On Infinite, Cubic, Vertex-Transitive Graphs with Applications to Totally Disconnected, Locally Compact Groups

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Submitted: Sep 8, 2021; Accepted: Oct 14, 2022; Published: Dec 2, 2022

Abstract

We study groups acting vertex-transitively and non-discretely on connected, cubic graphs (regular graphs of degree 3). Using ideas from Tutte's fundamental papers in 1947 and 1959, it is shown that if the action is edge-transitive, then the graph has to be a tree. When the action is not edge-transitive Tutte's ideas are still useful and can, amongst other things, be used to fully classify the possible two-ended graphs. Results about cubic graphs are then applied to Willis' scale function from the theory of totally disconnected, locally compact groups. Some of the results in this paper have most likely been known to experts but most of them are not stated explicitly with proofs in the literature.

Mathematics Subject Classifications: 05E18, 05C63, 20B27, 22D05, 20E08

1 Introduction

Tutte's papers on cubic graphs in 1947, [27], and 1959, [28], are rightly regarded as the starting point of the study of group actions on graphs as a separate discipline. In these

*Partly supported by the Early Postdoc.Mobility scholarship No. 175106 from the Swiss National Science Foundation and F.R.S.-FNRS postdoctoral researcher.

papers, he investigates arc-transitive group actions on finite, connected, cubic (3-regular) graphs. In many of his results the assumption that the graph is finite can be dropped and replaced with the assumption that the action is discrete, see e.g. [10]. Here, discrete means that vertex stabilisers are finite after dividing out the kernel of the action. For instance, it follows from Tutte's work that if Γ is a connected cubic graph and G acts arc-transitively and discretely on Γ, then G acts regularly on the set of s-arcs for some $s \leqslant 5$.

The present work is a study of the "other" case, i.e. of vertex-transitive actions on infinite, connected, cubic graphs that are non-discrete. Suppose we have such an action. What can be said about the graph? What can be said about the group?

If we, like Tutte did, restrict ourselves to arc-transitive actions, it turns out that the graph is completely determined: It is the cubic tree (the 3-regular tree), and assuming additionally that the group is closed in the permutation topology, the action is even 2-transitive on the set of ends (Theorem 8). Even in the case where the group acts vertex- and edge-transitively, but not arc-transitively, the graph has to be isomorphic to the cubic tree, but then the group fixes an end of the tree. If, in addition, it is assumed that the group is closed in the permutation topology, then it acts transitively on the remaining ends (Theorem 12).

The idea at the heart of Tutte's papers is to look at the action on s-arcs. This is also fundamental in our investigation of the case with infinite stabilisers. Many of our arguments are based directly on arguments from Tutte's two papers cited above and his later book [29]. If the action is not edge-transitive, then a simple argument shows that there can be only two orbits on the arcs (Section 3). We prove that in this case, the group acts transitively on the set of s-arcs starting with an arc from a given orbit and then visiting the two orbits alternatingly (Theorem 21). There are many more graphs than just the cubic tree that fall under this case, and their classification seems almost hopeless. However, in Theorem 26 we do classify all such graphs that have precisely two ends. This classification is already implicit in Trofimov's paper [25] (in Russian).

We give applications of our results: one in the theory of totally disconnected, locally compact groups (a brief introduction to the topic of totally disconnected, locally compact groups can be found at the start of Section 7) and one in graph theory, reproving a theorem of Trofimov, [25, Theorem 3.1].

It was shown by Abels in [1] that every compactly generated, totally disconnected, locally compact group acts vertex-transitively on a connected graph of finite degree such that the stabilisers of vertices are compact, open subgroups. Such a graph is commonly called a *Cayley–Abels graph* since it is clearly a generalisation of the Cayley graph of a finitely generated group. It is natural to ask about the relationship between the degree of a Cayley–Abels graph, in particular the minimal degree, and various properties of the group. In particular one can ask if there is something special about groups that have Cayley–Abels graphs with low degree. If the minimal degree of a Cayley–Abels graph is 2 then the structure of the group can be described, see [2, Theorem 4.1]. The results in Section 7 describe special properties of groups that have a Cayley–Abels graph of degree 3. We show that if a compactly generated, totally disconnected, locally compact group with a cubic Cayley–Abels graph has the property that every group element normalises a compact, open subgroup, then the group has a compact, open, normal subgroup (Corollary 35). This property was previously known for p-adic Lie groups, see [11], but is not true in general as shown in [3].

For a graph Γ we let Γ_2 denote the graph obtained by adding to Γ as edges all pairs $\{\alpha, \beta\}$ of vertices such that $d_\Gamma(\alpha, \beta) = 2$. The graph theoretic application is a short proof of a result by Trofimov [25, Theorem 3.1] saying that every connected, cubic graph Γ with vertex-transitive automorphism group and infinite vertex stabilisers is two-ended or the graph Γ_2 has a subgraph isomorphic to the cubic tree (Theorem 38).

