Degree and Regularity of Eulerian Ideals of Hypergraphs

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Submitted: Apr 9, 2022; Accepted: Aug 31, 2022; Published: Oct 7, 2022

Abstract

We define the Eulerian ideal of a k-uniform hypergraph and study its degree and Castelnuovo–Mumford regularity. The main tool is a Gröbner basis of the ideal obtained combinatorially from the hypergraph. We define the notion of parity join in a hypergraph and show that the regularity of the Eulerian ideal is equal to the maximum cardinality of such a set of edges. The formula for the degree involves the cardinality of the set of sets of vertices, T, that admit a T-join. We compute the degree and regularity explicitly in the cases of a complete k-partite hypergraph and a complete hypergraph of rank three.

Mathematics Subject Classifications: 13A02, 13P10, 13P25, 05E40; 05C65, 05C70

1 Introduction

Eulerian ideals of graphs were introduced in [13], motivated by the notion of vanishing ideals of projective toric sets parameterized by graphs, the study of which started in [14]. Both the Eulerian ideal of a graph and the vanishing ideal of the toric subset parameterized by a graph are homogeneous binomial ideals of the polynomial ring on the edges of the graph and yield one-dimensional, Cohen–Macaulay quotients. These properties are favorable to the study of the Castelnuovo–Mumford regularity and the degree of

*Partially supported by the Centre for Mathematics of the University of Coimbra - UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES

†Partially supported by the Centre for Mathematics of the University of Coimbra - UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES and by the Portuguese Republic / MCTES, the ESF, and the POR_Centro, through FCT – Fundação para a Ciência e a Tecnologia, I.P., within the scope of the grant 2021.05420.BD

these ideals. In the case of the vanishing ideal of the projective toric set parameterized by a graph, there has been substantial progress in the computation of these invariants (*cf.* [5, 6, 8, 10, 12, 15]) and results indicate that they involve the cardinality of the base field. On the other hand, by [13, Proposition 2.8] we known that in the case of the Eulerian ideal, the regularity and the degree do not depend on the base field and therefore they reflect the combinatorics in a clear way. The degree of the Eulerian ideal has a simple formula involving the number of connected components of G and the bipartite property (*cf.* [13, Proposition 2.11]). As for the formula of the regularity, the key combinatorial notion, introduced in [9], is that of *parity join*. These are subsets of edges of the graph which, given any even subgraph of the graph with an *even* number of edges (for example, two edge-disjoint triangles), have no more than half the number of edges of that subgraph in common with it (*cf.* Definition 14, below). Together with a re-interpretation of this invariant in terms of T-joins with cardinalities of fixed parity, these have enabled a combinatorial formula for the Castelnuovo–Mumford regularity of the Eulerian ideal (or, equivalently, the quotient it yields) for a general graph (*cf.* [9, Theorem 4.13]).

The purpose of the present article is to generalize the notions and results of [13, 9] to hypergraphs. To obtain a homogeneous ideal without changing the original definition, we will restrict to k-uniform hypergraphs, i.e., hypergraphs whose edges have cardinality equal to k. We describe a Gröbner basis for the Eulerian ideal of a k-uniform hypergraph and from this basis we derive combinatorial formulas for the degree and the Castelnuovo–Mumford regularity. Finally, we apply these to the explicit computation of these invariants in the case of complete hypergraphs.

The contents are the following. Sections 2 and 3 are devoted to preliminary material; we give the definition of the Eulerian ideal of a hypergraph (Definition 1), a characterization of the homogeneous binomials in the ideal (Proposition 3) and we define T-joins and even subsets in a hypergraph (Definition 5). In Section 4 we describe a Gröbner basis of the ideal (Theorem 10) which we use in Section 5 to describe the Hilbert function of the quotient by the ideal in terms of reduced parity joins (Definition 14 and Theorem 17). In Sections 6 and 7 we give combinatorial formulas for the invariants regularity (Theorem 24) and degree (Theorem 26). In Section 8, we compute explicitly these invariants for some families of complete hypergraphs.

2 The ideal

Let $\mathcal{H}$ be a hypergraph. More precisely, let $\mathcal{H} = (V_\mathcal{H}, E_\mathcal{H})$ where $V_\mathcal{H}$ is a set (of vertices) and $E_\mathcal{H}$ is a set of subsets of $V_\mathcal{H}$, the elements of which we call edges of $\mathcal{H}$. All hypergraphs in this article are assumed to be finite, with nonempty edge set and k-uniform, with $k \geqslant 2$, i.e., their edges have common cardinality equal to k. Simple graphs are identified with 2-uniform hypergraphs. Fix K a field and let

$$K[V_\mathcal{H}] = K[x_v : v \in V_\mathcal{H}] \quad \text{and} \quad K[E_\mathcal{H}] = K[t_e : e \in E_\mathcal{H}]$$

be polynomial rings. Define a homomorphism of graded rings, $\varphi\colon K[E_\mathcal{H}] \to K[V_\mathcal{H}]$, by $\varphi(t_e) = \prod_{v\in e} x_v$, for all $e \in E_\mathcal{H}$. Throughout we will use the multi-index notation for monomials, i.e., whenever $\alpha \in \mathbb{N}^{E_{\mathcal{H}}}$ is a nonnegative integer valued function of the set of edges, $\mathbf{t}^\alpha \in K[E_{\mathcal{H}}]$ will denote the monomial:

$$\mathbf{t}^\alpha = \prod_{e \in E_{\mathcal{H}}} t_e^{\alpha(e)}.$$

If $\alpha \in \mathbb{Z}^{E_{\mathcal{H}}}$ is an integer-valued function of the edge set of $\mathcal{H}$, we will use $\operatorname{supp}(\alpha)$ to denote its support, i.e., $\operatorname{supp}(\alpha) = \{e \in E_{\mathcal{H}} : \alpha(e) \neq 0\}$. Similarly for functions of the vertex set γ and monomials $\mathbf{x}^\gamma \in K[V_{\mathcal{H}}]$.

Definition 1. Let $\mathcal{H}$ be a k-uniform hypergraph. With the above notations, the Eulerian ideal of $\mathcal{H}$ is the ideal $I_{\mathcal{H}} = \varphi^{-1}(x_v^2 - x_w^2 : v, w \in V_{\mathcal{H}})$.

Since φ is a graded homomorphism and $I_{\mathcal{H}}$ is the preimage of a homogeneous binomial ideal, by a standard elimination argument, the Eulerian ideal of $\mathcal{H}$ is also a homogeneous binomial ideal (*Cf.*, for instance, [13, Proposition 2.2]). In the graph case, the name Eulerian ideal comes from the characterization of a Gröbner basis of this ideal in terms of the Eulerian subgraphs of the graph (*cf.* [9, Theorem 3.3]). In the hypergraph case, the role of Eulerian subgraphs will be taken by the even subsets of edges (*cf.* Definition 5), as we will see later. This is equivalent in the case of graphs and, in the case of hypergraphs, is a direct approach that avoids considering any of the several notions of connectivity. The motivation for this is the next result, which is a characterization of the binomials in the Eulerian ideal, along the lines of [13, Proposition 2.5]. Let us first fix some notation.

Definition 2. (i) If $C \subset E_{\mathcal{H}}$ and $v \in V_{\mathcal{H}}$, let $\deg_C(v)$ denote $\sum_{e \in C} |\{v\} \cap e|$. (ii) If $\alpha \in \mathbb{Z}^{E_{\mathcal{H}}}$, let

$$\operatorname{supp}_2(\alpha) = \{e \in E_{\mathcal{H}} : \alpha(e) \equiv_2 1\} \subset E_{\mathcal{H}}.$$

Proposition 3. *Let $\mathbf{t}^\alpha - \mathbf{t}^\beta \in K[E_{\mathcal{H}}]$ be homogeneous and let $C = \operatorname{supp}_2(\alpha - \beta)$. Then $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I_{\mathcal{H}}$ if and only if, for every $v \in V_{\mathcal{H}}$, $\deg_C(v)$ is even.*

Proof. Let $\varphi(\mathbf{t}^\alpha - \mathbf{t}^\beta) = \mathbf{x}^\gamma - \mathbf{x}^\mu$, for some $\gamma, \mu \in \mathbb{N}^{V_{\mathcal{H}}}$. Then, for every $v \in V_{\mathcal{H}}$,

$$\begin{aligned}\deg_C(v) &= \textstyle\sum_{e \in C} |\{v\} \cap e| \\ &\equiv_2 \textstyle\sum_{e \in E_{\mathcal{H}}} |\{v\} \cap e| \cdot (\alpha(e) - \beta(e)) \\ &\equiv_2 \gamma(v) - \mu(v).\end{aligned}$$

Suppose that $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I_{\mathcal{H}}$. Then $\mathbf{x}^\gamma - \mathbf{x}^\mu \in (x_v^2 - x_w^2 : v, w \in V_{\mathcal{H}})$ and therefore there exist $g_{vw} \in K[E_{\mathcal{H}}]$ such that

$$\mathbf{x}^\gamma - \mathbf{x}^\mu = \sum_{v,w \in E_{\mathcal{H}}} g_{vw}(x_v^2 - x_w^2).$$

Fix $v \in V_{\mathcal{H}}$. Setting all variables but x_v, in the expression above, equal to 1 we get:

$$x_v^{\gamma(v)} - x_v^{\mu(v)} = h(x_v^2 - 1)$$

for some $h \in K[x_v]$. Writing h as linear combination of monomials in x_v and working out the product on the left explicitly, we deduce that there exists $m \in \mathbb{Z}$ such that $\gamma(v) = \mu(v) + 2m$, as required.

