

Ordering circuits of matroids

Cameron Crenshaw James Oxley

Department of Mathematics
Louisiana State University
Baton Rouge, U.S.A.

`ccrens5@lsu.edu` `oxley@math.lsu.edu`

Submitted: Mar 15, 2022; Accepted: Oct 21, 2022; Published: Nov 18, 2022

© The authors. Released under the CC BY-ND license (International 4.0).

Abstract

The cycles of a graph give a natural cyclic ordering to their edge-sets, and these orderings are consistent in that two edges are adjacent in one cycle if and only if they are adjacent in every cycle in which they appear together. An orderable matroid is one whose set of circuits admits such a consistent ordering. In this paper, we consider the question of determining which matroids are orderable. Although we are able to answer this question for non-binary matroids, it remains open for binary matroids. We give examples to provide insight into the potential difficulty of this question in general. We also show that, by requiring that the ordering preserves the three arcs in every theta-graph restriction of a binary matroid M , we guarantee that M is orderable if and only if M is graphic.

Mathematics Subject Classifications: 05B35

1 Introduction

In a graph, the edges of each cycle have an ordering on them. But this is not true for the circuits of a matroid. The goal of this paper is to see to what extent we can distinguish graphic matroids by an ordering condition that mimics the ordering condition on the edges of the cycles of a graph.

A *reversible cyclic ordering* of a finite set X is an arrangement of the elements of X on the vertices of an n -gon with one element at each vertex. Elements x_1 and x_2 of X are *adjacent* in the ordering when the corresponding vertices of the n -gon lie on a common edge. Figure 1 shows an example of such an ordering $(x_1 x_2 \dots x_n)$. The same ordering can also be denoted, for example, by $(x_3 x_2 x_1 x_n \dots x_4)$. Throughout this paper, all orderings are assumed to be reversible cyclic orderings unless stated otherwise.

In a graph, there is an associated ordering on the edge set of each cycle. These orderings have the property that two edges are adjacent in an ordering of a given cycle

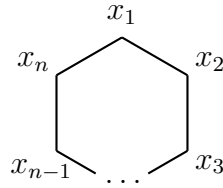


Figure 1: A reversible cyclic ordering.

if and only if they are adjacent in the ordering of every cycle in which the edges appear together.

Unlike the cycles of a graph, the circuits of a matroid are sets without inherent order. We give a matroid M an ordering by imposing an ordering on each of its circuits. Such an ordering of M is *consistent* if, for every pair $\{e, f\}$ of distinct elements of $E(M)$ and every pair $\{C, C'\}$ of circuits of M with $\{e, f\} \subseteq C \cap C'$, if e and f are adjacent in the ordering of C , then e and f are adjacent in the ordering of C' . A matroid is called *orderable* if it has a consistent ordering.

The notation for matroids in this paper follows [5] with one modification. We call a matroid N a *series extension* of a matroid M if N can be obtained from M by a (possibly empty) sequence of single-element series extensions; a *parallel extension* is defined analogously.

The primary goal of this work is characterizing orderable matroids. As noted above, our first examples of orderable matroids are graphic matroids.

Proposition 1. *If M is a graphic matroid, then M is orderable.*

However, orderability is not enough to distinguish graphic matroids from non-graphic matroids. Our main result specifies all non-binary orderable matroids. The infinitely many such matroids are all built from $U_{2,n}$ for some $n \geq 4$ by using two operations, which we now describe.

For a matroid M without coloops, a series extension of M is *balanced* if, for some integer k exceeding one, each element of M is replaced by k elements in series. We call k the *order* of the balanced series extension. The second operation is a generalization of the operation of adding an element in parallel to another. A *theta-graph* is a graph consisting of a pair of distinct vertices and three internally disjoint paths between them. Now, let P be a nonempty subset of a series class of a matroid M . Fix an element t of P , contract $P - t$, and relabel t as t' to obtain M' . Let N be the cycle matroid of a theta-graph with series classes $\{t'\}$, P , and P' , where $|P'| = |P|$. Finally, let M'' be the 2-sum of M' and N with basepoint t' . The operation transforming M into M'' is called *parallel-path addition*. The *size* of this addition is $|P|$; we call P and P' *parallel paths* of M'' , and say that M'' is obtained from M by adding P' in parallel to P . The following theorem is the main result of the paper.

Theorem 2. *Let M be a connected non-binary matroid. Then M is orderable if and only if it can be obtained from $U_{2,n}$ for some $n \geq 4$ by a sequence of the following operations:*

- (i) *balanced series extension; and*
- (ii) *parallel-path addition.*

When we come to consider binary orderable matroids, we encounter considerable difficulty. For example, as we show in the next section, F_7^* and $M^*(K_5)$ are not orderable, yet each has an orderable series extension. In view of this, it is natural to consider additional conditions that one can add to orderability in order to distinguish graphic matroids within binary matroids. The next theorem gives three equivalent such additional conditions.

Theorem 3. *The following are equivalent for a binary matroid M :*

- (i) *M is graphic.*
- (ii) *every minor of M is orderable.*
- (iii) *every series minor of M is orderable.*
- (iv) *every parallel minor of M is orderable.*

Although, as noted above, there are orderable binary matroids that are not graphic, we know of no counterexample to the following.

Conjecture 4. *A 3-connected orderable binary matroid is graphic.*

We have, however, made the following progress.

Theorem 5. *A 4-connected regular orderable matroid is graphic.*

Another condition one can add to orderability to distinguish graphic matroids within binary matroids involves the theta-graphs in a matroid M , where a *theta-graph* in M is a restriction of M that is isomorphic to the cycle matroid of a theta-graph. Equivalently, it is a restriction of M that is isomorphic to a series extension of $U_{1,3}$. The series classes of a theta-graph are called its *theta-arcs*. A subset B of a circuit C is a *block* if there is a listing b_1, b_2, \dots, b_k of the elements of B such that b_i and b_{i+1} are adjacent for all i in $[k - 1]$. A consistent ordering of a matroid M is a *theta-ordering* if every theta-arc of every theta-graph of M is a block in the ordering; M is *theta-orderable* if it has a theta-ordering.

Theta-orderability turns out to be equivalent to a concept introduced by Wagner [9]. For distinct circuits C and D of a matroid M , an *arc* of C is a minimal non-empty subset A of C such that $A \cup D$ contains at least two circuits. A set $\{A_1, A_2, A_3\}$ of arcs of a common circuit is *incompatible* if $A_1 \cap A_2 \cap A_3 \neq \emptyset$ and $A_i - (A_j \cup A_k) \neq \emptyset$ for all i, j , and k such that $\{i, j, k\} = \{1, 2, 3\}$. In Section 4, we prove the following characterization of theta-orderable binary matroids. The equivalence of (i) and (ii) is Wagner's main result [9].

Theorem 6. *The following are equivalent for a binary matroid M :*

- (i) M is graphic;
- (ii) M has no set of incompatible arcs; and
- (iii) M is theta-orderable.

The following characterization of theta-orderable non-binary matroids will also be proved in Section 4.

Theorem 7. *Let M be a connected non-binary matroid. Then M is theta-orderable if and only if M is a parallel extension of a balanced series extension of $U_{2,n}$ for some $n \geq 4$.*

In Section 2, after some preliminaries, we prove Theorem 3. The proof of our main result, Theorem 2, is in Section 3, and Theorem 5 is proved in Section 5.

2 Preliminaries

Our first proposition collects some basic properties of orderability. These properties will be used frequently and often implicitly. We omit their straightforward proofs.

Proposition 8. *Let M be a matroid.*

- (i) *If M is orderable, then $M \setminus e$ is orderable for all $e \in E(M)$.*
- (ii) *If $r(M) \leq 2$, then M is orderable.*
- (iii) *M is orderable if and only if the connected components of M are orderable.*
- (iv) *M is orderable if and only if $\text{si}(M)$ is orderable.*

Next, we note a partial converse to Proposition 1.

Proposition 9. *If M is an orderable binary matroid with a spanning circuit, then M is graphic.*

Proof. Let C be a spanning circuit of M and e be an element in C . Fix a consistent ordering of M , and take a standard binary representation of M with respect to the basis $C - e$. Now construct a graph G beginning with a cycle having edge set C , ordered consistently with the fixed ordering of M . Now, for each element f of $E(M) - C$, let C_f be the fundamental circuit of f with respect to $C - e$. Because $C_f - f$ is a block in the ordering, we may add an edge f to G as a chord of C so that it forms a cycle with edge set C_f . The result is a graph whose cycle matroid has ground set $E(M)$, has $C - e$ as a basis, and has the same fundamental circuits with respect to this basis as M . Since M and $M(G)$ are binary, we deduce that $M = M(G)$. \square

We now note a necessary condition for a matroid to be orderable, along with some consequences of this condition.

Proposition 10. *Let M be a simple matroid and X be a subset of $E(M)$ with $|X| \geq 3$. If there are elements e and f in $E(M) - X$ such that $X \cup e$ and $X \cup f$ are both circuits of M , then M is not orderable.*

Proof. Assume to the contrary that M has a consistent ordering. Notice that the ordering of $X \cup f$ is obtained from that of $X \cup e$ by replacing f with e . Let a and b be the elements in X that are adjacent to e . Using strong circuit elimination on $X \cup e$ and $X \cup f$, we obtain a circuit $C \subseteq X \cup \{e, f\}$ containing e but not a , and another $C' \subseteq X \cup \{e, f\}$ containing f but not b .

As C is not properly contained in either $X \cup e$ or $X \cup f$, it must contain both e and f . Further, M is simple, so $C \cap X$ is nonempty. Since a and b are the only elements in X adjacent to e or f , it follows that $C = \{e, f, b\}$. By symmetry, $C' = \{e, f, a\}$.

Circuit elimination applied to C and C' now yields a circuit D that does not contain e . Then $D \subseteq \{a, b, f\}$. Since $|X| \geq 3$, it follows that D is a proper subset of $X \cup f$, a contradiction. \square

Corollary 11. *Let M be a matroid of rank at least three and X be a circuit-hyperplane of M . If $E(M) - X$ is not a parallel class of M , then the matroid obtained from M by relaxing X is not orderable.*

Corollary 12. *The only orderable whirl is $U_{2,4}$.*

We now prove Theorem 3, whose proof relies on the next lemma and its corollary. The following technical property facilitates the statements of these results. A matroid M has the (e, f, g) -property if

- (i) M has a circuit containing $\{e, f, g\}$;
- (ii) e, f , and g are distinct; and
- (iii) M has a circuit D containing f but neither e nor g and, with the exception of at most one d in D , there is a circuit of M containing $\{e, f, g, d\}$.

