Morphology of small snarks

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Abstract

This paper classifies all snarks up to order 36 and explains the reasons for their uncolourability. The crucial part of our approach is a computer-assisted structural analysis of cyclically 5-connected critical snarks, which is justified by the fact that every other snark can be constructed from them by a series of simple operations while preserving uncolourability. Our results reveal that most of the analysed snarks are built from pieces of the Petersen graph and can be naturally distributed into a small number of classes having the same reason for uncolourability. This sheds new light on the structure of all small snarks. Based on our analysis, we generalise certain individual snarks to infinite families and identify a rich family of cyclically 5-connected critical snarks.

Mathematics Subject Classifications: 05C15, 05C75

1 Introduction

In this paper, we attempt to provide insight into the structure of snarks of small order. The ultimate aim is to contribute to understanding the nature of snarks in general. Snarks are — in essence — connected cubic graphs whose edges cannot be properly coloured with three colours. In recent years, snarks have been attracting considerable attention, mainly because this family might contain counterexamples to several profound and long-standing conjectures such as the cycle double cover conjecture, the 5-flow conjecture, or the Berge-Fulkerson conjecture [19, 25, 26]. Understanding the structure of snarks is, therefore, crucial for proving or disproving any of these conjectures.

Snarks are very difficult to find since almost all cubic graphs are hamiltonian and hence 3-edge-colourable [41]. This asymptotic behaviour manifests itself already at very small orders: on 26 vertices, the ratio of the number of cyclically 4-edge-connected snarks to the number of connected cubic graphs is below 6.2×10^{-7} (see [20]), and seems to exponentially decrease with increasing order. On the other hand, deciding whether a cubic graph is 3-edge-colourable or not is an NP-complete problem [22], which means that the class of snarks is still sufficiently rich: for instance, there are more than 400 million of cyclically 4-edge-connected snarks on at most 36 vertices [6, 20].

Snarks are also challenging to understand because the reasons that force the absence of 3-edge-colourings in cubic graphs are in general unknown. Numerous constructions of snarks have been presented by various authors (see [1, 23, 46, 32, 33, 31, 43] for some examples), often aiming at proving the existence of snarks possessing certain special properties. Despite this effort, very little is known about the intrinsic structure of snarks. For example, it is not known whether small edge-cuts in snarks are unavoidable: a conjecture of Jaeger and Swart [24] states that every snark contains a cycle-separating edge-cut comprising at most six edges, but this conjecture has been open for almost 40 years without any visible progress. Thus no general approach to classification of the entire family of snarks seems to be within reach.

The evidence drawn from the published lists of small snarks (see, for example, [6, 7, 8]) reveals that most of them are composed of several common construction components, typically obtained from the Petersen graph. This phenomenon has neither been formalised nor thoroughly studied yet, and it is unclear whether anything similar holds for snarks of large order. Although we believe that these questions are well worth of investigation, it does not seem that the currently available methods are powerful enough to attack them in full generality. Neither probabilistic nor constructive methods provide us with good insight into the structure of snarks. In particular, no uniform random model for snarks is currently known, which leaves us without strong theoretical tools for studying the typical behaviour of snarks. The NP-completeness of the problem of edge-colourability [22] also indicates that no quick progress is likely. This is why we focus on what we have at hand, which is the complete list of all cyclically 4-edge-connected snarks on up to 36 vertices, recently produced by Brinkmann et al. [6] and completed in [20] by employing an exhaustive computer search.

Suppose, for a moment, that we would like to move a step further and produce a list of all snarks on 38 vertices. An obvious way to do it would be to generate all cubic graphs on 38 vertices and discard those that are colourable. Unfortunately, there are just too many of them in comparison with the computing power presently available for research, even if we restrict ourselves to those with cyclic connectivity at least 4 and girth at least 5. Such an approach is therefore very unlikely to work. In this situation, it may be helpful to realise that a vast majority of known snarks contain an edge whose removal followed by the suppression of the resulting 2-valent vertices again leaves a snark. Conversely, most known snarks arise from a smaller snark by choosing a suitable pair of edges, subdividing each of them with one additional vertex, and connecting the resulting 2-valent vertices with a new edge; this operation is called an I-extension. The meaning of "suitable" in order for the operation of I-extension to be feasible is easy to explain (see Proposition 4), which suggests that this approach might be promising. The hard part of the problem, however, is the snarks that cannot be obtained by a series of I-extensions from a smaller snark in such a way that each member of the extension series is a snark. Such snarks indeed exist and have already been studied [11, 12, 38, 43]; in fact, they have been rediscovered several times [14, 16, 42]: they are known as *critical snarks* and are characterised by the property that for each edge, the inverse of I-extension produces a colourable graph.

Critical snarks are known to be cyclically 4-edge-connected with girth at least 5 [38, Proposition 4.8], and thus can be regarded as "proper snarks" by the usual standards. A decomposition theory developed by Chladný and Škoviera in [12] suggests that critical snarks that possess a cycle-separating 4-edge-cut can be explained via the reversal of the well-known operation of dot product. This is especially true for bicritical snarks, an important subclass of critical snarks: every bicritical snark containing a cycle-separating 4-edge-cut admits a decomposition into a unique collection of cyclically 5-edge-connected bicritical snarks, and conversely, it can be reconstructed from them by a repeated application of dot product [12, Theorem C].

By contrast, a decomposition process along 5-edge-cuts is much more complicated [10, 38, 39]. Moreover, it only works in one way and cannot be easily used for constructing snarks. Indeed, as argued in [38, p. 273], the original snark cannot be reconstructed from decomposition factors by using any collection of well-defined simple operations. Thus, from this point of view, the most fundamental and, at the same time, most enigmatic snarks are those that are critical and cyclically 5-edge-connected, also known under the term 5-*simple* [38]. Since all cyclically 4-edge-connected snarks can be obtained from them by applying I-extensions and dot products, it is natural to start the structural analysis of snarks by investigating *cyclically* 5-*edge-connected critical snarks*, that is, 5-simple snarks.

Therefore, our aim in this paper is to analyse and classify *all* 5-simple snarks of order not exceeding 36. The list of such snarks is known and contains exactly 2110 graphs. We have extracted it from the complete list of all nontrivial snarks of order up to 36 produced by Brinkmann et al. [6] in 2013. The list of all critical snarks of order up to 36 was previously compiled by Carneiro et al. [9]; however, those with cyclic connectivity at least 5 have not been singled out.

The method which we apply to the analysis of snarks is similar to what biologists have been doing for centuries in morphology. By discovering and investigating more and more species they have constantly been improving and refining the hierarchy of organisms, making it more complete with every new species examined. We aim to describe the structure of each 5-simple snark of order not exceeding 36 in a manner comprehensible to a human, with uncolourability readily verifiable by hand, as opposed to having a proof that relies on an exhaustive enumeration carried out by a computer. It transpires that the uncolourability of small snarks can be conveniently explained in terms of multipoles (subgraphs with dangling edges) and their interconnections. Snarks with a similar structure of multipoles, similar interconnections, and similar reasons for uncolourability are collected into families. As we analyse larger and larger snarks, the set of multipoles with known colouring properties grows. Whenever we encounter a snark whose uncolourability cannot be fully explained by previously discovered multipoles, we analyse it, extend the list of known multipoles, and employ it in the further analysis.

The advantage of this approach is evident from the fact that each of the families resulting from our analysis can be easily turned into an infinite class of snarks in a straight-

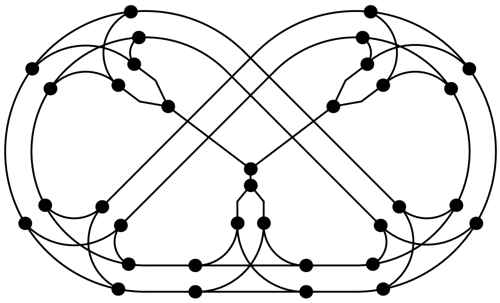


Figure 1: A remarkable critical snark of order 34

forward manner: it suffices to substitute the basic construction blocks, usually taken from the Petersen graph, with those obtained from larger snarks in a similar manner.

The results of our analysis are summarised in Section 6. Amongst the snarks up to 30 vertices we have not discovered any example that could not be easily explained in terms of multipoles arising from the Petersen graph. A new phenomenon arises on 32 vertices with class denoted by 32-A and schematically depicted in Figure 18. The 7-pole M_{11} contained in these snarks is perhaps the most intriguing specimen of all — it is the only multipole that we have not been able to generalise.

On 34 and 36 vertices, there are several families of interest not described before; all of them contain several disjoint copies of 5-poles, each consisting of a pair of 5-cycles sharing two edges. These 5-poles can be obtained from the Petersen graph by removing a path of length 2 and are known as negators.

Perhaps the most aesthetically pleasing family is illustrated in Figure 25. The smallest snark from this family can also be regarded as a cleverly arranged tangle of 6-cycles complemented by a 6-pole arising from the Petersen graph by removing a 6-cycle; it is displayed in Figure 1.

Despite our success in explaining the uncolourability of the investigated graphs, we have not accomplished that much in terms of truly understanding their criticality. In particular, we can easily generalise any of the described families of snarks into an infinite class of new snarks, but we know very little about which of these new snarks are critical or even bicritical. All our achievements in this direction are collected in Section 8. On the one hand, we observe that construction components for critical snarks need not come from critical snarks. On the other hand, we give examples of constructions for which obvious necessary conditions, such as the criticality of the constituting multipoles, are insufficient to ensure the resulting snarks' criticality. Despite that, we are able to describe a new rich infinite family of bicritical snarks. This family demonstrates that many (perhaps even all) of the families described in this paper can be turned into infinite families of bicritical snarks by imposing additional restrictions on the construction blocks.

Finally, we employ the results of our analysis to give a negative answer to a question posed by Chladný and Škoviera in [12, Problem 5.7] about pairs of edges essential for a dot product of bicritical snarks to be bicritical. According to the theory developed therein, an essential pair of edges must be non-removable (that is, its removal leaves a colourable graph). However, it was left open whether there exists a non-removable pair of edges that is not essential. An example of such a pair is provided in Section 9.

We conclude this section with a short list of definitions. We assume that the reader has the basic knowledge related to graph colourings and flows. For more information on this matter, we recommend consulting [26].

Our graphs are finite and may contain parallel edges and loops. A connected 2-regular graph is called a *cycle*. A cubic graph G is said to be *cyclically k-connected* (or *cyclically k-edge-connected*, to be more precise) if no set of fewer than k edges separates two cycles of G from each other. The *cyclic connectivity* of G, denoted by $\lambda_c(G)$, is the largest integer k such that G is cyclically k-connected. A connected uncolourable cubic graph is a *snark*. The graph which consists of two vertices joined by an edge and has a loop attached at each vertex is called the *dumbbell graph*; it is denoted by Db. Note that Db is a snark, according to our definition. It is well known that the smallest 2-connected snark is the Petersen graph, denoted here by Pg. Cyclically 4-connected snarks with girth at least 5 will be called *nontrivial*, and the remaining ones will be *trivial*.

2 Multipoles and their Tait colourings

Snarks are often described as combinations of graph-like structures called multipoles. In contrast to graphs, multipoles are permitted to contain dangling edges or even isolated edges, see e. g. [17, 33, 38]. Formally, a *multipole* is a pair $M = (V(M), E(M))$, where $V(M)$ is a set of vertices and $E(M)$ is a set of edges. Every edge $e \in E(M)$ has two *ends* which may, or may not, be incident with a vertex. An edge whose ends are incident with two distinct vertices is called a *link*. If only one end of an edge is incident with a vertex, then the edge is a *dangling edge*, and if neither end of an edge is incident with a vertex, it is called an *isolated edge*. A *semiedge* is an end of an edge that is incident with no vertex. The set of all semiedges of a multipole M is denoted by $S(M)$. A multipole with k semiedges is called a *k-pole*. The *order* $|M|$ of a multipole M is the number of its vertices. In this paper, we will only consider *cubic multipoles*, that is, multipoles where each vertex is incident with three edge ends.

It is often convenient to partition the set $S(M)$ into pairwise disjoint sets $S_1, \ldots, S_n$ called *connectors*. A *k-connector* is a connector comprising k semiedges. Although semiedges in a connector are unordered, sometimes it is useful to endow a connector with a linear order. Such a connector $S = (e_1, \ldots, e_k)$ is called an *ordered connector*. A multipole M with n connectors $S_1, S_2, \ldots, S_n$ such that $|S_i| = c_i$ for $i \in \{1, 2, \ldots, n\}$ is denoted by $M(S_1, S_2, \ldots, S_n)$ and called a *$(c_1, c_2, \ldots, c_n)$-pole*. If a connector S contains only one semiedge s, we will usually only write s in place of $\{s\}$.

The *junction* of semiedges e and f is performed by gluing the semiedges e and f, thereby producing a new edge joining the end-vertices of e and f. The junction of two connectors $S = \{e_1, e_2, \ldots, e_k\}$ and $T = \{f_1, f_2, \ldots, f_k\}$ of the same size k consists of performing k individual junctions of e_i and f_i for $i \in \{1, 2, \ldots, k\}$. If S and T are not ordered, we can enumerate their semiedges in an arbitrary order prior to performing the junction. Although different orderings may lead to several different multipoles, in a vast majority of cases our results do not depend on the order in which the junctions of semiedges of two connectors have been performed. A junction of two $(c_1, \dots, c_n)$-poles $M(S_1, \dots, S_n)$ and $N(T_1, \dots, T_n)$ consists of n individual junctions of S_i and T_i for each $i \in \{1, 2, \dots, n\}$. Hence, performing a junction of M and N requires to join the corresponding connectors S_i and T_i, while the order of semiedges in S_i and T_i may be arbitrary, unless the connectors are ordered.

A natural approach to explaining the uncolourability of a snark is to split it into a set of multipoles and to study interactions between their colourings. The aim is to show that any combination of colourings of the constituting multipoles gives rise to a conflict within the snark. By an *edge colouring* of a multipole M, we mean a mapping $\varphi\colon E(M) \to X$ from the edge set of M to a certain set X of colours. An edge colouring naturally induces a colouring of edge ends. If the ends of all edges incident with any vertex v of M receive distinct colours, the colouring is said to be *proper*. If $|X| = k$, the colouring is a *k-edge-colouring*.

Since multipoles in this paper are all cubic, colourings considered in this paper will mostly be proper 3-edge-colourings. This permits us to abbreviate the term "proper 3-edge-colouring" to just "colouring". We say that a multipole M is *colourable* whenever it has a colouring; otherwise, M is *uncolourable*.

A convenient set of colours for the study of snarks is provided by the set $\mathbb{K}$ of nonzero elements of the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$. An edge colouring using $\mathbb{K}$ as the colour set is often termed a *Tait colouring* because the usage of such colourings can be traced back to Tait's paper [45] on the Four-Colour Problem. One of the advantages of using $\mathbb{K}$ is that we can use addition in the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ to express the properties of a colouring. Indeed, a colouring of a multipole M is proper if and only if for every vertex v of M the three colours meeting at v sum to 0 in $\mathbb{Z}_2 \times \mathbb{Z}_2$. In other words, a proper 3-edge-colouring of a cubic multipole is a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow and vice versa. Now, if we regard a colouring φ of a multipole M as a flow, we can use the Kirchhoff law to conclude that $\sum_{e \in S(M)} \varphi(e) = 0$. This fact has a useful consequence commonly known as the parity lemma, first proved by Tutte [15] under the pseudonym of Blanche Descartes.

Lemma 1. (Parity lemma [15]) *Let M be a k-pole and let k_1, k_2, and k_3 be the numbers of semiedges of colour $(0, 1)$, $(1, 0)$, and $(1, 1)$, respectively. Then*

$$k_1 \equiv k_2 \equiv k_3 \equiv k \pmod 2.$$

3 Reducibility and criticality of snarks

We have already indicated why critical snarks are a natural place to start an investigation of the intrinsic structure of snarks. In this section, we discuss this matter in a greater detail and explain important relations between criticality of snarks, their nontriviality, and reducibility.

A natural way of approaching the idea of nontriviality of snarks is by asking whether the snark in question contains vertices that do not contribute to its uncolourability. According to the parity lemma, removing just one vertex from a snark leaves an uncolourable graph, so one has to remove at least two vertices to make the graph colourable. A pair of distinct vertices $\{u, v\}$ of a snark G will be called *removable* if $G - \{u, v\}$ is not 3-edge-colourable; otherwise it will be called *non-removable.* A snark G is *critical* if every pair of adjacent vertices in G is non-removable; it is called *bicritical* if every pair of distinct vertices in G is non-removable.

