{i,w(j)} P(w)]$. In other words, it has a copy of

$P(w)$ in each block corresponding to a 1-entry of $P(w)$, and all other blocks are zero. For instance, if t is the transposition that interchanges $(1, 2)$ in W_4 then

$$P(t)^{\otimes 2} = \begin{bmatrix} 0 & P(t) & 0 & 0 \\ P(t) & 0 & 0 & 0 \\ 0 & 0 & P(t) & 0 \\ 0 & 0 & 0 & P(t) \end{bmatrix}$$

as a block matrix. Now we are ready for the promised example.

Example 1 (Roberson–Schmidt). Take $(n, r) = (4, 2)$. Let $M = \sum_{w \in W_4} c_w P(w)^{\otimes 2}$, where $c_w = 1/5$ for each transposition $w \in W_4$, $c_1 = -1/5$ is the coefficient of the identity matrix, and $c_w = 0$ for all other $w \in W_4$. Then M has the block form shown in Figure 1, with its rows and columns indexed by $[4] \times [4]$ ordered lexicographically. This matrix is

Figure 1: Counterexample for $(n, r) = (4, 2)$

$$\begin{bmatrix} .4 & 0 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 & .2 \\ 0 & 0 & .2 & .2 & .2 & 0 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & .2 & 0 & 0 \\ 0 & .2 & 0 & .2 & 0 & 0 & .2 & 0 & .2 & 0 & 0 & 0 & 0 & 0 & .2 & 0 \\ 0 & .2 & .2 & 0 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & .2 & .2 & 0 & 0 & 0 \\ 0 & .2 & 0 & 0 & 0 & 0 & .2 & .2 & .2 & 0 & 0 & 0 & .2 & 0 & 0 & 0 \\ .2 & 0 & 0 & 0 & 0 & .4 & 0 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 & .2 \\ 0 & 0 & .2 & 0 & .2 & 0 & 0 & .2 & 0 & .2 & 0 & 0 & 0 & 0 & .2 & 0 \\ 0 & 0 & 0 & .2 & .2 & 0 & .2 & 0 & 0 & 0 & 0 & .2 & 0 & .2 & 0 & 0 \\ 0 & 0 & .2 & 0 & .2 & 0 & 0 & 0 & 0 & .2 & 0 & .2 & .2 & 0 & 0 & 0 \\ 0 & .2 & 0 & 0 & 0 & 0 & .2 & 0 & .2 & 0 & 0 & .2 & 0 & .2 & 0 & 0 \\ .2 & 0 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 & .4 & 0 & 0 & 0 & 0 & .2 \\ 0 & 0 & 0 & .2 & 0 & 0 & 0 & .2 & .2 & .2 & 0 & 0 & 0 & 0 & .2 & 0 \\ 0 & 0 & 0 & .2 & .2 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 & .2 & .2 & 0 \\ 0 & .2 & 0 & 0 & 0 & 0 & 0 & .2 & 0 & .2 & 0 & 0 & .2 & 0 & .2 & 0 \\ 0 & 0 & .2 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 & .2 & .2 & .2 & 0 & 0 \\ .2 & 0 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 & .4 \end{bmatrix}$$

doubly stochastic. We claim that it contains no positive Kronecker power diagonal, and thus by Proposition 2 does not lie in the convex hull of Γ. Notice that

$$M_{i,j} = .2 P(t_{i,j}) \qquad \text{for all } i \neq j$$

where $t_{i,j}$ is the transposition interchanging i, j. Each of these blocks has a unique positive diagonal. By the observation preceding this example, if M had a positive Kronecker power diagonal, it would be of the form $P(t)^{\otimes 2}$, for some transposition t. But none of the blocks $M_{i,i}$ on the main block diagonal contains a positive diagonal corresponding to any transposition, so the claim is established.

We note that Proposition 1 implies that the analogue of Birkhoff's theorem holds for a given r if and only if the $P(w)^{\otimes r}$, $w \in W_n$, are the *only* vertices of the convex region Ω. The existence of Example 1 suggests the following interesting problem.

Question 2. Determine the vertex set of the convex polytope Ω.

By Proposition 1, all the points in Γ are vertices of Ω, but Example 1 shows that there can be others. In fact, for $(n, r) = (4, 2)$, calculations by Roberson and Schmidt reveal that Ω has 162 vertices, far more than the 24 in Γ.

2 Interpretation of the main result in case $r = 1$

We work over $\Bbbk$ (an arbitrary unital commutative ring) from now on, unless explicitly stated otherwise. Theorem 1 gives two bases of $\mathrm{im}(\Phi)$ which appear to be new, even for $r = 1$. We wish to explore this case in detail, as it turns out that the set of permutations in question has interesting structure.

We need to understand the set of $w \in W_n$ having an increasing subsequence of length $n - 1$. It is easy to list all such w by a combinatorial process of filling in n slots. We will use the shorthand notation $w_1 w_2 \cdots w_n$ for the sequence $(w(1), w(2), \ldots, w(n))$ for $w \in W_n$. To construct a permutation $w_1 w_2 \cdots w_n$ on the list, that is, one having an increasing subsequence of length $n - 1$, pick a number $k \in [n]$ and a slot $j \in [n]$, and place k in the jth slot. The remaining elements, i.e., those in $[n] \setminus \{k\}$, are placed in the remaining slots in increasing order. As there are n choices for the number and n choices for its slot, there are n^2 items in the list.