Lemma 13. *If a matroid M has the (e, f, g) -property, then f is not adjacent to both e and g in a consistent ordering of M .*

Proof. Suppose M has the (e, f, g) -property and f is adjacent to both e and g . Then, in the circuit D of condition (iii), f is adjacent to elements d_1 and d_2 of $D - f$. But M has a circuit containing $\{e, f, g, d_i\}$ for some i in $\{1, 2\}$, a contradiction. \square

Corollary 14. *Let C be a circuit of a matroid M . Suppose there is an element c of C so that M has the (e, c, g) -property for every choice of e and g in $C - c$. Then M does not have a consistent ordering.*

Proof of Theorem 3. Since graphic matroids are orderable and the class of graphic matroids is minor-closed, (i) implies (ii)-(iv). Let \mathcal{S} be the set

$$\{F_7, F_7^*, M^*(K_5), M^*(K_{3,3}), M^*(K'_{3,3}), M^*(K''_{3,3}), M^*(K'''_{3,3}), R_{10}\}.$$

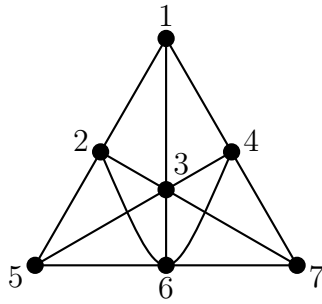


Figure 2: The matroid F_7 .

By results of Tutte [8] and Bixby [1, 2], \mathcal{S} contains all binary matroids that are excluded minors, excluded series minors, or excluded parallel minors for the class of graphic matroids. Thus we can prove that (i) follows from each of (ii)-(iv) by showing that none of the matroids in \mathcal{S} is orderable.

Let F_7 be labelled as in Figure 2. Using the element 1 in the circuit $\{1, 2, 3, 4\}$, Corollary 14 gives that F_7 has no consistent ordering.

Let F_7^* be labelled as in Figure 3. Consider the circuits $C_1 = \{1, 2, 3, 4\}$, $C_2 = \{1, 3, 5, 7\}$, and $C_3 = \{2, 4, 5, 7\}$. The ordering of a four-element circuit is uniquely determined by a single pair of non-adjacent elements, and the automorphism group of F_7^* is doubly transitive. Thus we may assume that C_1 has the ordering $(1\ 2\ 3\ 4)$.

Since 1 and 3 are not adjacent in C_1 , it follows that C_2 has the ordering $(1\ 5\ 3\ 7)$. Thus 5 and 7 are non-adjacent, so C_3 has the ordering $(2\ 5\ 4\ 7)$. However, the elements of the set $\{1, 2, 5\}$ are now pairwise adjacent, so the circuit $\{1, 2, 5, 6\}$ cannot be ordered. Thus F_7^* has no consistent ordering.

Let $M^*(K_5)$ be labelled as in Figure 4, and assume that $M^*(K_5)$ has a consistent ordering. Let C be the circuit $\{1, 2, 3, 4\}$. By symmetry, we may assume its ordering is $(1\ 2\ 3\ 4)$. This ordering and the circuit $\{1, 2, 4, 7, 8, 9\}$ give that 1 and 8 are not adjacent, so the circuit $\{0, 1, 5, 8\}$ must be ordered $(0\ 1\ 5\ 8)$. Similarly, the circuit $\{1, 2, 3, 5, 6, 7\}$

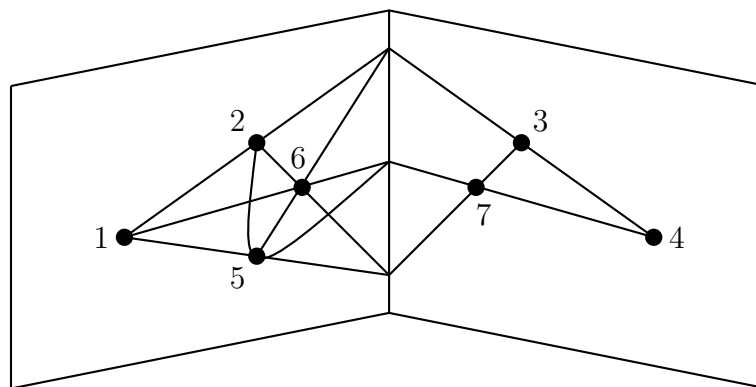


Figure 3: The matroid F_7^* .

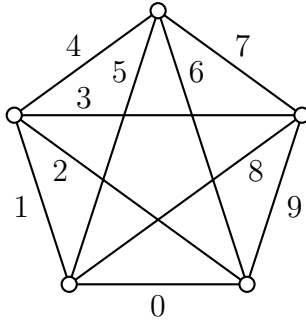


Figure 4: The graph K_5 .

gives that 2 and 6 are not adjacent, so $\{0, 2, 6, 9\}$ must be ordered $(0\ 2\ 9\ 6)$. Now 0 is adjacent to 1, 6, and 8 in the circuit $\{0, 1, 4, 6, 7, 8\}$, a contradiction.

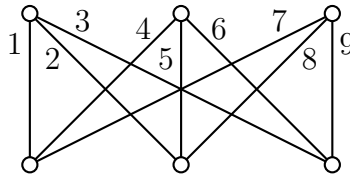


Figure 5: The graph $K_{3,3}$.

Let $M^*(K_{3,3})$ be labelled as in Figure 5. We shall use Corollary 14 letting c be the element 1 in the circuit $C = \{1, 3, 5, 8\}$ of $M^*(K_{3,3})$. The cases $\{e, g\} = \{3, 5\}$ and $\{e, g\} = \{3, 8\}$ are symmetric, and the circuits C and $\{1, 3, 5, 7, 9\}$ certify that M has the $(3, 1, 5)$ -property with $D = \{1, 4, 8, 9\}$. The circuit $\{1, 5, 6, 8, 9\}$ certifies that M has the $(5, 1, 8)$ -property with $D = \{1, 2, 6, 9\}$. Corollary 14 now implies that $M^*(K_{3,3})$ has no consistent ordering.

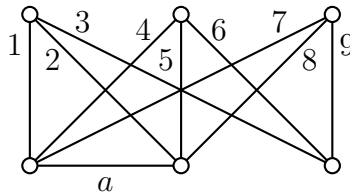


Figure 6: The graph $K'_{3,3}$.

The next two cases will also use Corollary 14. Let $M^*(K'_{3,3})$ be labelled as in Figure 6, and let c be the element 3 in the circuit $C = \{3, 6, 7, 8\}$ of $M^*(K'_{3,3})$. The cases $\{e, g\} = \{6, 7\}$ and $\{e, g\} = \{6, 8\}$ are symmetric, and the circuit $\{2, 3, 5, 6, 7, 8\}$ certifies that M has the $(6, 3, 7)$ -property with $D = \{1, 2, 3\}$. The circuit C certifies that M has the $(7, 3, 8)$ -property with $D = \{3, 6, 9\}$. Corollary 14 now implies that $M^*(K'_{3,3})$ has no consistent ordering.

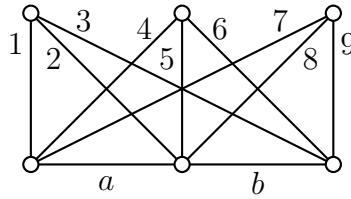


Figure 7: The graph $K''_{3,3}$.

Let $M^*(K''_{3,3})$ be labelled as in Figure 7, and let c be the element 1 in the circuit $C = \{1, 3, 5, 8, a, b\}$ of $M^*(K''_{3,3})$. When $\{e, g\} \subseteq C - 3$, the circuit C certifies that M has the $(e, 1, g)$ -property with $D = \{1, 2, 3\}$. Each of the remaining cases uses $D = \{1, 4, 7, a\}$. The cases $\{e, g\} = \{3, 5\}$ and $\{e, g\} = \{3, 8\}$ are symmetric, and the circuit $\{1, 3, 5, 7, 9, a, b\}$ certifies that M has the $(3, 1, 5)$ property. The circuits $\{1, 3, 4, 6, 8, a, b\}$ and $\{1, 3, 5, 7, 9, a, b\}$ certify the $(3, 1, a)$ -property. Finally, the circuit $\{1, 3, 4, 6, 8, a, b\}$ certifies the $(3, 1, b)$ -property. Corollary 14 now implies that $M^*(K''_{3,3})$ has no consistent ordering.

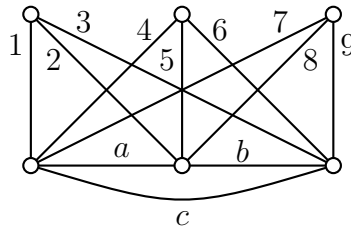


Figure 8: The graph $K'''_{3,3}$.

Let $M^*(K'''_{3,3})$ be labelled as in Figure 8. We begin by noting that there must be at least one adjacent pair in the set $\{1, 4, 7\}$ due to the circuit $\{1, 4, 7, a, c\}$. By symmetry, we may assume that 1 and 4 are adjacent.

Combining this adjacent pair with the three-element circuits, we get that 2145 is a block in the circuit $\{1, 2, 4, 5, 9, b, c\}$. Therefore 4 is not adjacent to 9, b , or c . This means that, in the circuit $\{3, 4, 5, 9, b, c\}$, we must have 4 adjacent to 3. Using the three-element circuit $\{4, 5, 6\}$, we now have that 4 is adjacent to 1, 3, and 6. Therefore the circuit $\{1, 3, 4, 6, 8, a, b\}$ cannot be ordered consistently, and $M^*(K'''_{3,3})$ has no consistent ordering.

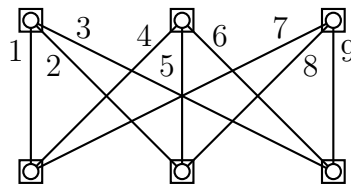


Figure 9: A graft corresponding to R_{10} .