Apart from Trofimov's paper [25], the only other paper directly treating vertex-transitive, cubic graphs such that vertex-stabilisers are infinite is the paper [22] by Nebbia. There, Nebbia only considers actions on the cubic tree but with that assumption he derives results similar to some of the results in Sections 3 and 4.

Many of the results in Sections 3, 4, 5 and Appendix A will doubtlessly be known to experts in the area of group actions on graphs, but these don't seem to be explicitly stated with proofs in the literature. Originally the results in Section 7 were intended for the paper [2] on Cayley–Abels graphs, but the authors found that a reasonably self-contained and detailed presentation would be too long to be included there. This gave rise to the present article. The authors hope that their work will attract readers interested in group actions on graphs as well as readers interested in the study of totally disconnected, locally compact groups. In order to accommodate both of these groups, this paper is written in an expository style. Therefore, arguments that might otherwise have been left out are presented in full details. Every effort has also been made to make the paper as self-contained as possible: Apart from basic theory of graphs and permutation groups, the only results used in an essential way in the proofs are the classification of two-ended, highly arc-transitive digraphs with prime in- and out-degrees from [21] and a formula from [19] for the scale function on a totally disconnected, locally compact group.

2 Notation and preliminary remarks

2.1 Graphs

The graphs we consider have neither loops nor multiple edges. Thus a graph Γ (undirected) is defined as a pair $(\mathrm{V}\Gamma, \mathrm{E}\Gamma)$, where $\mathrm{V}\Gamma$ is the set of *vertices* and $\mathrm{E}\Gamma$, the set of *edges*, is a set of two element subsets of $\mathrm{V}\Gamma$. A graph Γ' is a *subgraph* of Γ if $\mathrm{V}\Gamma' \subseteq \mathrm{V}\Gamma$ and $\mathrm{E}\Gamma' \subseteq \mathrm{E}\Gamma$. An *arc* in a graph Γ is an ordered pair (α, β) such that $\{\alpha, \beta\} \in \mathrm{E}\Gamma$. The set of all arcs in Γ is denoted by $\mathrm{A}\Gamma$. Two vertices α and β are said to be *adjacent*, or *neighbours*, if $\{\alpha, \beta\}$ is an edge. The set of neighbours of a vertex $\alpha \in \mathrm{V}\Gamma$ is denoted by $\Gamma(\alpha)$. The *degree* of a vertex in a graph is the cardinality of its set of neighbours. A graph is said to be *regular* if all vertices have the same degree d, and we say that d is the degree of the graph. If the degree of every vertex is finite, then the graph is said to be *locally finite*.

We also consider digraphs (directed graphs). A *digraph* consists of a vertex set $\mathrm{V}\Gamma$ and a subset $A\Gamma \subseteq V\Gamma \times V\Gamma$ that does not intersect the diagonal. Elements of $V\Gamma$ are called *vertices* and elements of $A\Gamma$ are called *arcs*. The *underlying undirected graph* of a digraph Γ has the same vertex set as Γ and the set of edges is the set of all pairs $\{\alpha, \beta\}$, where (α, β) or (β, α) is an arc in Γ. For a vertex α in a digraph Γ we define the sets of *in-* and *out-neighbours* as $\mathrm{in}(\alpha) = \{\beta \in V\Gamma \mid (\beta, \alpha) \in A\Gamma\}$ and $\mathrm{out}(\alpha) = \{\beta \in V\Gamma \mid (\alpha, \beta) \in A\Gamma\}$, respectively. The cardinality of $\mathrm{in}(\alpha)$ is the *in-degree* of α and the cardinality of $\mathrm{out}(\alpha)$ is the *out-degree* of α. A digraph is *regular* if any two vertices have the same in-degree and also the same out-degree.

For an integer $s \geqslant 0$ an *s-arc* in Γ (here Γ can be an undirected graph or a digraph) is an $(s+1)$-tuple $(\alpha_0, \ldots, \alpha_s)$ of vertices such that for every $0 \leqslant i \leqslant s-1$ the ordered pair (α_i, α_{i+1}) is an arc in Γ, and $\alpha_{i-1} \neq \alpha_{i+1}$ for all $1 \leqslant i \leqslant s-1$. Infinite arcs come in three different shapes: There are 1-way ∞-arcs, $(\ldots, \alpha_{-1}, \alpha_0)$ and $(\alpha_0, \alpha_1, \ldots)$, and then there are 2-way ∞-arcs $(\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots)$. In all cases we insist that (α_i, α_{i+1}) is an arc in Γ, and $\alpha_{i-1} \neq \alpha_{i+1}$ for all i.