Conversely, suppose that $\gamma(v) - \mu(v)$ is even, for every $v \in V_{\mathcal{H}}$. Then there exists $\rho \in \{0,1\}^{V_{\mathcal{H}}}$ such that $\gamma = 2\gamma' + \rho$ and $\mu = 2\mu' + \rho$ for some $\gamma', \mu' \in \mathbb{N}^{V_{\mathcal{H}}}$. Let us fix $u \in V_{\mathcal{H}}$ and let us work with the relations $x_v^2 \equiv x_u^2$ modulo the ideal $(x_v^2 - x_w^2 : v, w \in V_{\mathcal{H}})$. Then

$$\mathbf{x}^\gamma - \mathbf{x}^\mu \equiv x_u^{2|\gamma'|}\mathbf{x}^\rho - x_u^{2|\mu'|}\mathbf{x}^\rho,$$

where $|\gamma'| = \sum_{v \in V_{\mathcal{H}}} \gamma'(v)$ and $|\mu'| = \sum_{v \in V_{\mathcal{H}}} \mu'(v)$. As $\mathbf{x}^\gamma - \mathbf{x}^\mu$ is homogeneous (because $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is), we get $|\gamma'| = |\mu'|$ and thus the last binomial above is equal to zero. We conclude that $\varphi(\mathbf{t}^\alpha - \mathbf{t}^\beta) \in (x_v^2 - x_w^2 : v, w \in V_{\mathcal{H}})$, i.e., $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I_{\mathcal{H}}$. □

Example 4. Let $\mathcal{H}$ be the 3-uniform hypergraph on $V_{\mathcal{H}} = \{1,2,3,4,5\}$ with

$$E_{\mathcal{H}} = \{\{1,2,4\},\{2,3,4\},\{1,3,4\},\{1,2,5\},\{2,3,5\},\{1,3,5\}\}, \tag{1}$$

depicted in Figure 1 as the faces of a triangular bipyramid. Consider the set of faces

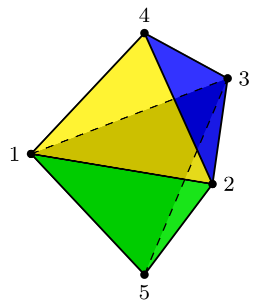

Figure 1: A 3-uniform graph.

incident to vertex 2,

$$J = \{\{1,2,4\},\{2,3,4\},\{1,2,5\},\{2,3,5\}\}. \tag{2}$$

One checks easily that $\deg_J(v)$ is even, for every $v \in V_{\mathcal{H}}$. Hence, by Proposition 3, if $J = A \sqcup B$ is any partition of J into parts of cardinality 2, the binomial

$$\textstyle\prod_{e \in A} t_e - \prod_{e \in B} t_e$$

is an element of $I_{\mathcal{H}}$. For example, $t_{124}t_{234} - t_{125}t_{235} \in I_{\mathcal{H}}$. There are 6 binomials coming from these choices. In a minimal generating set, which one can compute using Macaulay2, there are 8 binomials of this type (of the 18 we get by varying J as the set of edges incident to 1, 2 or 3, only 8 are linearly independent) and there are 5 other binomials of the form $t_e^2 - t_f^2$, for $e, f \in E_{\mathcal{H}}$.

3 T-joins

The characterization of the binomials in $I_{\mathcal{H}}$ given in Proposition 3 leads us to the notion of T-join in hypergraphs. This notion will play a key role in the computation of the regularity and the degree of $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$.

Definition 5. (i) If $T \subset V_{\mathcal{H}}$ and $J \subset E_{\mathcal{H}}$, we say that J is a T-join if and only if

$$\deg_J(v) = \textstyle\sum_{e\in J} |\{v\} \cap e| \text{ is odd } \iff v \in T.$$

In particular, a $\emptyset$-join is a subset of edges $J \subset E_{\mathcal{H}}$ such that $\deg_J(v)$ is even, for every $v \in V_{\mathcal{H}}$. These subsets of edges are called *even.* (ii) Let us denote the set of all even subsets of $E_{\mathcal{H}}$ by $\mathcal{E}(E_{\mathcal{H}})$. (iii) Let us denote by $\mathcal{T}(V_{\mathcal{H}})$ the set of all $T \subset V_{\mathcal{H}}$ for which there exists at least a T-join.

Given $T \subset V_{\mathcal{H}}$, the question of existence of a T-join is pertinent. When $T = \emptyset$ or when T is the set of vertices of a single edge, then a T-join always exists, namely the empty set and the singleton of the edge in question, respectively. In general, not every subset $T \subset V_{\mathcal{H}}$ admits a T-join. It is well-known that for $k = 2$, i.e., for graphs, if a T-join exists for a given $T \subset V_{\mathcal{H}}$ then $|J \cap E_{\mathcal{C}}|$ must be even, for every connected component $\mathcal{C}$ of the graph $\mathcal{H}$. The converse also holds (*cf. [7, Proposition 12.7]*). The situation for odd k is different. For example in the hypergraph of Example 4 the set of edges $J = \{\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$, corresponding to the pyramid on the top, is a $\{4\}$-join.

The following is a minimal set of restrictions on the elements of $\mathcal{T}(V_{\mathcal{H}})$, which we will use in the remainder of this article.

Proposition 6. *Let $T \in \mathcal{T}(V_{\mathcal{H}})$ and let J be a T-join. (i) If k is even then $|T|$ is even. (ii) If k is odd then $|T| \equiv_2 |J|$. In particular, when k is odd there are no elements of $\mathcal{E}(E_{\mathcal{H}})$ with odd cardinality.*

Proof. Since

$$\textstyle\sum_{v\in V_{\mathcal{H}}} \deg_J(v) = \sum_{v\in T} \deg_J(v) + \sum_{v\notin T} \deg_J(v) = k|J|,$$

if k is even, then $|T| \equiv_2 0$ and if k is odd, then $|T| \equiv_2 |J|$. □

Many properties of T-joins in graphs extend to hypergraphs. One example is the relation of T-joins with the symmetric difference of sets, which we denote by $\triangle$. Namely, the fact that if J_1 is a T_1-join and J_2 is a T_2-join then $J_1 \triangle J_2$ is a $(T_1 \triangle T_2)$-join. This follows from the observation that, for every $v \in V_{\mathcal{H}}$,

$$\deg_{J_1\triangle J_2}(v) = \textstyle\sum_{e\in J_1} |\{v\} \cap e| + \sum_{e\in J_2} |\{v\} \cap e| - 2\sum_{e\in J_1\cap J_2} |\{v\} \cap e|.$$

In particular, $\mathcal{E}(E_H)$, the set of even subsets of $E_{\mathcal{H}}$ is closed under the symmetric difference.

4 A Gröbner basis

Definition 7. (i) Denote $\mathbf{t} = \{t_e^2 - t_f^2 : e, f \in E_{\mathcal{H}}\}$. (ii) A binomial $\mathbf{t}^\alpha - \mathbf{t}^\beta$, is said an Eulerian binomial if $\mathbf{t}^\alpha$ and $\mathbf{t}^\beta$ are square-free, coprime, of same degree and $\operatorname{supp}(\alpha) \sqcup \operatorname{supp}(\beta) \subset E_{\mathcal{H}}$ belongs to $\mathcal{E}(E_{\mathcal{H}})$. Let $\mathscr{E}$ denote the finite set of all Eulerian binomials. (iii) Let us denote $\mathscr{G} = \mathbf{t} \cup \mathscr{E}$.

Note that, here, the even subset $\operatorname{supp}(\alpha) \sqcup \operatorname{supp}(\beta) \subset E_{\mathcal{H}}$ has even cardinality, since $\mathbf{t}^\alpha$ and $\mathbf{t}^\beta$ are square-free, coprime and of the same degree.