Example 2. If $n = 4$, carrying out the above procedure yields the following grid of sequences, in the shorthand notation:

$$
\begin{array}{cccc}
\underline{1}234 & 2\underline{1}34 & 23\underline{1}4 & 234\underline{1} \\
\underline{2}134 & 1\underline{2}34 & 13\underline{2}4 & 134\underline{2} \\
\underline{3}124 & 1\underline{3}24 & 12\underline{3}4 & 124\underline{3} \\
\underline{4}123 & 1\underline{4}23 & 12\underline{4}3 & 123\underline{4}
\end{array}
$$

in which we have underlined the number placed in the chosen slot.

Notice that the identity permutation appears n times on the main diagonal, and the $n - 1$ elements on the superdiagonal are the same as the corresponding ones on the subdiagonal. So our list overcounts by $2(n-1)$ items. Omitting the duplicates, we obtain a list of $n^2 - 2n + 2$ permutations, which is the (well known) dimension of $\mathrm{im}(\Phi)$ in the $r = 1$ case.

The structure of this set of permutations is revealed by writing the permutations not as sequences, but instead as products of disjoint cycles.

Example 3. The corresponding elements in Example 2 written in the *cycle notation* (e.g., $(4, 3, 2)$ means $4 \mapsto 3 \mapsto 2 \mapsto 4$) are:

$$
\begin{array}{cccc}
(1) & (1,2) & (1,2,3) & (1,2,3,4) \\
(2,1) & (1) & (2,3) & (2,3,4) \\
(3,2,1) & (3,2) & (1) & (3,4) \\
(4,3,2,1) & (4,3,2) & (4,3) & (1)
\end{array}
$$

where we write (1) for the identity permutation.

Observe that every element consists of a single cycle of consecutive numbers, and all such cycles appear. Elements along the diagonals have the same cycle length, the cycle length increasing by one each step as the diagonal distance increases away from the main diagonal. Cycles which are in symmetric positions about the diagonal are mutual inverses. Finally, we observe that the picture is compatible with restriction, because we obtain the grid for $n-1$ by deleting the last row and column of the grid for n.

Let w_0 be the longest element (with respect to the usual Coxeter length function) in W_n. Note that w_0, as a sequence, is the reverse of the identity sequence (it swaps $(1, n)$, $(2, n-1)$, etc).

We define a *consecutive k-cycle* to be either a k-cycle which maps each integer in the interval $[i, i+k-1]$ to its successor modulo k, or its inverse. Notice that all the cycles in Example 3 are consecutive.

Proposition 4. *The set of $w \in W_n$ having an increasing subsequence of length $n-1$ is the same as the set $\mathrm{Cons}(n)$ of all consecutive cycles in W_n, and thus $\{P(w) \mid w \in \mathrm{Cons}(n)\}$, $\{P(ww_0) \mid w \in \mathrm{Cons}(n)\}$ are the bases in Theorem 1 in case $r = 1$.*

Proof. The first claim is proved by induction on n. Assuming the desired equality in the statement has been established for $n-1$, one easily checks that the additional $2n-3$ consecutive cycles which move n coincide with the non-diagonal elements in the last row and column of the grid, which shows that the equality holds when $n-1$ is replaced by n.

Hence $\{P(w) \mid w \in \mathrm{Cons}(n)\}$ is the first basis in Theorem 1. The second basis is obtained by reversing the order of each sequence in the grid, which obviously interchanges increasing and decreasing subsequences. Let $\overleftarrow{w}$ denote the reverse of $w \in W_n$. Then $\overleftarrow{w} = ww_0$. It follows that $\{P(ww_0) \mid w \in \mathrm{Cons}(n)\}$ is the second basis in Theorem 1. $\square$

The two bases in Proposition 4 appear to be new. Compare e.g. with the bases in $[8, 11, 12, 15, 20, 26]$. The first basis in Proposition 4 has something of the same flavor as that in $[12, \text{Cor. } 1]$, which is also indexed by a (different) set of cycles and is also compatible with restriction.

3 Proof of Theorem 1

The main task of the proof is to rewrite the basis of $\ker(\Phi)$ given in $[4, \text{Thm. } 7.4]$ (which is written in terms of certain Murphy basis elements at $v = 1$) in terms of the Kazhdan–Lusztig basis. Our main technical tool is the paper of Geck $[14]$, which works out the

relation between the two approaches in the context of the Iwahori–Hecke algebra $\mathcal{H}$ associated to the symmetric group. One may wish to compare our proof with the proof of [37, Thm. 1], which is also based on Geck's paper, although both the results and proofs are different.

Let S be the set of adjacent transpositions in W_n and write $W = W_n$ in this section. Let $l : W \to \mathbb{Z}_{\geqslant 0}$ be the usual length function with respect to S. The Iwahori–Hecke algebra $\mathcal{H} = \mathcal{H}(W, S)$ is the $\mathbb{Z}[v, v^{-1}]$-algebra (v an indeterminate) with basis $\{T_w \mid w \in W\}$ (where $T_1 = 1$) and with multiplication given by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w) + 1, \\ T_{sw} + (v - v^{-1})T_w & \text{if } l(sw) = l(w) - 1 \end{cases}$$

for all $w \in W$, $s \in S$.