A *path* of length $s \geqslant 0$ in a graph Γ is a subgraph with vertex set $\{\alpha_0, \ldots, \alpha_s\}$ and edge set $\{\{\alpha_0, \alpha_1\}, \ldots, \{\alpha_{s-1}, \alpha_s\}\}$ such that the vertices $\alpha_0, \ldots, \alpha_s$ are distinct. The vertices α_0 and α_s are called the *end-vertices* of the path and we speak of an $\alpha_0 - \alpha_s$ path. Paths can also be infinite. A *ray* in a graph Γ is a subgraph with vertex set $\{\alpha_0, \alpha_1, \ldots\}$ and edge set $\{\{\alpha_0, \alpha_1\}, \{\alpha_1, \alpha_2\}, \ldots\}$ such that all the vertices $\alpha_0, \alpha_1, \ldots$ are distinct. A *line* is a subgraph with vertex set $\{\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots\}$ and edge set $\{\ldots, \{\alpha_{-1}, \alpha_0\}, \{\alpha_0, \alpha_1\}, \{\alpha_1, \alpha_2\}, \ldots\}$ such that all the vertices $\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots$ are distinct. We will refer to paths, rays and lines by listing the vertices in the natural order. A path P with vertex set $\{\alpha_0, \ldots, \alpha_s\}$ and edge set $\{\{\alpha_0, \alpha_1\}, \ldots, \{\alpha_{s-1}, \alpha_s\}\}$ will thus be denoted by $P = \alpha_0, \ldots, \alpha_s$ and similarly for rays and lines. A sequence $\alpha_0, \ldots, \alpha_s$ is a *path* in a digraph Γ if and only if it is a path in the underlying undirected graph. Rays and lines in digraphs are defined analogously.

A *cycle* in a graph is a subgraph with vertex set $\{\alpha_0, \ldots, \alpha_{s-1}\}$ such that $\alpha_0, \ldots, \alpha_{s-1}$ is a path and in addition it contains the edge $\{\alpha_{s-1}, \alpha_0\}$. A cycle is often denoted with a sequence $\alpha_0, \ldots, \alpha_{s-1}, \alpha_s$ such that $\alpha_s = \alpha_0$. A cycle in a digraph is a subdigraph such that the corresponding subgraph in the underlying undirected graph is a cycle.

A graph Γ is said to be *connected* if for every two distinct vertices α and β in Γ there exists an $\alpha - \beta$ path in Γ. The *graph theoretical distance* between vertices α and β in a connected graph is defined as the length of a shortest $\alpha - \beta$ path and is denoted with $d_\Gamma(\alpha, \beta)$. A digraph is connected if its underlying undirected graph is connected and the distance between two vertices in a connected digraph is the same as the distance between the corresponding vertices in the underlying undirected graph.

An *end* of a graph Γ is an equivalence class of rays: Two rays R_1 and R_2 are said to be equivalent if there is a third ray R_3 that intersects both R_1 and R_2 in infinitely many vertices. (For further information about ends of graphs, in particular in relation to the automorphism group, the reader is referred to [17].) In the special case when the graph Γ is a tree, two rays belong to the same end if and only if their intersection is a ray. The set of ends of Γ is denoted with $\Omega\Gamma$. When Γ is a digraph we define the ends of Γ in terms of the ends of the underlying undirected graph.

An alternative way to describe the ends of a graph Γ is to look at connected components of graphs of the type $\Gamma\setminus\Phi$, where Φ is a finite set of vertices (the graph you get by removing all the vertices in Φ and all edges incident with them). If $R = \alpha_0, \alpha_1, \ldots$ is a ray in Γ, then we say that a set B of vertices in Γ *contains a subray* of R if there is a number N such that $\alpha_N, \alpha_{N+1}, \ldots \in B$. Now it is easy to see that two rays R_1 and R_2 belong to the same end if and only if for every finite set Φ of vertices there is a connected component C of $\Gamma \setminus \Phi$ that contains a subray of both R_1 and R_2. If the rays R_1 and R_2 belong to different ends of Γ, then there is a finite set of vertices Φ such that there are distinct components C_1 and C_2 of $\Gamma \setminus \Phi$ containing subrays of R_1 and R_2, respectively. In that case we say that Φ *separates* the ends that R_1 and R_2 belong to. When the graph Γ is connected and locally finite then Γ has just one end if and only if $\Gamma \setminus \Phi$ has just one infinite component for every finite subset $\Phi \subseteq \mathrm{V}\Gamma$. A connected, locally finite graph has precisely two ends if and only if whenever Φ is a finite subset of $\mathrm{V}\Gamma$ then the number of infinite components of $\Gamma \setminus \Phi$ is at most 2 and there is a finite subset $\Phi_0 \subseteq \mathrm{V}\Gamma$ such that $\Gamma \setminus \Phi_0$ has two infinite components.