Let M be the graft matroid of $K_{3,3}$ where the graft element e_γ corresponds to the set of boxed vertices in Figure 9. Then $M \cong R_{10}$. Using Corollary 14 again, let c be the element 1 in the circuit $C = \{1, 2, 4, 5\}$ of M . When $\{e, g\} = \{2, 4\}$, the circuit $\{1, 2, 4, 6, 8, 9\}$ certifies the $(2, 1, 4)$ -property when $D = \{1, 3, 4, 6\}$. When $\{e, g\} = \{2, 5\}$, the circuit $\{1, 2, 5, 6, 7, 9\}$ certifies the $(2, 1, 5)$ -property with $D = \{1, 3, 7, 9\}$. Finally, when $\{e, g\} = \{4, 5\}$, the circuit $\{1, 4, 5, 6, 7, e_\gamma\}$ certifies the $(4, 1, 5)$ -property with $D = \{1, 6, 8, e_\gamma\}$. Corollary 14 now implies that R_{10} has no consistent ordering. \square

We conclude this section with a pair of examples that indicate the potential difficulty of characterizing orderable binary matroids.

Example 15. This example describes a 12-element orderable series extension of F_7^* , which we refer to as O_1 . Thus, the pair O_1 and F_7^* demonstrates that the class of binary orderable matroids is not closed under the taking of series minors. Let F_7^* be labelled as in Figure 3. We obtain O_1 by adding $1'$, $2'$, and $7'$ in series with 1, 2, and 7, respectively, and adding $4'$ and $4''$ in series with 4. Figure 10 gives a consistent ordering of the circuits of O_1 .

$$\begin{array}{cccc} (1\ 5\ 1'\ 2'\ 6\ 2) & (1\ 5\ 1'\ 7'\ 3\ 7) & (2\ 6\ 2'\ 7'\ 3\ 7) & (3\ 4\ 5\ 4'\ 6\ 4'') \\ (1\ 4'\ 2'\ 1'\ 4\ 3\ 4''\ 2) & (1\ 7\ 4\ 1'\ 7'\ 4''\ 6\ 4') & & (2\ 7\ 4\ 5\ 4'\ 2'\ 7'\ 4'') \end{array}$$

Figure 10: A consistent ordering of O_1 .

Example 16. Let K_5 be labelled as in Figure 4. We obtain a regular, non-graphic matroid O_2 from $M^*(K_5)$ by adding elements $0'$ and $2'$ in series with 0 and 2, respectively. Figure 11 gives a consistent ordering of O_2 .

$$\begin{array}{cccc} (4\ 6\ 5\ 7) & (2'\ 1\ 2\ 6\ 5\ 8\ 9) & (0'\ 1\ 0\ 9\ 3\ 4\ 6) & (2'\ 1\ 2\ 4\ 7\ 8\ 9) \\ (3\ 7\ 8\ 9) & (2\ 1\ 2'\ 3\ 7\ 5\ 6) & (0'\ 1\ 0\ 8\ 7\ 4\ 6) & (0'\ 1\ 0\ 9\ 3\ 7\ 5) \\ (0'\ 2'\ 3\ 4\ 2\ 0\ 8\ 5) & (0'\ 2'\ 9\ 0\ 2\ 4\ 7\ 5) & & (2'\ 0'\ 6\ 2\ 0\ 8\ 7\ 3) \\ (2\ 1\ 2'\ 3\ 4) & (0\ 1\ 0'\ 5\ 8) & (0\ 2\ 6\ 0'\ 2'\ 9) & (3\ 4\ 6\ 5\ 8\ 9) \end{array}$$

Figure 11: A consistent ordering of O_2 .

3 A Characterization of Non-Binary Orderable Matroids

In this section, we prove Theorem 2. We begin by finding the orderable series extensions of uniform matroids and their consistent orderings. These results allow us to characterize the non-binary orderable matroids that are 3-connected, from which we obtain the full characterization using the canonical tree decomposition of Cunningham and Edmonds [3].

A uniform matroid is binary if and only if it is graphic. Thus, the binary uniform matroids are certainly orderable, as are those whose rank is at most two. Proposition 10 implies this list is complete.

Corollary 17. *A uniform matroid is orderable if and only if it is binary or has rank at most two.*

The next two results deduce the structure of a consistent ordering of a series extension of a non-binary uniform matroid, and show that such an ordering can be used to consistently order the underlying uniform matroid. For a non-coloop element e of a matroid M , we denote the series class of M containing e by S_e or sometimes by $S_e(M)$.

Let M be a matroid with a consistent ordering. Suppose X and Y are disjoint subsets of a circuit C of M . We say X and Y are *adjacent* if there is an adjacent pair of elements x and y , where x belongs to X and y belongs to Y . Let \mathcal{B} be the union of a set of blocks that belong to a common circuit of M . If there is a listing B_1, B_2, \dots, B_k of the blocks in \mathcal{B} such that B_i and B_{i+1} are adjacent for all i in $[k - 1]$, then \mathcal{B} is a *section*. Finally, let S be a series class of M . If a block of M is contained in S and is maximal with this property, then it is called an *S -block*.

Lemma 18. *Let M be an orderable series extension of a non-binary uniform matroid $U_{r,n}$ and fix a consistent ordering of M . Let C be a circuit of M , and let x and y be elements of C from distinct series classes of M .*

- (i) *If a section K in C is adjacent to a pair of S_x -blocks, then K must contain an S_y -block.*
- (ii) *Every series class S of M has the same number of S -blocks.*

Proof. For (i), suppose to the contrary that there is a section K in C that contains no S_y -block and is adjacent to a pair of distinct S_x -blocks. As M is non-binary, $2 \leq r \leq n - 2$ and there is a circuit D_x of M that contains K and S_x but avoids S_y . Let $D_y = (D_x - S_x) \cup S_y$. Observe that, since M is a series extension of $U_{r,n}$, the set D_y is a circuit. The consistency of D_y with C implies that K is not adjacent to S_y -blocks in D_y , but the consistency of D_y with D_x gives that K can only be adjacent to S_y -blocks in D_y , a contradiction.

We now deduce (ii) from (i). Let S be a series class of $E(M)$ for which the number of S -blocks is as large as possible. We may assume this number exceeds one. In a circuit C containing S , let K be a minimal section that is adjacent to a pair of distinct S -blocks. Note that the number of such minimal sections in C equals the number of S -blocks. Let S' be a series class of M contained in C that is distinct from S . Part (i) implies there is an S' -block in K and, as K contains no S -blocks, (i) further implies that there is exactly one S' -block in K . Thus there are the same number of S' -blocks as S -blocks. Part (ii) now follows. \square

Proposition 19. *Let $U_{r,n}$ be a non-binary uniform matroid. If a series extension of $U_{r,n}$ is orderable, then so is $U_{r,n}$.*

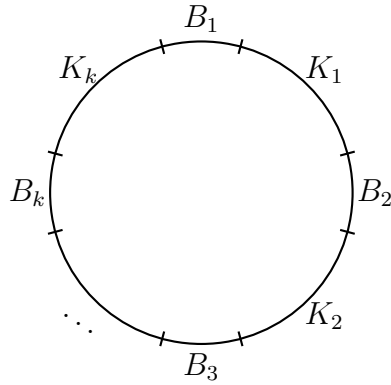


Figure 12: The circuit C in the proof of Proposition 19.

Proof. Let M be an orderable series extension of $U_{r,n}$ and fix a consistent ordering of M . By Lemma 18(ii), there is an integer $k \geq 1$ such that every series class of M is divided into exactly k blocks. If $k = 1$, the result follows immediately, so assume $k \geq 2$.

Let $[n]$ be the ground set of $U_{r,n}$. Consider the circuit C of M that contains the set $\{1, 2, \dots, r + 1\}$. Label the S_1 -blocks in C as B_1, B_2, \dots, B_k , such that B_i and B_{i+1} abut a section K_i that does not contain S_1 -blocks, as in Figure 12.

Applying Lemma 18(i), we see that each section K_i contains exactly one S_j -block for all j in $\{2, 3, \dots, r + 1\}$. Thus, $B_i \cup K_i$ defines a permutation of $\{1, 2, \dots, r + 1\}$ that begins with 1. We show this permutation is the same for all i .

Without loss of generality, suppose the block in K_1 adjacent to B_1 is an S_2 -block. If the block in K_2 adjacent to B_2 is an S_j -block with $j \neq 2$, then the S_j -blocks in K_1 and K_2 abut a section that contains no S_2 -block, contradicting Lemma 18(i). Thus the block in K_2 adjacent to B_2 is an S_2 -block. Repeating this argument gives that $B_1 \cup K_1$ and $B_2 \cup K_2$ define the same permutation on $\{1, 2, \dots, r + 1\}$. It follows that $B_i \cup K_i$ defines the same permutation on $\{1, 2, \dots, r + 1\}$ for all i in $[n]$. Thus $B_i \cup K_i \cup B_{i+1}$ defines the same reversible cyclic ordering on $\{1, 2, \dots, r + 1\}$ for all i in $[n]$; it is this reversible cyclic ordering that we extract from C and use to order the circuit $\{1, 2, \dots, r + 1\}$ in $U_{r,n}$.

In this way, every circuit of $U_{r,n}$ is ordered using the corresponding circuit of M . Since the ordering of M is consistent, so too is the ordering it gives to $U_{r,n}$. \square

Theorem 20. *Let $U_{r,n}$ be a non-binary uniform matroid of rank at least three. If M is a matroid with a series minor isomorphic to $U_{r,n}$, then M is not orderable.*

Proof. By [5, Proposition 5.4.2], we may write $U_{r,n} = M \setminus X / Y$ where each element of Y is in series with an element of $M \setminus X$ not in Y . By Corollary 17, the matroid $U_{r,n}$ is not orderable. Therefore, by Proposition 19, neither is its series extension $M \setminus X$. Thus, M is not orderable. \square

Recall that, in a balanced series extension N of a matroid M without coloops, each element of M is replaced by k elements in series for some positive integer k .

Lemma 21. *Let N be a balanced series extension of a matroid M . If M is orderable, then so is N .*

Proof. A consistent ordering of N may be obtained from a consistent ordering of M by extracting a linear order from each ordered circuit of M , then repeating this linear order k times in the corresponding circuit of N . Specifically, if x_1, x_2, \dots, x_n is a linear order for the elements of a circuit C of M , and $x_i^1, x_i^2, \dots, x_i^k$ are the k elements in series in N that replace the element x_i , then

$$(x_1^1 x_2^1 \dots x_n^1 x_1^2 x_2^2 \dots x_n^2 \dots x_1^k x_2^k \dots x_n^k)$$

is the ordering of the circuit of N corresponding to C . □

The following proposition specializes some of the results about uniform matroids to $U_{2,n}$ with $n \geq 4$. These rank-two uniform matroids will serve as the foundation from which all non-binary orderable matroids are built.