Remark 1. We follow the notational conventions of [14, 27] here; in particular the generators T_s satisfy the "balanced" quadratic eigenvalue relation $(T_s + v^{-1})(T_s - v) = 0$. To get back to the setup in the older articles [22, 35, 36] one needs to set $q = v^2$ and replace T_s by vT_s (which defines an isomorphism between the two versions).

There is a unique ring involution $\mathbb{Z}[v, v^{-1}] \to \mathbb{Z}[v, v^{-1}]$, written $a \mapsto \overline{a}$, such that $\overline{v} = v^{-1}$. This extends to a ring involution $\jmath : \mathcal{H} \to \mathcal{H}$ such that

$$\jmath\left(\textstyle\sum_{w \in W} a_w T_w\right) = \textstyle\sum_{w \in W}(-1)^{l(w)}\overline{a}_w T_w \tag{4}$$

for any $a_w \in \mathbb{Z}[v, v^{-1}]$. There is also a unique $\mathbb{Z}[v, v^{-1}]$-algebra automorphism

$$\dagger : \mathcal{H} \to \mathcal{H} \text{ such that } T_s \mapsto T_s^\dagger = -T_s^{-1} \quad (s \in S). \tag{5}$$

We have $T_w^\dagger = (-1)^{l(w)} T_{w^{-1}}^{-1}$, for any $w \in W$. The maps $\jmath, \dagger$ commute. Define a map $^- : \mathcal{H} \to \mathcal{H}$, $h \mapsto \overline{h}$, where $\overline{h} = \jmath(h^\dagger) = \jmath(h)^\dagger$. The map $^-$ is a ring involution of $\mathcal{H}$ such that

$$\overline{\textstyle\sum_{w \in W} a_w T_w} = \textstyle\sum_{w \in W} \overline{a}_w T_{w^{-1}}^{-1} \quad (a_w \in \mathbb{Z}[v, v^{-1}]). \tag{6}$$

By [27, Thm. 5.2], for any $w \in W$, there exist unique C_w, C_w' in $\mathcal{H}$ such that

$$\begin{aligned} \overline{C_w} &= C_w \quad \text{and} \quad C_w \equiv T_w \mod \mathcal{H}_{>0} \\ \overline{C_w'} &= C_w' \quad \text{and} \quad C_w' \equiv T_w \mod \mathcal{H}_{<0} \end{aligned} \tag{7}$$

where

$$\mathcal{H}_{>0} := \textstyle\sum_{w \in W} v\mathbb{Z}[v]T_w, \qquad \mathcal{H}_{<0} := \textstyle\sum_{w \in W} v^{-1}\mathbb{Z}[v^{-1}]T_w.$$

Then $\{C_w \mid w \in W\}$, $\{C_w' \mid w \in W\}$ are both bases of $\mathcal{H}$. These are the "Kazhdan–Lusztig bases" first introduced in [22]. It was proved in [22, Thm. 1.1] that

$$C_w = T_w + \sum_{y < w}(-1)^{l(w)+l(y)}\overline{p_{y,w}}\, T_y, \qquad C_w' = T_w + \sum_{y < w} p_{y,w}\, T_y \tag{8}$$

for any $w \in W$, where both sums are over the set of $y \in W$ such that $y < w$ in the Bruhat–Chevalley order on $W = W_n$ and $p_{y,w}$ is in $v^{-1}\mathbb{Z}[v^{-1}]$. It follows that $C_w = (-1)^{l(w)}\jmath(C'_w)$.

Now we recall Murphy's bases. As usual, we write $\lambda \vdash n$ to indicate that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition of n (meaning that $\lambda \in \mathbb{Z}^k$, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0$, and $\sum \lambda_i = n$). If $\lambda \vdash n$, set

$$\mathbb{T}(\lambda) = \{\text{standard Young tableaux of shape } \lambda\}$$

where as usual the numbers in a standard tableau are strictly increasing along the rows and down the columns. (See [13] for details.) In [35] (see also [36]) Murphy introduces two bases

$$\{x_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathbb{T}(\lambda), \lambda \vdash n\}, \quad \{y_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathbb{T}(\lambda), \lambda \vdash n\}$$

of $\mathcal{H}$, indexed by pairs of standard Young tableaux of the same shape. For $\lambda \vdash n$ and $\mathfrak{s}, \mathfrak{t} \in \mathbb{T}(\lambda)$,

$$\begin{aligned} x_\lambda &:= \textstyle\sum_{w \in W_\lambda} v^{l(w)} T_w \quad \text{and} \quad x_{\mathfrak{s}\mathfrak{t}} := T_{d(\mathfrak{s})} x_\lambda T_{d(\mathfrak{t})^{-1}} \\ y_\lambda &:= \textstyle\sum_{w \in W_\lambda} (-v)^{-l(w)} T_w \quad \text{and} \quad y_{\mathfrak{s}\mathfrak{t}} := T_{d(\mathfrak{s})} y_\lambda T_{d(\mathfrak{t})^{-1}} \end{aligned} \tag{9}$$

where W_λ is the usual Young subgroup associated to λ and $d(\mathfrak{t}) := y$, for a tableau $\mathfrak{t}$, if $y \in W$ is the unique element of W such that $y\mathfrak{t} = \mathfrak{t}^\lambda$. Here $\mathfrak{t}^\lambda$ is the tableau in which the the numbers $1, \ldots, n$ appear in their natural order, written as in order across rows from the top row to the bottom one. Notice that W_λ is the row-stabilizer of $\mathfrak{t}^\lambda$.