2.2 Groups

Let G be a group acting (from the right) on a set Ω. Denote the image of a point $\alpha \in \Omega$ under an element $g \in G$ by αg. The *orbit* of an element $\alpha \in \Omega$ is the set $\alpha G = \{\alpha g \mid g \in G\}$. The action is said to be *transitive* if $\alpha G = \Omega$ for one, and hence every, $\alpha \in \Omega$ or, in other words, for any two points α, β in Ω there exists an element $g \in G$ such that $\alpha g = \beta$. The *stabiliser* of $\alpha \in \Omega$ is the subgroup $G_\alpha = \{g \in G \mid \alpha g = \alpha\}$. For a set $A \subseteq \Omega$ the *pointwise stabiliser* of A is the subgroup $G_{(A)} = \{g \in G \mid \alpha g = \alpha \text{ for all } \alpha \in A\}$. The kernel of the action is the subgroup $K = G_{(\Omega)}$. When $K = \{1\}$ we say that the action is *faithful* and then we can think of G as a permutation group of Ω, i.e. a subgroup of $\mathrm{Sym}(\Omega)$ (the group of all permutations of the set Ω).

An action of a group G on a set Ω is called *semi-regular* (or *free*) if $G_\alpha = \{1\}$ for all points $\alpha \in \Omega$ and *regular* if it is semi-regular and transitive. An action of a group G on a set Ω is said to be *discrete* if for every $\alpha \in \Omega$ the image of G_α in $\mathrm{Sym}(\Omega)$ is finite. If the action is discrete, it is possible to find a finite subset $A \subseteq \Omega$ such that $G_{(A)} = K$, where K is the kernel of the action.

When Γ and Δ are graphs, or digraphs, a *graph morphism* from Γ to Δ is a map $\varphi\colon \mathrm{V}\Gamma \to \mathrm{V}\Delta$ such that if $(\alpha, \beta) \in \mathrm{A}\Gamma$, then $(\varphi(\alpha), \varphi(\beta)) \in \mathrm{A}\Delta$. If Γ is a graph or a digraph and $\varphi : \mathrm{V}\Gamma \to \mathrm{V}\Gamma$ is a bijective map, then φ is an *automorphism* of Γ if φ induces a bijection $\mathrm{A}\Gamma \to \mathrm{A}\Gamma$. The set of all automorphisms of Γ is a group, the *automorphism group of* Γ, denoted by $\mathrm{Aut}(\Gamma)$. We will think of $\mathrm{Aut}(\Gamma)$ and subgroups of $\mathrm{Aut}(\Gamma)$ as permutation groups on $\mathrm{V}\Gamma$.

A graph or a digraph Γ is *vertex-transitive* if its automorphism group acts transitively on the vertex set. Vertex-transitive graphs are always regular and it is well-known that a connected, vertex-transitive graph either has no ends, one end, two ends or uncountably many ends, see [7, Corollary 4]. We say that Γ is *edge-transitive* or *arc-transitive* if the automorphism group acts transitively on the edge set or the arc set, respectively. If the automorphism group of Γ acts transitively on the set of s-arcs in Γ, then we say that Γ

is *s-arc-transitive*. When the automorphism group is s-arc-transitive for all s, it is said that Γ is *highly arc-transitive*.

Consider now a group G that acts vertex-transitively on a graph Γ of degree d. Let $\alpha \in V\Gamma$. The stabiliser G_α clearly leaves $\Gamma(\alpha)$, the set of neighbours of α, invariant and thus induces an action on $\Gamma(\alpha)$. The kernel of this action is $K_\alpha = G_\alpha \cap G_{(\Gamma(\alpha))}$ and the quotient G_α/K_α embeds as a subgroup into $\mathrm{Sym}(d)$, the symmetric group on a set with d elements. Let now α' be another vertex of Γ. By assumption there exists $g \in G$ with $\alpha g = \alpha'$. The actions of G_α on $\Gamma(\alpha)$ and $G_{\alpha'}$ on $\Gamma(\alpha')$ are conjugate via g. Thus, the following definition is independent of the choice of α.

Definition 1. Let Γ be a graph of degree d on which a group G acts vertex-transitively. Let $\alpha \in V\Gamma$. The *local action* of G on Γ is the conjugacy class of the finite group G_α/K_α, seen as a subgroup of $\mathrm{Sym}(d)$.

Usually we will say that the local action is the subgroup G_α/K_α of $\mathrm{Sym}(d)$, or that G acts locally like G_α/K_α, and omit the mention of the conjugacy class.

When σ is an equivalence relation on the vertex set of a graph Γ we can form the *quotient graph* Γ/σ. Its vertex set is the set of σ-classes, and if A and B are distinct σ-classes, then $\{A, B\}$ is an edge in Γ/σ if and only if there is a vertex $\alpha \in A$ and a vertex $\beta \in B$ such that $\{\alpha, \beta\}$ is an edge in Γ. If G is a subgroup of $\mathrm{Aut}(\Gamma)$, then Γ/G denotes the quotient graph of Γ with respect to the equivalence relation whose classes are the G-orbits on the vertex set. If σ is a *G-congruence* (i.e. for every $g \in G$ it holds that αg is equivalent to βg if and only if α is equivalent to β), then G has a natural action on the σ-classes and thus an action on the quotient graph Γ/σ by automorphisms.