Proposition 8. $\mathscr{G} = \mathbf{t} \cup \mathscr{E} \subset I_{\mathcal{H}}$.

Proof. Let $\mathbf{t}^\alpha - \mathbf{t}^\beta \in \mathscr{G}$ and denote $C = \operatorname{supp}_2(\alpha - \beta)$. By Proposition 3, we must show that C is even. If $\mathbf{t}^\alpha - \mathbf{t}^\beta \in \mathbf{t}$ then $C = \emptyset$ and therefore $\deg_\emptyset(v) = 0$, for every $v \in V_{\mathcal{H}}$ (*cf.* Definition 2). If $\mathbf{t}^\alpha - \mathbf{t}^\beta \in \mathscr{E}$ then

$$\operatorname{supp}_2(\alpha - \beta) = \operatorname{supp}(\alpha) \sqcup \operatorname{supp}(\beta)$$

which, by definition, is even. $\square$

From now on, fix a total order on the set $E_{\mathcal{H}}$ and consider the associated graded reverse lexicographic order on $K[E_{\mathcal{H}}]$. The next result gives a sufficient condition for a binomial to reduce to zero modulo the set of binomials $\mathscr{G} = \mathbf{t} \cup \mathscr{E}$.

Proposition 9. *Let* $\mathbf{t}^\alpha - \mathbf{t}^\beta \in K[E_{\mathcal{H}}]$ *be homogeneous. If* $\operatorname{supp}_2(\alpha - \beta)$ *is even, then, with respect to the graded reverse lexicographic order,* $\mathbf{t}^\alpha - \mathbf{t}^\beta$ *reduces to zero modulo* $\mathscr{G} = \mathbf{t} \cup \mathscr{E}$.

Proof. We will use complete induction on the degree of $\mathbf{t}^\alpha - \mathbf{t}^\beta$. If $\deg(\mathbf{t}^\alpha - \mathbf{t}^\beta) = 0$, then $\mathbf{t}^\alpha - \mathbf{t}^\beta = 0$ and there is nothing to show. Assume that $\deg(\mathbf{t}^\alpha - \mathbf{t}^\beta) > 0$. Let $\mathbf{t}^\delta = \gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta)$ and assume that $\mathbf{t}^\delta \neq 1$. Let us write $\mathbf{t}^\alpha = \mathbf{t}^\delta \mathbf{t}^\gamma$ and $\mathbf{t}^\beta = \mathbf{t}^\delta \mathbf{t}^\mu$. Then

$$\operatorname{supp}_2(\alpha - \beta) = \operatorname{supp}_2(\gamma - \mu).$$

As $\deg(\mathbf{t}^\gamma - \mathbf{t}^\mu) < \deg(\mathbf{t}^\alpha - \mathbf{t}^\beta)$, by induction, $\mathbf{t}^\gamma - \mathbf{t}^\mu \xrightarrow{\mathscr{G}} 0$. We may thus restrict to the case when $\mathbf{t}^\alpha$ and $\mathbf{t}^\beta$ are coprime.

Assume, without loss of generality, that the leading term of $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is $\mathbf{t}^\alpha$. If $\mathbf{t}^\alpha$ and $\mathbf{t}^\beta$ are both square-free, then $\mathbf{t}^\alpha - \mathbf{t}^\beta \in \mathscr{E}$ and we have finished. Suppose that $\mathbf{t}^\alpha$ is square-free but there exists $\mathbf{t}^\epsilon \neq 1$ such that $\mathbf{t}^\beta = (\mathbf{t}^\epsilon)^2 \mathbf{t}^\eta$, for suitable square-free $\mathbf{t}^\eta$. Let $\mathbf{t}^\sigma$ denote the product of the least d edges in $\operatorname{supp}(\alpha)$, where $d = \deg(\mathbf{t}^\epsilon)$, and consider $\mathbf{t}^\alpha \mathbf{t}^{-\sigma} - \mathbf{t}^\eta \mathbf{t}^\sigma$. Since $\deg(\mathbf{t}^\alpha) = \deg(\mathbf{t}^\beta)$ we deduce that $\mathbf{t}^\alpha \mathbf{t}^{-\sigma} - \mathbf{t}^\eta \mathbf{t}^\sigma$ is homogeneous. Furthermore, since $\mathbf{t}^\alpha \mathbf{t}^{-\sigma}$ and $\mathbf{t}^\eta \mathbf{t}^\sigma$ are square-free, coprime monomials and since

$$\operatorname{supp}(\alpha - \sigma) \sqcup \operatorname{supp}(\eta + \sigma) = \operatorname{supp}(\alpha) \sqcup \operatorname{supp}(\eta) = \operatorname{supp}_2(\alpha - 2\epsilon - \eta)$$

which, by assumption, is even, we conclude that $\mathbf{t}^\alpha \mathbf{t}^{-\sigma} - \mathbf{t}^\eta \mathbf{t}^\sigma$ is, in fact, an Eulerian binomial. As $\mathbf{t}^\sigma$ is the product of a set of least edges in $\operatorname{supp}(\alpha)$, its leading term is

$\mathbf{t}^\alpha\mathbf{t}^{-\sigma}$, which divides $\mathbf{t}^\alpha$, the leading term of $\mathbf{t}^\alpha - (\mathbf{t}^\epsilon)^2\mathbf{t}^\sigma$. Using this Eulerian binomial in a one-step reduction, we get:

$$\mathbf{t}^\alpha - (\mathbf{t}^\epsilon)^2\mathbf{t}^\eta \xrightarrow{\mathscr{E}} \mathbf{t}^\eta\big[(\mathbf{t}^\sigma)^2 - (\mathbf{t}^\epsilon)^2\big].$$

Hence we may reduce to the case that $\mathbf{t}^\alpha$ is not square-free (since the leading term of $(\mathbf{t}^\sigma)^2 - (\mathbf{t}^\epsilon)^2$ is certainly not square-free). Suppose then that there exists $e \in E_{\mathcal{H}}$ such that $t_e^2 \mid \mathbf{t}^\alpha$. Let $f \in E_{\mathcal{H}}$ be an edge, less than e, such that $t_f \mid \mathbf{t}^\beta$. Then $t_e^2 - t_f^2$ has leading term t_e^2. Using this element of $\mathbf{t}$ in a one-step reduction, we get:

$$\mathbf{t}^\alpha - \mathbf{t}^\beta \xrightarrow{\mathbf{t}} t_f(t_f\mathbf{t}^\alpha t_e^{-2} - \mathbf{t}^\beta t_f^{-1}).$$

If $\gamma, \mu \in \mathbb{N}^{E_{\mathcal{H}}}$ are such that $\mathbf{t}^\gamma = t_f\mathbf{t}^\alpha t_e^{-2}$ and $\mathbf{t}^\mu = \mathbf{t}^\beta t_f^{-1}$ then

$$\operatorname{supp}_2(\gamma - \mu) = \operatorname{supp}_2(\alpha - \beta).$$

Hence, by induction $\mathbf{t}^\gamma - \mathbf{t}^\mu \xrightarrow{\mathscr{G}} 0$. □

Next, let us use Propositions 3 and 9 to show that $\mathscr{G} = \mathbf{t} \cup \mathscr{E}$ is a Gröbner basis for the Eulerian ideal with respect to the graded reverse lexicographic order. This result generalizes [9, Theorem 3.3].

Theorem 10. *With respect to the graded reverse lexicographic order, $\mathscr{G} = \mathbf{t} \cup \mathscr{E}$ is a Gröbner basis of the Eulerian ideal $I_{\mathcal{H}}$.*

Proof. We will use Buchberger's criterion and we start by showing that $I_{\mathcal{H}} = (\mathscr{G})$. By Proposition 8, $(\mathscr{G}) \subset I_{\mathcal{H}}$. To prove the reverse inclusion, we will use the fact that $I_{\mathcal{H}}$ is generated by homogeneous binomials. Let $\mathbf{t}^\alpha - \mathbf{t}^\beta$ be a homogeneous binomial in $I_{\mathcal{H}}$ and let $C = \operatorname{supp}_2(\alpha - \beta)$. Then, by Proposition 3, C is even and hence, by Proposition 9, $\mathbf{t}^\alpha - \mathbf{t}^\beta$ reduces to zero modulo $\mathscr{G} = \mathbf{t} \cup \mathscr{E}$, which implies that $\mathbf{t}^\alpha - \mathbf{t}^\beta \in (\mathscr{G})$.