Remark 2. The notation here differs slightly from Murphy's in [35, 36]. Because of renormalization (see Remark 1) what he writes as T_w corresponds to $v^{l(w)} T_w$ in our notation. Also, the order of the products defining $x_{\mathfrak{s}\mathfrak{t}}, y_{\mathfrak{s}\mathfrak{t}}$ is reversed here, as we deal with left modules while he works with right ones. Our conventions are chosen to agree with those in Geck's paper [14].

Recall [13, 41] or [23, §5.1.4] that the Robinson–Schensted–Knuth (RSK for short) correspondence gives a bijection

$$\bigsqcup_{\lambda \vdash n} \left(\mathbb{T}(\lambda) \times \mathbb{T}(\lambda)\right) \to W$$

mapping pairs of standard tableaux of the same shape to permutations. Write $\pi_\lambda(\mathfrak{s}, \mathfrak{t})$ for the image of a pair $(\mathfrak{s}, \mathfrak{t})$ of standard tableaux of shape λ. Given $w \in W$, the pair $(\mathfrak{s}, \mathfrak{t})$ such that $\pi_\lambda(\mathfrak{s}, \mathfrak{t}) = w$ is explicitly constructed by the insertion algorithm [13, 23, 41]: $\mathfrak{s}$ is obtained by inserting the numbers in the sequence $(w(1), w(2), \ldots, w(n))$ into an initially empty tableau, and $\mathfrak{t}$ records the order in which the positions of $\mathfrak{s}$ were filled.

For any $\mathfrak{s}, \mathfrak{t} \in \mathbb{T}(\lambda)$, any $\lambda \vdash n$, the ring involution $\jmath : \mathcal{H} \to \mathcal{H}$ defined in (4) satisfies

$$\jmath(x_\lambda) = y_\lambda \text{ and thus } \jmath(x_{\mathfrak{s}\mathfrak{t}}) = \pm y_{\mathfrak{s}\mathfrak{t}}. \tag{10}$$

Hence by [14, Cor. 4.3], it follows that Geck's element $\tilde{y}_{\mathfrak{s}\mathfrak{t}} := T_{d(\mathfrak{s})} C_{w_\lambda} T_{d(\mathfrak{t})^{-1}}$ satisfies the identity

$$\tilde{y}_{\mathfrak{s}\mathfrak{t}} = \pm v^{l(w_\lambda)} y_{\mathfrak{s}\mathfrak{t}}. \tag{11}$$

The element $w_\lambda = \pi_{\lambda'}(\mathfrak{t}^{\lambda'}, \mathfrak{t}^{\lambda'})$ here is the longest word in W_λ, where

$$\lambda' \text{ denotes the transpose partition of } \lambda.$$

By [14, Cor. 5.6], the two-sided Kazhdan–Lusztig cell $\mathfrak{R}(\lambda)$ indexed by any $\lambda \vdash n$ is given by

$$\mathfrak{R}(\lambda) = \{\pi_{\lambda'}(\mathfrak{s}, \mathfrak{t}) \mid \mathfrak{s}, \mathfrak{t} \in \mathbb{T}(\lambda')\}. \tag{12}$$

For any $w \in W$, Geck writes $\lambda_w = \lambda \iff w \in \mathfrak{R}(\lambda)$. We prefer instead to label cells by their *RSK-shape*, written $\mathrm{RSK}(w)$, which we define to be the common shape of the associated pair of tableaux in the RSK-correspondence. Thus

$$\lambda_w = \lambda \iff \mathrm{RSK}(w) = \lambda'; \quad \text{i.e., } \mathrm{RSK}(w) = \lambda'_w. \tag{13}$$

Recall the *dominance order* $\trianglerighteq$ on partitions, a partial order, defined by

$$\lambda = (\lambda_1, \ldots, \lambda_m) \trianglerighteq \mu = (\mu_1, \ldots, \mu_l)$$

if $\sum_{i \leqslant k} \lambda_i \geqslant \sum_{i \leqslant k} \mu_i$ for all k. Write $\lambda \trianglelefteq \mu$ if and only if $\mu \trianglerighteq \lambda$, and $\lambda \triangleleft \mu$ if and only if $\lambda \trianglelefteq \mu$ but $\lambda \neq \mu$, etc. Recall that transposition reverses the dominance order: $\lambda \trianglerighteq \mu \iff \lambda' \trianglelefteq \mu'$. Reformulating the statement of [14, Cor. 4.11] in light of [14, Cor. 5.11] in these terms gives the following.

Proposition 5 (Geck). *Let $\lambda \vdash n$. For any $\mathfrak{s}, \mathfrak{t}$ in $\mathbb{T}(\lambda)$, there exists a unique element $w \in \mathfrak{R}(\lambda)$ of RSK-shape λ', such that*

$$\tilde{y}_{\mathfrak{s}\mathfrak{t}} = C_w \quad + \sum_{x:\, \mathrm{RSK}(x) = \lambda'} a_x C_x \quad + \sum_{x:\, \mathrm{RSK}(x) \triangleleft \lambda'} b_x C_x$$

where $a_x \in v\mathbb{Z}[v]$, $b_x \in \mathbb{Z}[v, v^{-1}]$ for all x.