It is easy to see directly from the definitions that if g is an automorphism of a graph Γ, then two rays R_1 and R_2 belong to the same end if and only if the rays $R_1 g$ and $R_2 g$ belong to the same end. Therefore the relation of being in the same end is an $\mathrm{Aut}(\Gamma)$-congruence on the set of rays in Γ and the automorphism group of Γ acts on the set of ends of Γ. We also see that if a finite set Φ of vertices separates some two ends ω_1 and ω_2, then the set Φg separates the ends $\omega_1 g$ and $\omega_2 g$.

2.3 Convergent sequences of permutations

In this section the notions of *convergence of sequences of permutations* and *closed groups of permutations* are introduced. Here we avoid introducing a topology, but in Section 7 a topology on a permutation group is described and the convergence introduced here is convergence in that topology.

Definition 2. Let $\{g_i\}$ be a sequence of permutations of some set Ω. We say that the sequence *converges* to a permutation g of Ω if for every point $\alpha \in \Omega$ there exists a number $N_\alpha \geqslant 0$ such that $\alpha g_i = \alpha g$ for all $i \geqslant N_\alpha$.

A group G of permutations of some set Ω is said to be *a closed permutation group* (or a *closed* subgroup of $\mathrm{Sym}(\Omega)$) if, whenever $\{g_i\}$ is a sequence of permutations in G converging to a permutation g of Ω, then $g \in G$.

It is easy to show that the automorphism group of a graph (or a digraph) Γ is closed. It is also easy to see that if the action is discrete, then every convergent sequence is eventually constant. The following lemma will be used in Sections 4 and 5 and is the reason why these terms are introduced here.

Lemma 3. (Cf. [18, Lemma 1]) *Let Γ be a connected, locally finite graph (or digraph). Let $G \leqslant \mathrm{Aut}(\Gamma)$ be a closed subgroup such that its action on Γ is highly arc-transitive. If $(\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots)$ and $(\ldots, \beta_{-1}, \beta_0, \beta_1, \ldots)$ are 2-way ∞-arcs in Γ, then there is an element $g \in G$ such that $\alpha_i g = \beta_i$ for all i. In particular, the action of G on the set of 1-way ∞-arcs of Γ of type $(\ldots, \alpha_{-1}, \alpha_0)$, and on the set of 1-way ∞-arcs of type $(\alpha_0, \alpha_1, \ldots)$ is transitive.*

Proof. Let $(\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots)$ and $(\ldots, \beta_{-1}, \beta_0, \beta_1, \ldots)$ denote 2-way ∞-arcs in Γ. Since G acts highly arc-transitively on Γ, there is for each $i \geqslant 0$ an element g_i such that $(\alpha_{-i}, \ldots, \alpha_i)g_i = (\beta_{-i}, \ldots, \beta_i)$. Let $B_i(\alpha)$ denote the set of all vertices in Γ at distance at most i from the vertex α. Because the graph Γ is assumed to be locally finite, the sets $B_i(\alpha)$ are all finite. All elements in the sequence $\{g_i\}$ map the vertex α_0 to the vertex β_0 and thus map $B_i(\alpha_0)$ to $B_i(\beta_0)$. Since $B_1(\alpha_0)$ is finite, there are only finitely many possibilities for the maps we get by restricting the g_i's to $B_1(\alpha_0)$. Hence there is a subsequence C_1 of the sequence $\{g_i\}$ such that the restriction of all the elements in this subsequence to $B_1(\alpha_0)$ is the same. Let i_1 be a number such that g_{i_1} is in C_1. There are also only finitely many possibilities for the restriction of the permutations in the sequence $\{g_i\}$ to $B_2(\alpha_0)$ and thus we get a subsequence C_2 of C_1 such that restrictions of the elements in C_2 to $B_2(\alpha_0)$ are all identical. Choose i_2 such that $i_2 > i_1$ and g_{i_2} is in C_2. Continuing in this way we get a subsequence $\{g_{i_j}\}$ of our original sequence so that if $j, j' \geqslant i$ and α is a vertex in $B_i(\alpha_0)$ then $\alpha g_{i_j} = \alpha g_{i_{j'}}$. Hence we can define a permutation g of the vertex set of Γ by saying that αg is equal to αg_{i_j} for j equal to the distance in Γ between α_0 and α. One easily sees that g is well-defined and is an automorphism of Γ. Clearly the sequence $\{g_{i_j}\}$ converges to g and $\alpha_i g = \beta_i$ for all i. The assumption that G is a closed permutation group guarantees that $g \in G$. This shows that G acts transitively on the set of 2-way ∞-arcs.

In a connected highly arc-transitive graph a 1-way ∞-arc of either type can always be extended to a 2-way ∞-arc and thus the statement about the transitivity of the action on the sets of 1-way ∞-arcs follows from the transitivity of the action on 2-way ∞-arcs. □

3 Three cases

Our exploration starts with the use of the local action and the action on the edges and arcs to divide vertex-transitive, non-discrete group actions on cubic graphs into three separate cases.