Proposition 22. *Let M be an orderable series extension of $U_{2,n}$ for some $n \geq 4$, and fix a consistent ordering of M . Then*

- (i) *for all series classes S of M , every S -block of the ordering consists of a single element; and*
- (ii) *M is a balanced series extension of $U_{2,n}$.*

Proof. Statement (ii) follows from combining (i) with Lemma 18(ii), so it suffices to show (i). Let $E(U_{2,n}) = [n]$. Suppose, to the contrary, that M has an S_1 -block B of size at least two.

Applying Lemma 18(i), we have that B is adjacent to both an S_2 -block and an S_3 -block in the circuit of M containing $\{1, 2, 3\}$. Let 1_2 be the element of B adjacent to the S_2 -block and let 1_3 be the element of B adjacent to the S_3 -block, where 1_2 and 1_3 are necessarily distinct. In the circuit of M containing $\{1, 2, 4\}$, Lemma 18(i) now gives that B is adjacent to both an S_2 -block and an S_4 -block. Consistency dictates that 1_2 is again adjacent to the S_2 -block. Therefore 1_3 is now adjacent to the S_4 -block.

Now consider the circuit of M containing $\{1, 3, 4\}$. Consistency with the two aforementioned circuits requires that 1_3 be adjacent to both an S_3 -block and an S_4 -block. As $|B| \geq 2$, this is a contradiction. □

The next theorem identifies all orderable matroids that are 3-connected and non-binary.

Theorem 23. *If M is a 3-connected non-binary orderable matroid, then $M \cong U_{2,n}$ for some $n \geq 4$.*

The next two results will be used in the proof of this theorem.

Proposition 24. *If M is an orderable matroid, then M has no minor isomorphic to $U_{3,5}$.*

Proof. Assume instead that $M \setminus X/Y \cong U_{3,5}$, with X coindependent and Y independent. Then $M^*/X \setminus Y \cong U_{2,5}$ where M^*/X has rank two. Thus, after deleting a set Z of loops from M^*/X , we obtain a parallel extension of $U_{2,n}$ for some $n \geq 5$. This makes $M \setminus (X \cup Z)$ an orderable series extension of $U_{n-2,n}$, contradicting Theorem 20. \square

Proposition 25. *If M is an orderable matroid, then M has no minor isomorphic to \mathcal{W}^3 .*

The proof of this proposition will rely on the next lemma and its corollary. This second pair of results will use the following modification of the (e, f, g) -property. A matroid M has the *series (e, f, g) -property* if

- (i) M has a circuit containing $\{e, f, g\}$;
- (ii) $S_f(M)$ is distinct from both $S_e(M)$ and $S_g(M)$; and
- (iii) M has a circuit D containing f but not $\{e, g\}$ and, for each d in D , there is a circuit of M containing $\{e, f, g, d\}$.

Note that e and g may be equal in this definition.

Lemma 26. *Suppose that M has the series (e, f, g) -property and that N is a series extension of M . Then, in a consistent ordering of N , if $S_e(N) \neq S_g(N)$, then no $S_f(N)$ -block is adjacent to both an $S_e(N)$ -block and an $S_g(N)$ -block; and, if $S_e(N) = S_g(N)$, then no $S_f(N)$ -block is adjacent to two $S_e(N)$ -blocks.*

Proof. Let D be the circuit of M whose existence is guaranteed by condition (iii). Let D' be the circuit of N corresponding to D , and let B_f be an $S_f(N)$ -block. Notice D must have an element d not in $\{e, f, g\}$, so $D' - (S_e(N) \cup S_f(N) \cup S_g(N))$ is nonempty. If $S_e(N) = S_g(N)$ and B_f is adjacent to two $S_e(N)$ -blocks, then e is not in D , so B_f is not adjacent to any elements of $D' - B_f$, a contradiction. Now suppose $S_e(N) \neq S_g(N)$ and, without loss of generality, suppose e is in D but g is not. If B_f is adjacent to an $S_e(N)$ -block and an $S_g(N)$ -block, then all of the elements in $D' - B_f$ adjacent to B_f are in $S_e(N)$. This contradicts the fact that B_f is adjacent to an $S_e(N)$ -block and an $S_g(N)$ -block in a common circuit. \square

Corollary 27. *Let C be a circuit of a matroid M . Suppose that C contains an element c so that M has the series (e, c, g) -property for every choice of e and g in $C - c$. Then no series extension of M is orderable.*

Proof of Proposition 25. Assume instead that $M \setminus X/Y \cong \mathcal{W}^3$, with X coindependent and Y independent. Let L be the set of loops of M^*/X , and let N denote $M^*/(X \cup L)$. Note that N is a loopless rank-3 extension of \mathcal{W}^3 , so $\text{si}(N)$ is 3-connected. Further, N is a parallel extension of $\text{si}(N)$, which makes N^* an orderable series extension of $\text{co}(N^*)$.

27.1. $\text{si}(N)$ is ternary.

Number n of elements	Restrictions of O_7 with n elements
0	$U_{0,0}$
1	$U_{1,1}$
2	$U_{2,2}$
3	$U_{2,3}, U_{3,3}$
4	$U_{2,4}, U_{3,4}, U_{2,3} \oplus U_{1,1}$
5	$U_{2,4} \oplus U_{1,1}, P(U_{2,3}, U_{2,3}), U_{2,4} \oplus_2 U_{2,3}$
6	$P(U_{2,4}, U_{2,3}), M(K_4), \mathcal{W}^3$
7	O_7

Figure 13: Choices for K , the complement of $\text{si}(N)$ in $PG(2, 3)$.

To see this, first note that, as N^* is orderable, it has no $U_{3,5}$ -minor by Proposition 24. Thus, $\text{si}(N)$ has no $U_{2,5}$ -minor. As $\text{si}(N)$ is 3-connected and its rank and corank each exceed two, [5, Proposition 12.2.15] gives that $\text{si}(N)$ has no $U_{3,5}$ -minor. The rank of F_7^* exceeds three, so $\text{si}(N)$ also has no F_7^* -minor.

Finally, suppose $\text{si}(N)$ has an F_7 -minor. Then $\text{si}(N)|_Z \cong F_7$ for some set Z . As F_7 has no \mathcal{W}^3 -minor, $\text{si}(N)$ has an element e not in Z . Then $\text{si}(N)/e$ has a $U_{2,5}$ -restriction, a contradiction. We conclude that $\text{si}(N)$ has no F_7 -minor. Thus 27.1 holds.

By 27.1, $\text{si}(N)$ has the form $PG(2, 3) - K$, where K is a restriction of O_7 , the complement of \mathcal{W}^3 in $PG(2, 3)$. The matroid O_7 is obtained from $M(K_4)$ by adding a point freely to an existing 3-point line; the fifteen restrictions of O_7 are given in Figure 13. In the remainder of the proof, we eliminate each possibility for K .

If $K = U_{0,0}$, then $\text{si}(N) = PG(2, 3)$. Let $\text{si}(N)$ be labelled as in Figure 14. Suppose N^* has a consistent ordering, and let B_x, B_y , and B_z be S_{x^-}, S_{y^-} , and S_{z^-} -blocks in a common circuit C of N^* , where x, y , and z are elements of $E(\text{si}(N))$. Assume also that B_y is adjacent to B_x and B_z . Then, by Lemma 26, $\text{co}(N^*)$ does not have the series (x, y, z) -property. We show next that

27.2. x, y , and z are collinear in $\text{si}(N)$, and $x \neq z$.

Suppose x, y , and z are not collinear in $\text{si}(N)$. Then one easily finds circuits of $\text{co}(N^*)$ that verify the series (x, y, z) -property in $\text{co}(N^*)$, a contradiction. Similarly, when $x = z$ there are circuits of $\text{co}(N^*)$ that verify the series (x, y, z) -property in $\text{co}(N^*)$, a contradiction. Thus, 27.2 holds.

By symmetry, we may assume that C is the circuit $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of $\text{co}(N^*)$; let C' be the corresponding circuit of N^* . Consider an S_1 -block B in C' . The block B is adjacent to an S_e - and S_f -block for some e and f in $C - 1$. By 27.2, the elements 1, e , and f are collinear in $\text{si}(N)$; without loss of generality, say $e = 2$ and $f = 3$. Let B_3 be the S_3 -block adjacent to B . By repeatedly applying 27.2, we have that B_3 is adjacent to an S_2 -block B_2 , the block B_2 is adjacent to another S_1 -block B_1 , the block B_1 is adjacent to another S_3 -block, and so on. It follows that C' has a proper subset X of elements not adjacent to any element of $C' - X$, a contradiction.

If $K = U_{2,4}$, then $\text{si}(N) = AG(2, 3)$. Figure 15 gives two labelled copies of $\text{si}(N)$

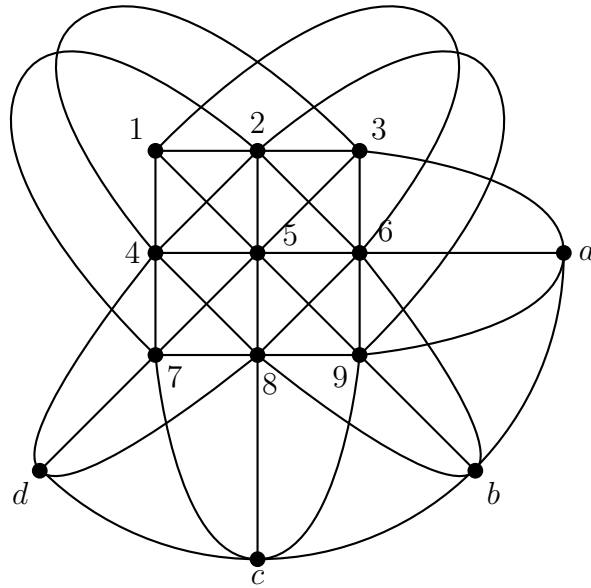


Figure 14: The matroid $PG(2, 3)$.

in order to illustrate some of the symmetries of this matroid. Using Corollary 27, let c be the element 1 in the circuit $C = \{1, 2, 3, 4, 5, 6\}$ of $\text{co}(N^*)$. When $e = g = 2$, the circuits C and $\{1, 2, 3, 7, 8, 9\}$ certify that $\text{co}(N^*)$ has the series $(2, 1, 2)$ -property with $D = \{1, 3, 4, 6, 7, 9\}$. Since $\text{co}(N^*)$ has a doubly transitive automorphism group, it follows that $\text{co}(N^*)$ has the series $(e, 1, e)$ -property for each e in C . When $\{e, g\} = \{2, 3\}$, the circuits C and $\{1, 2, 3, 7, 8, 9\}$ certify that $\text{co}(N^*)$ has the series $(2, 1, 3)$ -property with $D = \{1, 3, 4, 6, 7, 9\}$. The circuits C , $\{1, 2, 4, 5, 7, 8\}$, and $\{1, 2, 4, 6, 8, 9\}$ certify that $\text{co}(N^*)$ has the series $(2, 1, 4)$ -property with $D = \{1, 3, 4, 6, 7, 9\}$. A symmetric set of circuits certifies that $\text{co}(N^*)$ has the series $(e, 1, g)$ -property for each independent set $\{e, 1, g\}$ contained in C . By Corollary 27, N^* is not orderable.