If $\mathbf{t}^\alpha - \mathbf{t}^\beta \in \mathscr{G}$ then $\operatorname{supp}_2(\alpha - \beta)$ is even. To prove that $S(f, g)$ reduces to zero modulo $\mathscr{G}$ this property of $f, g \in \mathscr{G}$ will suffice. Assume, without loss of generality, that $f = \mathbf{t}^\alpha - \mathbf{t}^\beta$ and $g = \mathbf{t}^\gamma - \mathbf{t}^\mu$ with $\operatorname{lt}(f) = \mathbf{t}^\alpha$, $\operatorname{lt}(g) = \mathbf{t}^\gamma$. Let us denote

$$C_1 = \operatorname{supp}_2(\alpha - \beta), \quad C_2 = \operatorname{supp}_2(\gamma - \mu)$$

and let $\mathbf{t}^\delta = \gcd(\mathbf{t}^\alpha, \mathbf{t}^\gamma)$. The S-polynomial of f and g is equal to $\mathbf{t}^{\alpha+\mu-\delta} - \mathbf{t}^{\beta+\gamma-\delta}$, which is, of course, a homogeneous binomial. Let

$$\begin{aligned} C &= \operatorname{supp}_2(\alpha + \mu - \delta - \beta - \gamma + \delta) \\ &= \operatorname{supp}_2(\alpha - \beta) \triangle \operatorname{supp}_2(\gamma - \mu) \\ &= C_1 \triangle C_2. \end{aligned}$$

Since $C_1, C_2 \subset E_{\mathcal{H}}$ are even subsets, we deduce that C is even. Therefore, by Proposition 9, $S(f, g)$ reduces to zero modulo $\mathscr{G}$. □

5 The Hilbert function

As is well-known, a Gröbner basis of $I_{\mathcal{H}}$ enables an explicit characterization of a monomial basis for the quotient $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$. We will use this idea to determine a combinatorial formula for the Hilbert function and series of $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$. The result we obtain here is a generalization of the corresponding result for the case of graphs [11, Theorem 2.7]. Despite that $K = \mathbb{Z}/3$ in [11], because the emphasis there is on parameterized codes over graphs, this result holds over any field, as will be shown here. The key combinatorial invariant involved is the notion of *parity join* for hypergraphs, introduced below, which was first defined in [9], for graphs. This notion is related to the notion of *join* (*cf.* [3]).

Definition 11. Let $I \subset K[x_1, \ldots, x_n]$ be a homogeneous ideal in a polynomial ring endowed with a choice of a monomial order. Given $d \geqslant 0$, let $\mathcal{B}_d(I)$ denote the set of degree d monomials that do not belong to the initial ideal of I.

It is well known that the cosets with representative in $\mathcal{B}_d(I)$ form a K-basis for the degree d component of $K[x_1, \ldots, x_n]/I$. (*Cf.*, for example, [2, Theorem 2.6]). In particular, $\dim\big(K[x_1, \ldots, x_n]/I\big)_d = |\mathcal{B}_d(I)|$. From now on, let us denote by $\ell \in E_{\mathcal{H}}$ the least edge of $\mathcal{H}$. Consider $(I_{\mathcal{H}}, t_\ell^2)$. The quotient $K[E_{\mathcal{H}}]/(I_H, t_\ell^2)$ is an Artinian ring, because $t_e^2 \in (I_{\mathcal{H}}, t_\ell^2)$, for every $e \in E_{\mathcal{H}}$. We will use this quotient to characterize the Hilbert function of $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$. Let us first make use of the Gröbner basis, $\mathscr{G}$, of $I_{\mathcal{H}}$ given in Definition 7, to give an explicit characterization of the set $\mathcal{B}_d(I_{\mathcal{H}}, t_\ell^2)$.

Lemma 12. *If $\mathscr{G}$ is the Gröbner basis of $I_{\mathcal{H}}$ given in Definition 7, then*

$$\mathcal{B}_d(I_{\mathcal{H}}, t_\ell^2) = \big\{\mathbf{t}^\gamma \in K[E_{\mathcal{H}}] : \deg(\mathbf{t}^\gamma) = d \text{ and } \operatorname{lt}(g) \nmid \mathbf{t}^\gamma, \text{ for all } g \in \mathscr{G} \cup \big\{t_\ell^2\big\}\big\}.$$

Proof. Given that $\mathscr{G}$ is a Gröbner basis of $I_{\mathcal{H}}$ with respect to the graded reverse lexicographic order and no leading term of an element of $\mathscr{G}$ is divisible by t_ℓ we deduce that $\mathscr{G} \cup \{t_\ell^2\}$ is a Gröbner basis for $(I_{\mathcal{H}}, t_\ell^2)$, i.e., the ideal of leading terms of $(I_{\mathcal{H}}, t_\ell^2)$ is generated by the leading terms of $\mathscr{G} \cup \{t_\ell^2\}$ and the result follows. $\square$

Example 13. Let us go back to the 3-uniform hypergraph of Example 4. Fix the order of $E_{\mathcal{H}}$ as given in (1), so that the last variable is $t_\ell = t_{135}$. The following is a list of the sets $\mathcal{B}_d(I_{\mathcal{H}}, t_\ell^2)$, for $d \geqslant 0$, computed using Macaulay2.

d	$\mathcal{B}_d(I_{\mathcal{H}}, t_\ell^2)$
0	$\{1\}$
1	$\{t_{124},\ t_{234},\ t_{134},\ t_{125},\ t_{235},\ t_{135}\}$
2	$\{t_{124}t_{235},\ t_{124}t_{135},\ t_{234}t_{135},\ t_{134}t_{135},\ t_{125}t_{235},\ t_{125}t_{135},\ t_{235}t_{135}\}$
3	$\{t_{124}t_{235}t_{135},\ t_{125}t_{235}t_{135}\}$
$\geqslant 4$	$\emptyset$

Since, for every $e \neq \ell$, $t_e^2 - t_\ell^2 \in \mathscr{G}$ has leading term equal to t_e^2, from Lemma 12, we deduce that the elements of $\mathcal{B}(I_{\mathcal{H}}, t_\ell^2)$ are square-free monomials, as we can check directly in this example. Hence all the monomials in $\mathcal{B}_d(I_{\mathcal{H}}, t_\ell^2)$ are in bijection with a certain set of subsets of d edges of $\mathcal{H}$.

Definition 14. (i) If $J \subset E_{\mathcal{H}}$, then J is said a *parity join* if

$$|J \cap C| \leqslant \tfrac{|C|}{2},$$

for all $C \in \mathcal{E}(E_{\mathcal{H}})$, with $|C|$ *even.* (ii) Let us denote

$$\mu_{\mathrm{p}}(\mathcal{H}) = \max\left\{|J| : J \text{ is a parity join}\right\}.$$

(iii) With respect to the total order of $E_{\mathcal{H}}$ we are fixing, J is called a *reduced parity join* if J is a parity join and, for every nonempty $C \in \mathcal{E}(E_{\mathcal{H}})$ such that $|C|$ is even and $|J \cap C| = \frac{|C|}{2}$, J contains the least edge of C. We denote the set of reduced parity joins of cardinality d by $\mathcal{J}_d^{\mathrm{r}}(\mathcal{H})$.

We will see next that $\mathcal{B}_d(I_{\mathcal{H}}, t_\ell^2)$ and $\mathcal{J}_d^{\mathrm{r}}(\mathcal{H})$ are in bijection. The maximum cardinality of a parity join, $\mu_{\mathrm{p}}(\mathcal{H})$, is related to an important invariant of $\mathcal{K}[E_{\mathcal{H}}]/I_{\mathcal{H}}$, the Castelnuovo–Mumford regularity; which is the topic of the next section.

Theorem 15. *The map $\mathcal{B}_d(I_{\mathcal{H}}, t_\ell^2) \to \mathcal{J}_d^{\mathrm{r}}(\mathcal{H})$ defined by $\mathbf{t}^\gamma \mapsto \operatorname{supp}(\gamma)$ is well-defined and a bijection.*

Proof. Let $C \subset E_{\mathcal{H}}$ be an even set of even cardinality. Suppose that $J = \operatorname{supp}(\gamma)$ satisfies $|J \cap C| > \frac{|C|}{2}$. Let $\mathbf{t}^\alpha$ be the product of the first $\frac{|C|}{2}$ edges in $J \cap C$ and let $\mathbf{t}^\beta$ denote the product of the remaining edges of C. Then $\mathbf{t}^\alpha - \mathbf{t}^\beta \in \mathscr{E}$ has leading term equal to $\mathbf{t}^\alpha$. This is also the case if $|J \cap C| = \frac{|C|}{2}$ and J does not contain the least edge in C. Both cases lead to a contradiction. Since $\mathbf{t}^\gamma$ is square free and thus $|J| = \deg(\mathbf{t}^\gamma) = d$, we deduce that $J \in \mathcal{J}_d^{\mathrm{r}}(\mathcal{H})$. Hence the map is well-defined. Using again the square-free property of the elements of $\mathcal{B}_d(I_{\mathcal{H}}, t_\ell^2)$ we see that the map is injective.