The concept of criticality of snarks can also be developed by interpreting 3-edge-colourings of a cubic graph as nowhere-zero flows. This direction has been explored by several authors, see for example [9, 13, 14, 16, 18]. Following da Silva et al. [13] we define a graph to be *k-flow-edge-critical* if it does not admit a nowhere-zero k-flow but the graph obtained by the contraction of any edge does. We further define a graph to be *k-flow-vertex-critical* if it does not admit a nowhere-zero k-flow but the graph obtained by the identification of any two distinct vertices does. These definitions apply to snarks with $k = 4$ since no snark admits a nowhere-zero 4-flow. If we take into account the fact that contracting an edge has the same effect on the existence of a nowhere-zero flow as identifying its end-vertices, 4-flow-edge-critical snarks and 4-flow-vertex-critical snarks are natural counterparts of critical and bicritical snarks, respectively. Nevertheless, it has been only recently shown [34] that, in spite of different formal definitions, flow-critical snarks are the same as critical snarks.

Theorem 2. [34] *A snark is 4-flow-edge-critical if and only if it is critical. A snark is 4-flow-vertex-critical if and only if it is bicritical.*

Consider a pair of adjacent vertices u and v forming an edge e of a cubic graph G, and let $G \sim e$ denote the cubic graph homeomorphic to $G - e$. The operation that transforms G into $G \sim e$ is called an *edge reduction*, and its reverse is called an *edge extension* or an I-*extension.* To perform an edge extension of a cubic graph G' one picks in G' two edges e_1 and e_2 (not necessarily distinct), subdivides each of them with a new vertex, and adds an edge e joining the two new vertices (if $e_1 = e_2$, this results in a digon, that is, a pair of parallel edges). The resulting graph is denoted by $G'(e_1, e_2)$.

It turns out that removing a pair of adjacent vertices u and v from a snark has the same effect on colourability as reducing the edge e joining them. This fact was first observed in [38].

Proposition 3. [38] *Let G be a snark and $e = uv$ an edge of G. Then $G \sim e$ is 3-edge-colourable if and only if $G - \{u, v\}$ is 3-edge-colourable.*

Proposition 3 implies that a snark G is critical if and only if $G \sim e$ is colourable for each edge e. Every non-critical snark thus contains an edge whose reduction leaves a smaller snark. If the resulting snark is still not critical, we can repeat the process and continue until we eventually produce a critical snark. Reversing this process shows that every snark can be constructed from a critical snark by a series of edge extensions, with all intermediate graphs being snarks that contain a subdivision of the initial critical snark. All this tells us that critical snarks can be regarded as basic building blocks of all snarks.

In order to make the extension process work, it is important to know under what conditions an edge extension $G(e_1, e_2)$ of a snark G is again a snark. The answer requires one more definition. A pair $\{e_1, e_2\}$ of edges of a snark G is said to be *removable* if $G - \{e_1, e_2\}$ is not 3-edge-colourable.

Proposition 4. *Let G be a snark, and let e_1 and e_2 be distinct edges of G. Then the edge extension $G(e_1, e_2)$ of G is a snark if and only if the pair $\{e_1, e_2\}$ is removable.*

Proof. If $\{e_1, e_2\}$ is a removable pair of edges of G, then $G - \{e_1, e_2\}$ is uncolourable. Since $G - \{e_1, e_2\} \subseteq G(e_1, e_2)$, we conclude that so is $G(e_1, e_2)$. Thus $G(e_1, e_2)$ is a snark.

Conversely, let $G(e_1, e_2)$ be a snark and suppose to the contrary that $\{e_1, e_2\}$ is nonremovable, that is, $G - \{e_1, e_2\}$ is colourable. Let e_1', e_1'' and e_2', e_2'' be the edges of $G(e_1, e_2)$ obtained by subdividing e_1 and e_2, respectively. Every 3-edge-colouring φ of $G - \{e_1, e_2\}$ forces at least one of the pairs $\{e_1', e_1''\}$ and $\{e_2', e_2''\}$ to receive distinct colours, say $\varphi(e_1') = a$ and $\varphi(e_1'') = b$, otherwise φ would induce a colouring of G. By the parity lemma, the other pair also receives colours a and b. Thus the edge of $G(e_1, e_2)$ added across e_1 and e_2 can be coloured $a + b$ to produce a 3-edge-colouring of $G(e_1, e_2)$, which is a contradiction. $\square$

Another possibility to capture the notion of nontriviality of snarks is to identify edge-cuts whose removal from a snark produces an uncolourable component. The aim is to generalise the well-known fact that snarks with short cycles and small edge-cuts are just trivial modifications of smaller snarks. For this purpose Nedela and Škoviera [38] proposed the following definitions. Let G be a snark which can be expressed as a junction $M * N$ of two k-poles M and N for some $k \geqslant 0$. If one of M and N, say M, is uncolourable, we can extend M to a snark $\bar{M}$ of order not greater than $|G|$ by adding to M a small number of vertices and edges; possibly $\bar{M} = G$. By creating the graph $\bar{M}$ we have reduced a snark G to a new snark called a *k-reduction* of G. (Note that an edge reduction $G \sim e$ is a special case of a 4-reduction.) A k-reduction $\bar{M}$ of G is *proper* if $|\bar{M}| < |G|$. If G admits a proper k-reduction for some $k \geqslant 0$, the essence of the uncolourability of the smaller snark is the same as the one that can be found in G. A snark is *k-irreducible*, for $k \geqslant 1$, if it has no proper m-reduction for any $m < k$. A snark is *irreducible* if it is k-irreducible for every $k > 0$, that is, if it admits no proper reductions at all. Observe that a k-irreducible snark is also r-irreducible for every $r \leqslant k$.

The following theorem, proved in [38], puts the concept of criticality of snarks into the perspective of various ranks of irreducibility. Among others, it tells us that there are, surprisingly, only finitely many different degrees of irreducibility, with bicritical snarks holding the highest position.

Theorem 5. [38] *Let G be a snark. Then the following statements hold.*

(i) *If $1 \leqslant k \leqslant 4$, then G is k-irreducible if and only if it is either cyclically k-connected or the dumbbell graph.*

(ii) *If $k \in \{5, 6\}$, then G is k-irreducible if and only if it is critical.*

(iii) *If $k \geqslant 7$, then G is k-irreducible if and only if it is bicritical.*

Theorem 5 implies that irreducible snarks coincide with bicritical ones, and that critical snarks are just one step away from being irreducible. Critical snarks that are not bicritical, called *strictly critical*, appear to be very rare. This can be observed already among snarks of small order: there are exactly 55172 critical snarks of order not exceeding 36, but only 846 of them are strictly critical, just slightly over 1.5 percent [5, 34].

Another important consequence of Theorem 5 is that every critical snark is cyclically 4-edge-connected and has girth at least 5, see [38, Proposition 4.1]. Thus critical snarks are nontrivial by all generally accepted standards.

Order	$\lambda_c = 4$	$\lambda_c = 5$	$\lambda_c = 6$	Total
10	0	1	0	1
18	2	0	0	2
20	0	1	0	1
22	0	2	0	2
24	0	0	0	0
26	103	8	0	111
28	31	1	1	33
30	104	11	0	115
32	16	13	0	29
34	38827	1503	0	40330
36	14063	568	1	14548

Table 1: Numbers of critical snarks by connectivity

Table 1 indicates that among the critical snarks those with cyclic connectivity 4 significantly prevail. It transpires, however, that critical snarks whose cyclic connectivity equals 4 can be reasonably well understood through the concept of a dot product. Given two snarks G and H, their *dot product* $G\,.\,H$ is defined as follows (see [2, 23]): Choose two independent edges $e = ab$ and $f = cd$ in G and two adjacent vertices u and v in H. Let a', b', and v be the neighbours of u, and let c', d', and u be the neighbours of v. Remove the edges e and f from G and the vertices u and v from H. Finally, connect a to a', b to b', c to c', and d to d'. It is well known that $G\,.\,H$ is indeed a snark and that it is cyclically 4-edge-connected provided that both G and H are (see [2, Theorem 2]).

Note that the added edges aa', bb', cc', and dd' of $G\,.\,H$ form a cycle-separating 4-edge-cut called the *principal* 4-edge-cut of $G\,.\,H$. Thus the cyclic connectivity of a dot product snark cannot exceed 4. Various authors observed that the converse holds as well: every snark that contains a cycle-separating 4-edge-cut can be expressed as a dot product of two smaller snarks (see [10, 21] for an example).

Theorem 6. [10, 21] *Every cycle-separating* 4*-edge-cut* S *in a snark* G *gives rise to a decomposition of* G *into a dot product* $G = G_1\,.\,G_2$ *in such a way that the principal cut of* $G_1\,.\,G_2$ *coincides with* S*. Moreover, if* $G - S$ *is* 3*-edge-colourable, then* G_1 *and* G_2 *are uniquely determined by* S*.*

The previous theorem raises the following natural question: What can be said about the decomposition $G = G_1 . G_2$ when G is critical or bicritical? The following three theorems, proved in [12], provide answers.

Theorem 7. [12] *Let G and H be snarks different from the dumbbell graph. Then $G . H$ is critical if and only if H is critical, G is nearly critical, and the pair of edges of G involved in this dot product is essential in G.*

A *nearly critical snark* is one where every pair of adjacent vertices is non-removable except possibly the pairs of endvertices of the edges $e = ab$ and $f = cd$ of G involved in the dot product. Being *essential* is a rather technical local property which will be defined and discussed in Section 9.

For dot products of bicritical snarks, we only have a partial result, nevertheless, one of fundamental importance. Its essence is that the class of bicritical snarks is closed under decompositions into a dot product.

Theorem 8. [12] *Let G and H be snarks different from the dumbbell graph. If $G . H$ is bicritical, then both G and H are bicritical. Moreover, the pair of edges of G involved in this dot product is essential in G.*

Let G be a bicritical snark that contains a cycle-separating 4-edge-cut. By Theorems 6 and 8, we can decompose G into a dot product $G = G_1 . G_2$ of two smaller bicritical snarks different from the dumbbell graph. If one of these graphs again contains a cycle-separating 4-edge-cut, we can continue the process. After a finite number of steps, we eventually obtain a collection $H_1, H_2, \ldots, H_r$ of cyclically 5-edge-connected bicritical snarks which cannot be further decomposed. Note that the decomposition process is not uniquely determined. The edge-cuts used on the way to a final collection of cyclically 5-edge-connected bicritical snarks may intersect in a very complicated fashion, and choosing one particular 4-edge-cut at a certain stage of the decomposition process may exclude certain cuts from a later use in the decomposition. This concerns especially the cuts which do not exist in the original snark but might be, and often are, created during the process. It is, therefore, rather unexpected that the following theorem is true.

Theorem 9. [12] *Every bicritical snark G different from the dumbbell graph can be decomposed into a collection $\{H_1, \ldots, H_n\}$ of cyclically 5-connected bicritical snarks such that G can be reconstructed from them by repeated dot products. Moreover, such a collection is unique up to isomorphism and ordering of the factors.*

The assumption of Theorem 9 requiring a snark G to be bicritical cannot be relaxed. Indeed, in [12, Section 12] it is shown that there exist critical snarks with substantially different decompositions, and even with decompositions having different numbers of factors.

The results discussed in the preceding paragraphs can be summarised as follows. For every snark G, there exists a sequence of edge reductions

$$G_0 = G, G_1 = G_0 \sim e_1, \ldots, G_r = G_{r-1} \sim e_r$$

such that each G_i is a snark and the terminal member G_r of the sequence is a critical snark.

If G_r is bicritical, then it has a unique decomposition into a collection of cyclically 5-edge-connected critical snarks (all of which are even bicritical). Therefore G can be reconstructed from a collection of cyclically 5-edge-connected critical snarks by using repeated dot products and by a series of edge extensions of snarks.

If G_r is strictly critical, the situation is more complicated because G_r can possibly be expressed as a dot product $H_1 \,.\, H_2$ where H_2 is critical, but H_1 is only nearly critical. Nevertheless, as shown in [12, Section 6], strictly critical snarks whose cyclic connectivity equals 4 can still be reasonably well understood. This brings us back to cyclically 5-edge-connected critical snarks even in the latter case.

4 Methods of analysis

Exhaustive computer search performed by Brinkmann et al. [6] reveals that there are very few 5-simple snarks with girth greater than 5 on up to 36 vertices. These can be put aside and discussed separately. The remaining ones have girth 5 and hence contain a 5-cycle. If C is a 5-cycle in a snark G and $e = uv$ and $e' = u'v'$ are two edges of C, then according to Kászonyi [28] (see also Bradley [4]) the number of 3-edge-colourings of $G \sim e$ is the same as that of $G \sim e'$. In particular, the pairs $\{u, v\}$ and $\{u', v'\}$ are either both non-removable or both removable. It follows that connected components of the subgraph K formed by the union of all 5-cycles of G are subgraphs that play a fundamental structural role in G. Any such component will be called a 5-*cycle cluster* of G.

The smallest 5-cycle clusters can be found in the Petersen graph. We call them *basic* or *Petersen clusters*. It is convenient to view them as cubic multipoles with a natural partition of their semiedges into connectors, which is determined by how they were constructed from the Petersen graph. Clearly, every 5-cycle cluster in a cyclically 5-connected snark must have at least five semiedges and girth 5. To start our analysis, we have identified all 5-cycle clusters of order up to 10 with at least five semiedges. There are seven such 5-cycle clusters, six of which are Petersen clusters.

- The *pentagon* $\mathbf{P}$ is the smallest 5-cycle cluster. It consists of a single cycle of length 5 together with 5 dangling edges forming its unique connector (see Figure 2a). It can be constructed from the Petersen graph by removing any 5-cycle.

- The *dyad* $\mathbf{D}$ (or *Petersen negator*) is a 5-cycle cluster consisting of two 5-cycles sharing a path of length 2. It can be constructed by removing a path uwv of length 2 from the Petersen graph. The natural distribution of semiedges into connectors makes it a $(2, 2, 1)$-pole $\mathbf{D}(I, O, r)$, with 2-connectors I and O containing the dangling edges formerly incident with u and v respectively, and the 1-connector containing the only dangling edge r formerly incident with w (see Figure 2b). The dyad has 7 vertices, 8 edges, and 5 semiedges.

- The *triad* $\mathbf{T}$ is a 5-cycle cluster formed by three 5-cycles C_1, C_2, and C_3 such that C_1 and C_2 have exactly one edge in common while C_3 contains the common edge of C_1 and C_2 and one additional edge of each C_1 and C_2. It can be constructed from the Petersen graph by removing one vertex and severing an edge not incident with it. The natural distribution of semiedges into connectors turns it into a $(2,3)$-pole $\mathbf{T}(B,C)$, shown in Figure 2c, where the connector B corresponds to the severed edge and the connector C corresponds to the removed vertex. The triad has 9 vertices, 11 edges, and 5 semiedges.

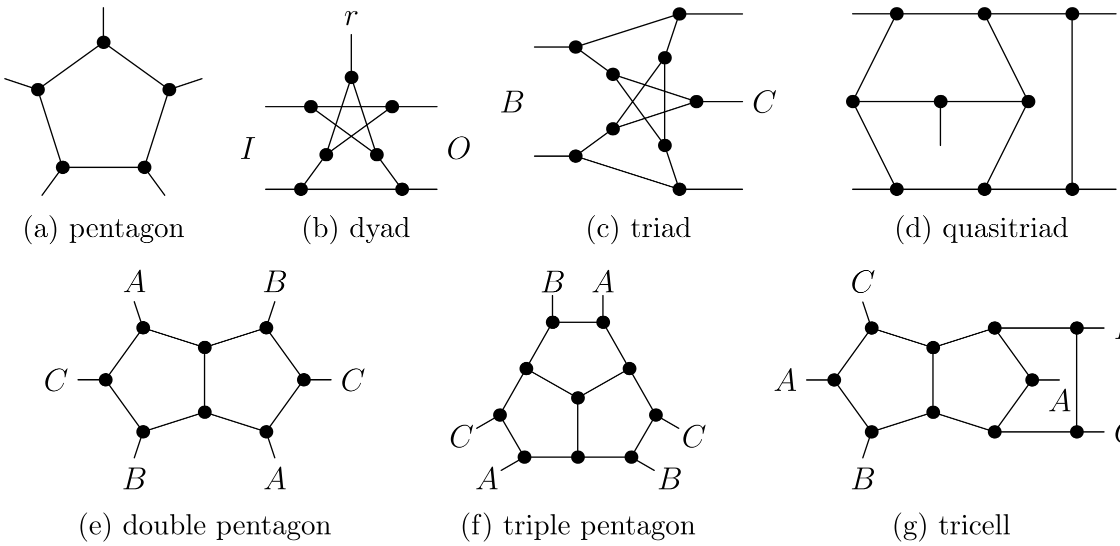
- The *quasitriad* $\mathbf{qT}$ is a 5-cycle cluster consisting of three 5-cycles C_1, C_2, and C_3 such that C_1 and C_2 share two edges and C_3 shares one edge with each C_1 and C_2. One can simply check that performing the junction of a pair of semiedges and connecting the remaining three semiedges to a new vertex always yields a cycle of length smaller than 5. Hence, the quasitriad is not a Petersen cluster. Like triad, quasitriad has 9 vertices, 11 edges, and 5 dangling edges. The quasitriad contains exactly one pair of dangling edges at distance 1, distinguishing it from the triad containing two such pairs.