Suppose G acts vertex-transitively on a connected cubic graph Γ. The local action of G on Γ is a conjugacy class of subgroups of the symmetric group $\mathrm{Sym}(3)$. There are four possibilities: the trivial group, the cyclic group of order 2, the cyclic group of order 3, and the whole group $\mathrm{Sym}(3)$. First note that if the stabiliser of a vertex acts locally like the trivial group, then, since Γ is connected, we see that the stabiliser of a vertex acts trivially on the graph. In the case where the stabiliser of a vertex acts locally like a cyclic group of order 3 we see similarly that the subgroup fixing some pair of adjacent vertices is trivial. In both cases the action is discrete.

If the action of G is non-discrete, we are left with the possibilities that the group acts locally either like the cyclic group of order 2 or like the full symmetric group. In the second case it is clear that the group acts both edge- and arc-transitively on Γ.

Assume that G acts locally like a cyclic group of order 2. It is possible that the group G acts edge-transitively, and then its action is not arc-transitive, but it is also possible that G has two orbits on the edges of Γ. Let us briefly analyse the latter case.

Let α be a vertex of Γ and let β denote the neighbour of α that is fixed by G_α. Colour the edges in the orbit $\{\alpha, \beta\}G$ *red* and the edges in the other edge orbit are coloured *blue*. Each vertex in Γ is therefore the end-vertex of precisely one red edge and precisely two blue edges and this colouring is preserved by the action of G. We let arcs in Γ inherit the colour from the edge that gives rise to them.

Definition 4. Let Γ denote a connected, cubic graph. Suppose a group G acts vertex-transitively on Γ, but has two orbits on the edges of Γ. The colouring of the edges and arcs of Γ described above is called *expedient*.

Remove all the blue edges from Γ. As each vertex is the end vertex of only one red edge, we get a vertex-transitive graph of degree 1. Let $\{\alpha, \beta\}$ be a red edge. If $g \in G$ and $\alpha g = \beta$, then $\beta g = \alpha$. From this it is apparent that G must act transitively on the red arcs. Removing the red edges from Γ we get a vertex-transitive graph of degree 2. Each connected component is therefore either a finite cycle or a line, and the connected components are all isomorphic. Since the graph is vertex-transitive and the stabiliser of a vertex acts locally like the cyclic group of order 2, we see that G acts transitively on the blue arcs. Hence the group has two orbits on the arcs of Γ.

The outcome of the above discussion is that when we have a connected, cubic graph Γ and a subgroup G acting vertex-transitively and non-discretely, then there are three possible cases:

Case **A**: The stabiliser of a vertex is infinite and the group acts locally like the symmetric group on three elements. The group G acts transitively on both the set of edges and the set of arcs of Γ.

Case **B**: The stabiliser of a vertex is infinite and acts locally like a cyclic group of order two. The group G acts transitively on the edges, but not on the arcs.

Case **C**: The stabiliser of a vertex is infinite and acts locally like a cyclic group of order two, and the group has two orbits on the edges and two orbits on the arcs. When discussing Case **C** we will use an expedient colouring of the edges.

We say that a cubic graph Γ satisfies the conditions in one of the cases, if Γ and the action of $\mathrm{Aut}(\Gamma)$ satisfy the conditions.

Remark 5. In [22] Nebbia studies faithful, vertex-transitive group actions with infinite vertex stabilisers on the cubic tree and describes the same division into cases as above.

Let us start by looking at examples.

Example 6.

1. The cubic tree T_3 satisfies the conditions in Case **A**. It is shown in the next section that in Case **A** the graph must be the cubic tree.

2. Let $\Gamma = T_3$. Define Γ_+ as the digraph we get by orienting the edges of Γ so that at each vertex there is one incoming arc and two outgoing arcs. The graph Γ and the action of $\mathrm{Aut}(\Gamma_+)$ on Γ satisfy the conditions in Case **B**. In the next section it is shown that in Case **B** the graph is equal to T_3.

3. Colour each edge in T_3 either red or blue so that each vertex is adjacent to one red edge and two blue edges. Let G denote the subgroup of the automorphism group of T_3 that preserves this colouring. The stabiliser in G of a vertex α is infinite and the action of G on T_3 satisfies the conditions in Case **C**.

4. The arc-graph $A(\Gamma)$ of a graph Γ (undirected) has as its vertex set the set of arcs of Γ and two arcs (α, β) and (γ, δ) are adjacent in $A(\Gamma)$ if and only if $\beta = \delta$ or $(\gamma, \delta) = (\beta, \alpha)$. Considering the case $\Gamma = T_3$ we see that each vertex in T_3 gives rise to a triangle in $A(T_3)$ and the triangles for a pair of adjacent vertices are joined by a single edge. This graph is cubic and satisfies the conditions in Case **C**. If we colour the edges in the triangles blue and the other edges red, then we have an expedient colouring. One can also describe this graph as the graph one gets by replacing each vertex with a triangle and then putting in edges between triangles representing adjacent vertices. Clearly it is also possible to replace each vertex in the n-regular tree by an n-gon in this way to get a cubic graph such that the conditions in Case **C** are satisfied.