Recall what it means to "specialize $v \mapsto \xi$ in $\Bbbk$". If $\xi \in \Bbbk$ is invertible, we regard $\Bbbk$ as a $\mathbb{Z}[v, v^{-1}]$-algebra by means of the (unique) ring homomorphism $\mathbb{Z}[v, v^{-1}] \to \Bbbk$ sending $v^{\pm 1} \mapsto \xi^{\pm 1}$. Let $\mathcal{H}_\Bbbk := \Bbbk \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H}$ be the $\Bbbk$-algebra obtained by extending scalars via this morphism. By abuse of notation, we identify symbols such as T_w, C_w, $y_{\mathfrak{s}\mathfrak{t}}$, etc with their respective images $1 \otimes T_w$, $1 \otimes C_w$, $1 \otimes y_{\mathfrak{s}\mathfrak{t}}$, etc in $\mathcal{H}_\Bbbk$. As in Dipper and James [9], the left ideal $M^\lambda := \mathcal{H}x_\lambda$ is a "permutation module" indexed by $\lambda \vdash n$. If we specialize $v \mapsto 1$ in $\Bbbk$ then $\mathcal{H}_\Bbbk \cong \Bbbk[W_n]$ and M^λ is isomorphic to the usual permutation module for $\Bbbk[W_n]$.

At this point, there are two possibilities for how to proceed, depending on whether we prefer to specialize now or later. Rather than favor one over the other, we discuss both.

Theorem 2. *Suppose that $r < n - 1$. Let $\alpha(n, r) := (n - r, 1^r)$ be the partition $(n - r, 1, \ldots, 1)$ with 1 repeated r times.*

    (a) *Under specialization $v \mapsto 1$ in $\Bbbk$, the set $\{C_x \mid \mathrm{RSK}(x) \not\trianglerighteq \alpha(n, r)\}$ is a $\Bbbk$-basis of the annihilator of the $\Bbbk[W_n]$-action on $\mathbf{V}^{\otimes r}$.*

(b) *Over $\mathbb{Z}[v, v^{-1}]$, the set $\{C_x \mid \mathrm{RSK}(x) \not\trianglelefteq \alpha(n,r)\}$ is a $\mathbb{Z}[v, v^{-1}]$-basis of the annihilator of the $\mathcal{H}$-action on $M^{\alpha(n,r)}$.*

Proof. (a) Let $\Delta := \{a \in \mathbb{k}[W_n] \mid a \text{ annihilates } \mathbf{V}^{\otimes r}\}$. By [4, Thm. 7.4], the set

$$\{y_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathbb{T}(\lambda), \ \lambda \not\trianglelefteq \alpha(n,r)'\} = \{y_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathbb{T}(\lambda), \ \lambda' \not\trianglelefteq \alpha(n,r)\}$$

is a $\mathbb{k}$-basis of Δ. Let

$$a \quad = \sum_{\mathfrak{s}, \mathfrak{t} \in \mathbb{T}(\lambda): \ \lambda' \not\trianglelefteq \alpha(n,r)} a_{\mathfrak{s}\mathfrak{t}} y_{\mathfrak{s}\mathfrak{t}} \quad (a_{\mathfrak{s}\mathfrak{t}} \in \mathbb{k})$$

be an arbitrary element of Δ. As we are working at $v = 1$, we have $y_{\mathfrak{s}\mathfrak{t}} = \pm \tilde{y}_{\mathfrak{s}\mathfrak{t}}$ by equation (11). By Proposition 5, each $y_{\mathfrak{s}\mathfrak{t}}$ appearing on the right hand side of the above equality belongs to the $\mathbb{k}$-span of $\{C_x \mid \mathrm{RSK}(x) \trianglelefteq \lambda'\}$, for some $\lambda' \not\trianglelefteq \alpha(n,r)$. But

$$\lambda' \not\trianglelefteq \alpha(n,r) \text{ and } \mathrm{RSK}(x) \trianglelefteq \lambda' \implies \mathrm{RSK}(x) \not\trianglelefteq \alpha(n,r),$$

so a is in the span of $\{C_x \mid \mathrm{RSK}(x) \not\trianglelefteq \alpha(n,r)\}$. So that set spans Δ. Linear independence is clear, so it is a basis. This proves (a).

(b) Now let $\Delta := \{a \in \mathcal{H} \mid a \text{ annihilates } M^{\alpha(n,r)}\}$. By [10, §8], which extends the main result of [4] to $\mathcal{H}$, essentially the same set

$$\{y_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathbb{T}(\lambda), \ \lambda \not\trianglelefteq \alpha(n,r)'\} = \{y_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathbb{T}(\lambda), \ \lambda' \not\trianglelefteq \alpha(n,r)\}$$

is a $\mathbb{Z}[v, v^{-1}]$-basis of Δ. The rest of the argument is almost exactly the same as for part (a), except that coefficients are in $\mathbb{Z}[v, v^{-1}]$. The power of v in equation (11) causes no trouble, as v is invertible. $\qquad\square$

Remark 3. (i) Parts (a), (b) of Theorem 2 are connected by [4, Thm. 7.4(c)], which says that when $v \mapsto 1$ in $\mathbb{k}$, the annihilators of $\mathbf{V}^{\otimes r}$ and $M^{\alpha(n,r)}$ coincide. It follows that (b) implies (a) in Theorem 2. On the other hand, we proved (a) directly without assuming (b), based on the main result of [4], and (a) is really all we need. (ii) There is no Hopf algebra structure on $\mathcal{H}$ properly deforming that of $\mathbb{Z}[W_n]$, so there is no interesting "q-analogue" of the diagonal action of $\mathbb{k}[W_n]$ on $\mathbf{V}^{\otimes r}$; thus it doesn't make sense to ask for the annihilator of $\mathbf{V}^{\otimes r}$ in the context of part (b).