- The *double pentagon* $\mathbf{dP}$ is a 5-cycle cluster containing two 5-cycles sharing an edge. It can be obtained from the Petersen graph by removing two adjacent vertices u and v and severing an edge e at distance 2 from uv. The natural distribution of semiedges turns it into a $(2,2,2)$-pole $\mathbf{dP}(A,B,C)$, shown in Figure 2e, where the connectors A and B correspond to the vertices u and v respectively, and C corresponds to the edge e. The double pentagon has 8 vertices, 9 edges, and 6 dangling edges.

- The *triple pentagon* $\mathbf{tP}$ is a 5-cycle cluster consisting of three 5-cycles, each pair having two edges in common. It can be constructed from the Petersen graph by severing three pairwise non-adjacent edges lying on a 6-cycle in an alternating order, which makes the triple pentagon a $(2,2,2)$-pole $\mathbf{tP}(A,B,C)$ as depicted in Figure 2f. Note that the three severed edges cannot be extended to a perfect matching of the Petersen graph. The triple pentagon has 10 vertices, 12 edges, and 6 dangling edges.

- The *tricell* $\mathbf{tC}$ is a 5-cycle cluster containing three 5-cycles C_1, C_2 and C_3, where C_1 and C_2 share one edge, C_2 and C_3 share two edges, and C_1 and C_3 are disjoint. Like the triple pentagon, it arises from the Petersen graph by severing three pairwise non-adjacent edges, but in this case, these edges do not lie on a 6-cycle. The 3-cell has a natural representation as a $(2,2,2)$-pole $\mathbf{tC}(A,B,C)$ as shown in Figure 2g. It has 10 vertices, 12 edges, and 6 dangling edges. In contrast to the triple pentagon, the tricell has one dangling edge whose distance to each other dangling edge is at least 2.

The key step of our analysis of 5-simple snarks is the identification of 5-cycle clusters. They are easy to find by a limited-depth breadth-first search in any given graph; we do it by employing a computer program. Identifying the 5-cycle clusters is not sufficient since many snarks contain a significant number of vertices belonging to no clusters. This is



(a) pentagon (b) dyad (c) triad (d) quasitriad

(e) double pentagon (f) triple pentagon (g) tricell

Figure 2: All 5-cycle clusters on up to 10 vertices with girth 5 and at least 5 dangling edges.

why for every given snark, we determine the structure of its 5-cycle clusters, the vertices not contained in clusters, and the connections between them. This allows us to distribute almost all 5-simple snarks into a small number of classes depending on which basic clusters they contain and how they are interconnected. Finally, for each class we theoretically explain why their members are not colourable.

It is worth mentioning that quasitriads, triple pentagons, and tricells do not occur in the analysed snarks; thus, they are listed mainly for completeness. While the latter two clusters have not been observed only because the analysed critical snarks are too small, quasitriads cannot occur in critical snarks at all. Indeed, one can easily check that the quasitriad is a *colour-closed* 5*-pole* in the sense of [38], which means that in every snark of the form $G = \mathbf{qT} * M$ the complementary 5-pole M must be uncolourable. It follows that G has a proper 5-reduction, and such a snark cannot be critical by Theorem 5.

5 Commonly used multipoles

In this section, we develop tools for determining the reasons why the analysed graphs fail to admit a 3-edge-colouring. Since most of our graphs are built from 5-cycle clusters and a small number of additional vertices, the crucial point is to analyse the colouring properties of the basic 5-cycle clusters and their combinations. Since our arguments only use the fact that the Petersen graph is a snark, one can replace the Petersen graph with a larger snark to construct a multipole in a similar manner and with similar colouring properties as the given basic 5-cycle cluster. Regarding the basic 5-cycle clusters as special cases of these multipoles enables us to generalise small snarks to infinite classes that cover almost all 5-simple snarks up to order 36.

We start with several technical definitions that are necessary for this purpose.

Consider a multipole $M(S_1, \dots, S_n)$ with connectors $S_1, \dots, S_n$ and a proper 3-edge-colouring φ. Define the *flow through a connector* S_i of M to be the value

$$\varphi_*(S_i) = \sum_{e \in S_i} \varphi(e).$$

Note that $\varphi_*(S_i)$ may happen to be 0. A connector S_i is called *proper* if $\varphi_*(S_i) \neq 0$ for every colouring φ of M; it is called *improper* if $\varphi_*(S_i) = 0$ for every colouring φ of M. A multipole is called *proper* if all of its connectors are proper; similarly, it is *improper* if all its connectors are improper.

Now let M be an arbitrary k-pole and let $S(M) = \{e_1, e_2, \dots, e_k\}$ be the set of its semiedges. Although semiedges in connectors are generally not ordered, we fix the order given by increasing indices to avoid ambiguity. For any 3-edge-colouring φ of M and any subset $T = \{f_1, f_2, \dots, f_l\}$ of $S(M)$ we set $\varphi(T) = (\varphi(f_1), \varphi(f_2), \dots, \varphi(f_l))$, where the order of semiedges in T agrees with the chosen order of $S(M)$. We define the *colouring set* of M to be the set

$$\mathrm{Col}(M) = \{\varphi(S(M)) \mid \varphi \text{ is a Tait colouring of } M\}.$$

Two k-poles M and N are called *colour-disjoint* if $\mathrm{Col}(M) \cap \mathrm{Col}(N) = \emptyset$. Note that if M and N are colour-disjoint, then $M * N$ is a snark, and vice versa.

Many constructions of snarks can be conveniently described in terms of multipole substitution. This is a very natural concept, and as such, it has previously occurred in the literature under various disguises and different names, see for example [17].

Consider two k-poles M_1 and M_2. We say that M_1 is *colour-contained* in M_2 if $\mathrm{Col}(M_1) \subseteq \mathrm{Col}(M_2)$. If $\mathrm{Col}(M_1) = \mathrm{Col}(M_2)$, we say that M_1 and M_2 are called *colour-equivalent*. Now, let $G = M * N$ be a snark expressed as a junction of two k-poles M and N, and let M' be a k-pole colour-contained in M. We say that the graph $G' = M' * N$ is obtained from the snark G by a *substitution* of M' for M. Observe that G' is again a snark: if G' was colourable, then any colouring of G' could be modified to a colouring of G in a straightforward manner.

Substitution is a generic method of constructing new snarks from old ones. It can be used in two ways: either to produce smaller snarks with a more transparent structure or to create larger snarks from snarks already known. Applying substitution requires having suitable pairs of multipoles. The rest of this section describes several examples of such pairs which occur in 5-simple snarks. As we shall see, most of them arise from generalisations of the basic 5-cycle clusters.

5.1 Negators

Let G be a snark and let uwv be a path of length two in G. Removing uwv from G leaves a multipole M whose semiedges can be naturally distributed into two 2-connectors and one 1-connector, each consisting of the semiedges formerly incident with the same vertex of the path. Let $I = \{e_1, e_2\}$ and $O = \{e_3, e_4\}$ be the connectors consisting of the semiedges formerly incident with u and v, respectively, and let $R = \{e_5\}$ be the

1-connector containing the remaining semiedge of M. If G is the Petersen graph, then $M(I, O, r)$ is the dyad.

Observe that, for each colouring of M, the total flow through precisely one of its 2-connectors is zero — otherwise, the colouring could be extended to a colouring of G. The flow through the other 2-connector is the same as that through the 1-connector due to the parity lemma. In other words, the colouring set of M is a subset of

$$\mathcal{C} = \{(x, x, a, b, a + b), (a, b, x, x, a + b) \in \mathbb{K}^5 \mid a \neq b\}.$$

It means that M behaves like an inverting gadget which inverts matching colours in the *input connector* I to mismatching colours in the *output connector* O, and vice versa. For this reason, M is referred to as a *negator* and is denoted by $\mathrm{Neg}(G; u, v)$. The edge e_5 does not contribute to the inverting property of M, and therefore it is called *residual*.

The notation $\mathrm{Neg}(G; u, v)$ is ambiguous if u and v have more than one common neighbour w. This can only happen if the girth of G does not exceed 4, in which case G is a trivial snark. As we are primarily interested in nontrivial snarks, this ambiguity will be marginal in our work.

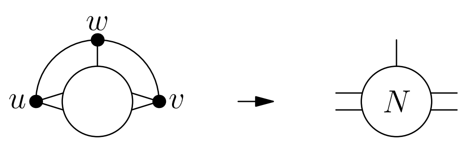


Figure 3: The snark G and a symbolic representation of the negator $N = \mathrm{Neg}(G; u, v)$

A negator whose colouring set is identical with $\mathcal{C}$ is called *perfect*, otherwise it is called *imperfect*. For an imperfect negator N, it is possible that one of its 2-connectors is improper, which means that the other 2-connector is proper. If such a negator N additionally admits all such colourings, it is called *semiperfect*. The following theorem proved by Máčajová and Škoviera [33] implies that each negator is either perfect, semiperfect, or is uncolourable, and provides a characterisation of perfect and semiperfect negators.

Theorem 10. [33] *Let $N = \mathrm{Neg}(G; u, v)$ be a negator and let w a common neighbour of u and v. If N is colourable, then it is either perfect or semiperfect. Moreover, the following hold.*

(i) *N is perfect if and only if each of the pairs $\{u, w\}$ and $\{v, w\}$ of adjacent vertices is non-removable.*

(ii) *N is semiperfect if and only if at least one of the pairs $\{u, w\}$ and $\{v, w\}$ is removable.*

The smallest (connected) example of a negator can be constructed from the Petersen graph. Since the Petersen graph is 2-arc transitive, there is, up to isomorphism, only one way of removing a path of length 2, and thus there is, up to isomorphism, only one Petersen negator — the dyad. Obviously, the dyad is a perfect negator.

The parity lemma implies that every $(2,2,1)$-pole which is colour-disjoint from a perfect negator must be a proper $(2,2,1)$-pole. The smallest such $(2,2,1)$-pole is the path of length 2 with dangling edges retained and distributed into connectors in the usual way. It consists of two endvertices u and v and their common neighbour w. We will denote it by $P_2(I,O,r)$, where the connector I corresponds to the dangling edges incident with u, O corresponds to the dangling edges incident with v, and the remaining semiedge r arises from the dangling edge incident with w. The colouring set of P_2 is

$$\mathrm{Col}(P_2) = \{(a,b,c,d,e) \in \mathbb{K}^5 \mid a \neq b,\ c \neq d,\ a+b+c+d+e = 0\}.$$

5.2 Proper (2,3)-poles

Take an arbitrary snark G and choose a vertex v and an edge e in G. Form a $(2,3)$-pole $T(B,C)$ by removing v and severing e. Let B be the set of semiedges arising from severing e and let C be the set of semiedges formerly incident with v. Observe that if G is the Petersen graph and e is not incident with v, then the result is the triad. For connectivity reasons, we only consider proper $(2,3)$-poles that arise from a snark G where the removed vertex v and edge e are not incident, although the colouring properties of $T(B,C)$ hold in the general case as well.

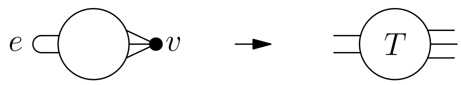


Figure 4: The snark G and a symbolic representation of a proper $(2,3)$-pole T

We claim that T is a proper $(2,3)$-pole. Suppose not. Then T admits a colouring φ such that the flow through one of the connectors is zero. The parity lemma then implies that $\varphi_*(B) = \varphi_*(C) = 0$ and allows extending the colouring of T to a colouring of the original snark G, which is impossible. Therefore T is proper. The colouring set of each proper $(2,3)$-pole T is clearly a subset of

$$\mathcal{C} = \{(a_1,a_2,b_1,b_2,b_3) \in \mathbb{K}^5 \mid a_1+a_2 = b_1+b_2+b_3 \neq 0\}.$$

If a proper $(2,3)$-pole T admits all colourings such that the flow through each of the connectors is nonzero — that is to say, if its colouring set coincides with $\mathcal{C}$ — then T is called *perfect*; otherwise it is *imperfect*. As one can expect, the triad is a perfect proper $(2,3)$-pole, which can easily be verified by hand or with the help of a computer. An example of an imperfect proper $(2,3)$-pole will be discussed in Section 5.8.

Let us consider the multipole M removed from a snark G in order to construct the proper $(2,3)$-pole $T(B,C)$. Clearly, M coincides with a disconnected $(2,3)$-pole $M_{ev}(B,C)$ whose connectors are $B = (b_1,b_2)$ and $C = (c_1,c_2,c_3)$, where b_1 and b_2 are the ends of an isolated edge and c_1, c_2, and c_3 are ends of three dangling edges incident with one common vertex (see Figure 13). Its colouring set is

$$\mathrm{Col}(M_{ev}) = \{(x,x,a,b,c) \in \mathbb{K}^5 \mid a+b+c = 0\}.$$

5.3 Proper (3,3)-poles

Let u and v be two vertices of a snark G. Construct a $(3,3)$-pole $M(I,O)$ by removing u and v from G and putting the three semiedges formerly incident with u and v into the connectors I and O, respectively. The $(3,3)$-pole M is proper: Kirchhoff's law implies that $\varphi_*(I) = \varphi_*(O)$ for every 3-edge-colouring φ of M, and if any of these values equals zero, then φ can be extended to a colouring of G, which is a contradiction. Again, for connectivity reasons, we only consider proper $(3,3)$-poles where the vertices u and v are not adjacent.

In the Petersen graph, there is only one way of removing a pair of non-adjacent vertices because the graph is 2-arc-transitive and has diameter 2. The result is a unique proper $(3,3)$-pole M_8 of order 8. It is easy to see that M_8 can be constructed from the dyad by attaching the residual semiedge to a new vertex incident with two additional dangling edges, one contributing to the input and the other one to the output.

5.4 Even (2,2,2)-poles

An *even* $(2,2,2)$*-pole* is one where the number of connectors having nonzero total flow is even. The term for this multipole was coined by Goldberg [21], who was the first to study this kind of multipoles. An even $(2,2,2)$-pole can be constructed as follows. Take a snark G and a vertex v with neighbours u_1, u_2 and u_3. Remove v, u_1, u_2, and u_3, and for each $i \in \{1,2,3\}$ let $S_i = \{e_i, f_i\}$ be the set containing the semiedges formerly incident with u_i that do not arise from $u_i v$. We claim that the resulting $(2,2,2)$-pole $H(S_1,S_2,S_3)$ is even. Kirchhoff's law tells us that a flow cannot have a nonzero total flow through exactly one connector. Therefore, it is sufficient to show that the flow through at least one of the connectors S_1, S_2, and S_3 is zero. However, if $\varphi_*(S_i) = c_i \neq 0$ for every $i \in \{1,2,3\}$, then c_1, c_2, and c_3 are three distinct colours, so φ can be extended to a proper colouring of G, which is a contradiction. Therefore $H(S_1,S_2,S_3)$ is indeed an even $(2,2,2)$-pole.

A simple example of an even $(2,2,2)$-pole is a *hexagram*, which arises from a cycle of length six, with a dangling edge at every vertex, by forming each connector from a pair of opposite dangling edges of the 6-cycle (see Figure 5). It is derived from the Petersen graph. Note that every even $(2,2,2)$-pole can be constructed from a negator by deleting the residual semiedge along with its end-vertex.

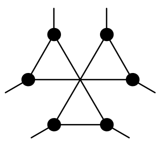


Figure 5: Hexagram — an even $(2,2,2)$-pole of order 6.

Denote by $V_4(S_1,S_2,S_3)$ the complementary $(2,2,2)$-pole we removed from the snark G to create H. It consists of a vertex v and its three neighbours u_1, u_2 and u_3, each of them with two dangling edges attached. The connector S_i contains the semiedges belonging to the dangling edges incident with u_i (for $i \in \{1,2,3\}$). The colouring set of V_4 is the set

$$\mathrm{Col}(V_4) = \{(a_1, b_1, a_2, b_2, a_3, b_3) \in \mathbb{K}^6 \mid a_1 + b_1 + a_2 + b_2 + a_3 + b_3 = 0, (\forall i)(a_i \neq b_i)\}.$$

Large even $(2,2,2)$-poles can be constructed from smaller ones as follows. Let M be an arbitrary 3-pole, possibly containing loops and parallel edges, with three pairwise distinct vertices v_1, v_2, and v_3, each being incident with one dangling edge. Replace each vertex v of M with an even $(2,2,2)$-pole H_v and each edge (including the dangling ones) with the $(2,2)$-pole consisting of two isolated edges which have their ends in different connectors. The result is a $(2,2,2)$-pole $H_M(S_1, S_2, S_3)$, where each connector S_i consists of two dangling edges that replace the dangling edge incident with v_i. We claim that H_M is an even $(2,2,2)$-pole. Assume that $\varphi_*(S_i) = a \neq 0$ for some connector S_i of H_M. Observe that each even $(2,2,2)$-pole H must have an even number of connectors with flow a through it. By applying this argument inductively, one can readily conclude that there exists another connector S_j of H_M with $\varphi_*(S_j) = a$. Consequently, $\varphi_*(S_k) = 0$ for $k \in \{1,2,3\} - \{i,j\}$, and therefore the number of connectors of H_M with nonzero flow is either zero or two. In other words, H_M is an even $(2,2,2)$-pole.