Let us now prove that the map is surjective. Let $J \subset E_{\mathcal{H}}$ be a reduced parity join and let $\mathbf{t}^\gamma$ be the product of the edges in J. Since $\mathbf{t}^\gamma$ is square-free, it suffices to show that $\mathbf{t}^\gamma$ is not divisible by the leading term of an element in $\mathscr{E} \subset \mathscr{G}$. Let us consider an Eulerian binomial $\mathbf{t}^\alpha - \mathbf{t}^\beta$ and assume, without loss of generality, that $\operatorname{lt}(\mathbf{t}^\alpha - \mathbf{t}^\beta) = \mathbf{t}^\alpha$. Then, this means that $\operatorname{supp}(\beta)$ contains the least edge of the corresponding even set

$$C = \operatorname{supp}(\alpha) \sqcup \operatorname{supp}(\beta) \subset E_{\mathcal{H}}.$$

Assume, with a view to a contradiction that $\mathbf{t}^\alpha \mid \mathbf{t}^\gamma$. Then $|J \cap C| \geqslant \deg(\mathbf{t}^\alpha) = \frac{|C|}{2}$. Since J is a parity join, we deduce that $|J \cap C| = \frac{|C|}{2}$ and therefore $J \cap C = \operatorname{supp}(\alpha)$. But then, as J is a reduced parity join, the least edge of C must be in J, which is to say that it belongs to $\operatorname{supp}(\alpha)$. This is a contradiction. $\square$

Before we use Theorem 15 to give a combinatorial formula for the Hilbert function of $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$ we need to show that t_ℓ^2 is a regular element for this quotient. We will show that, in fact, any monomial has this property. The proof can be taken almost *verbatim* from the proof of [13, Proposition 2.1].

Lemma 16. *If $\mathbf{t}^\gamma \in K[E_{\mathcal{H}}]$ is a monomial, then $\mathbf{t}^\gamma$ is $\big(K[E_{\mathcal{H}}]/I_{\mathcal{H}}\big)$-regular.*

Proof. Given the definition of $I_{\mathcal{H}}$ (*cf.* Definition 1), it suffices to show that if x_u is any variable in $K[V_{\mathcal{H}}]=K[x_v : v\in V_{\mathcal{H}}]$ and $f\in K[V_{\mathcal{H}}]$, then

$$x_u f\in (x_v^2-x_w^2 : v,w\in V_{\mathcal{H}}) \iff f\in (x_v^2-x_w^2 : v,w\in V_{\mathcal{H}}).$$

Fix a total order of the vertices of $\mathcal{H}$ for which $u\in V_{\mathcal{H}}$ is the least vertex. Then

$$\mathfrak{g}=\left\{x_v^2-x_u^2 : v\in V_{\mathcal{H}}, v\neq u\right\}$$

is a Gröbner basis of $(x_v^2-x_w^2 : v,w\in V_{\mathcal{H}})$ with the respect to the graded reverse lexicographic order. Assume $x_u f\in (x_v^2-x_w^2 : v,w\in V_{\mathcal{H}})$. Then $x_u f$ reduces to zero modulo $\mathfrak{g}$. As no leading term of an element of $\mathfrak{g}$ is divisible by x_u, we conclude that x_u is a factor of every one-step reduction in the division algorithm, which is to say that f reduces to zero modulo $\mathfrak{g}$ and hence $f\in (x_v^2-x_w^2 : v,w\in V_{\mathcal{H}})$. □

Theorem 17. $\dim_K\big(K[E_G]/I_{\mathcal{H}}\big)_d=\sum_{j\geqslant 0}|\mathcal{J}^{\mathrm{r}}_{d-2j}(\mathcal{H})|$.

Proof. Since t_ℓ^2 is $\big(K[E_{\mathcal{H}}]/I_{\mathcal{H}}\big)$-regular, the following is a short exact sequence of graded homomorphisms of $K[E_{\mathcal{H}}]$-modules:

$$0\to \frac{K[E_{\mathcal{H}}]}{I_{\mathcal{H}}}[-2]\xrightarrow{\cdot t_\ell^2}\frac{K[E_{\mathcal{H}}]}{I_{\mathcal{H}}}\to\frac{K[E_{\mathcal{H}}]}{(I_{\mathcal{H}},t_\ell^2)}\to 0. \tag{3}$$

Using Theorem 15 and the above, we deduce that

$$\dim_K\big(K[E_G]/I_{\mathcal{H}}\big)_d=\dim_K\big(K[E_G]/I_{\mathcal{H}}\big)_{d-2}+|\mathcal{J}^{\mathrm{r}}_d(\mathcal{H})|.$$

Iterating, we obtain $\dim_K\big(K[E_G]/I_{\mathcal{H}}\big)_d=\sum_{j\geqslant 0}|\mathcal{J}^{\mathrm{r}}_{d-2j}(\mathcal{H})|$. □

Corollary 18. *$K[E_{\mathcal{H}}]/I_{\mathcal{H}}$ is a 1-dimensional and Cohen–Macaulay ring.*

Proof. For $d>\mu_{\mathrm{p}}(\mathcal{H})$ the set $\mathcal{J}^{\mathrm{r}}_d(\mathcal{H})$ is empty. Therefore, by Theorem 17, the Hilbert polynomial is a nonzero constant and hence $\dim K[E_H]/I_H=1$. Since any monomial $\mathbf{t}^\gamma$ is $\big(K[E_{\mathcal{H}}]/I_{\mathcal{H}}\big)$-regular, we deduce that $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$ is Cohen–Macaulay. □

Corollary 19. *Let $s=|E_{\mathcal{H}}|$ and, for every $0\leqslant d\leqslant s$, let $j_d=|\mathcal{J}^{\mathrm{r}}_d(\mathcal{H})|$ denote the number of reduced parity joins of cardinality d. The Hilbert series of $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$, in the variable z, is equal to:*

$$\frac{1+sz+j_2z^2+\cdots+j_sz^s}{1-z^2} \tag{4}$$

Proof. Let us denote by $F(z)$ the Hilbert series of $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$ and by M the quotient $\frac{K[E_{\mathcal{H}}]}{(I_{\mathcal{H}},t_\ell^2)}$. By (3) and Theorem 15,

$$(1-z^2)F(z)=\textstyle\sum_{d\geqslant 0}\big(\dim_K M_d\big)z^d=\sum_{d\geqslant 0}j_dz^d.$$

Since $j_0=1$, $j_1=|E_{\mathcal{H}}|=s$ and $j_d=0$, for all $d>s$, the formula follows. □

6 Regularity

As we saw in the previous section, the Hilbert function of $K[E_{\mathcal{H}}]/I_H$ is related to reduced parity joins. We will show that the Castelnuovo–Mumford regularity of this graded ring is related to the maximum cardinality of a reduced parity join. However there is a way to give this result in terms of the simpler notion of parity join and $\mu_{\mathrm{p}}(\mathcal{H})$ and that is what we will do. The way to do this is to exploit the relation between parity joins and T-joins. The next result, which we use in the computation of the maximum cardinality is the analogue for hypergraphs of [9, Lemmas 4.6 and 4.12].

Proposition 20. *Let $T \in \mathcal{T}(V_{\mathcal{H}})$ and let J be a T-join. (i) There exist T-joins J_1, J_2 with $|J_1| \not\equiv_2 |J_2|$ if and only if there exists $C \in \mathcal{E}(E_{\mathcal{H}})$ with $|C|$ odd. (ii) J is a parity join if and only if J has minimum cardinality among all T-joins J' such that $|J'| \equiv_2 |J|$.*

Proof. (i) Let $J_1, J_2 \subset E_{\mathcal{H}}$ be T-joins with $|J_1| \not\equiv_2 |J_2|$. Then $C = J_1 \triangle J_2 \subset E_{\mathcal{H}}$ is an even set of odd cardinality. Conversely, if $C \in \mathcal{E}(E_{\mathcal{H}})$ and $|C|$ is odd then $J \triangle C$ is another T-join and $|J \triangle C| \not\equiv_2 |J|$.