The following Lemma will be applied to deduce the Corollary to Theorem 2 that follows, which in turn is used in proving Theorem 1.

Lemma. *For any subset U of W_n, the image of $\{T_w \mid w \in W_n \setminus U\}$ is a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathcal{H}/\Delta$, where Δ is the submodule spanned by $\{C_x \mid x \in U\}$. The corresponding statement holds over $\mathbb{k}$ upon specialization $v \mapsto 1$.*

Proof. This is essentially the same idea as Möbius inversion over the poset W_n under the Bruhat–Chevalley order, using the unitriangular relation (8) between the bases $\{C_x \mid x \in W_n\}$, $\{T_x \mid x \in W_n\}$. By inverting the unitriangular matrix giving the basis transition in (8), we see that

$$T_w = C_w + \sum_{y < w} b_{y,w} C_y \quad (b_{y,w} \in \mathbb{Z}[v, v^{-1}]).$$

This implies a similar relation holds in the image $\mathcal{H}/\Delta$, that is,

$$\tilde{T}_w = \tilde{C}_w + \sum_{y<w,\, y\notin U} b_{y,w}\tilde{C}_y \quad (b_{y,w} \in \mathbb{Z}[v,v^{-1}])$$

where we set $\tilde{T}_w := T_w + \Delta$, $\tilde{C}_w := C_w + \Delta$ in the quotient. Clearly, the set $\{\tilde{C}_x \mid x \notin U\}$ is a basis of $\mathcal{H}/\Delta$. Inverting again, we see that

$$\tilde{C}_w = \tilde{T}_w + \sum_{y<w,\, y\notin U} d_{y,w}\tilde{T}_y \quad (d_{y,w} \in \mathbb{Z}[v,v^{-1}]).$$

Thus, $\mathcal{H}/\Delta$ is spanned by $\{\tilde{T}_w \mid w \in W_n \setminus U\}$. We leave the proof of linear independence of that set to the reader. $\quad\square$

Corollary. *The image of $\{T_x \in \mathcal{H} \mid \mathrm{RSK}(x) \trianglerighteq \alpha(n,r)\}$ is a $\mathbb{Z}[v,v^{-1}]$-basis of the quotient $\mathcal{H}/\Delta$, where Δ is the annihilator of $M^{\alpha(n,r)}$. Under specialization $v \mapsto 1$ in $\Bbbk$, the image of $\{x \in W_n \mid \mathrm{RSK}(x) \trianglerighteq \alpha(n,r)\}$ is a $\Bbbk$-basis of $\Bbbk[W_n]/\Delta$, where $\Delta = \ker(\Phi)$ is the annihilator of $\mathbf{V}^{\otimes r}$.*

Proof. The first statement follows from part (b) of Theorem 2, by taking U in the Lemma to be the set of x in W_n such that $\mathrm{RSK}(x) \ntrianglerighteq \alpha(n,r)$. The second statement follows from the first, as T_x becomes x upon specialization $v \mapsto 1$. Alternatively, it follows from part (a) of Theorem 2, by making the same choice for U and specializing in the Lemma. $\quad\square$

We can now give the proof of the main result.

Proof of Theorem 1. By the Corollary, $\{P(w)^{\otimes r} \mid w \in W_n, \mathrm{RSK}(w) \trianglerighteq \alpha(n,r)\}$ is a $\Bbbk$-basis of $\mathrm{im}(\Phi) \cong \Bbbk[W_n]/\ker(\Phi)$. By [4, Lem. 6.2],

$$\lambda \trianglerighteq \alpha(n,r) \iff \lambda_1 \geqslant n - r$$

and by Schensted's theorem [40] (see also [41]) we have

$$\mathrm{RSK}(w) = \lambda \implies \mathrm{IS}(w) = \lambda_1$$

where $\mathrm{IS}(w)$ is the length of the longest increasing subsequence of $(w(1), w(2), \ldots, w(n))$. Putting these facts together shows that

$$\lambda = \mathrm{RSK}(x) \trianglerighteq \alpha(n,r) \iff \mathrm{IS}(\lambda) \geqslant n - r$$

which gives the first basis in Theorem 1.