If $K = U_{2,4} \oplus U_{1,1}$, then $\text{si}(N) = AG(2, 3) \setminus 9$ with $AG(2, 3)$ labelled as in Figure 15. Using Corollary 27 again, let c be the element 1 in the circuit $C = \{1, 2, 3, 7, 8\}$ of $\text{co}(N^*)$. When $e = g = 2$, the circuits C and $\{1, 2, 4, 6, 8\}$ certify that $\text{co}(N^*)$ has the series $(2, 1, 2)$ -property with $D = \{1, 3, 4, 6, 7\}$. A symmetric set of circuits certifies that $\text{co}(N^*)$ has the series $(e, 1, e)$ -property for each e in $C - 1$.

When $\{e, g\} = \{2, 3\}$, the circuits C and $\{1, 2, 4, 6, 8\}$ certify that $\text{co}(N^*)$ has the series $(2, 1, 3)$ -property with $D = \{1, 3, 4, 6, 7\}$. From Figure 15, we see that the cases $\{e, g\} = \{2, 7\}$ and $\{e, g\} = \{3, 8\}$ are symmetric; and the circuits C and $\{1, 2, 4, 5, 7, 8\}$ certify that $\text{co}(N^*)$ has the series $(2, 1, 7)$ -property with $D = \{1, 3, 4, 5, 8\}$. The cases $\{e, g\} = \{2, 8\}$ and $\{e, g\} = \{3, 7\}$ are also symmetric, and the circuits C and $\{1, 2, 4, 6, 8\}$ certify that $\text{co}(N^*)$ has the series $(2, 1, 8)$ -property with $D = \{1, 3, 4, 6, 7\}$. Finally, the circuits $\{1, 2, 4, 5, 7, 8\}$ and $\{1, 3, 5, 6, 7, 8\}$ certify that $\text{co}(N^*)$ has the series $(7, 1, 8)$ -property with $D = \{1, 2, 3, 4, 5, 6\}$. Corollary 27 now implies N^* is not orderable.

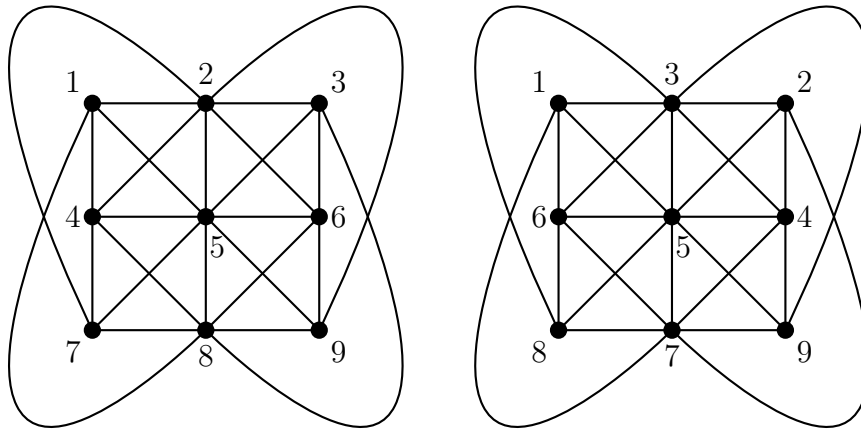


Figure 15: Two geometric representations of the matroid $AG(2, 3)$.

The next five cases make frequent use of Proposition 22(ii). The strategy is to contract strategic parallel classes of N to get parallel extensions of $U_{2,4}$. These parallel extensions are dual to orderable series extensions of $U_{2,4}$, and Proposition 22(ii) implies that the parallel classes of such a parallel extension have the same size. For each case, we view $\text{si}(N)$ as a restriction of the labelled copy of $PG(2, 3)$ in Figure 14. For each element e in $E(N)$, let p_e be the size of the parallel class of N containing e .

If $K = U_{1,1}$, then $\text{si}(N) = PG(2, 3) \setminus d$. The following equations are obtained by applying Proposition 22(ii) in the minors $N/\text{cl}(\{a\})$, $N/\text{cl}(\{b\})$, and $N/\text{cl}(\{c\})$, respectively:

$$p_b + p_c = p_1 + p_2 + p_3 = p_4 + p_5 + p_6 = p_7 + p_8 + p_9;$$

$$p_a + p_c = p_1 + p_5 + p_9 = p_2 + p_6 + p_7 = p_3 + p_4 + p_8;$$

$$p_a + p_b = p_3 + p_6 + p_9 = p_2 + p_5 + p_8 = p_1 + p_4 + p_7.$$

Combining these equations, we obtain

$$3(p_a + p_b) + 3(p_a + p_c) + 3(p_b + p_c) = 3(p_1 + p_2 + \cdots + p_9),$$

which implies

$$2(p_a + p_b + p_c) = p_1 + p_2 + \cdots + p_9,$$

and therefore

$$3(p_a + p_b + p_c) = |E(N)|.$$

We conclude that exactly one-third of the elements of $E(N)$ lie on the line $\{a, b, c\}$. By symmetry, the same is true of the lines $\{1, 6, 8\}$, $\{3, 5, 7\}$, and $\{2, 4, 9\}$, so now four disjoint lines each account for one-third of the elements in N , a contradiction.

If $K = U_{2,3}$, then $\text{si}(N) = PG(2, 3) \setminus \{b, c, d\}$. The following equations are obtained by applying Proposition 22(ii) in the minors $N/\text{cl}(\{1\})$, $N/\text{cl}(\{2\})$, and $N/\text{cl}(\{3\})$, respectively:

$$p_2 + p_3 + p_a = p_4 + p_7 = p_6 + p_8 = p_5 + p_9 = \frac{1}{4}(|E(N)| - p_1); \quad (1)$$

$$p_1 + p_3 + p_a = p_4 + p_9 = p_5 + p_8 = p_6 + p_7 = \frac{1}{4}(|E(N)| - p_2); \quad (2)$$

$$p_1 + p_2 + p_a = p_5 + p_7 = p_6 + p_9 = p_4 + p_8 = \frac{1}{4}(|E(N)| - p_3).$$

Solving equations (1) and (2) for $|E(N)|$, we see that

$$p_1 + 4p_2 + 4p_3 = 4p_1 + p_2 + 4p_3,$$

so $p_1 = p_2$. Through additional substitutions, it follows that $p_i = p_j$ for each $i, j \neq a$. But now $p_a = 0$, a contradiction.

If $K = P(U_{2,3}, U_{2,4})$, then $\text{si}(N) = PG(2, 3) \setminus \{7, 8, 9, a, b, d\} \cong P_7$. From the minors $N/\text{cl}(\{1\})$ and $N/\text{cl}(\{3\})$ and Proposition 22(ii), we get the equations

$$p_2 + p_3 = p_4 + p_c = p_5 = p_6,$$

$$p_1 + p_2 = p_6 + p_c = p_4 = p_5.$$

It follows that $p_c = 0$, a contradiction.

If $K = \mathcal{W}^3$, then $\text{si}(N) = PG(2, 3) \setminus \{6, 8, 9, b, c, d\} \cong O_7$. From the minors $N/\text{cl}(\{7\})$ and $N/\text{cl}(\{5\})$ we get the equations

$$p_1 + p_4 = p_3 + p_5 = p_2 = p_a,$$

$$p_3 + p_7 = p_4 + p_a = p_1 = p_2.$$

It follows that $p_4 = 0$, a contradiction.

If $K = O_7$, then $\text{si}(N) = PG(2, 3) \setminus \{6, 8, 9, a, b, c, d\} \cong \mathcal{W}^3$. From the minors $N/\text{cl}(\{2\})$ and $N/\text{cl}(\{4\})$ we get the equations

$$p_1 + p_3 = p_4 = p_5 = p_7,$$

$$p_1 + p_7 = p_2 = p_3 = p_5,$$

so $p_1 = 0$, a contradiction.

For the next six cases, we continue to view N as a parallel extension of a restriction of $PG(2, 3)$, with $PG(2, 3)$ labelled as in Figure 14. However, we now represent the deletion of an element e from $PG(2, 3)$ by setting p_e to be 0. Each of these cases is eliminated using the following assertion.

27.3. Let N be a restriction of $PG(2, 3)$ such that

- (i) $p_c = p_d = 0$;
- (ii) $p_x \neq 0$ for each x in $\{a, b, 1\}$;
- (iii) $\text{si}(N/\text{cl}(\{x\})) \cong U_{2,4}$ for each x in $\{a, b, 1\}$; and
- (iv) p_2 and p_3 are not both zero.

Then N^* is not orderable.

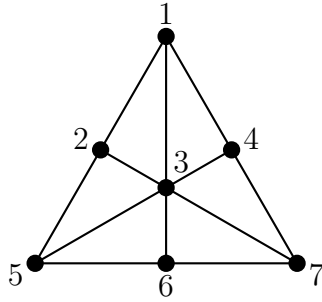


Figure 16: The matroid F_7^- .