(ii) Reasoning as before we deduce that

$$\{J \triangle C : C \in \mathcal{E}(E_{\mathcal{H}}) \text{ and } |C| \text{ is even}\}. \tag{5}$$

is the set of all T-joins, $J' \subset E_{\mathcal{H}}$, with $|J'| \equiv_2 |J|$. If $C \in \mathcal{E}(E_{\mathcal{H}})$ with $|C|$ even, then

$$|J \cap C| \leqslant \tfrac{|C|}{2} \iff |J \triangle C| \geqslant |J|.$$

Therefore J is a parity join if and only if $|J|$ is the minimum cardinality of an element of (5). $\square$

Let us denote the power set of $E_{\mathcal{H}}$ by $\mathcal{P}(E_{\mathcal{H}})$. Since a set of edges may be identified with the corresponding monomial of $K[E_{\mathcal{H}}]$, the graded reverse lexicographic order induces a total order on $\mathcal{P}(E_{\mathcal{H}})$. Let us define it explicitly. Let $J_1 \neq J_2 \in \mathcal{P}(E_{\mathcal{H}})$ and denote by $\mathbf{t}^\alpha \neq \mathbf{t}^\beta$ the square-free monomials obtained as the products of the edges of J_1 and J_2, respectively. Then $\mathbf{t}^\alpha \succ \mathbf{t}^\beta$, in the graded reverse lexicographic order, if and only if either

$$\deg(\mathbf{t}^\alpha) \succ \deg(\mathbf{t}^\beta) \iff |J_1| > |J_2|$$

or $\operatorname{supp}(\beta)$ contains the least edge of $\operatorname{supp}(\alpha - \beta)$, i.e., J_2 contains the least edge of the symmetric difference $J_1 \triangle J_2$.

Definition 21. If $J_1 \neq J_2 \in \mathcal{P}(E_{\mathcal{H}})$ are elements of the power set of $E_{\mathcal{H}}$, we set $J_1 \succ J_2$ if $|J_1| > |J_2|$ or, if $|J_1| = |J_2|$ and J_2 contains the least edge in $J_1 \triangle J_2$.

It is clear that the partial order $\succeq$ defined by the above is a total order and, in particular, every (finite) subset of $\mathcal{P}(E_{\mathcal{H}})$ has a minimum element

Proposition 22. *Let $T \in \mathcal{T}(V_{\mathcal{H}})$ and let J be a T-join. Then J is a reduced parity join if and only if J is the minimum with respect to $\succeq$ of the set*

$$\{J' : J' \textit{ is a } T\textit{-join and } |J'| \equiv_2 |J|\}. \tag{6}$$

Proof. Let J' be another T-join with $|J'| \equiv_2 |J|$ and denote $C = J' \triangle J$ the corresponding even cardinality element of $\mathcal{E}(E_{\mathcal{H}})$. Then

$$|J \cap C| = \tfrac{|C|}{2} \iff |J'| = |J|.$$

Hence, if J is a reduced parity join and $|J'| = |J|$ then J contains the last edge of C which implies that $J' \succ J$. Therefore J is the minimum of (6). Conversely if J is the minimum of this set then, by Proposition 20, J is a parity join. Suppose that there exists $C \in \mathcal{E}(E_{\mathcal{H}})$ nonempty and such that $|J \cap C| = \frac{|C|}{2}$. Set $J' = J \triangle C$. Then J' is another T-join and $|J'| = |J|$. Then, since $J' \succ J$, by definition, J contains the least edge of C. We conclude that J is a reduced parity join. $\square$

Proposition 20 gives the existence of a reduced parity join of cardinality equal to that of any given parity join. In particular, it allows to compute $\mu_{\mathrm{p}}(\mathcal{H})$, as the maximum cardinality of a reduced parity join (*cf.* Definition 14).

Corollary 23. *$\mu_{\mathrm{p}}(\mathcal{H})$ is the maximum cardinality of a reduced parity join.*

The next theorem is a combinatorial formula for the regularity of $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$. This result is the generalization for hypergraphs of [9, Theorem 4.13].

Theorem 24. *The regularity of $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$ is equal to $\mu_{\mathrm{p}}(\mathcal{H}) - 1$.*

Proof. Since $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$ is a 1-dimensional, Cohen–Macaulay graded module, its regularity is equal to its index of regularity, i.e., the smallest degree r such that $H(d) = P(d)$, for all $d \geqslant r$, where H and P denote the Hilbert function and the Hilbert polynomial of $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$, respectively (*cf.* [1, Corollary 4.8]). In turn, the index of regularity is equal to the degree of the Hilbert series plus one (*cf.* [16, Corollary 5.1.9]). Since, by Corollary 19, the degree of the Hilbert series of $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$ is equal to the maximum cardinality of a reduced parity join minus two and, in turn, using Corollary 23, the regularity of $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$ is equal to $\mu_{\mathrm{p}}(\mathcal{H}) - 1$. $\square$

7 Degree

Since $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$ is a 1-dimensional $K[E_{\mathcal{H}}]$-module its degree coincides with its Hilbert polynomial, which, as the Hilbert series, (4), indicates is related to the cardinality of the set of reduced parity joins, $\sqcup_{d=0}^{s} \mathcal{J}_d^{\mathrm{r}}(\mathcal{H})$. In the case of graphs, because we know exactly which sets belong to $\mathcal{T}(V_{\mathcal{H}})$, the degree of this module may be given in terms of the connected components of the graph and the bipartite property. (*Cf.* [13, Proposition 2.11] and [9, Proposition 4.10].) In the general case, an alternative combinatorial characterization of the degree involves the cardinality of $\mathcal{T}(V_{\mathcal{H}})$.

Proposition 25. *The map $\sqcup_{d=0}^{s} \mathcal{J}_d^{\mathrm{r}}(\mathcal{H}) \to \mathcal{T}(V_{\mathcal{H}})$ sending a reduced parity join, $J \subset E_{\mathcal{H}}$, to the set $\{v \in V_{\mathcal{H}} : \deg_J(v) \text{ is odd}\}$ is a surjection. If all elements of $\mathcal{E}(E_{\mathcal{H}})$ have even cardinality then it is a bijection, otherwise, it is 2-to-1.*

Proof. Fix $T \in \mathcal{T}(E_{\mathcal{H}})$. By Proposition 22, there exists a reduced parity join J that maps to T. If all elements of $\mathcal{E}(E_{\mathcal{H}})$ have even cardinality then, by Proposition 20 the set (6) is the full set of T-joins and, using again Proposition 22, we deduce that there exists only one T-join which is a reduced parity join. If there exists $C \in \mathcal{E}(E_{\mathcal{H}})$ of odd cardinality then J and $J \triangle C$ are T-joins with $|J \triangle C| \not\equiv_2 |J|$. Hence we may apply Proposition 22 twice to find two reduced parity joins mapping to T. As any other T-join must belong to one of the two corresponding sets of T-joins (the even cardinality ones and the odd cardinality ones) and, by Proposition 22, only one in each set is a reduced parity join, we deduce that the preimage of T consists of exactly two reduced parity joins. □

Theorem 26. *If $s = |E_{\mathcal{H}}|$, then the degree of $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$ is equal to $\frac{1}{2}\sum_{d=0}^{s} |\mathcal{J}_d^{\mathrm{r}}(\mathcal{H})|$. Moreover, if no element of $\mathcal{E}(E_{\mathcal{H}})$ has odd cardinality, then the degree is $\frac{1}{2}|\mathcal{T}(V_{\mathcal{H}})|$, otherwise the degree is $|\mathcal{T}(V_{\mathcal{H}})|$.*

Proof. Since $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$ is 1-dimensional, the degree of this $K[E_{\mathcal{H}}]$-module may be obtained by multiplying its Hilbert series by $(1 - z)$ and setting $z = 1$. Using the rational form of the Hilbert series given in (4), we deduce that

$$\deg K[E_{\mathcal{H}}]/I_{\mathcal{H}} = \tfrac{1}{2}\textstyle\sum_{d=0}^{s} j_d = \tfrac{1}{2}\sum_{d=0}^{s} |\mathcal{J}_d^{\mathrm{r}}(\mathcal{H})|.$$

The rest of the statement follows from Proposition 25. □

8 Complete Hypergraphs

In this section we compute the degree and the regularity of $K[E_{\mathcal{H}}]/I_H$ when $\mathcal{H}$ is a complete k-partite graph or a 3-uniform complete hypergraph. The computations rely on the fact that in these cases the set $\mathcal{T}(V_{\mathcal{H}})$ and the set of T-joins, for any given $T \in \mathcal{T}(V_{\mathcal{H}})$, can be analyzed explicitly, without using a notion of hypergraph connectivity.

Definition 27. A hypergraph is called a complete k-partite hypergraph if the vertex set, $V_{\mathcal{H}}$, is endowed with a k-partition, $V_{\mathcal{H}} = \sqcup_{i=1}^{k} V_i$, with $|V_i| = a_i > 0$, such that $E_{\mathcal{H}} = \{\{v_1, \ldots, v_k\} : v_i \in V_i,$ for every $i = 1, \ldots, k\}$. Let us denote a complete k-partite hypergraph by $\mathcal{K}^k_{a_1,\ldots,a_k}$.