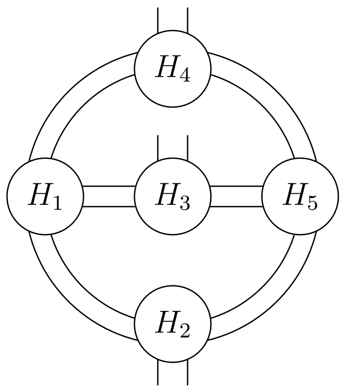


Figure 6: An even $(2,2,2)$-pole constructed from five smaller ones

For illustration, consider the cubic graph B_2 consisting of two vertices and three parallel edges joining them, subdivide each edge with a new vertex, and attach a dangling edge to all three 2-valent vertices, thereby producing a 3-pole M on five vertices. If we apply the construction described above to this 3-pole, we obtain an even $(2,2,2)$-pole $H_M(S_1, S_2, S_3)$ composed from five smaller ones. The result is represented in Figure 6, with the $(2,2,2)$-poles denoted by H_i for $i \in \{1,\ldots,5\}$. Now, if for each of the five even $(2,2,2)$-poles we take the hexagonal $(2,2,2)$-pole derived from the Petersen graph and perform the junction $H_M(S_1, S_2, S_3) * V_4(S_1, S_2, S_3)$ we obtain the snark displayed in Figure 1. Clearly, an analogous construction can be performed starting from any cubic graph in place of B_2.

5.5 Isaacs $(3,3)$-poles

Flower snarks are a well-known and in a certain sense exceptional family of snarks introduced by Isaacs in [23]. Their basic building block is the $(3,3)$-pole $Y(I, O)$ with two ordered connectors $I = (i_1, i_2, i_3)$ and $O = (o_1, o_2, o_3)$ shown in Figure 7. It can be constructed from the complete bipartite graph $K_{3,3}$ by deleting two vertices from the same partite set, forming the connectors in the usual way, and ordering their semiedges in such a way that $\{i_1, o_2\}$, $\{i_2, o_1\}$, and $\{i_3, o_3\}$ are adjacent pairs of dangling edges.

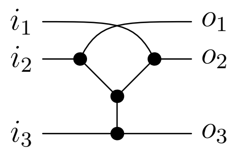


Figure 7: The Isaacs $(3, 3)$-pole Y

It is known [23, Theorem 4.1.1] that the colouring set of the Isaacs $(3, 3)$-pole Y is the set $\mathcal{C}$ consisting of all 6-tuples $(a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{K}^6$ satisfying the following conditions:

(i) $a_3 \neq b_3$

(ii) If $(a_1, a_2, a_3, b_1, b_2, b_3) \in \mathcal{C}$, then $(b_1, b_2, b_3, a_1, a_2, a_3,) \in \mathcal{C}$.

(iii) The first three entries a_1, a_2, and a_3 never coincide.

(iv) If $\{a_1, a_2, a_3\} = \{a, a, b\}$ for some $a, b \in \mathbb{K}$, then $\{b_1, b_2, b_3\} = \{c, c, b\}$, where $\{a, b, c\} = \{1, 2, 3\}$.

(v) If $\{a_1, a_2, a_3\} = \{1, 2, 3\}$, then $(b_1, b_2, b_3) = (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$, where σ is an odd permutation of the index set $\{1, 2, 3\}$.

The Isaacs flower snark J_n, where $n \geqslant 3$ is odd, can be produced by taking the disjoint union on n copies $Y(I_i, O_i)$ of $Y(I, O)$, where $i \in \{1, 2, \ldots, k\}$, and by joining O_i to I_{i+1} following the order of semiedges in the connectors; of course, the subscripts in the definition are taken modulo n. The fact that J_k is not 3-edge-colourable readily follows Conditions (iii)-(v) stated above.

Let $Y_k(I_1, O_k)$ denote the $(3, 3)$-pole arising similarly from the union of k disjoint copies $Y(I_i, O_i)$ of $Y(I, O)$ and by performing the junction of O_i and I_{i+1} only for $i \in \{1, 2, \ldots, k-1\}$. It is known (and easy to see) that $\mathrm{Col}(Y_{2m}) = \mathrm{Col}(Y_2)$ for every integer $m \geqslant 1$, see [38].

Note that all the connectors considered in this subsection are ordered, since the arguments of the uncolourability of the Isaacs snarks require this fixed order of the semiedges in the connectors of Y (see Item (v) above).

5.6 Proper $(2, 2, 1)$-poles of type NN

Take two negators $N_1(I_1, O_1, r_1)$ and $N_2(I_2, O_2, r_2)$ and perform the junction of the connectors O_1 and I_2. Add one vertex v incident with the semiedges r_1, r_2 and one new dangling edge producing a semiedge r_3 (see Figure 8). Denote the resulting $(2, 2, 1)$-pole $M(I_1, O_2, r_3)$ by $\mathrm{NN}(N_1, N_2)$.

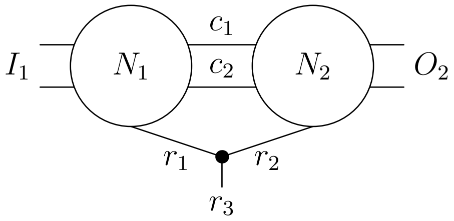


Figure 8: A $(2, 2, 1)$-pole of type NN

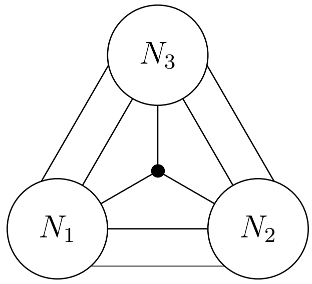


Figure 9: A generalised Loupekine snark

We prove that $\mathrm{Col}(M) \subseteq \mathrm{Col}(P_2)$, and if N_1 and N_2 are perfect negators, then $\mathrm{Col}(M) = \mathrm{Col}(P_2)$. Let φ be a colouring of M. The flow through the connectors O_1 and I_2 has to be zero, otherwise $\varphi(r_1) = \varphi_*(O_1) = \varphi_*(I_2) = \varphi(r_2)$, a contradiction. Consequently, the flows through I_1 and O_2 have to be nonzero. This implies that $\varphi(S(M)) \in \mathrm{Col}(P_2)$. Now assume that N_1 and N_2 are perfect negators. Let φ be an assignment of colours to the dangling edges of M such that $\varphi(S(M)) \in \mathrm{Col}(P_2)$, that is, $\varphi_*(I_1) = a \neq 0$, $\varphi_*(O_2) = b \neq 0$ and $\varphi(r_3) = a + b \neq 0$. We extend φ to a colouring of the multipole M. Set $\varphi(r_1) = a$, $\varphi(r_2) = b$, and $\varphi(c_1) = \varphi(c_2) = a$ for the edges c_1 and c_2 connecting N_1 to N_2. Both the negators N_1 and N_2 have admissible colours on their dangling edges, and since they are perfect, we can extend φ to the entire $(2, 2, 1)$-pole M.

Since P_2 is a proper $(2, 2, 1)$-pole, so is M. Taking both the negators N_1 and N_2 from the Petersen graph — that is, dyads — we obtain a proper $(2, 2, 1)$-pole $P_{NN} = \mathrm{NN}(\mathbf{D}, \mathbf{D})$ of order 15 with $\mathrm{Col}(P_{NN}) = \mathrm{Col}(P_2)$.

It is worth mentioning that every snark G which arises from a snark H by a substitution of $\mathrm{NN}(N_1, N_2)$ for P_2 can be alternatively described as follows. In the snark G, the complementary 5-pole M' connected to $\mathrm{NN}(N_1, N_2)$ is a snark H with P_2 removed. It means that M' is the negator $N_3 = \mathrm{Neg}(H; u, v)$, where u and v are the endvertices of P_2. Thus the new snark G has the structure of a Loupekine snark with the Petersen negators replaced with N_1, N_2, and N_3 (see Figure 9).

5.7 Proper $(2, 2, 1)$-poles of type TT

Take two proper $(2, 3)$-poles $T_1(B_1, C_1)$ and $T_2(B_2, C_2)$ and perform the junction of C_1 to C_2. Pick one of the newly created edges, subdivide it with a vertex v and attach a dangling edge to v, producing a semiedge r (see Figure 10). Denote the $(2, 2, 1)$-pole $M(B_1, B_2, r)$ constructed in this way by $\mathrm{TT}(T_1, T_2)$.

We prove that $\mathrm{Col}(\mathrm{TT}(T_1, T_2)) \subseteq \mathrm{Col}(P_2)$ with equality attained if both T_1 and T_2 are perfect (see Section 5.2 for the definition). Since T_1 and T_2 are proper, any colouring φ of M satisfies $\varphi_*(B_1) = a \neq 0$ and $\varphi_*(B_2) = b \neq 0$, so $\varphi(S(M)) \in \mathrm{Col}(P_2)$. Assume that T_1 and T_2 are perfect. Let φ be an assignment of colours to the dangling edges of M such that $\varphi(S(M)) \in \mathrm{Col}(P_2)$. Then there exist distinct colours a and b such that $\varphi_*(B_1) = a$, $\varphi_*(B_2) = b$, and $\varphi(r) = a + b \neq 0$. Assign the edges joining v to T_1 and T_2 colours b and a, respectively. Now, if we colour the remaining two edges of M joining T_1 to T_2 with

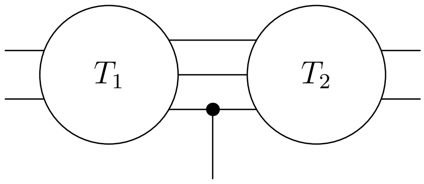


Figure 10: A $(2,2,1)$-pole of type TT

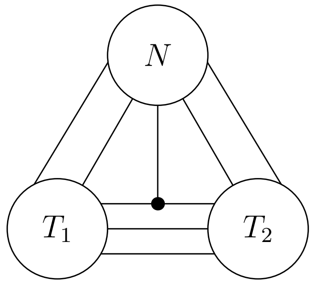


Figure 11: Structure of a snark consisting of two proper $(2,3)$-poles T_1 and T_2 and one negator N

colours a and b arbitrarily, we obtain admissible assignments of colours for the semiedges of both T_1 and T_2. Since both T_1 and T_2 are perfect, the assignments extend to colourings of T_1 and T_2 and hence to a colouring of the entire M.

Since P_2 is a proper $(2,2,1)$-pole, so is $\mathrm{TT}(T_1,T_2)$. If both T_1 and T_2 are obtained from the Petersen graph — that is, if they are triads — we get a perfect proper $(2,2,1)$-pole $P_{TT} = \mathrm{TT}(\mathbf{T},\mathbf{T})$ of order 19 with $\mathrm{Col}(P_{TT}) = \mathrm{Col}(P_2)$.

Observe that removing a copy of P_2 from the snark G yields a negator N. Hence the substitution of $\mathrm{TT}(T_1,T_2)$ for P_2 produces a snark consisting of two proper $(2,3)$-poles T_1 and T_2 and one negator N as depicted in Figure 11.

5.8 Improper $(2,3)$-poles of type NT

Let $N(I,O,r)$ be a negator and $T(B,C)$ be a proper $(2,3)$-pole. Perform the junction of O and B, subdivide one of the dangling edges belonging to the 3-connector C of T, say e, with a new vertex v, and attach the residual semiedge r of N to v (see Figure 12). The resulting $(2,3)$-pole $M(I,C')$ has its 2-connector I inherited from N while its 3-connector C' has two semiedges e_1 and e_2 inherited from the output connector C of T, and the third semiedge e_3 arises from the subdivision of e with v. Denote the $(2,3)$-pole M by $\mathrm{NT}(N,T)$.

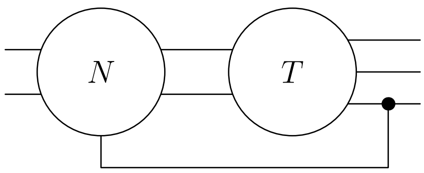


Figure 12: A $(2,3)$-pole of type NT

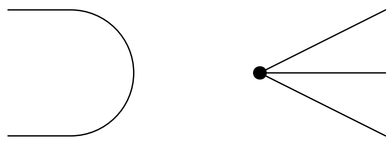


Figure 13: The $(2,3)$-pole M_{ev}

We show that $\mathrm{Col}(\mathrm{NT}(N,T)) \subseteq \mathrm{Col}(M_{ev})$, where M_{ev} is a $(2,3)$-pole defined in Section 5.2 and depicted in Figure 13. We also show that if both N and T are perfect, then $\mathrm{Col}(\mathrm{NT}(N,T)) = \mathrm{Col}(M_{ev})$.

Let φ be a colouring of the multipole M. Since T is a proper $(2,3)$-pole, $\varphi_*(B) = \varphi_*(O) \neq 0$ and thus $\varphi_*(I) = 0$. The parity lemma implies that $\varphi_*(C') = 0$ and therefore

$\varphi(S(M)) \in \mathrm{Col}(M_{ev})$.

Assume that N is a perfect negator and T is a perfect proper $(2,3)$-pole. Let φ be an assignment of colours to the dangling edges of M such that $\varphi_*(I) = 0$ and $\varphi(C') = (\varphi(e_1), \varphi(e_2), \varphi(e_3)) = (a, b, c)$ where $a + b + c = 0$. To produce a colouring of $M(I, C')$, set $\varphi(r) = a$, and assign the colours b and c to the dangling edges of B in any order. Then $\varphi_*(O) = \varphi(r) = a = \varphi_*(C)$, and since N and T are both perfect, this assignment extends to a colouring of M, as required. Therefore $\varphi(S(M)) \in \mathrm{Col}(M_{ev})$ in this case.

As we have seen, the flow through each of the connectors of $\mathrm{NT}(N, T)$ is always zero, which means that $\mathrm{NT}(N, T)$ is an improper $(2,3)$-pole. The improper $(2,3)$-pole $P_{NT} = \mathrm{NT}(\mathbf{D}, \mathbf{T})$ is obtained by taking the negator and the proper $(2,3)$-pole from the Petersen graph. It has 17 vertices and $\mathrm{Col}(P_{NT}) = \mathrm{Col}(M_{ev})$.

If we distribute the semiedges of the connector C' of the improper $(2,3)$-pole $M(I, C')$ into a 2-connector and a 1-connector, we obtain a semiperfect negator $M'(I, C' - \{s\}, s)$ where I is its improper connector, and $s \in C'$ takes the role of a residual semiedge. Furthermore, if the residual semiedge s of M' is adjoined to I to make a 3-connector, a proper $(3,2)$-pole $M''(I \cup \{s\}, C' - \{s\})$ is obtained. Indeed, for every colouring φ of M'' we have $\varphi_*(I \cup \{s\}) = \varphi_*(I) + \varphi(s) = 0 + \varphi(s) \neq 0$. However, M'' is an imperfect $(3,2)$-pole, because the two semiedges of I always receive the same colour.

Note that if $G = M_{ev} * T_1$ is a snark for a suitable 5-pole T_1, then T_1 is a proper $(2,3)$-pole with respect to the natural distribution of its semiedges into connectors. Hence, the substitution of an improper $(2,3)$-pole $\mathrm{NT}(N, T_2)$ for M_{ev} yields a snark consisting of two proper $(2,3)$-poles T_1 and T_2, and a negator N as shown in Figure 11.

5.9 Proper (2,2,2)-poles of type TTT

Take three proper $(2,3)$-poles $T_i(B_i, C_i)$, where $i \in \{1,2,3\}$, and a $(3,3,3)$-pole $W = W(D_1, D_2, D_3)$ formed from a single vertex w with three dangling edges by adding three isolated edges in such a way that each isolated edge contributes to different 3-connectors of W. Perform the junctions C_i to D_i for each $i \in \{1,2,3\}$ to obtain a $(2,2,2)$-pole $M(B_1, B_2, B_3)$ shown in Figure 14. Denote the result by $\mathrm{TTT}(T_1, T_2, T_3)$.