5. Start with a $2n$-gon. For each pair α, β of opposite vertices in that $2n$-gon take a new $2n$-gon, select some pair δ, γ of opposite vertices in the new $2n$-gon and add edges $\{\alpha, \delta\}$ and $\{\beta, \gamma\}$. Then look at pairs of opposite vertices in the new $2n$-gons, where the vertices have degree 2. For each such pair get a new $2n$-gon and join some pair of opposite vertices in the new $2n$-gon to the pair in the old $2n$-gon by a pair of edges as above. Continue like this *ad infinitum* until you have a cubic graph (see Figure 1). The stabiliser of a vertex in the automorphism group of this new graph is clearly infinite and the conditions in Case **C** are satisfied. Contracting each and everyone of the $2n$-gons leaves us with the n-regular tree.

6. Let Γ be a connected quartic graph (i.e. a regular graph of degree 4). Suppose G is a subgroup of $\mathrm{Aut}(\Gamma)$ acting vertex-transitively and locally like D_4, the dihedral group with 8 elements, in its natural action on a 4 element set. For a vertex α there is a natural G_α-congruence σ_α on $\Gamma(\alpha)$ with two classes $\Sigma_{\alpha,1}$ and $\Sigma_{\alpha,2}$.

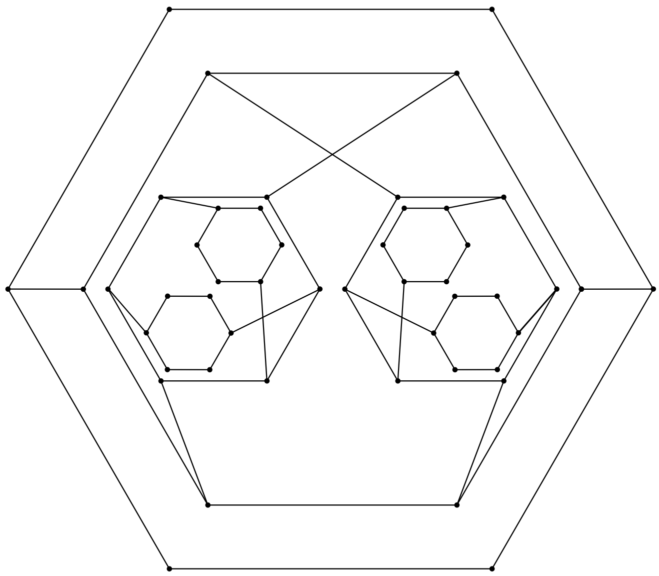

Figure 1: A partial picture of a graph satisfying the conditions in Case **C**, see Part 5 of Example 6 with $n = 3$.

We use Γ to construct a new cubic graph Γ' that G acts on such that the conditions in Case **C** are satified. Informally one can say that we get Γ' by splitting each vertex in Γ into two vertices and putting in a new edge between the two halves. Formally, the construction is as follows: The vertex set of Γ' is the indexed set $\{\Sigma_{\alpha,i} \mid \alpha \in V\Gamma, i \in \{1,2\}\}$. For each $\alpha \in V\Gamma$ the pair $\{\Sigma_{\alpha,1}, \Sigma_{\alpha,2}\}$ is an edge in Γ'. If α, β are adjacent vertices in Γ such that $\beta \in \Sigma_{\alpha,i}$ and $\alpha \in \Sigma_{\beta,j}$, then $\{\Sigma_{\alpha,i}, \Sigma_{\beta,j}\}$ is an edge in Γ'. This construction gives us a connected, cubic graph. The action of G on Γ induces an action of G on Γ'. Clearly G has precisely two orbits on the edges of Γ' and if stabilisers in G of vertices in Γ are infinite, then the action of G on Γ' will satisfy the conditions in Case **C**.

Conversely, start with a connected cubic graph and a subgroup G of its automorphism group so that the conditions in Case **C** are satisfied. Take an expedient colouring of the edges and contract each red edge. Then we get a quartic graph and the local action of G on this graph is D_4.

An example of a quartic graph Γ like the one described above is the graph with vertex set $\{1,2\} \times \mathbb{Z}$, and the edge set is the set of all pairs $\{(i,j), (i',j')\}$ with $j' = j + 1$. The construction then gives the graph Γ' in Figure 2. This graph has two ends.

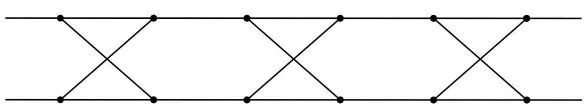

Figure 2: A two-ended graph satisfying the conditions in Case **C**, see Part 6 of Example 6.

The connection described above between cubic graphs with a group action satisfying the conditions in Case **C** and quartic graphs that have a vertex-transitive group action such that the local action is D_4 is further explored in the following example, the proof of Theorem 26 and in Appendix A.