The existence of the second basis in Theorem 1 follows from the first, using another observation of Schensted, that $\mathrm{DS}(w) = \mathrm{IS}(\overleftarrow{w})$ for any w in W_n, where $\mathrm{DS}(w)$ is the length of the longest decreasing subsequence of $(w(1), \ldots, w(n))$ and $\overleftarrow{w} = (w(n), \ldots, w(1))$ is the reverse of w, already considered in the proof of Proposition 4. The map

$$\sum a_w w \mapsto \sum a_w \overleftarrow{w} = \sum a_w w w_0 \quad (a_w \in \Bbbk)$$

given by right multiplication by w_0 defines a linear involution of $\Bbbk[W_n]$ carrying $\{w \mid \mathrm{DS}(w) \geqslant k\}$ onto $\{w \mid \mathrm{IS}(w) \geqslant k\}$, for any k. It induces a linear involution on $\mathrm{im}(\Phi)$ which is given by right multiplication by the matrix $P(w_0)^{\otimes r}$; this clearly interchanges the two bases. $\qquad\square$

Remark 4. Theorem 1 has a counterpart for the $\Bbbk$-submodule $\mathbf{V}^{\otimes r} \otimes \mathbf{v}_n$ of $\mathbf{V}^{\otimes(r+1)}$, which we identify with $\mathbf{V}^{\otimes r}$. The restriction of the diagonal action of W_n to the subgroup

$$W_{n-1} = \{w \in W_n \mid w(n) = n\} \subset W_n$$

gives an action on $\mathbf{V}^{\otimes r}$ fixing $\mathbf{v}_n$. Let $\Phi' : \Bbbk[W_{n-1}] \to \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r})$ be the corresponding representation. Then *the set of $P(w)^{\otimes r}$ indexed by $w \in W_{n-1}$ having an increasing (resp., decreasing) subsequence of length $n - 1 - r$ is a basis of* $\mathrm{im}(\Phi')$. The proof is nearly identical with that of Theorem 1; we leave the details to the reader.

4 Connections with Schur–Weyl duality

Let $\mathcal{A} = \Bbbk[W_n]$ be the group algebra of W_n. The diagonal action of W_n makes $\mathbf{V}^{\otimes r}$ into an $\mathcal{A}$-module. Let

$$\mathcal{A}' = \mathrm{End}_{\mathcal{A}}(\mathbf{V}^{\otimes r}) = \{\varphi \in \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r}) \mid \varphi(\alpha t) = \alpha\varphi(t), \alpha \in \mathcal{A}, t \in \mathbf{V}^{\otimes r}\},$$

the commutant of $\mathcal{A}$. Then $\mathbf{V}^{\otimes r}$ is also an $\mathcal{A}'$-module, with $\varphi \in \mathcal{A}'$ acting by $\varphi \cdot t = \varphi(t)$ for any $t \in \mathbf{V}^{\otimes r}$. Each $\alpha \in \mathcal{A}$ induces an $\mathcal{A}'$-homomorphism $f_\alpha : \mathbf{V}^{\otimes r} \to \mathbf{V}^{\otimes r}$ defined by $f_\alpha(t) = \alpha t$. Now consider the bicommutant (double centralizer)

$$\mathcal{A}'' = \mathrm{End}_{\mathcal{A}'}(\mathbf{V}^{\otimes r}) = \{\psi \in \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r}) \mid \psi f = f\psi, \text{ for all } f \in \mathcal{A}'\}$$

where the multiplication here is functional composition. Then the map

$$\Phi : \mathcal{A} \to \mathrm{End}_{\mathcal{A}'}(\mathbf{V}^{\otimes r}) = \mathcal{A}'', \quad \alpha \mapsto f_\alpha. \tag{14}$$

is an $\Bbbk$-algebra homomorphism. It is abstractly the same map as the representation Φ considered in (3), with restricted codomain.

If $\Bbbk$ is a field of characteristic zero or characteristic larger than n, then $\mathbf{V}^{\otimes r}$ is semisimple as an $\mathcal{A}$-module and Jacobson's density theorem [18] (see also [19, §4.3] or [25, Chap. XVII, Theorem 3.2]) implies that Φ is surjective. By the main result of [3] (see also [10, §6]), Φ is surjective in general, for any unital commutative ring $\Bbbk$.

Let $\mathcal{P}(r, n)$ be the partition algebra [21, 29–31] over $\Bbbk$ on $2r$ vertices with parameter n. It has a basis indexed by the set partitions (equivalence relations) on the set $\{1, \ldots, r, 1', \ldots, r'\}$; basis elements are often depicted by diagrams on $2r$ vertices labeled by elements of that set, with a path connecting two vertices if and only if they lie in the same subset of the set partition. The action of $\mathcal{P}(r, n)$ on $\mathbf{V}^{\otimes r}$ is described explicitly in [17], to which we refer for basic properties of partition algebras. Let $\Psi : \mathcal{P}(r, n) \to \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r})$ be the representation afforded by the action.

Proposition 6 (Schur–Weyl duality). *The commutant $\mathcal{A}' = \mathrm{End}_{W_n}(\mathbf{V}^{\otimes r})$ is the image of the representation $\Psi : \mathcal{P}(r,n) \to \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r})$. The bicommutant $\mathcal{A}'' = \mathrm{im}(\Phi)$ is the centralizer algebra $\mathrm{End}_{\mathcal{P}(r,n)}(\mathbf{V}^{\otimes r})$.*

Proof. The fact that $\mathcal{A}' = \mathrm{End}_{W_n}(\mathbf{V}^{\otimes r})$ is the image of Ψ is [17, Thm. 3.6]; the combinatorial proof given there is valid over any $\Bbbk$. The aforementioned surjectivity of Φ then implies the second claim. $\qquad\square$

If we now assume that $\Bbbk = \mathbb{R}$, the following shows that the study of the set of nonnegative invariants in $\mathrm{End}_{\mathcal{P}(r,n)}(\mathbf{V}^{\otimes r})$ reduces to the study of the set Ω of doubly stochastic elements of $\mathrm{im}(\Phi)$.