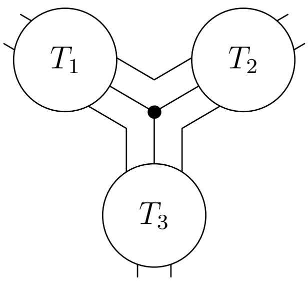


Figure 14: A $(2,2,2)$-pole of type TTT

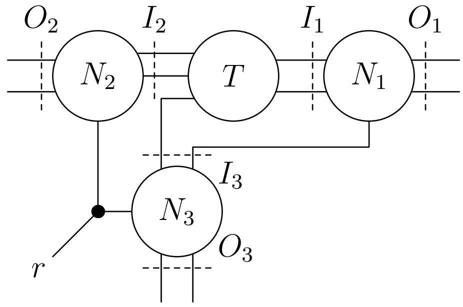


Figure 15: A $(2,2,2,1)$-pole of type 3NT

We prove that $\mathrm{Col}(\mathrm{TTT}(T_1, T_2, T_3)) \subseteq \mathrm{Col}(V_4)$, with equality attained whenever all of T_1, T_2, and T_3 are perfect (see Section 5.4 for the definition of V_4). Let φ be a colouring of $M(B_1, B_2, B_3)$. Since B_i is a connector of a proper $(2,3)$-pole, we see that $\varphi_*(B_i) \neq 0$ for all $i \in \{1,2,3\}$. This fact readily implies that $\varphi(S(M)) \in \mathrm{Col}(V_4)$. Next, assume that T_1, T_2, and T_3 are perfect. Consider an assignment φ of colours to the semiedges of M such that $c_i = \varphi_*(B_i) \neq 0$ for $i \in \{1,2,3\}$, and $c_1 + c_2 + c_3 = 0$. Let e_1, e_2, and e_3 be those semiedges from C_1, C_2, and C_3, respectively, that are incident with the vertex w. If we assign the same colour, chosen arbitrarily, to all the remaining semiedges of the connectors C_i and set $\varphi(e_i) = c_i$ for $i \in \{1,2,3\}$, we get an admissible colouring of the semiedges for each perfect $(2,3)$-pole T_i. Therefore φ can be extended to the entire multipole M. This shows that $\mathrm{TTT}(T_1, T_2, T_3)$ is a proper $(2,2,2)$-pole, and if T_1, T_2, and T_3 are all perfect, then $\mathrm{Col}(\mathrm{TTT}(T_1, T_2, T_3)) = \mathrm{Col}(V_4)$. By choosing the triads for all three constituting proper $(2,3)$-poles we obtain a proper $(2,2,2)$-pole $P_{TTT} = \mathrm{TTT}(\mathbf{T}, \mathbf{T}, \mathbf{T})$ of order 28 with $\mathrm{Col}(P_{TTT}) = \mathrm{Col}(V_4)$.

Note that after removing the $(2,2,2)$-pole V_4 from any snark G we obtain an even $(2,2,2)$-pole. It follows that the snark resulting from a substitution of $\mathrm{TTT}(T_1, T_2, T_3)$ for V_4 consists of an even $(2,2,2)$-pole joined to a proper $(2,2,2)$-pole of type TTT.

5.10 Panchromatic (2,2,2,1)-poles of type 3NT

Take three negators $N_i(I_i, O_i, r_i)$, for $i \in \{1,2,3\}$), and one proper $(2,3)$-pole $T(B, C)$. Arrange them as depicted in Figure 15 and denote the resulting $(2,2,2,1)$-pole $M(O_1, O_2, O_3, r)$ by $3\mathrm{NT}(N_1, N_2, N_3, T)$.

Let $M_7(\{e_1, e_2\}, I, O, r)$ denote a $(2,2,2,1)$-pole consisting of two components, an isolated edge, whose semiedges e_1 and e_2 constitute the first 2-connector, and a path of length 2 with the standard distribution of semiedges into two 2-connectors and a 1-connector. We show that $\mathrm{Col}(M) \subseteq \mathrm{Col}(M_7)$. Moreover, if the negators N_1, N_2, N_3, and the proper $(2,3)$-pole T are all perfect, then $\mathrm{Col}(M) = \mathrm{Col}(M_7)$.

Let φ be a colouring of M, let e be the edge joining I_3 and the 3-connector C of T, and let $\varphi(e) = a$. Since the connector I_1 is joined to the proper $(2,3)$-pole T, there exist an element $b \in \mathbb{K}$ such that $\varphi_*(I_1) = b$, whence $\varphi_*(O_1) = 0$ and $\varphi(r_1) = b$.

Suppose that $a + b = \varphi_*(I_3) \neq 0$. Since N_3 is a negator, $\varphi(r_3) = a + b$. On the other hand, the parity lemma applied to T implies that $\varphi_*(I_2) = \varphi_*(B) + \varphi(e) = a + b \neq 0$. However, N_2 is a negator, so $\varphi(r_2) = a + b = \varphi(r_3)$. Thus both r_2 and r_3 receive the same colour, which is impossible because they are adjacent. Hence, $\varphi_*(I_3) = 0$ and $a = b$.

Now, $\varphi_*(O_3) \neq 0$ so $\varphi(e) = b = \varphi(r_1) = \varphi_*(I_1)$. If we apply the parity lemma to T again, we conclude that $b = \varphi_*(C) = \varphi_*(I_2) + \varphi(e) = \varphi_*(I_2) + b$. Therefore $\varphi_*(I_2) = 0$ whence $\varphi_*(O_2) \neq 0$. Summing up, $\varphi_*(O_1) = 0$ while $\varphi_*(O_2)$, $\varphi_*(O_3)$, and $\varphi(r)$ are all nonzero. By the parity lemma, the connectors of M receive from φ all four values of $\mathbb{Z}_2 \times \mathbb{Z}_2$, which justifies calling M *panchromatic*. Furthermore, one can immediately see that $\varphi(S(M)) \subseteq \mathrm{Col}(M_7)$.

Now assume that all the multipoles N_1, N_2, N_3, and T are perfect. If φ is an assignment of colours to the dangling edges of M such that $\varphi(S(M)) \in \mathrm{Col}(M_7)$, we can extend it to a colouring of dangling edges of each of the 5-poles N_1, N_2, N_3, and T in such a way that the flows through their connectors are the same as the flows in the proof that

$\text{Col}(M) \subseteq \text{Col}(M_7)$. Such a colouring is admissible for all of the connectors of multipoles N_1, N_2, N_3, and T, and since they are perfect, it can be extended to the entire M.

By plugging in dyads and triads we obtain a panchromatic $(2,2,2,1)$-pole $P_{3NT} = 3\text{NT}(\mathbf{D},\mathbf{D},\mathbf{D},\mathbf{T})$ of order 31 with $\text{Col}(P_{3NT}) = \text{Col}(M_7)$.

Panchromatic $(2,2,2,1)$-poles can be used to create an interesting family of snarks based on 4-regular graphs. Let G be a 4-regular graph. For every vertex v of G take a panchromatic $(2,2,2,1)$-pole X_v and for each edge uv join a connector of X_u to a connector of X_v mimicking the adjacency of G. Of course, this can be accomplished only when G has a perfect matching, because a 1-connector must be joined to another 1-connector. If the edge-chromatic number of G is greater than 4, then the resulting graph $\tilde{G}$ is a snark — otherwise a 3-edge-colouring of $\tilde{G}$ would induce a proper 4-edge-colouring of G using all elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ as colours. This idea first appears in [30].

5.11 Superpentagons

Let $C_5 = C_5(e_0,\ldots,e_4)$ denote the 5-pole consisting of a 5-cycle having vertices $v_0,\ldots,v_4$, arranged cyclically, with five semiedges $e_0,\ldots,e_4$ attached to them correspondingly. We define a *superpentagon* to be any 5-pole M with $\text{Col}(M) \subseteq \text{Col}(C_5)$. Substituting a superpentagon M for a 5-cycle K in a snark G produces another snark G'. It may be worth mentioning that substituting a superpentagon for a 5-cycle is equivalent to a 5-product of G with a snark $\bar{M}$ obtained from M by joining M to a 5-cycle in a Petersen-like manner (that is, as a pentagram). For the definition of a 5-product of snarks we refer the reader to [10, last paragraph of Section 3] or [12, pp. 51-52].

It is a well known fact, proved in [38, Lemmas 6.2-6.5], that for an arbitrary 5-pole M with $\text{Col}(M) \subseteq \text{Col}(C_5)$ only two possibilities can occur: either $\text{Col}(M) = \emptyset$ or $\text{Col}(M) = \text{Col}(C_5)$. In the latter case, we call M a *perfect superpentagon*. A familiar example of a perfect superpentagon has 15 vertices and can be obtained from the Isaacs flower snark J_5 by removing the unique 5-cycle C of J_5 and changing the cyclic order e_0,e_1,e_2,e_3,e_4 of the resulting dangling edges to e_0,e_2,e_4,e_1,e_3.

We now describe a superpentagon $Q(f_0,\ldots,f_4)$ that can be observed in the analysed graphs. Let $T = T(B,C)$ be a proper $(2,3)$-pole with $B = \{d_1,d_2\}$. By distributing the semiedges of B into two 1-connectors we obtain a $(3,1,1)$-pole $T(C,d_1,d_2)$, which is proper as well. Next, let $U = U(S_1,S_2,r)$ be a $(3,3,1)$-pole with $S_i = \{e_i,f_i,g_i\}$, where $i \in \{1,2\}$, which consists of one vertex v incident with three dangling edges whose semiedges are e_1, e_2 and r, and two isolated edges with semiedges f_1, f_2 and g_1, g_2, respectively (see copies U_2 and U_3 of U in Figure 16). Finally, let $R = R(I,O)$ be a proper $(3,3)$-pole.

To construct $Q(f_0,\ldots,f_4)$, proceed as follows (see Figure 16):

- Set $f_0 = e_0$.
- For $i \in \{1,4\}$ substitute a copy T_i of $T(C,d_1,d_2)$ for the vertex v_i of C_5. Define f_i to be the copy of the semiedge d_1 in T_i.

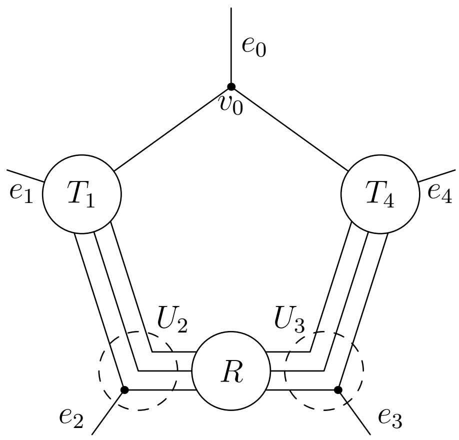


Figure 16: A superpentagon

- For $i \in \{2, 3\}$ substitute a copy U_i of $U(S_1, S_2, r)$ for the vertex v_i of C_5. Define f_i to be the copy of the semiedge r in U_i.
- Substitute the proper $(3, 3)$-pole $R = R(I, O)$ the for the edge v_2v_3 of C_5. Join I to the copy of the connector S_2 in U_2, and further join O to the copy of S_1 in U_3.
- Join the copy of S_1 in U_2 to the 3-connector of T_1. Join the copy of S_2 in U_3 to the 3-connector of T_4.
- Attach the copy of d_2 in both T_1 and T_4 to v_0.

It is easy to check that $Q(f_0, \ldots, f_4)$ is a superpentagon. Moreover, if the involved $(3, 3)$-pole R is colourable and both proper $(2, 3)$-poles corresponding to T_1 and T_4 are perfect, then Q is a perfect superpentagon.

The smallest example that can be produced by this method has 29 vertices. It is constructed by choosing T to be the triad and R to be the proper $(3, 3)$-pole on eight vertices. Clearly, this superpentagon is proper.

6 Results of analysis

Having specified all necessary tools, we present the results of our analysis. This section only describes bicritical snarks because critical snarks that are not bicritical deserve special attention; they will be treated in a separate section.

For a given snark G we start by identifying all 5-cycle clusters as explained in Section 4. Subsequently we check whether G contains any of the 5-poles $P_{NN} = \mathrm{NN}(\mathbf{D}, \mathbf{D})$, $P_{NT} = \mathrm{NT}(\mathbf{D}, \mathbf{T})$, $P_{TT} = \mathrm{TT}(\mathbf{T}, \mathbf{T})$, $P_{TTT} = \mathrm{TTT}(\mathbf{T}, \mathbf{T}, \mathbf{T})$, and $P_{3NT} = \mathrm{3NT}(\mathbf{D}, \mathbf{D}, \mathbf{D}, \mathbf{T})$ which we have described in Section 5. As we have proved in Section 5, the multipoles P_{NN}, P_{NT}, P_{TT}, P_{TTT}, and P_{3NT} are colour-equivalent to P_2, M_{ev}, P_2, V_4, and M_7, respectively. Here we make use of the fact that both $\mathbf{D}$ and $\mathbf{T}$ are perfect. The corresponding pairs of colour-equivalent multipoles are listed in Table 2.

Whenever a snark G contains any of these multipoles, it can be constructed from a smaller snark G' by a suitable substitution. The order of G' is easy to compute: if $|G| \leqslant 36$, then $|G'| \leqslant 24$. There are not so many snarks of order at most 24, so it is easy to check whether G' is isomorphic to one of them. It may be useful to note that in all cases where this procedure could be applied, the snark G' was nontrivial, although sometimes its cyclic connectivity was smaller (necessarily 4) or G' was reducible.

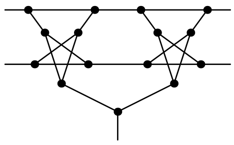
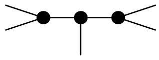
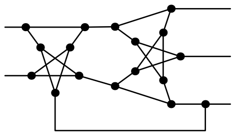
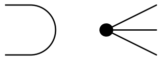
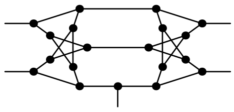
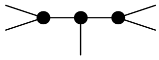
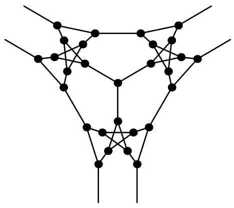
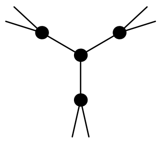
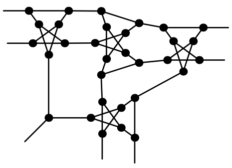
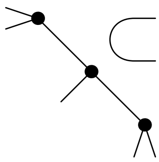
multipole		order	smallest equivalent		property
P_{NN}		15		P_2	proper
P_{NT}		17		M_{ev}	improper
P_{TT}		19		P_2	proper
P_{TTT}		28		V_4	proper
P_{3NT}		31		M_7	panchromatic

Table 2: Common multipoles and their smallest colour-equivalent counterparts

If we discard the snarks arising from smaller snarks by one of the substitutions mentioned above, some snarks will remain. We distribute them, for each particular order, into several classes. Each class can be characterised by specific junctions of suitable multipoles. Most of these multipoles are just 5-cycle clusters, but several more interesting ones have emerged, too (for example, M_{11} in Figure 18). In contrast to 5-cycle clusters, they have been analysed by hand. We present each class as an infinite family of snarks containing the desired small snarks.

Up to isomorphism, there is only one Petersen negator N_P, the dyad **D**, and only one Petersen proper $(2,3)$-pole T_P, the triad **T**. However, when performing a junction of two connectors containing more than one semiedge, the result may depend on the particular order of semiedges in the connectors. Typically, the order does not affect our uncolourability arguments because they involve the connectors in their entirety, the only exception being the Isaacs snarks. Nevertheless, choosing different order of semiedges in connectors may lead to several non-isomorphic variations of the multipoles P_{NN}, P_{NT}, P_{TT}, P_{TTT}, and P_{3NT}. However, for simplicity, Table 2 displays only one variation from each of them. It would be possible to take this into account in our classification, but such a level of detail would only obscure the analysis without tangible benefits. Perhaps the only case where one might be interested in distinguishing non-isomorphic variants occurs when these multipoles are used as construction blocks for larger snarks with specific properties.

By applying the approach mentioned above, we have analysed all cyclically 5-connected bicritical snarks with at most 36 vertices. The list of such snarks was obtained from [5]. The results are summarised in Table 3. The uncolourability of certain snarks can be explained in several different ways. Consequently, they are included in more than one of our classes, which explains why the numbers in Table 3 do not add up.

Order	10	20	22	24	26	28	30	32	34	36	38
NN substitution	0	0	2	0	0	0	10	11	26	10	$\geqslant 39$
TT substitution	0	0	0	0	8	0	0	0	84	69	$\geqslant 3$
NT substitution	0	0	0	0	8	0	0	0	1084	396	$\geqslant 17$
TTT substitution	0	0	0	0	0	0	0	0	22	0	$\geqslant 0$
Superpentagon subst.	0	0	0	0	0	0	0	0	72	0	$\geqslant 0$
Other	1	1	0	0	0	1	1	2	215	9	$\geqslant 4$
TOTAL	1	1	2	0	8	1	11	13	1503	484	$\geqslant 56$

Table 3: Classification of all cyclically 5-connected bicritical snarks of order up to 36

Isaacs flower snarks

We have defined the Isaacs flower snarks J_n, where $n \geqslant 3$ is odd, in Section 5.5. An alternative approach to describing them uses substitutions starting from the Petersen graph. The flower snark J_3 arises from the Petersen graph by substituting a triangle for a vertex. For each $k \geqslant 2$, the snark J_{2k+1} can be constructed from J_{2k-1} by substituting the $(3,3)$-pole Y_4 for a copy of the $(3,3)$-pole Y_2 contained in it, both $(3,3)$-poles having the same colouring set (for the definition of Y_i see Section 5.5). It is well known that the flower snarks J_n are cyclically 6-edge-connected for all $n \geqslant 7$ and bicritical for $n \geqslant 5$ (see [38, Proposition 4.7]). The snark J_3 is the only trivial Isaacs snark.