The next result is the analogue of the well-known result that a bipartite graph contains no cycles of odd cardinality. It is also an important characteristic of a complete k-partite hypergraph as regards the results of the previous sections.

Proposition 28. *If $\mathcal{H} = \mathcal{K}^k_{a_1,\ldots,a_k}$ and $C \in \mathcal{E}(E_{\mathcal{H}})$, then $|C|$ is even.*

Proof. If $C \subset \mathcal{E}(E_{\mathcal{H}})$, then, since every edge in C contains a single vertex of V_1,

$$|C| = \textstyle\sum_{v \in V_1} \sum_{e \in C} |\{v\} \cap e| = \sum_{v \in V_1} \deg_C(v).$$

Since $\deg_C(v)$ is even, for every $v \in V_{\mathcal{H}}$, this implies that $|C|$ is even. □

Proposition 29. *Let* $\mathcal{H} = \mathcal{K}^k_{a_1,\ldots,a_k}$ *and* $T \subset V_{\mathcal{H}}$. $T \in \mathcal{T}(V_{\mathcal{H}})$ *if and only if* $T_i = T \cap V_i$, *for* $i = 1, \ldots, k$, *have cardinalities of equal parity. Moreover, if* T *satisfies this condition, then the minimum cardinality of a* T*-join is* $\max_{i=1}^k |T_i|$.

Proof. If $T \in \mathcal{T}(V_{\mathcal{H}})$ and if J is a T-join then, for every $i = 1, \ldots, k$,

$$|J| = \textstyle\sum_{v \in T_i} \deg_J(v) + \sum_{v \in V_i \setminus T_i} \deg_J(v) \equiv_2 |T_i|. \tag{7}$$

Hence $|T_i|$, for $i = 1, \ldots, k$, have equal parity. Conversely, let $T \subset V_{\mathcal{H}}$ be such that $T_i = T \cap V_i$, for $i = 1, \ldots, k$, have cardinalities of equal parity. Denote $r = \max_{i=1}^k |T_i|$. Consider the sequence of elements of T_i written as $w^i_1, \ldots, w^i_r$, where, the first $|T_i|$ in this sequence are the members of T_i (without repetitions) and, if $|T_i| < r$, the last $r - |T_i|$ of them are equal to w^i_1. Consider the set of r edges

$$J = \left\{ \left\{ w^1_i, \ldots, w^k_i \right\} : i = 1, \ldots, r \right\}.$$

If $v \notin T$ then, clearly $\deg_J(v) = 0$. If $v = w^i_j \in T_i$, with $2 \leqslant j \leqslant |T_i|$ then $\deg_J(v) = 1$. If $v = w^i_1 \in T_i$ then, because $|T_i| \equiv_2 r$, $\deg_J(v) = r - |T_i| + 1 \equiv_2 1$. We deduce that J is a T-join of cardinality $r = \max_{i=1}^k |T_i|$. It remains to be proved that this is the minimum cardinality of a T-join. If J' is any T-join, then, applying the equality in (7) to J', we get

$$|J'| \geqslant \textstyle\sum_{v \in T_i} \deg_{J'}(v) \geqslant |T_i|,$$

for every $i = 1, \ldots, k$. We conclude that $|J'| \geqslant \max_{i=1}^k |T_i|$. $\square$

Theorem 30. *If* $\mathcal{H} = \mathcal{K}^k_{a_1,\ldots,a_k}$, *then*

$$\begin{cases} \log_2\bigl(\deg K[E_{\mathcal{H}}]/I_H\bigr) = \bigl(\sum_{i=1}^k a_i\bigr) - k \\ \operatorname{reg} K[E_{\mathcal{H}}]/I_{\mathcal{H}} = \max\{a_1, \ldots, a_k\} - 1. \end{cases}$$

Proof. As for the degree, using Theorem 26, we only need to show that

$$\log_2 |\mathcal{T}(V_{\mathcal{H}})| = \bigl(\textstyle\sum_{i=1}^k a_i\bigr) - k + 1. \tag{8}$$

But this is now straightforward from Proposition 29. Let us now deal with the statement on the regularity of $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$. By Theorem 24 we must show that

$$\mu_{\mathrm{p}}(\mathcal{H}) = \max\{a_1, \ldots, a_k\}.$$

Let $J \subset E_{\mathcal{H}}$ be a parity join and let $T \in \mathcal{T}(V_{\mathcal{H}})$ be the set of vertices, $v \in V_{\mathcal{H}}$, such that $\deg_J(v)$ is odd. Since by Proposition 28 there are no elements of $\mathcal{E}(E_{\mathcal{H}})$ of odd cardinality, using Proposition 20, we deduce that $|J|$ is the minimum cardinality of a T-join. Since we showed in Proposition 29 that the minimum cardinality of a T-join is $\max_{i=1}^k |T_i|$, where, recall, $T_i = T \cap V_i$, we conclude that $|J| \leqslant \max\{a_1, \ldots, a_k\}$. This shows that

$$\mu_{\mathrm{p}}(\mathcal{H}) \leqslant \max\{a_1, \ldots, a_k\}.$$

To prove the opposite inequality, it suffices to show that there exists a parity join of cardinality $\max\{a_1,\ldots,a_k\}$. Arguing as before, this is the same as showing that there exists $T \in \mathcal{T}(V_{\mathcal{H}})$ such that $\max\{a_1,\ldots,a_k\} = \max_{i=1}^k |T_i|$. Let $1 \leqslant m \leqslant k$ be such that $a_m = \max\{a_1,\ldots,a_k\}$. If a_m is even, fix $T = V_m$, if a_m is odd, then, choosing $v_i \in V_i$, for all $i \neq m$, set $T = \{v_1,\ldots,v_{m-1}\} \cup V_m \cup \{v_{m+1},\ldots,v_k\}$. In both cases $T \in \mathcal{T}(V_{\mathcal{H}})$ and $a_m = \max_{i=1}^k |T_i|$. □

Definition 31. A hypergraph is called complete of rank k if $|V_{\mathcal{H}}| \geqslant k$ and $E_{\mathcal{H}}$ is the set of cardinality k subsets of $V_{\mathcal{H}}$. Let us denote a complete hypergraph of rank k, with n vertices, by $\mathcal{K}_n^k$.

Recall that we are assuming throughout this article that $k \geqslant 2$.

Proposition 32. *Let $\mathcal{H} = \mathcal{K}_n^k$. (i) If $n = k$ then $\mathcal{T}(V_{\mathcal{H}}) = \{\emptyset, V_{\mathcal{H}}\}$. (ii) If $n > k$ and k is odd, then $\mathcal{T}(V_{\mathcal{H}}) = \mathcal{P}(V_{\mathcal{H}})$. (iii) If $n > k$ and k is even, then $\mathcal{T}(V_{\mathcal{H}})$ is equal to the set of elements of $\mathcal{P}(V_{\mathcal{H}})$ of even cardinality.*

Proof. (i) If $n = k$ and $V_{\mathcal{H}} = \{v_1,\ldots,v_n\}$ then $E_{\mathcal{H}} = \{\{v_1,\ldots,v_n\}\}$ and therefore a T-join exists if and only if $T = \emptyset$ or $T = \{v_1,\ldots,v_n\} = V_{\mathcal{H}}$. (ii) Assume that $n > k$ and k is odd. To show that $\mathcal{T}(V_{\mathcal{H}}) = \mathcal{P}(V_{\mathcal{H}})$ it suffices to show that $\{v\} \in \mathcal{T}(V_{\mathcal{H}})$, for all $v \in V_{\mathcal{H}}$. Fix $v_0 \in V_{\mathcal{H}}$ and let V be any subset of $V_{\mathcal{H}} \setminus \{v_0\}$ with cardinality k. Consider:

$$J = \{\{v_0\} \cup J' : J' \subset V \text{ and } |J'| = k-1\} \subset E_{\mathcal{H}}.$$