Corollary. *Assume that $\Bbbk = \mathbb{R}$. Then the set of all nonnegative matrices in $\mathrm{im}(\Phi) = \mathrm{End}_{\mathcal{P}(r,n)}(\mathbf{V}^{\otimes r})$ identifies with the set of nonnegative scalar multiples of the set Ω of doubly stochastic matrices in $\mathrm{im}(\Phi)$.*

Proof. If M is a matrix in $\mathrm{im}(\Phi)$, it may be written as a linear combination of elements of the set Γ of $P(w)^{\otimes r}$, for $w \in W_n$. The rows and columns of the $P(w)^{\otimes r}$ all sum to 1; hence the rows and columns of M all sum to the same value. If the entries of M are nonnegative, then so is the common value s of the row and column sums. If $s \neq 0$ then $s^{-1}M$ is doubly stochastic, so M is a positive multiple of that doubly stochastic matrix. If $s = 0$ then $M = [0]$ must be the zero matrix, which is also zero times a doubly stochastic matrix.

Conversely, suppose that $D \in \mathrm{im}(\Phi)$ is doubly stochastic. Then it is a nonnegative matrix in $\mathrm{End}_{\mathcal{P}(r,n)}(\mathbf{V}^{\otimes r})$, hence the same is true of any nonnegative scalar multiple. $\qquad\square$

Remark 5. Return to general $\Bbbk$. In the situation of Remark 4, there is an action of the "half" partition algebra $\mathcal{P}(r + \frac{1}{2}, n)$ on $\mathbf{V}^{\otimes r} \cong \mathbf{V}^{\otimes r} \otimes \mathbf{v}_n$, where $\mathcal{P}(r + \frac{1}{2}, n)$ is the subalgebra of $\mathcal{P}(r + 1, n)$ spanned by all diagrams with an edge connecting vertices r, r'; see [17] for details. All of the results in this section generalize to the half partition algebra. In particular, the bicommutant of the action of W_{n-1} is equal to

$$\mathrm{im}(\Phi') = \mathrm{End}_{\mathcal{P}(r+\frac{1}{2}, n)}(\mathbf{V}^{\otimes r}).$$

Again, we leave the details to the interested reader. Remark 4 gives a basis of this algebra.

5   Equations for $\mathrm{im}(\Phi)$ and Ω

There is another symmetric group $\mathfrak{S}_r$ acting on $\mathbf{V}^{\otimes r}$, by place-permutation, and its commutant algebra is the Schur algebra $\mathrm{End}_{\mathfrak{S}_r}(\mathbf{V}^{\otimes r})$ studied in [16,32], etc. We write it as $\mathfrak{S}_r$ to emphasize that the actions of $\mathfrak{S}_r$ and W_n on tensors are very different. Write $(i_1 \cdots i_r)^\sigma$ for the effect of place-permuting $i_1 \cdots i_r$ according to $\sigma \in \mathfrak{S}_r$. By [3, Prop. 3.2], combined with the fact that $\mathcal{P}(r,n)$ is generated by $\mathfrak{S}_r$ and the elements p_1, $p_{3/2}$ in the notation of [17, (1.10)], an $n^r \times n^r$ matrix $X = [x_{i_1 \cdots i_r, \, j_1 \cdots j_r}]$ belongs to the bicommutant $\mathcal{A}'' = \mathrm{im}(\Phi)$ if and only if

(i) $x_{i_1 \cdots i_r, \, j_1 \cdots j_r} = x_{(i_1 \cdots i_r)^\sigma, \, (j_1 \cdots j_r)^\sigma}$, for all $\sigma \in \mathfrak{S}_r$.

(ii) $x_{i_1 \cdots i_r, \, j_1 \cdots j_r} = 0$ if $(i_1 = i_2$ but $j_1 \neq j_2)$ or $(i_1 \neq i_2$ but $j_1 = j_2)$.

(iii) $\sum_{i=1}^{n} x_{i\,i_2 \cdots i_r, \, j_1 \cdots j_r} = \sum_{j=1}^{n} x_{i_1 \cdots i_r, \, j\,j_2 \cdots j_r}$, for all $i_1, \ldots, i_r, \, j_1, \ldots, j_r$.

Condition (i) is the condition that X is in the Schur algebra, and (iii) is equivalent to X commuting with $J_n \otimes I_n^{\otimes(r-1)}$, where $J_n = [1]$ is the $n \times n$ matrix of all 1's and $I_n = [\delta_{ij}]$ is the $n \times n$ identity matrix. Thanks to (i), conditions (ii), (iii) can be place-permuted to any other places.

Finally, if $\Bbbk = \mathbb{R}$, including the additional conditions

(iv) $x_{i_1 \cdots i_r, \, j_1 \cdots j_r} \geqslant 0$,

(v) $\sum_{i_1 \cdots i_r} x_{i_1 \cdots i_r, \, j_1 \cdots j_r} = 1 = \sum_{j_1 \cdots j_r} x_{i_1 \cdots i_r, \, j_1 \cdots j_r}$

(for all $i_1 \cdots i_r$, $j_1 \cdots j_r$) along with conditions (i)–(iii) gives a description of the set Ω of doubly stochastic elements of $\mathrm{im}(\Phi) = \mathrm{End}_{P_r(n)}(\mathbf{V}^{\otimes r})$.

Acknowledgements

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