There are three nontrivial Isaacs flower snarks up to order 36, namely J_5, J_7, and J_9. The snark J_5 on 20 vertices contains a single cycle-separating 5-edge-cut; the next one, J_7 on 28 vertices, is the smallest cyclically 6-connected snark, and J_9 is also cyclically 6-connected. In the rest of this classification, the Isaacs snarks will not be mentioned anymore.

Order 22

The only cyclically 5-connected bicritical snarks of order 22 are two Loupekine snarks. They are schematically represented in Figure 9. In full, they can be found, for example, in [38, Figure 18]. Both of them contain the 5-pole P_{NN}, so they can be constructed from the Petersen graph by a substitution of P_{NN} for P_2.

Order 26

There are eight cyclically 5-connected bicritical snarks of order 26. All of them contain a proper $(2, 2, 1)$-pole P_{NT} and also a proper $(2, 2, 1)$-pole P_{TT}. Hence they can be obtained by a substitution from the Petersen graph. Also, each of them contains the uncolourable 7-pole of order 25 from Figure 17 consisting of one dyad N connected to two triads T_1 and T_2. Steffen and others [32, 43] used this multipole to construct cyclically 5-connected snarks with small order and large resistance.

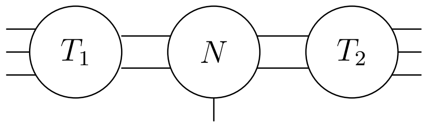


Figure 17: An uncolourable 7-pole

Order 30

On 30 vertices, there is one cyclically 5-connected snark of girth 6, the double-star snark discovered by Isaacs [23]. It can be described as a 5-product of J_5 with itself, or as a junction of two superpentagons of order 15 mentioned in Section 5.11. All the remaining snarks of order 30 arise from the Blanuša snarks by substituting P_{NN} for P_2: six snarks from the Type 1 Blanuša snark and four snarks from the Type 2. By the Type 1 Blanuša snark, we mean a nontrivial snark of order 18 discovered in 1946 by Blanuša [3], and by the Type 2 Blanuša snark we mean the other nontrivial snark on 18 vertices, the Blanuša double as it is called in [40], where the history and properties of these two snarks are discussed in detail. Note that this substitution increases cyclic connectivity.

Order 32

There are 13 cyclically 5-connected bicritical snarks of order 32. From among them, 11 contain the 5-pole P_{NN}. All of them can be constructed from the flower snark J_5 by a substitution. The remaining two constitute Class 32-A which is described in detail below.

Class 32-A

The two snarks of this class consist of three Petersen negators $N_i(I_i, O_i; r_i)$ for $i \in \{1, 2, 3\}$ and one 7-pole M_{11}, combined as shown in Figure 18. In general, the negators N_1, N_2, and N_3 can be taken from an arbitrary snark. The 7-pole M_{11} can be derived from the flower snark J_3 by removing one vertex and severing two edges, as indicated in Figure 19.

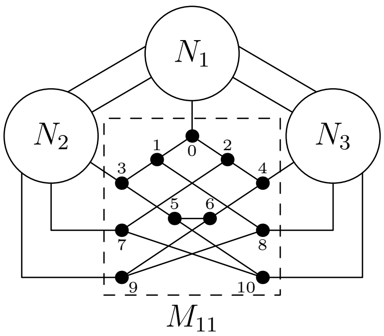


Figure 18: The structure of Class 32-A

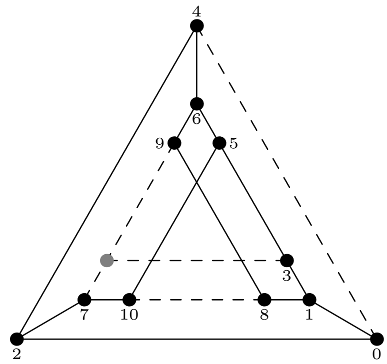


Figure 19: The 7-pole M_{11} constructed from J_3

We now explain why every graph G represented by Figure 18 is a snark. By contradiction, suppose that G has a 3-edge-colouring φ. To avoid ambiguity assume that, in G, the connector O_2 is joined to I_1 and the connector O_1 is joined to I_3. Let $a = \varphi(r_1)$. One of the connectors of N_1, say, I_1, must have zero total flow. Then $\varphi_*(O_1) = a = \varphi_*(I_3)$. For the negator N_3 we get that $\varphi_*(O_3) = 0$ and $\varphi(r_3) = a$. Now, consider the 7-pole M_{11}. The semiedges e_1 and e_2 connected to the connector O_3 have the same colour and so have the semiedges e_3 and e_4 connected to r_3 and r_1, respectively. The parity lemma implies that the sum of the flows through the remaining three semiedges e_5, e_6, and e_7 is zero. Therefore, we can perform junctions of e_1 with e_2, e_3 with e_4 and add one new vertex incident with e_5, e_6 and e_7, giving rise to a graph H with edges coloured by φ. However, the graph H is isomorphic to the flower snark J_3, which is a contradiction.

Note that we require that M_{11} can be extended to a snark in two symmetric ways, so we cannot replace it with an arbitrary 7-pole constructed from a snark by removing a vertex and cutting two edges.

Order 34

Among the 1503 bicritical cyclically 5-connected snarks of order 34, we have found 26 that contain the 5-pole P_{NN}. They can be constructed from the Loupekine snarks of order 22 (both types) by a substitution for P_2. The 5-pole P_{NT} is contained in 1084 snarks arising from the Blanuša snarks (both types) by substituting P_{NT} for the $(2,3)$-pole M_{ev} (see Table 2). The next 84 snarks contain the 5-pole P_{TT}. They can be constructed from the Blanuša snarks (both types) by substituting P_{TT} for P_2. Further 72 snarks arise from the Petersen graph by substituting a superpentagon described in Section 5.11 for a pentagon.

After analysing the structure of the remaining snarks we have decided to categorise them into six classes. The classification for order 34 is summarised in Table 4.

Type of a snark	Number of snarks
Containing P_{NN}	26
Containing P_{NT}	1084
Containing P_{TT}	84
Containing P_{TTT}	22
Containing superpentagon	72
Class 34-A	21
Class 34-B	$18 + 18\ (P_{TT}) + 54\ (P_{NT}) = 90$
Class 34-C	$162 + 18\ (P_{NT}) = 180$
Class 34-D	5
Class 34-E	7
Class 34-F	2
TOTAL	1503

Table 4: Structure of cyclically 5-connected bicritical snarks of order 34.

Class 34-A

Take two negators N_1 and N_2 and two proper $(2,3)$-poles T_1 and T_2, and construct a 9-pole M_1 as shown in Figure 20. We prove that M_1 is uncolourable. Suppose to the contrary that M_1 admits a colouring φ. Let e denote the dangling edge whose endvertex does not belong to any of the multipoles N_1, N_2, T_1, and T_2. If a connector of a negator is joined to a proper connector, then its other connector must have zero flow through it. In our case, this is true for both N_1 and N_2. After applying the parity lemma to the $(2,2,1)$-pole connecting N_1 to N_2, which contains one vertex with three dangling edges and one isolated edge, we obtain $\varphi(e) = 0$. This is a contradiction.

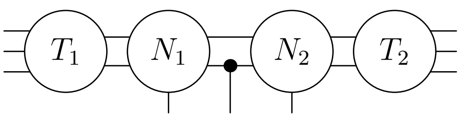


Figure 20: Uncolourable 9-pole M_1 (Class 34-A)

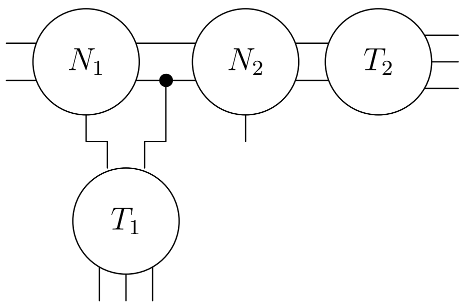


Figure 21: Uncolourable 9-pole M_2 (Class 34-B)

Among the studied snarks of order 34, there are 21 snarks containing M_1. In all of them, M_1 is built from dyads and triads and therefore has 33 vertices.

Class 34-B

Assume that we have two negators N_1 and N_2 and two proper $(2,3)$-poles T_1 and T_2 which are assembled to a 9-pole M_2 as depicted in Figure 21. We prove that M_2 is uncolourable. Let v denote the vertex of M_2 not belonging to any of N_1, N_2, T_1, and T_2, and let e be the edge between v and T_1. Suppose to the contrary that there is a colouring φ of M_2. Since a connector of N_2 is joined with a proper connector, the flow through its other connector is zero. If we apply the parity lemma first to the $(2,2,1)$-pole containing v (which is joined to N_1, N_2, and T_1) and then to N_1, we can conclude that the flow through e is the same as the flow through the residual semiedge of N_1. These values force a zero flow through a connector of T_1, which is impossible. Hence, M_2 is uncolourable.

The multipole M_2 contained in the studied snarks of order 34 consists of dyads and triads and has 33 vertices. We have identified 90 snarks containing this 9-pole. Of them, 54 snarks contain also the 5-pole P_{NT}, and 18 snarks contain the 5-pole P_{TT}.

Class 34-C

Let G be a snark. Delete a path uv from G, sever an edge $e \neq uv$ of G, and denote the resulting $(2,2,2)$-pole by $R(A,B,C)$, where the connectors A and B contain the two semiedges formerly incident with u and v, and C contains two semiedges that arise from severing e. Should R be contained in a cyclically 5-connected graph, e cannot be incident with any of u and v. The crucial property of R is that it admits no colouring φ such that $\varphi_*(A) \neq 0$ and $\varphi_*(C) = 0$. Clearly, any such colouring could be extended to the entire snark G. Note that if we choose the Petersen graph for G, we obtain the double pentagon as the $(2,2,2)$-pole R.

Take the $(2,2,2)$-pole $R(A,B,C)$, a negator $N(I,O)$, and two proper $(2,3)$-poles $T_1(B_1,C_1)$ and $T_2(B_2,C_2)$. Join B_1 to A, C to I, O to B_2, and denote the resulting 9-pole M_3 (see Figure 22). Assume that M_3 has a colouring φ. The negator N is connected to the proper $(2,3)$-pole T_2, hence $\varphi_*(I) = \varphi_*(C) = 0$. Since T_1 is also proper, we have $\varphi_*(B_1) = \varphi_*(A) \neq 0$. This colouring is impossible for R—a contradiction.

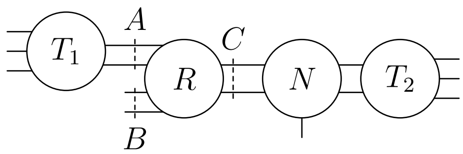


Figure 22: Uncolourable 9-pole M_3 (Class 34-C)

There are 72 bicritical 5-connected snarks of order 34 belonging to this class (18 of them also contain P_{NT}). In all of them, the negators are dyads, proper $(2,3)$-poles are triads, and $R(A,B,C)$ is the double pentagon.

Class 34-D

Take four negators $N_i(I_i,O_i,r_i)$ for $i \in \{1,2,3,4\}$, connect them as shown in Figure 23, and denote the resulting graph by G_4. We prove that G_4 is a snark. If it is not, then it has a colouring φ. Without loss of generality, we may assume that $\varphi_*(O_1) = 0$. Let $\varphi(uw_1) = a \in \mathbb{K}$; then $\varphi_*(I_2) = \varphi_*(O_1) + \varphi(uw_1) = a \neq 0$, so $\varphi_*(O_2) = 0$. Analogously, we get $\varphi_*(O_3) = \varphi_*(O_4) = 0$. By applying the parity lemma repeatedly, we get $\varphi(w_3u) = \varphi_*(I_4) = \varphi(r_4) = \varphi(r_2) = \varphi_*(I_2) = a$, which means that there is a colour conflict at u.

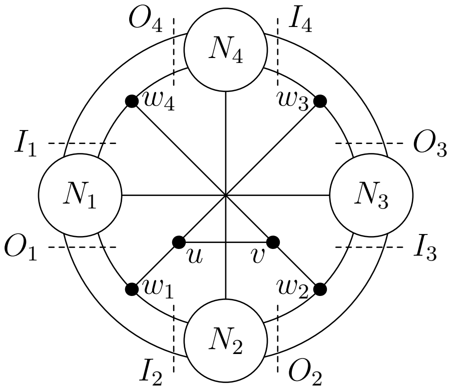


Figure 23: The structure of Class 34-D snarks

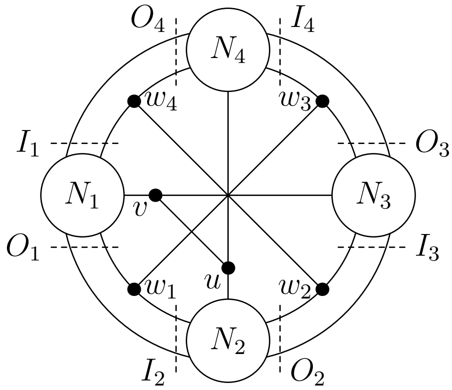


Figure 24: The structure of Class 34-E snarks

We have identified six snarks having this structure, with all four negators taken from the Petersen graph. They all are permutation snarks, which means that they admit a 2-factor consisting of two induced cycles. For more information about permutation snarks, one can consult [6, 35, 36].

Class 34-E

The four negators N_1, N_2, N_3 and N_4 can also be arranged in a different way shown in Figure 24. The proof of uncolourability is similar to the one for the previous class. There are six snarks of order 34 with this structure. The 12 snarks constituting the classes 34-D and 34-E form a complete set of all cyclically 5-connected permutation snarks of order 34, see Brinkmann et al. [6].

Class 34-F

To construct snarks in this class, we take five even $(2,2,2)$-poles H_1, H_2, H_3, H_4, and H_5, and assemble a larger $(2,2,2)$-pole H as explained in Section 5.4 and indicated in Figure 6. Since H is an even $(2,2,2)$-pole, $H * V_4$ is a snark. Class 34-F consists of snarks of the form $H * V_4$ whose scheme can be seen in Figure 25. Each even $(2,2,2)$-pole H_i with $i \in \{2,3,4\}$ has one connector whose semiedges are joined to a vertex. Since adding such a vertex and a dangling edge to H_i produces a negator N_i, the structure of Class 34-F snarks can be alternatively described by using two even $(2,2,2)$-poles H_1 and H_5 and three negators N_2, N_3 and N_4 as depicted in Figure 26.

Among the studied snarks of order 34, there are two snarks of this class. Both use components derived from the Petersen graph: hexagons as even $(2,2,2)$-poles and dyads as negators.

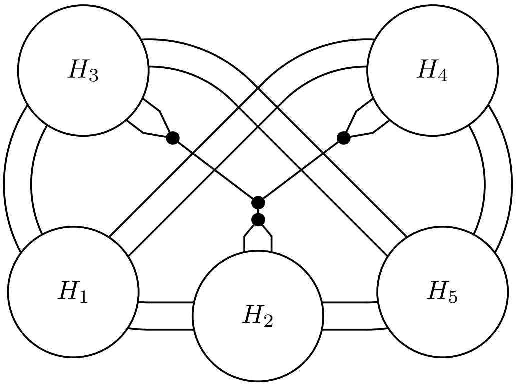


Figure 25: The structure of Class 34-F snarks

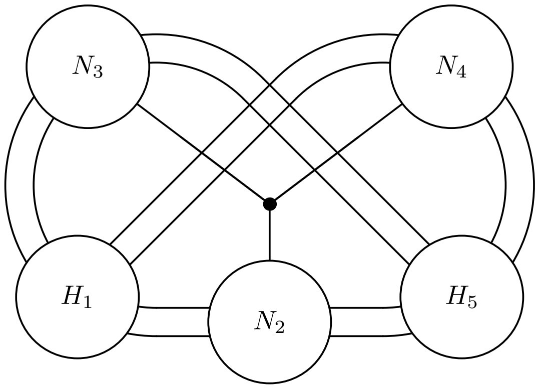


Figure 26: Negators in Class 34-F snarks

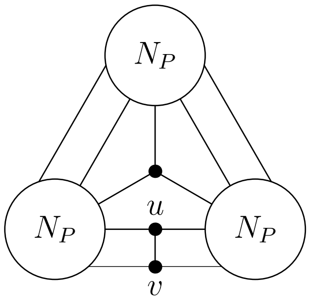


Figure 27: A scheme for two cyclically 5-connected reducible snarks of order 24.