As $\deg_J(v_0) = \binom{k}{k-1} = k$ is odd, $\deg_J(w) = \binom{k-1}{k-2} = k-1$ is even, for every $w \in V$, and $\deg_J(v) = 0$ for every $v \notin V \cup \{v_0\}$, we deduce that

$$\deg_J(v) \text{ is odd } \iff v \in \{v_0\}.$$

In other words, J is $\{v_0\}$-join. (iii) Assume that $n > k$ and k is even. Let $T \in \mathcal{T}(V_{\mathcal{H}})$ and let J be a T-join. By Proposition 6, $|T|$ is even. To prove that $\mathcal{T}(V_{\mathcal{H}})$ is equal to the set of subsets of $V_{\mathcal{H}}$ of even cardinality, it suffices to show that, for every pair of distinct vertices $v_1, v_2 \in V_{\mathcal{H}}$, $\{v_1, v_2\} \in \mathcal{T}(V_{\mathcal{H}})$. Let us fix $v_1, v_2 \in V_{\mathcal{H}}$. Choose $V \subset V_{\mathcal{H}} \setminus \{v_1, v_2\}$ of cardinality $k-1$. Consider

$$J = \{\{v_1, v_2\} \cup J' : J' \subset V \text{ and } |J'| = k-2\} \subset E_{\mathcal{H}}.$$

Arguing as before we deduce that J is a $\{v_1, v_2\}$-join. □

Corollary 33. *Let $\mathcal{H} = \mathcal{K}_n^k$. (i) If $n = k$, then $\deg K[E_{\mathcal{H}}]/I_H$ is equal to 1. (ii) If $n > k$, then $\deg K[E_{\mathcal{H}}]/I_H$ is equal to 2^{n-1}.*

Proof. (i) If $n = k$ then $|E_H| = 1$ hence $I_{\mathcal{H}} = (0)$ and $K[E_{\mathcal{H}}]$ is 1-dimensional. (ii) There are two cases. Assume first that k is odd. Then, by Proposition 6, no element of $\mathcal{E}(E_{\mathcal{H}})$ has odd cardinality. Therefore, by Theorem 26, the degree of $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$ is equal to $\frac{|\mathcal{T}(V_{\mathcal{H}})|}{2}$, which, by Proposition 32, is equal to 2^{n-1}. Let us now assume that k is even. Let $V \subset V_{\mathcal{H}}$

be a subset of vertices of cardinality $k+1$ and let J be the set of edges with vertices in V. Then

$$\deg_J(v) = \binom{k}{k-1} = k$$

is even, for every $v \in V$ while $\deg_J(v) = 0$, for every $v \in V_{\mathcal{H}} \setminus V$. In other words, J is an even subset of edges. Since $|J| = \binom{k+1}{k} = k+1$ we conclude that $\mathcal{E}(E_{\mathcal{H}})$ contains elements of odd cardinality. Hence the degree of $K[E_{\mathcal{H}}]/I_{\mathcal{H}}$ is $|\mathcal{T}(E_{\mathcal{H}})|$, which, by Proposition 32, is equal to 2^{n-1}. $\square$

Theorem 34. *Let $\mathcal{H} = \mathcal{K}_n^3$. If $n = 3$, then $\operatorname{reg} K[E_{\mathcal{H}}]/I_{\mathcal{H}} = 0$. If $n = 4$, then $\operatorname{reg} K[E_{\mathcal{H}}]/I_{\mathcal{H}} = 4$. If $n \geqslant 5$, then $\operatorname{reg} K[E_{\mathcal{H}}]/I_{\mathcal{H}} = \lfloor \frac{n+1}{3} \rfloor$.*

Proof. If $n = 3$ then $|E_{\mathcal{H}}| = 1$ and therefore, by Theorem 10, $I_{\mathcal{H}} = (0)$, and hence $\operatorname{reg} K[E_{\mathcal{H}}]/I_{\mathcal{H}} = 0$. Assume now that $n = 4$ and let $C \in \mathcal{E}(E_{\mathcal{H}})$. By Proposition 6, $|C|$ must be even and hence $|C| = 0$, 2 or 4. A set of two distinct edges yields two vertices of degree 2 and two of degree 1. A set of four edges is the whole of $E_{\mathcal{H}}$, which yields all vertices of degree 3. Hence we must have $|C| = 0$. In other words, $\mathcal{E}(E_{\mathcal{H}}) = \{\emptyset\}$ and, consequently, all subsets of $E_{\mathcal{H}}$ are parity joins. Therefore $\mu_{\mathrm{p}}(\mathcal{H}) = |E_{\mathcal{H}}| = 4$ and, by Theorem 24, $\operatorname{reg} K[E_{\mathcal{H}}]/I_{\mathcal{H}} = 3$.

Let us now assume that $n \geqslant 5$. As no element of $\mathcal{E}(E_{\mathcal{H}})$ has odd cardinality, by Proposition 20, parity joins coincide with minimum cardinality T-joins. We will use this to show that $\mu_{\mathrm{p}}(\mathcal{H}) = \lfloor \frac{n+1}{3} \rfloor + 1$. Note that, by Proposition 32, $\mathcal{T}(V_{\mathcal{H}}) = \mathcal{P}(V_{\mathcal{H}})$. Given $T \subset V_{\mathcal{H}}$ let us denote by $\tau(\mathcal{H}, T)$ the minimum cardinality of a T-join. If $T = \emptyset$ then $\tau(\mathcal{H}, T) = 0$. If $|T| = 1$, say $T = \{v_1\}$, then $\tau(\mathcal{H}, T) \leqslant 3$ since there exists a T-join of cardinality 3, as is shown in Figure 2. (In this figure, vertices in T are shown in black, other vertices in

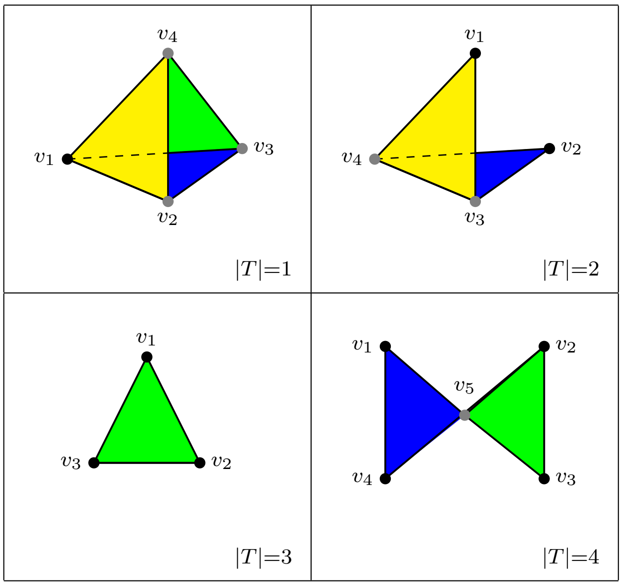

Figure 2: T-joins in K_n^3, for $0 \leqslant |T| \leqslant 7$.

gray.) Assume now $|T| \geqslant 2$ and let us use induction to prove that

$$\tau(\mathcal{H}, T) \leqslant \lfloor \tfrac{|T|+1}{3} \rfloor + 1.$$

For $|T| = 2, 3$ and 4 this is proved by showing there is a T-join of cardinality $\leqslant 2$. For each $|T|$, the corresponding T-join is indicated in Figure 2. Assume now that $|T| \geqslant 5$. Consider any three elements of T say v_1, v_2, v_3, let $T' = T \setminus \{v_1, v_2, v_3\}$ and, by induction, let J' be a T'-join of cardinality less than or equal to $\lfloor \frac{|T|+1}{3} \rfloor$. Then

$$J = J' \triangle \{\{v_1, v_2, v_3\}\}$$

is a T-join with cardinality less than or equal to $\lfloor \frac{|T|+1}{3} \rfloor + 1$.

Since $|T| \leqslant n$ and $n \geqslant 5$, so far we have shown that a minimum cardinality T-join has cardinality less than or equal to $\lfloor \frac{n+1}{3} \rfloor + 1$. Hence $\mu_{\mathrm{p}}(\mathcal{H}) \leqslant \lfloor \frac{n+1}{3} \rfloor + 1$. To show the opposite inequality we only need to prove that there exists $T \subset V_{\mathcal{H}}$ such that any T-join has cardinality at least $\lfloor \frac{n+1}{3} \rfloor + 1$. Let $T \subset V_{\mathcal{H}}$ be any set of $3\lfloor \frac{n+1}{3} \rfloor - 1$ vertices of $\mathcal{H}$ and let J be a T-join. Then, from

$$\textstyle\sum_{v \in T} \deg_J(v) + \sum_{v \notin T} \deg_J(v) = 3|J| \tag{9}$$

we get

$$3|J| \geqslant |T| = 3\lfloor \tfrac{n+1}{3} \rfloor - 1 \implies |J| \geqslant \lfloor \tfrac{n+1}{3} \rfloor.$$

Now, $|J| = \lfloor \frac{n+1}{3} \rfloor$ is impossible since (9) reduced modulo 2 yields

$$|J| \equiv_2 |T| \equiv_2 \lfloor \tfrac{n+1}{3} \rfloor - 1.$$

We conclude that $|J| \geqslant \lfloor \frac{n+1}{3} \rfloor + 1$. □

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