To see this, we first use the minors $N/\text{cl}(\{a\})$ and $N/\text{cl}(\{b\})$ to establish the equations

$$p_b = p_1 + p_2 + p_3 = p_4 + p_5 + p_6 = p_7 + p_8 + p_9,$$

$$p_a = p_2 + p_6 + p_7 = p_1 + p_5 + p_9 = p_3 + p_4 + p_8,$$

from which we obtain

$$3p_a = p_1 + p_2 + \cdots + p_9 = 3p_b,$$

so $p_a = p_b$, and $|E(N)| = 5p_a$. Now, $N/\text{cl}(\{1\})$ gives that

$$|E(N)| - p_1 = 4(p_2 + p_3 + p_a),$$

and substituting $5p_a$ for $|E(N)|$ produces

$$p_a = p_1 + 4(p_2 + p_3).$$

Finally, since $p_a = p_1 + p_2 + p_3$, we deduce that $p_2 + p_3 = 0$, a contradiction. Thus 27.3 holds.

The six options for K eliminated by 27.3 are the matroids $U_{2,2}$, $U_{3,3}$, $U_{3,4}$, $U_{2,3} \oplus U_{1,1}$, $P(U_{2,3}, U_{2,3})$, and $U_{2,3} \oplus_2 U_{2,4}$. It is straightforward to check that, for each K in this list, we may set classes of $PG(2, 3)$ equal to zero in such a way that the zeroed classes form a restriction isomorphic to K , and the conditions of 27.3 hold. For example, $U_{2,3} \oplus_2 U_{2,4}$ is produced when p_5, p_7, p_9, p_c , and p_d are the zeroed classes.

In the final case, $K = M(K_4)$ and $\text{si}(N) = F_7^-$. Label F_7^- as in Figure 16, and, for each e in $[7]$, let $S_e = S_e(N^*)$. Suppose N^* has a consistent ordering, and let B be an S_1 -block in the ordering. In N^* , there is a circuit corresponding to each circuit $\{1, 2, 3, 5, 7\}$, $\{1, 3, 4, 5, 7\}$, and $\{1, 3, 5, 6, 7\}$ of $\text{co}(N^*)$; let \mathcal{X} be the collection of these circuits of N^* . Similarly, let \mathcal{Y} be the collection of circuits of N^* corresponding to the circuits $\{1, 2, 3, 4\}$, $\{1, 4, 5, 6\}$, and $\{1, 2, 6, 7\}$ of $\text{co}(N^*)$.

Suppose B is adjacent to an S_e -block for some e in $\{2, 4, 6\}$. Then the consistency of the circuits in \mathcal{X} implies that B is adjacent to an S_e -block for every e in $\{2, 4, 6\}$. The circuits in \mathcal{Y} now imply that B is adjacent to an S_2 -, S_4 -, and S_6 -block. Further, B is not adjacent to an S_e -block for any e in $\{3, 5, 7\}$. It follows that, in the circuit of

N^* corresponding to $\{1, 2, 3, 5, 7\}$, the block B must be adjacent to a pair of S_2 -blocks, contradicting the fact that B is adjacent to both an S_2 -block and an S_4 -block in the circuit of N^* corresponding to $\{1, 2, 3, 4\}$.

We now know that, for each $e \in \{2, 4, 6\}$, the block B is not adjacent to an S_e -block. The circuits in \mathcal{Y} now imply that, in the circuit of N^* corresponding to $\{1, 2, 3, 5, 7\}$, the block B is adjacent to an S_e -block for every $e \in \{3, 5, 7\}$. This contradiction implies N^* is not orderable. \square

The next proposition is a result of Oxley [4] (see also [5, Corollary 12.2.18]). We will use it to prove Theorem 23.

Proposition 28. *A 3-connected non-binary matroid whose rank and corank exceed two has a minor isomorphic to one of \mathcal{W}^3 , P_6 , Q_6 , and $U_{3,6}$.*

Proof of Theorem 23. Assume that the theorem fails for M . Then $r(M) \geq 3$. As P_6 , Q_6 , and $U_{3,6}$ each have $U_{3,5}$ as a minor, Proposition 28 and Propositions 24 and 25 now imply that $r^*(M) \leq 2$, so $r^*(M) = 2$. As M is 3-connected, it follows that $M \cong U_{n-2,n}$ for some $n \geq 5$. Hence M has a $U_{3,5}$ -minor, a contradiction. \square

If $\{M_1, M_2, \dots, M_n\}$ is a set of a matroids, then a *matroid-labelled tree* with vertex set $\{M_1, M_2, \dots, M_n\}$ is a tree T such that

- (i) if e is an edge of T with endpoints M_i and M_j , then $E(M_i) \cap E(M_j) = \{e\}$, and $\{e\}$ is not a separator of M_i or M_j ; and
- (ii) $E(M_i) \cap E(M_j)$ is empty if M_i and M_j are non-adjacent.

The matroids M_1, M_2, \dots, M_n are called the *vertex labels* of T . Now suppose e is an edge of T with endpoints M_1 and M_2 . We obtain a new matroid-labelled tree T/e by contracting e and relabelling the resulting vertex with $M_1 \oplus_2 M_2$. As 2-sum is associative, T/X is well defined for all subsets X of $E(T)$.

Let T be a matroid-labelled tree with $V(T) = \{M_1, M_2, \dots, M_n\}$ and $E(T) = \{e_1, e_2, \dots, e_{n-1}\}$. Then T is a *tree decomposition* of a connected matroid M if

- (i) $E(M) = (E(M_1) \cup E(M_2) \cup \dots \cup E(M_n)) - \{e_1, e_2, \dots, e_{n-1}\}$;
- (ii) $|E(M_i)| \geq 3$ for all i unless $|E(M)| < 3$, in which case $n = 1$ and $M = M_1$; and
- (iii) M labels the single vertex of $T/E(T)$.

In this case, the elements $\{e_1, e_2, \dots, e_{n-1}\}$ are the *edge labels* of T . The next theorem of Cunningham and Edmonds [3] (see also [5, Theorem 8.3.10]) tells us that M has a *canonical tree decomposition*, unique to within relabelling of the edges.

Theorem 29. *Let M be a 2-connected matroid. Then M has a tree decomposition T in which every vertex label is 3-connected, a circuit, or a cocircuit, and there are no two adjacent vertices that are both labelled by circuits or are both labelled by cocircuits. Moreover, T is unique to within relabelling of its edges.*

Let T be a tree decomposition of a matroid M , and let N and p be a vertex label and edge label of T , respectively. For the remainder of this section, we define $M_{p,N}$ and $M'_{p,N}$ to be the matroids such that $M = M_{p,N} \oplus_2 M'_{p,N}$ with basepoint p , where $E(M_{p,N})$ contains the subset of $E(M)$ corresponding to the component of $T \setminus p$ containing N . Notice that if the vertex labels M_1 and M_2 lie in different components of $T \setminus p$, then $M_{p,M_1} = M'_{p,M_2}$.

In the next four lemmas, M is assumed to be a connected, orderable, non-binary matroid whose canonical tree decomposition is T .

Lemma 30. *Suppose that T has a vertex label U that is isomorphic to $U_{2,n}$ for some $n \geq 4$. Then, for all $e, f \in E(U)$,*

- (i) *e is an edge label of T , unless M is a parallel extension of $U_{2,n}$;*
- (ii) *all circuits of $M'_{e,U}$ containing e have the same size; and*
- (iii) *the circuits of $M'_{e,U}$ containing e have the same size as the circuits of $M'_{f,U}$ containing f .*

Proof. We may assume that M is not a parallel extension of $U_{2,n}$ otherwise (i) holds. For each element y of $E(U)$ that labels an edge of T , let C_y be a circuit of $M'_{y,U}$ that contains y . As M is not a parallel extension of $U_{2,n}$, we may assume that $|C_x| \geq 3$ for some element x . Let M'' be the matroid that is obtained from U by attaching each C_y via 2-sum. This matroid is a restriction of M having $C_x - x$ as a non-trivial series class. Moreover, M'' is a series extension of $U_{2,n}$ and it is orderable. Thus, by Proposition 22(ii), M'' is a balanced series extension of $U_{2,n}$. Hence (i) holds. Furthermore, $|C_x| = |C_y| \geq 3$ for all y in $E(U) - \{x\}$. Parts (ii) and (iii) now follow without difficulty. \square

The next lemma generalizes Lemma 30(ii) to arbitrary edges of T .

Lemma 31. *Suppose that T has a vertex label U that is isomorphic to $U_{2,n}$ for some $n \geq 4$, and suppose e is an edge label of T . Then the circuits of $M'_{e,U}$ that contain e all have the same size.*

Proof. Let N be the endpoint of e in the same component of $T \setminus e$ as U . If $U = N$, then the assertion holds by Lemma 30(ii), so assume otherwise. Let f be the label of the edge incident with U that lies on the path connecting U to N in T . Next, let T' be the subtree of $T \setminus \{e, f\}$ containing N , and let M' be the matroid with tree decomposition T' .

Fix a circuit C of M' that contains e and f . Observe that, for each circuit D of $M'_{e,N}$ that contains e , there is a circuit $(D - e) \cup (C - e)$ of $M'_{f,U}$ that contains f . By Lemma 30(ii), the quantity $|(D - e) \cup (C - e)|$ is the same for each choice of D , so every such circuit D has the same size. \square

Lemma 32. *The tree T has exactly one 3-connected non-binary vertex label, and this label is isomorphic to $U_{2,n}$ for some $n \geq 4$.*

Proof. As M is non-binary, it has at least one 3-connected non-binary vertex label N . For each element y of $E(N)$ that labels an edge of T , let C_y be a circuit of $M'_{y,N}$ that contains y . Let M'' be the matroid that is obtained from N by attaching each C_y via 2-sum. Then M'' is a restriction of M . Thus M'' is an orderable series extension of N . By Propositions 24, 25, and 28, $N \cong U_{2,n}$ for some $n \geq 4$. Now suppose T has a pair of 3-connected non-binary vertex labels $N_1 \cong U_{2,n_1}$ and $N_2 \cong U_{2,n_2}$ with $n_1, n_2 \geq 4$. Let e_1 and e_2 be the edge labels of T incident with N_1 and N_2 that lie on the path connecting N_1 and N_2 in T .