Class 34-F is contained in a larger class of snarks of the form $H * V_4$ where H is an arbitrary even $(2, 2, 2)$-pole. The $(2, 2, 2)$-poles H used in the construction can be composed from any odd number $k \geqslant 5$ of smaller even $(2, 2, 2)$-poles (for example, hexagrams) by using the method described in Section 5.4.

Order 36

Out of the 484 bicritical cyclically 5-connected snarks of this order, 396 snarks contain the 5-pole P_{NT} and all of them arise from the flower snark J_5 by a substitution for M_{ev}. In 69 snarks we have identified the 5-pole P_{TT} and all of them arise from J_5 by a substitution for P_2.

The 5-pole P_{NN} is contained in ten snarks. They are constructed by substitution from two smaller snarks of order 24 whose structure can be seen in Figure 27. Their structure is similar to Loupekine snarks; they only contain two additional vertices, u and v, joined by an edge. The pair of $\{u, v\}$ is removable, so these snarks are reducible, although after a substitution of P_{NN} for a path P_2, the resulting snarks become irreducible. In all ten cases, the $(2, 2)$-pole P_2 used for the substitution contains one of the vertices u or v.

Excluding the flower snark J_9, there remain eight snarks of order 36, which fall into two classes.

Class 36-A

Take three negators $N_i(I_i, O_i, r_i)$, where $i \in \{1,2,3\}$, with unordered connectors (as usual), arrange them cyclically and join each O_i to I_{i+1}, where the indices are reduced modulo 3. Subdivide one of the two edges connecting N_i to N_{i+1} with a new vertex v_i and attach a dangling edge e_i to v_i, thereby producing a cubic 6-pole. Turn it into a $(3,3)$-pole $M_{24}(I,O)$ by distributing the semiedges into two ordered connectors $I = (r_1, r_2, r_3)$ and $O = (e_3, e_1, e_2)$; see Figure 28.

Consider the $(3,3)$-pole $Y_3(I_Y, O_Y)$ consisting of three copies of the Isaacs $(3,3)$-pole Y, defined in Section 5.5. We prove that M_{24} and Y_3 are colour-disjoint, implying that $M_{24} * Y_3$ is a snark.

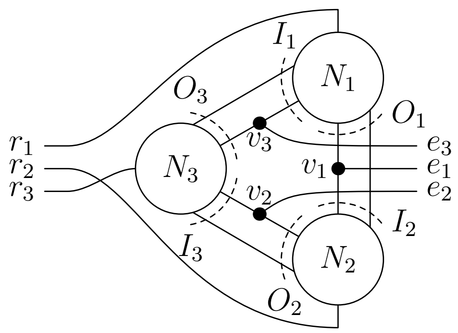


Figure 28: A $(3,3)$-pole M_{24} contained in class 36-A snarks

Let φ be a colouring of M_{24} and assume that $\varphi(I) = (a,b,c) \in \mathbb{K}^3$. It is not difficult to see that for the connectors of the three negators, we have either $\varphi_*(I_1) = \varphi_*(I_2) = \varphi_*(I_3) = 0$ or $\varphi_*(O_1) = \varphi_*(O_2) = \varphi_*(O_3) = 0$; the proof is similar to that for Class 34-D. If the former case occurs, then $\varphi_*(O_1) = \varphi(r_1) = a$ and the parity lemma implies that $\varphi(e_1) = \varphi_*(O_1) + \varphi_*(I_2) = a$. In a similar manner we get $\varphi(e_2) = b$ and $\varphi(e_3) = c$, whence $\varphi(O) = (c,a,b)$. If the latter case occurs, we similarly get that $\varphi(O) = (a,b,c)$. If follows that each element of $\mathrm{Col}(M_{24})$ is a 6-tuple (a,b,c,a',b',c') of elements of $\mathbb{K}$ where (a',b',c') arises from (a,b,c) by cyclically permuting its entries. (Obviously, certain 6-tuples of this form may be absent in $\mathrm{Col}(M_{24})$ when some of the negators are imperfect.) All of such 6-tuples are also contained in $\mathrm{Col}(Y_2)$, except (a,a,a,a,a,a). However, $(a,a,a,a,a,a) \notin \mathrm{Col}(Y_3)$. Since Y_2 and Y_3 are colour-disjoint, so are M_{24} and Y_3. Therefore $M_{24} * Y_3$ is a snark. Except the Isaacs flower snarks, this is the first family of snarks known to us where ordered connectors emerge in explaining uncolourability.

There are six snarks of order 36 belonging to Class 36-A.

Class 36-B

Take five negators $N_i(I_i, O_i, r_i)$, where $i \in \{1,2,3,4,5\}$, arrange them in a cyclic manner, and for $1 \leqslant i \leqslant 4$ join O_i to I_{i+1}. Let $I_1 = \{i_1, i_2\}$ and $O_5 = \{o_1, o_2\}$. Join i_1 to o_1, subdivide the resulting edge with a new vertex v, and attach to v a dangling edge with semiedge r_0. Finally, join r_0 to r_3, r_1 to r_5, i_2 to r_4, and o_2 to r_2 to obtain a cubic graph denoted by G, see Figure 29.

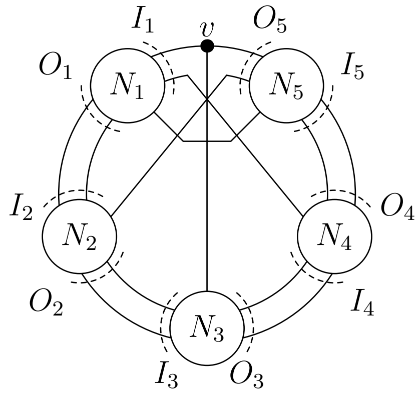


Figure 29: The structure of Class 36-B snarks

We show that G is a snark. Suppose to the contrary that G has a 3-edge-colouring φ. Since N_3 is a negator, the flow through one its connectors is nonzero. If $\varphi_*(O_3) \neq 0$, then there exists an element $a \in \mathbb{K}$ such that $\varphi_*(O_3) = a$. It follows that $\varphi_*(O_4) = 0$ and $\varphi(r_4) = a$. Moreover, $\varphi_*(I_2) \neq 0$, so $\varphi_*(I_1) = 0$. Now, the edge leading from N_4 to N_1 through I_1 has colour $\varphi(r_4) = a$, which implies that the edge connecting N_1 to v through I_1 has the same colour. However, the edge leading from N_3 to v has colour $\varphi(r_3) = a$, too, and this is a contradiction. If $\varphi_*(I_3) \neq 0$, a contradiction is derived similarly.

This family of snarks, built from five negators, can easily be generalised to a similar family built from a larger number of negators, namely $4k+1$, where $k \geqslant 1$. In this case we connect the vertex v to N_1 through I_1, to N_{4k+1} through O_{4k+1}, and to N_{2k+1} by using r_{2k+1}. We further join $i_2 \in I_1$ with r_{2k+2} and $o_2 \in O_{2k+1}$ with r_{2k}. The proof that the resulting graph G is a snark is similar to the one above. Clearly, G can be made cyclically 5-edge-connected by appropriately identifying pairs of residual semiedges.

Among the snarks of order 36, we have identified two snarks of this type.

7 Strictly critical snarks

We conclude our investigation of small cyclically 5-connected critical snarks by turning our attention to those that are strictly critical. In Section 3 we have explained that strictly critical snarks are of special interest, partially because some of them can be derived from non-critical snarks.

By Theorem 6.1 in [12], a strictly critical snark of order n exists if and only if $n \geqslant 32$. Among the snarks of order at most 36, there are only 84 cyclically 5-connected strictly critical snarks, all having 36 vertices. Of those, 77 arise from a non-critical snark of order 20 by a substitution of P_{NT} for M_{ev}. The structure of the remaining seven snarks is very similar: they all arise from the Petersen graph by a substitution of a suitable proper $(2,2,2)$-pole M for the $(2,2,2)$-pole V_4 consisting of one vertex and its three neighbours. The multipole M is constructed as follows. Let T_1, T_2, and T_3 be three perfect proper $(2,3)$-poles. Add three new vertices and produce M by connecting them to T_1, T_2, and T_3 in the manner indicated in Figure 30. For the reasons similar to those valid for the

TTT $(2,2,2)$-pole (Section 5.9), the multipole M is colour-equivalent to the $(2,2,2)$-pole V_4. However, if we remove from M any two of the three new vertices, the colouring set will not change. Consequently, no snark containing M is bicritical. The $(2,2,2)$-pole M contained in all seven remaining snarks consist of three triads.

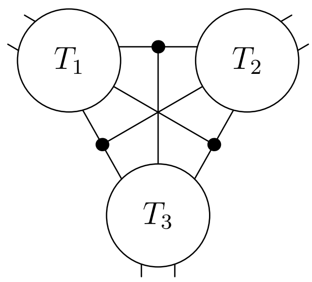


Figure 30: A $(2,2,2)$-pole contained in seven cyclically 5-connected strictly critical snarks of order 36

8 An infinite family of bicritical snarks

In the previous sections, we have presented a number of new constructions of snarks which mimic the structure of small snarks. We now deal with the problem of their bicriticality.

As mentioned in the discussion of the results for order 36, the base components used in constructions of bicritical snarks need not come from bicritical snarks. For example, ten bicritical snarks of order 36 arise from non-critical snarks of order 24 by a substitution of P_{NN} for P_2. Obviously, it is necessary to eliminate all pairs of removable vertices from the used construction blocks, but in general, we do not understand under what conditions it would be sufficient.

On the other hand, taking all construction blocks from bicritical snarks — a slightly stronger requirement than just the absence of removable pairs of vertices — does not ensure the resulting snark to be bicritical. For instance, a proper $(2,3)$-pole constructed even from a bicritical snark can be uncolourable (see results for order 26). Such a proper $(2,3)$-pole cannot be used in any substitution yielding a bicritical snark. In general, we do not know much about the circumstances under which proper $(2,3)$-poles are colourable or perfect. This is a significant difference from negators whose colouring properties are characterised by Theorem 10.

The purpose of this section is to illustrate that imposing certain additional requirements on the construction blocks can assure bicriticality of the resulting snark in a fairly general setting. The described requirements are not overly restrictive and it is possible that most construction blocks taken from bicritical snarks (of any given order) satisfy them.

For our demonstration, we have chosen snarks constructed by an NN-substitution (see Section 5.6). This is perhaps the simplest of the infinite families that we have described; nevertheless, a similar approach is likely to work for other families as well. We view

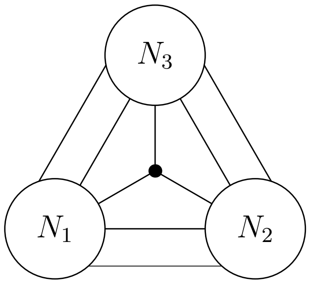


Figure 31: A schematic drawing of a snark $\mathrm{NNN}(N_1, N_2, N_3)$.

the snarks arising by an NN-substitution as consisting of three negators $N_i(I_i, O_i, r_i) = \mathrm{Neg}(G_i; u_i, v_i)$ for $i \in \{1, 2, 3\}$ arranged along a circle with an additional vertex attached to the residual semiedges (see Figure 31). We denote the resulting graph by $\mathrm{NNN}(N_1, N_2, N_3)$.

As discussed in the beginning of this section, our restrictions must be imposed on the construction blocks, not on the snarks they originate from. We call a negator $N = \mathrm{Neg}(G; u, v)$ *bicritical* if the multipole $G - \{x, y\}$ is colourable for any two distinct vertices x, $y \in V(N)$. The following proposition shows that this property is necessary.

Proposition 11. *If N_1, N_2, and N_3 are negators such that $G = \mathrm{NNN}(N_1, N_2, N_3)$ is a bicritical snark, then all of them are bicritical.*

Proof. Choose two arbitrary vertices x and y of the negator $N_1 = N(G_1; u_1, v_1)$. Since G is a bicritical snark, the multipole $G - \{x, y\}$ is colourable. The $(2, 2, 1)$-pole $\mathrm{NN}(N_2, N_3)$ is colour-contained in P_2, so replacing $\mathrm{NN}(N_2, N_3)$ with P_2 gives rise to a multipole $G_1 - \{x, y\}$ which is colourable as well. It follows that N_1 is bicritical, as claimed. A similar argument holds for N_2 and N_3 as well. □

We do not know whether the property stated in Proposition 11 — or a stronger property that all three negators are taken from bicritical snarks — is sufficient. We have constructed all negators from bicritical cyclically 5-connected snarks on up to 30 vertices. Using them, we have created all possible snarks of class NNN that contain at most two different negators; there are approximately 600,000 such snarks. With the help of a computer, we have verified that all of them are bicritical.

In order to specify a sufficient condition, we introduce the following rather technical property of negators. We will henceforth assume that the negators in question are constructed from snarks of girth at least 5, in order to avoid ambiguity of notation and certain marginal cases.

Definition 12. Let G be a snark of girth at least 5. A negator $N(\{i_1, i_2\}, \{o_1, o_2\}, r) = \mathrm{Neg}(G; u, v)$ is called *feasible* if it is bicritical and possesses the following properties:

(i) For every pair of vertices $\{x, y\}$ where $x \in \{u, v\}$ and $y \in V(N)$ and for any two dangling edges e and f of the 6-pole $G - \{x, y\}$ formerly incident with x there exists a colouring φ of the 6-pole $G - \{x, y\}$ such that $\varphi(e) = \varphi(f)$.

(ii) For any vertex $y \in V(N)$ there exist colourings φ and ψ of the 8-pole $N - y$ such that $\varphi(i_1, i_2, o_1, o_2, r) = (a, a, b, b, a)$ and $\psi(i_1, i_2, o_1, o_2, r) = (a, a, b, b, b)$ for any two distinct colours $a, b \in \mathbb{K}$.

Here we regard each multipole $M = G - \{x, y\}$ as a 6-pole — if the vertices x and y are adjacent, we keep the edge xy as an isolated edge in M.

If G is bicritical and $xy \in E(G)$, Property (i) is easily satisfied. Indeed, the 6-pole M is colourable because G is bicritical; by the parity lemma, the two dangling edges formerly incident with x have the same colour, which can be assigned to the isolated edge in M so that all three semiedges formerly incident with x have the same colour for a suitable colouring of M.

If G is bicritical but $xy \notin E(G)$, from the parity lemma we still deduce that, in every 3-edge-colouring of M, two of the semiedges formerly incident with x must have the same colour. However, Property (i) requires more: the two semiedges having the same colour can be arbitrarily prescribed. We have tested all cyclically 5-connected bicritical snarks of order up to 36; only six of them support a negator violating Property (i). We describe them in Section 9.

From the definition, it is unclear whether a feasible negator has to be perfect. Amongst the negators constructed from nontrivial snarks up to order 28, there is no example of an imperfect feasible negator.

Property (i) of feasible negators is related to a similar technical property introduced by Chladný and Škoviera [12] in their study of criticality and bicriticality of dot products of snarks.

Definition 13. [12] A pair $\{e, f\}$ of edges of a snark G is called *essential* if it is non-removable and, moreover, if for every 2-valent vertex v of $G - \{e, f\}$, the graph obtained from $G - \{e, f\}$ by suppressing v is colourable.

The following lemma relates essentiality to feasibility.

Lemma 14. *Let $N = \mathrm{Neg}(G; u, v)$ be a negator, $x \in \{u, v\}$, and $y \in V(N)$. Assume that for every edge e incident with x in G there is an edge f incident with y in G such that $\{e, f\}$ is an essential pair of edges in G. Then the negator N satisfies Property* (i) *of Definition 12.*

Proof. Let e_1, e_2, and e_3 denote the edges incident with x listed in an arbitrary order. From our assumption, it follows that there is an edge f incident with y such that $\{e_1, f\}$ is an essential pair of edges in G. Therefore, the multipole $G - \{e_1, f\}$ with x suppressed has a colouring φ. If we cut the edge resulting from the suppression of x into two dangling edges corresponding to e_2 and e_3 and remove y, we get the $(3, 3)$-pole $G - \{x, y\}$ with a colouring in which the dangling edges corresponding to e_2 and e_3 have the same colour. As e_2 and e_3 can be chosen arbitrarily, we see that the negator N satisfies Property (i) of Definition 12. □

We have tested negators for Property (ii). Although there are many negators violating (ii), more than 90% of all negators created from bicritical cyclically 5-connected snarks with at most 34 vertices are feasible. For instance, for every such snark G of order 34 there exist $34 \times 3 = 102$ possible negators which can be constructed from G (ignoring isomorphisms that might arise); the number of the feasible ones among them ranges from 74 to 102.

Theorem 15. *If N_1, N_2, and N_3 are any three feasible perfect negators, then $G = \mathrm{NNN}(N_1, N_2, N_3)$ is a bicritical snark.*

Proof. For $j \in \{1, 2, 3\}$ let $N_j = \mathrm{Neg}(G_j; u_j, v_j)$. Let $I_j = \{i_j, i'_j\}$ and $O_i = \{o_j, o'_j\}$ be the connectors of N_j, let r_j be its residual edge, and let w_j be the the common neighbour of u_j and v_j in G_j. Choose any two distinct vertices x and y of G. We show that the multipole $G - \{x, y\}$ is colourable, implying that G is bicritical.

Case 1. The vertices x and y belong to the same negator, say N_1. Set $M = \mathrm{NN}(N_2, N_3)$, so that $G = N_1 * M$. Since M is colour-equivalent to P_2, we can substitute P_2 for M, thereby producing the snark G_1 from which the negator N_1 has been constructed. Since N_1 is bicritical, $G_1 - \{x, y\}$ is colourable. Now, we reconstruct $G - \{x, y\}$ from $G_1 - \{x, y\}$ by substituting M for P_2. Applying colour-equivalence of M and P_2 again, we can conclude that $G - \{x, y\}$ is colourable, as required.

Case 2. Assume that x and y belong to different negators, say, $x \in V(N_1)$ and $y \in V(N_2)$. Remove the vertices v_1 and x from the snark G_1 and denote the semiedges formerly incident with v_1 by e_1, e_2, and e_3 in such a way that e_3 is incident with w_1. According to Property (i) of the feasible negator N_1, there exists a colouring φ_1 of $G_1 - \{v_1, x\}$ such that $\varphi_1(e_1) = \varphi_1(e_2)$. Let $a = \varphi_1(e_3)$ and $b = \varphi_1(u_1w_1)$; obviously $a \neq b$ (see Figure 32). We can simply restrict the colouring φ_1 to a colouring of the multipole $N_1 - x$ for which $\varphi_{1*}(I_1) = \varphi_1(u_1w_1) = b$, $\varphi_1(r_1) = a + b$ and $\varphi_{1*}(O_1) = 0$.

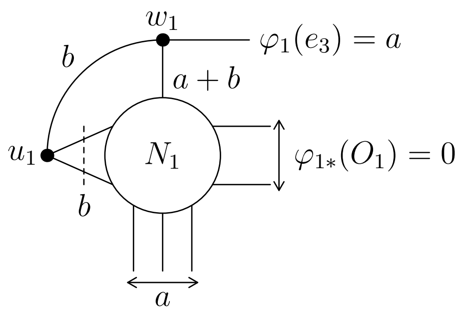


Figure 32: The colouring φ_1 of N_1.

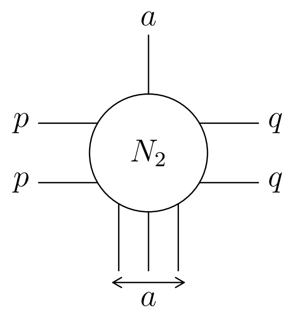


Figure 33: The colouring φ_2 of N_2.

Let $p = \varphi_1(o_1)$. The negator N_2 is feasible, so according to Property (ii), there exists a colouring φ_2 of $N_2 - y$ such that $\varphi_2(r_2) = a$, $\varphi_2(i_2) = \varphi_2(i'_2) = p$, and $\varphi(o_2) = \varphi(o'_2) = q$. If $p = a$, then $q = b$; if $p \neq a$ then $q = a$. The colourings φ_1 and φ_2 are compatible and can be combined to form a colouring φ of the multipole $M = \mathrm{NN}(N_1, N_2) - \{x, y\}$ depicted in Figure 34. Since N_3 is perfect, the colouring φ of M can be extended to the entire 6-pole $G - \{x, y\}$.

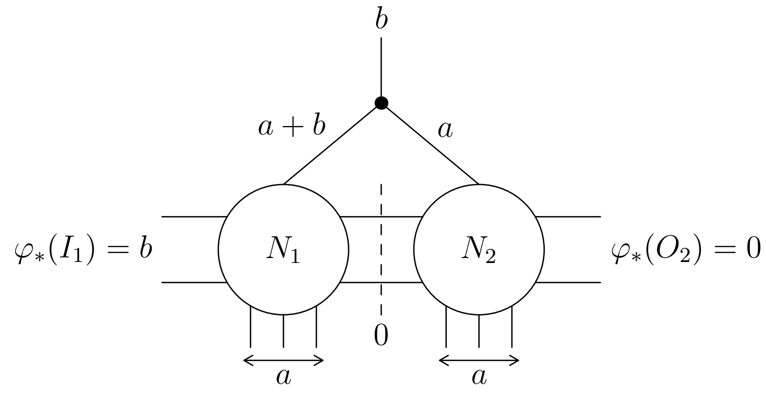


Figure 34: Colouring of the 11-pole $\mathrm{NN}(N_1, N_2) - \{x, y\}$

Case 3. If one of the vertices x and y, say y, does not belong to any of the negators, it must be the vertex attached to the residual semiedges of N_1, N_2, and N_3. Let $x \in V(N_1)$. Property (ii) applied to the feasible negator N_1 guarantees that there exists a colouring φ_1 of $N_1 - x$ such that $\varphi_1(i_1) = \varphi_1(i_1') = a$ and $\varphi_1(r_1) = \varphi(o_1) = \varphi_1(o_1') = b \neq a$. Since N_2 is perfect, it admits a colouring φ_2 such that $\varphi_{2*}(I_2) = 0$ and $\varphi_{2*}(O_2) \neq 0$. Finally, N_3 is also perfect, and hence it admits a colouring φ_3 that is compatible with both φ_1 and φ_2. It follows that the partial colourings φ_1, φ_2, and φ_3 can be combined to colour the entire 6-pole $G - \{x, y\}$. This completes the proof of the theorem. □

In order to create an infinite class of bicritical snarks, we need an infinite family of feasible negators. As one can expect, such negators can be obtained from the family of Isaacs flower snarks (see Section 5.5 for the definition).

Proposition 16. *Every negator constructed from the Isaacs flower snark J_k, where $k \geqslant 5$ is odd, is feasible.*

Proof. According to [12, Example 5.5], every pair of non-adjacent edges of J_k, with $k \geqslant 5$, is essential. Therefore, by Lemma 14, each negator created from such a snark possesses Property (i) of a feasible negator. We establish Property (ii) by induction on k.

For the induction basis, we have checked that all negators derived from J_5, J_7, and J_9 satisfy Property (ii); we did it with the help of a computer.

Consider the Isaacs snark J_k for an odd $k \geqslant 11$. Remove an arbitrary path uwv from J_k to produce a negator $N = \mathrm{Neg}(J_k; u, v)$; its dangling edges will be denoted by i_1, i_2, o_1, o_2, and r in the usual way. Next, remove from N an arbitrary vertex x and denote the resulting 8-pole by M. The path uwv intersects at most three consecutive copies of the Isaacs $(3, 3)$-pole Y, and the removal of the vertex x corrupts at most one other copy of Y. Consequently, there are at least four consecutive copies of Y in M that remain intact; they form a $(3, 3)$-pole isomorphic to Y_4. Replace this $(3, 3)$-pole with a $(3, 3)$-pole Y_2 consisting of two copies of Y, and denote the resulting 8-pole by M'. Clearly, M' is isomorphic to the multipole obtained from J_{k-2} by the removal of a certain path of length 2 and an additional vertex. By the induction hypothesis, there exists a colouring of M'

in which the dangling edges corresponding to i_1, i_2, o_1, o_2, and r have colours exactly as desired for either φ or ψ from Property (ii). Since the multipoles Y_4 and Y_2 are colour-equivalent (see Section 5.5), the desired colours can also be assigned to the semiedges i_1, i_2, o_1, o_2, r of M. Hence, every negator constructed from the Isaacs snark J_k, except J_3, satisfies Property (ii). Consequently, it is feasible, as claimed. □

Using feasible negators from the Isaacs snarks J_k, with $k \geqslant 5$, we can construct an infinite class of bicritical snarks. All such snarks are cyclically 5-connected. Since 5-cycles can only occur in negators constructed from J_5, avoiding such negators will lead to bicritical snarks of girth 6.

9 Non-removable edges that are not essential

As anticipated in the previous section, we wish to investigate snarks containing negators that violate Property (i) of Definition 12. Property (i) of a feasible negator is related to the concept of an essential pair of edges, which in turn plays a crucial role in factorisation of a critical snark into a dot product of two smaller snarks (see Theorem 7).

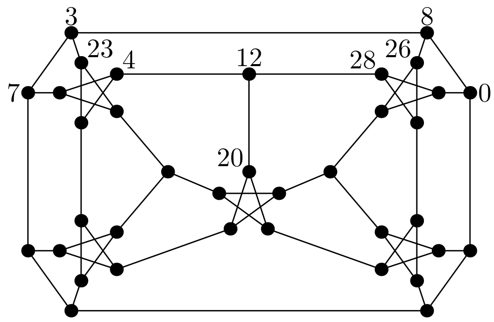


Figure 35: The snark G_{36}.

Amongst the 5-simple snarks of order at most 36, exactly six snarks support a negator violating Property (i); all of them belong to the class NNN, and all of them are bicritical. They consist of two dyads and one negator constructed from a reducible snark of order 24 (see Figure 27). One of these snarks, denoted by G_{36}, is illustrated in Figure 35. If we remove the pair of vertices 12 and 3 (or 12 and 8), we get a 6-pole M such that for each colouring of M, the dangling edges incident with the vertices 20 and 4 (or 20 and 28) have different colours; this property was verified by exhaustive computer search. If we construct a negator from the snark G_{36} by removing a path of length 2 starting from the vertex 12, the result violates Property (i).

The snark G_{36} has another interesting property. If we take the 6-pole $G_{36} - \{3, 12\}$ and perform the junction of the semiedges (4) and (20), we get an uncolourable 4-pole (because the two joined semiedges have different colours in every possible 3-edge-colouring). Furthermore, we can add one vertex incident with semiedges (8) and (23); the resulting uncolourable multipole is isomorphic to $G_{36} - \{12\text{-}28, 3\text{-}7\}$ with the vertex 12 suppressed. This implies that the pair of edges $\{12\text{-}28, 3\text{-}7\}$ is not essential in G_{36}. On the other hand, with the help of a computer we have found a colouring that proves that this pair of edges is non-removable. This solves Problem 5.7 proposed by Chladný and Škoviera in [12] by showing that there exists a pair of non-removable edges in an irreducible snark which is not essential. The same holds for the pairs of edges $\{12\text{-}28, 3\text{-}23\}$, $\{12\text{-}4, 8\text{-}0\}$ and $\{12\text{-}4, 8\text{-}2\}$.

10 Beyond order 36

We conclude our paper by analysing the currently known 5-simple snarks of order 38. We also explain how our results can be used to generate such snarks of higher orders.

At present, there are $19, 775, 768$ known nontrivial snarks of order 38 (see [5], section Snarks). Of them, 56 are 5-simple snarks, all being bicritical. In the latter set, we have identified the 5-pole P_{NN} in 39 snarks, the 5-pole P_{NT} in 22 snarks, and the 5-pole P_{TT} in 7 snarks, while there are 10 snarks containing both P_{NN} and P_{NT} and 6 snarks containing both P_{NN} and P_{TT}. In three snarks we have found the panchromatic $(2, 2, 2; 1)$-pole $M_{3NT} = 3\mathrm{NT}(\mathbf{D}, \mathbf{D}, \mathbf{D}, \mathbf{T})$. All of them arise from the Petersen graph by a substitution of M_{3NT} for the colour-equivalent multipole M_7 (which is a path of length two and one edge, see Table 2). The only remaining snark gives rise to an infinite family, which we are just about to describe.

Class 38-A

Take four negators $N_i(I_i, O_i; r_i)$, for $i \in \{1, 2, 3, 4\}$, and one proper $(2, 3)$-pole $T(B, C)$, connect them as shown in Figure 36, and denote the resulting graph by G.

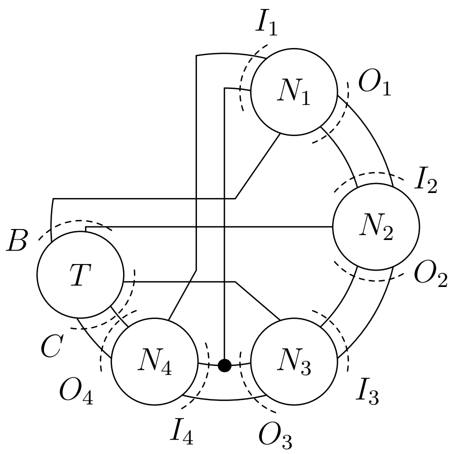


Figure 36: The structure of Class 38-A snarks

We show that G is a snark. The connector B is proper, so the edges r_1 and r_2 contained in B have different colours, say $\varphi(r_1) = a$ and $\varphi(r_2) = b$, where $a \neq b$. It follows that $\varphi_*(O_1) = \varphi_*(I_2) = 0$, for otherwise we would have $a = b$. Moreover, $\varphi_*(I_1) = a$ and $\varphi_*(O_2) = \varphi_*(I_3) = b$. Since N_3 is a negator, we infer that $\varphi(r_3) = b$ and $\varphi_*(O_3) = 0$.

Knowing the colour of three semiedges of the proper $(2,3)$-pole T, we can use the Kirchhoff law to determine the flow through the remaining two semiedges, which coincides with the flow through O_4. We have $\varphi_*(O_4) = \varphi(r_1) + \varphi(r_2) + \varphi(r_3) = a + b + b = a \neq 0$, so $\varphi_*(I_4) = 0$. Therefore $\varphi(r_4) = a = \varphi_*(I_1)$, which forces the flow value on the edge of I_1 different from r_4 to be zero. This contradiction proves that G is a snark.

The one remaining 5-simple snark of order 38 consists of four dyads and one triad. One can clearly see that permuting the semiedges in the connector C of the triad leads to other 5-simple snarks of order 38. Moreover, permutations of the semiedges in the connectors of dyads may also give rise to additional non-isomorphic 5-simple snarks of order 38.

Class 42-A

This family illustrates the fact mentioned in Section 4 that, unlike quasitriads, both triple pentagons and tricells may occur in 5-simple snarks. Among the six 5-cycle clusters on at most 10 vertices shown in Figure 37, the quasitriad is the only one that cannot occur in a 5-simple snark.

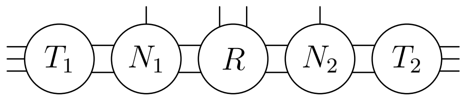


Figure 37: An infinite class of snarks containing triple pentagons and tricells.

Let R be a $(2,2,2)$-pole obtained from a snark G by severing three pairwise nonadjacent edges; if G is the Petersen graph, then R is either the triple pentagon or the tricell, depending on the choice of the three edges. Let N_1 and N_2 be two negators, and T_1 and T_2 two proper $(2,3)$-poles. Construct a 10-pole M as depicted in Figure 37. The 10-pole M is uncolourable: for each $i \in \{1,2\}$ the flow between N_i and R has to be zero, which is impossible for R. Thus, to obtain a snark, it is sufficient to perform junctions of the ten semiedges of M. If we take all the multipoles from the Petersen graph, we obtain several 5-simple snarks of order 42. Somewhat surprisingly, it is also possible to perform the junctions in such a way that the outcome will be cyclically 5-edge-connected but not critical.

Constructions and analysis of larger critical snarks

Infinite families described in this paper can be used to generate new 5-simple snarks of orders greater than 36. We briefly sketch the ideas that can be used to construct reasonable amounts of such snarks.

To construct a member of an infinite family, we choose snarks for the construction of the desired multipoles. Cyclically 5-connected critical snarks are a sensible choice; nevertheless, we have seen examples where the multipoles were constructed from non-critical snarks or snarks with smaller cyclic connectivity. These snarks might produce new members of the families, but it requires more computational time. One way or another, we still need to check whether the resultant snarks are cyclically 5-connected and critical, which is not guaranteed just by the membership in any of our families. Another possibility to construct new families of 5-simple snarks is to use ideas which have occurred in our proofs of uncolourability.

Currently, we are not able to construct a complete list of snarks of order 38 (or more). If a complete list of all the cyclically 5-connected critical snarks of a given order $n \geqslant 38$ eventually becomes available, it is possible to employ our methods to analyse them. We expect that, at least for reasonably small n, a large number of these snarks will fall into one of the already described families, and the remaining snarks give rise to new infinite families of snarks. With increasing n, however, the approach based on the analysis of 5-cycle clusters will become less efficient. For instance, we would not be able to identify important multipoles, such as negators or proper $(2, 3)$-poles, arising from Isaacs flower snarks. It might thus be helpful to extend the analysis of the computer-generated snarks by including the search for certain subgraphs of flower snarks, for instance, the iterated Isaacs $(3, 3)$-poles Y_k, which can be regarded as clusters of 6-cycles. In general, however, no method is known that would be appropriate for analysing cyclically 6-connected snarks. Despite the efforts of Karabáš et al. [27], a decomposition theorem for cyclically 6-connected snarks, similar to decomposition theorems for lower connectivities proved in [10, 38], remains unknown. Furthermore, small cyclically 6-connected snarks different from the Isaacs flower snarks seem to be very difficult to find: the smallest known example was constructed in 1996 by Kochol in [29] and has 118 vertices.

Acknowledgements

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