By Lemma 30(ii), the circuits of M'_{e_1,N_1} containing e_1 all have size k and the circuits of M'_{e_2,N_2} containing e_2 all have size ℓ , where k and ℓ are integers exceeding one. Let $\{e_1, x, y\}$ be a circuit of N_1 . By Lemma 30(i), x and y are also edge labels of T ; let C_x be a circuit of M'_{x,N_1} containing x , and C_y be a circuit of M'_{y,N_1} containing y . Then $k = |C_x| = |C_y|$ by Lemma 30(iii). Now there is a circuit of M'_{e_2,N_2} containing e_2 that also contains $C_x - x$ and $C_y - y$. Thus, $\ell \geq 2(k - 1) + 1$. A symmetric argument gives that $k \geq 2(\ell - 1) + 1$, and substitution yields that $k \leq 1$, a contradiction. \square

The next lemma rules out 3-connected binary vertex labels that are not circuits or cocircuits. It uses the following result of Seymour [6].

Proposition 33. *Let M be a 3-connected binary matroid with at least four elements. If $e \in E(M)$, then M has an $M(K_4)$ -minor using e .*

Lemma 34. *No vertex of T is labelled by a 3-connected binary matroid with at least four elements.*

Proof. Suppose B is such a vertex label of T , let U be the unique vertex label with $U \cong U_{2,n}$ and $n \geq 4$ given by Lemma 32, and say $E(U) = \{e_1, e_2, \dots, e_n\}$. Let $p \in E(B)$ and $e_1 \in E(U)$ be the labels of the edges incident with B and U , respectively, that lie on the path connecting B to U in T . By Proposition 33, B has a minor isomorphic to $M(K_4)$ that uses p .

This minor can be written in the form $B/I \setminus I^*$, where I is independent in B and I^* is coindependent in B . This makes B/I a rank-three binary matroid with $M(K_4)$ as a restriction, so after deleting the loops from B/I , we obtain a parallel extension of either $M(K_4)$ or F_7 . Dually, after deleting the coloops from $B \setminus I^*$, we obtain a series extension of $M(K_4)$ or F_7^* . Thus B has a restriction N_1 using p that is a series extension of $M(K_4)$ or F_7^* .

Suppose q is an edge label of T that is used in N_1 , and choose a circuit C_q of $M'_{q,B}$ that contains q . Form the matroid N_2 from N_1 by replacing q with $C_q - q$ in $E(N_1)$ for each q in $E(N_1) - p$ that is an edge label of T . Then N_2 is a series extension of $M(K_4)$ or F_7^* that appears as a restriction of $M_{p,B}$. Now, for each i in $\{2, 3\}$, let C_{e_i} be a circuit of $M'_{e_i,U}$ that contains e_i . Then $M'_{p,B}$ has a circuit C_p that contains p and both $C_{e_2} - e_2$ and $C_{e_3} - e_3$. Form the matroid N from N_2 by taking the 2-sum of N_2 and C_p across the basepoint p . Then N is a restriction of M that is a series extension of $M(K_4)$ or F_7^* . For each element x of $M(K_4)$ or F_7^* , let S_x be $S_x(N)$. By Lemma 31, every circuit of N_2 that contains p has the same size.

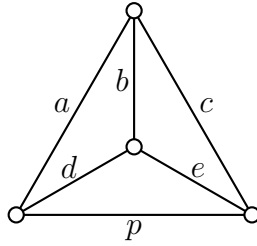


Figure 17: K_4 in the proof of Lemma 34.

Suppose first that N is a series extension of $M(K_4)$ with K_4 labelled as in Figure 17. Thus, every circuit of N that contains S_p has the same size. Since all circuits of N containing S_p have the same size, $|S_d| + |S_e| = |S_a| + |S_b| + |S_e|$, so

$$|S_d| = |S_a| + |S_b|. \tag{3}$$

Similarly, $|S_a| + |S_c| = |S_d| + |S_b| + |S_c|$, so

$$|S_a| = |S_d| + |S_b|. \tag{4}$$

Equations (3) and (4) imply that $|S_b| = 0$, a contradiction.

Now suppose that N is a series extension of F_7^* with F_7^* labelled as in Figure 18. Since the circuits of N containing S_p must have the same size,

$$|S_2| + |S_5| = |S_4| + |S_7|,$$

$$|S_2| + |S_6| = |S_3| + |S_7|,$$

and

$$|S_5| + |S_6| = |S_3| + |S_4|.$$

Together, these equations imply that

$$|S_2| = |S_7|. \tag{5}$$

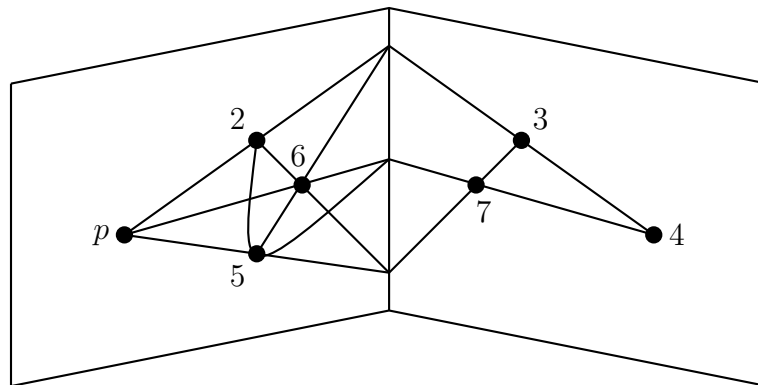


Figure 18: F_7^* in the proof of Lemma 34.

Fix a consistent ordering of M . This induces a consistent ordering of N . Consider the circuit $C = S_p \cup S_2 \cup S_3 \cup S_4$ of N . Notice that M has, as a restriction, a series extension U' of $U_{2,n}$ whose ground set contains C . Specifically, $C = S_{e_1} \cup S_{e_2} \cup S_{e_3}$, where S_{e_i} is $S_{e_i}(U')$.

Let t be an arbitrary member of the series class S_2 of N . In U' , the element t belongs to the class S_{e_1} , so $\{t\}$ is an S_{e_1} -block in the ordering of C by Proposition 22(i). Lemma 18(i) implies that t is adjacent to some element $x \in S_{e_2}$ and some $y \in S_{e_3}$; notice that, in N , the elements x and y both belong to S_p . Thus, every element of S_2 is adjacent to a pair of elements from S_p in C . In particular, t is not adjacent to any element of S_2 or of S_3 . Now observe that t is adjacent to this same pair $\{x, y\}$ in the circuit $S_p \cup S_2 \cup S_5 \cup S_6$ of N , so t is also not adjacent to any element of S_6 . It follows that t is adjacent to a pair of elements from S_7 in the circuit $S_2 \cup S_3 \cup S_6 \cup S_7$ of N . Therefore $|S_2| < |S_7|$, contradicting (5). \square

Proposition 35. *Let M'' be obtained from M by parallel-path addition. Then M is orderable if and only if M'' is orderable.*

Proof. In forming M'' from M , let P' be added in parallel to P . As M'' has M as a restriction, M is orderable if M'' is. Conversely, fix a consistent ordering of M and let C'' be a circuit of M'' . If C'' does not meet P' , give C'' the same ordering in M'' that it has in M . Otherwise, C'' contains P' and either $C'' = P \cup P'$, or there is a circuit C of M such that $C = (C'' - P') \cup P$. In the latter case, give C'' the same ordering in M'' that C has in M by replacing every element $p \in P$ by the corresponding element $p' \in P'$.

If $C'' = P \cup P'$, take a circuit D of M containing P . Let B_1, B_2, \dots, B_k be the P -blocks of D , numbered sequentially as they appear in a traversal of the ordering of D in M . For each i in $[k]$, let $B'_i = \{p' : p \in B_i\}$. Now, order C'' as $B_1, B'_1, B_2, B'_2, \dots, B_k, B'_k$. It is straightforward to check that this gives a consistent ordering of M'' . \square

We are now ready to prove the main result of the paper, which was given as Theorem 2 in the introduction and is restated here for convenience.

Theorem 36. *Let M be a connected non-binary matroid. Then M is orderable if and only if it can be obtained from $U_{2,n}$ for some $n \geq 4$ by a sequence of the following operations:*

- (i) *balanced series extension; and*
- (ii) *parallel-path addition.*

Proof. By Lemmas 21 and 35, a matroid obtained from $U_{2,n}$ by the given operations is certainly orderable, so it remains to show the converse.

We may assume that M is simple, as adding an element in parallel is a parallel-path addition of size one. If $M \cong U_{2,n}$, the result holds, so assume otherwise. Let T be the canonical tree decomposition of M . Lemmas 32 and 34 imply that there is a single vertex label U of T for which $U \cong U_{2,n}$ and $n \geq 4$, and every vertex of $T - U$ is labelled by a circuit or a cocircuit. By Lemma 30(i), each e in $E(U)$ labels an edge of T . Let T'_e be the component of $T \setminus e$ that does not have U as a vertex. As M is simple, the leaves of T are labelled by circuits. Therefore, if every T'_e has only one vertex, then M is a series

extension of $U_{2,n}$, and the result holds by Proposition 22(ii). We show that, if this is not the case, then each T'_e can be reduced to a single vertex labelled by a circuit via a sequence of deletions that can be undone by parallel-path additions.

Suppose T'_e has at least two vertices. Since only one vertex of T'_e is adjacent to U , not all vertices of T'_e are leaves of T . We now observe that

36.1. T'_e has a vertex v that

- (i) is adjacent to a leaf of T ; and
- (ii) has exactly one neighbor that is not a leaf of T

If L is the set of leaves of T , such a vertex v can be found as a leaf of $T - L$. Since the leaves of T are labelled by circuits and T is canonical, v is labelled by a cocircuit C^* . Lemma 31 now implies that the circuits that label the leaves of T adjacent to C^* all have the same size, and every element of C^* must be used as a basepoint labelling an edge of T .

We can delete all but one of the leaves, C say, of T that are adjacent to C^* , along with the corresponding basepoints in C^* , since the circuit that labels each deleted leaf can be added via a parallel-path addition. As C^* is now a pair of parallel elements, we can delete the leaf labelled C and relabel v with C . At this point, v is a leaf, and is either adjacent to U , in which case the work on this subtree is complete, or v is adjacent to another vertex of T'_e labelled by a circuit C' . In the latter case, keep T canonical by contracting the edge of T between v and C' and labelling the resulting vertex with the circuit that is the 2-sum of C and C' .

Provided the modification of T'_e continues to have at least two vertices, condition 36.1 continues to hold, and the process described in the previous paragraph can be repeated. Thus, we may assume T'_e consists of a single vertex labelled by a circuit. By applying this pruning process on the other subtrees attached to U , the tree T is reduced to the decomposition tree of a balanced series extension of $U_{2,n}$. Thus, M can be obtained from a balanced series extension of $U_{2,n}$ by a sequence of parallel-path additions. \square

4 Theta-Orderability

Recall that theta-orderability of a matroid requires a consistent ordering of the matroid with respect to the theta-graphs of that matroid. Each of the elementary properties of orderability given in Proposition 8 also holds for theta-orderability. Their straightforward proofs are omitted.

Proposition 37. *Let M be a matroid.*

- (i) *If M is theta-orderable, then $M \setminus e$ is theta-orderable for all e in $E(M)$.*
- (ii) *If $r(M) \leq 2$, then M is theta-orderable.*
- (iii) *M is theta-orderable if and only if the connected components of M are theta-orderable.*

(iv) M is theta-orderable if and only if $\text{si}(M)$ is theta-orderable.

Next we prove Theorem 6, a characterization of graphic theta-orderable matroids.

Proof of Theorem 6. It is clear that a graphic matroid is theta-orderable. Moreover, Wagner [9] proved that a matroid is graphic if and only if it has no set of incompatible arcs. Now suppose that M has a circuit C and a set $\{A_1, A_2, A_3\}$ of incompatible arcs of C . It remains to show that M is not theta-orderable. Our proof of this is a straightforward modification of Wagner's proof that no graphic matroid has a set of incompatible arcs [9, Lemma 2]. Assume that M is theta-orderable. Because each of A_1, A_2 , and A_3 is an arc, for each i in $\{1, 2, 3\}$, there is a theta-graph of M in which A_i is a theta-arc. As M is theta-orderable, A_i is a block in a consistent ordering of M . As $\{A_1, A_2, A_3\}$ is an incompatible set, there are distinct elements e_1, e_2 , and e_3 of C such that $e \in A_1 \cap A_2 \cap A_3$ and $e_i \in A_i - (A_j \cup A_k)$ for all $\{i, j, k\} = \{1, 2, 3\}$. For each h in $\{2, 3\}$, the set $A_1 \cup A_h$ is a block in C in which e appears between e_1 and e_h . Then e does not appear between e_2 and e_3 in $A_2 \cup A_3$, a contradiction. \square

To prove Theorem 7, we will establish the following equivalent version of it.

Theorem 38. *A simple connected non-binary matroid is theta-orderable if and only if it is a balanced series extension of $U_{2,n}$ for some $n \geq 4$.*

The proof of this theorem will use the next lemma and a corollary of it.

Lemma 39. *Let M be a connected non-binary orderable matroid, and let S be a sequence of balanced series extensions and parallel-path additions by which M is obtained from $U_{2,n}$ for some $n \geq 4$. Suppose that the operation s_1 immediately precedes the operation s_2 in S . Then*

- (i) *if s_1 and s_2 are balanced series extensions of orders m_1 and m_2 , then s_1 and s_2 may be replaced by a single balanced series extension of order $m_1 m_2$; and*
- (ii) *if s_1 is a parallel-path addition of size k , and s_2 is a balanced series extension of order m , then, in S , the order of the operations s_1 and s_2 can be reversed provided s_1 is replaced by a corresponding parallel-path addition of size km .*

Proof. Part (i) is immediate. For part (ii), let P_1 be the k -element set that is added in parallel to the subset P_2 of a series class at step s_1 . After the balanced series extension in step s_2 is performed, P_1 and P_2 become parallel paths P'_1 and P'_2 of size mk . Thus, the same result is obtained by first performing a balanced series extension of order m , then adding the mk -element set P'_1 in parallel to the subset P'_2 of a series class. \square

The following is an immediate consequence of the last lemma.

Corollary 40. *Let M be a connected non-binary orderable matroid. Then M is obtained from a balanced series extension of $U_{2,n}$ for some $n \geq 4$ by a sequence of parallel-path additions.*

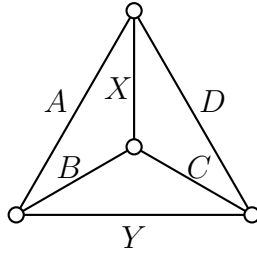


Figure 19: K_4 in the proof of Proposition 43.

Proof of Theorem 38. First, for $n \geq 4$, the matroid $U_{2,n}$ and its series extensions have no theta-graphs. Therefore, consistent orderings of these matroids are also theta-orderings.

Conversely, suppose M is a simple connected non-binary orderable matroid. By Corollary 40, for some $n \geq 4$, we can obtain M from $U_{2,n}$ by a balanced series extension B followed by a sequence of parallel-path additions. It now suffices to show that the sequence of parallel-path additions is empty.

Suppose to the contrary that P' is a set added in parallel to a subset P of a series class S of B . Note $|P| \geq 2$ since M is simple. Now, by Proposition 22, each S -block in a consistent ordering of B contains a single element. As B is a restriction of M , this implies that the elements of P are not a block in a consistent ordering of M . Since M has a theta-graph with P and P' as theta-arcs, this is a contradiction. \square

5 Characterizing 3-Connected Orderable Binary Matroids

This section proves the following partial result towards Conjecture 4. Theorem 5 is an immediate consequence of this result.

Theorem 41. *A 4-connected binary orderable matroid with no series minor isomorphic to F_7^* is graphic.*

Our proof will require the next three results, the first of which is due to Seymour [7]. Two elements are *opposite* in $M(K_4)$ if they form a matching in the K_4 .

Theorem 42. *Let M be a 4-connected binary matroid and let e and f be elements of M . Suppose there is no $M(K_4)$ -minor of M in which e and f are opposite elements. Then there is a graph G with $M = M(G)$ or $M^*(G)$, and e and f are adjacent edges in G .*

Proposition 43. *In a consistent ordering of a series extension M of $M(K_4)$, if two elements correspond to opposite elements in the $M(K_4)$, then they are not adjacent.*

Proof. Let A, B, C, D, X , and Y be the series classes of M , labelled as in Figure 19. Take elements x in X and y in Y , and suppose x and y are adjacent in the given consistent ordering of M .

In the circuit $A \cup X \cup C \cup Y$, we have that y is adjacent to at most one member of C . Therefore, in $B \cup C \cup Y$, there must be an element, b_y , of B that is adjacent to y . Similarly,

in $A \cup X \cup B$, there must be an element, b_x , of B adjacent to x . Now, in $B \cup X \cup D \cup Y$, we have the block $b_x x y b_y$, so no member of D is adjacent to y . By symmetry, no member of A is adjacent to y . Since y is adjacent to at most one element in Y , it follows that there is no second element of $A \cup D \cup Y$ adjacent to y , a contradiction. \square

Lemma 44. *Suppose M is a binary matroid with no series minor isomorphic to F_7^* . If e and f are opposite elements in an $M(K_4)$ -minor of M , then e and f are not adjacent in any consistent ordering of M .*

Proof. Assume that M has a consistent ordering in which e and f are adjacent. Let N be an $M(K_4)$ -minor of M in which e and f are opposite elements, and write $N = M \setminus X / Y$ with X coindependent and Y independent. Then $N^* = M^* / X \setminus Y$, where $r(M^* / X) = r(N^*) = 3$. Since M^* / X is binary and $N^* \cong M(K_4)$, if L is the set of loops of M^* / X , then $M^* / X \setminus L$ is a parallel extension of either $M(K_4)$ or F_7 . It follows that $M \setminus (X \cup L)$ is a series extension of $M(K_4)$ or F_7^* . By assumption, M has no series minor isomorphic to F_7^* , so $M \setminus (X \cup L)$ is a series extension of $M(K_4)$. However, e and f are adjacent in the consistent ordering of $M \setminus (X \cup L)$ inherited from M and correspond to opposite elements in N , a contradiction by Proposition 43. \square

We now prove the main result of this section.

Proof of Theorem 41. Let M be a 4-connected binary orderable matroid that does not have F_7^* as a series minor. Take a consistent ordering of M and assume M is not graphic. Suppose M is cographic, letting $M = M^*(G)$ for some graph G . Take an edge e of G with endpoints u and v . Let $(x_1 x_2 \cdots x_n e)$ be the ordering on the edges meeting u , and let $(e y_1 y_2 \cdots y_m)$ be the ordering on the edges meeting v . Then we may assume the ordering on the bond that is the symmetric difference of these two vertex bonds is $(x_1 x_2 \cdots x_n y_1 y_2 \cdots y_m)$, so x_n and y_1 are adjacent. Combining Lemma 44 and Theorem 42, we now have that x_n and y_1 share an endpoint in G . Hence, $\{e, x_n, y_1\}$ is a triangle in G , a contradiction as M is 4-connected.

We may now assume that M is not cographic. Let e and f be adjacent elements of M . By Theorem 42, e and f appear as opposite elements in some $M(K_4)$ -minor of M . Lemma 44 now gives a contradiction. \square

Acknowledgements

The authors thank Jim Geelen for suggesting that [7] may be helpful in resolving Conjecture 4.

References

- [1] Bixby, R. E., A strengthened form of Tutte's characterization of regular matroids, *J. Combin. Theory Ser. B* **20** (1976), 216–221.
- [2] Bixby, R. E., Kuratowski's and Wagner's theorems for matroids, *J. Combin. Theory Ser. B* **22** (1977), 31–53.

- [3] Cunningham, W. H., A combinatorial decomposition theory, Ph.D. thesis, University of Waterloo, 1973.
- [4] Oxley, J. G., On nonbinary 3-connected matroids, *Trans. Amer. Math. Soc.* **300** (1987), 663–679.
- [5] Oxley, J. G., *Matroid Theory*, 2nd ed., Oxford University Press, New York, 2011.
- [6] Seymour, P. D., Minors of 3-connected matroids, *European J. Combin.* **6** (1985), 375–382.
- [7] Seymour, P. D., Adjacency in binary matroids, *European J. Combin.* **7** (1986), 171–176.
- [8] Tutte, W. T., Matroids and graphs, *Trans. Amer. Math. Soc.* **90** (1959), 527–552.
- [9] Wagner, D. K., A circuit characterization of graphic matroids, *J. Combin. Theory Ser. B* **118** (2016), 284–290.
- [10] Whitney, H., 2-isomorphic graphs, *Amer. J. Math.* **55** (1933), 245–254.