7. Take Γ to be the Diestel-Leader graph $\mathrm{DL}(2,2)$; for a a description of this graph see Appendix B or e.g. [8] and [36]. This is a one-ended quartic graph such that the local action is D_4 and stabilisers in the automorphism group of vertices are infinite. The cubic graph constructed by the method above satisfies the conditions in Case **C** and the graph has only one end. The underlying undirected graph of the digraph in Example 1 in [18] is the graph $\mathrm{DL}(2,2)$. The construction in [18] can be adapted to provide more examples of cubic graphs satisfying the conditions in Case **C**, see Appendix B.

The following proposition provides further examples of graphs and groups satisfying the conditions in Case **C**. The construction is similar to the construction used in Example 6(4), but instead of a regular tree we start with a graph that has a suitable group action.

Proposition 7. *Let G be a group acting vertex-transitively on a connected, locally finite graph Γ of degree d. Suppose that the local action of G on Γ is the dihedral group D_d, i.e. with $2d$ elements, in its usual action on a set with d elements. Then, there is a connected cubic graph Δ satisfying the following.*

1. *The group G has a vertex-transitive action on Δ.*

2. *If the action of G on Γ is not discrete, then the action on Δ is not discrete and the conditions of Case* **C** *are satisfied.*

3. *There is a G-congruence σ on $\mathrm{V}\Delta$ such that $\Gamma = \Delta/\sigma$ and the subgraph in Δ spanned by each σ-class is a d-gon with blue edges.*

Proof. Define the graph Δ as follows: The vertex set is the set of arcs of Γ. Two vertices in Δ (i.e. arcs in Γ) are connected by a red edge if they are reverse to each other (as arcs in Γ). For a fixed vertex α in Γ we choose an element $r_\alpha \in G_\alpha$ so that r_α acts on $\Gamma(\alpha)$ as a d-cycle. If a vertex β is adjacent to α, then we say that $\{(\alpha,\beta),(\alpha,\beta r_\alpha)\}$ is a blue edge in Δ and so are all the elements in the G-orbit of $\{(\alpha,\beta),(\alpha,\beta r_\alpha)\}$. (Loosely speaking we can say that Δ is the graph we get if we replace each vertex in Γ with a d-cycle with blue edges, and then connect the d-cycles corresponding to a pair of adjacent vertices in Γ with a single red edge.)

Clearly the graph Δ we constructed is a cubic graph on which G acts vertex-transitively. If the action on Γ is not discrete, then the action of G on Δ is not discrete and the conditions in Case **C** are satisfied.

The classes of the equivalence relation σ in Part 3 are just the blue d-cycles. □

The construction in the proof of Proposition 7 is "reversible": Suppose G acts vertex-transitively on a cubic graph Δ such that the conditions in Case **C** are satisfied. Consider the “blue subgraph” we get by removing all the red edges from Γ. This subgraph is 2-regular, so either there is a number $d \geqslant 3$ such that each connected component is a d-gon or every component is a line. In the first case, by contracting in Γ each connected component of the blue subgraph to a vertex, we get a graph on which G acts vertex-transitively and its local action is the dihedral group with $2d$ elements. In the second case we could also contract each blue line to a vertex and get a graph on which G acts vertex-transitively. But in this case the graph would not be locally finite and the group would act locally like the infinite dihedral group D_∞ (the automorphism group of the graph that has $\mathbb{Z}$ as vertex set and edges $\{i, i+1\}$ for all i).

4 Vertex- and edge-transitive actions

In this section we show that in Cases **A** and **B** the graph Γ is the cubic tree.

In [28] Tutte proves that if a group acts vertex- and arc-transitively on a connected, cubic graph that is s-arc-transitive, but not $(s+1)$-arc-transitive, then the group acts regularly on the set of s-arcs. If stabilisers of vertices are infinite, then Tutte's result implies that the group acts highly arc-transitively on the graph, which in turn implies that the graph must be the cubic tree. This is known to people working on group actions on graphs and is for instance mentioned in the introduction of [10]. For the readers' convenience a full proof is included below.

Theorem 8. *Let G be a group acting vertex- and arc-transitively and non-discretely on a connected cubic graph Γ. Then Γ is a tree. Moreover, for all $s \geqslant 1$, the action of G on the set of s-arcs is transitive. If, in addition, the image of G is a closed subgroup of* $\mathrm{Aut}(\Gamma)$*, then the action of G on the ends of Γ is 2-transitive (i.e. G acts transitively on the set of ordered pairs of distinct ends).*

The proof follows the proof of the main theorem in [28]. First we introduce some notation.

Definition 9. An s-arc S' is a *predecessor* of an s-arc S if there exists an $(s+1)$-arc $(\alpha_0, \ldots, \alpha_{s+1})$ with $S = (\alpha_0, \ldots, \alpha_s)$ and $S' = (\alpha_1, \ldots, \alpha_{s+1})$. We also say that S is a *successor* of S'.

An s-arc S is said to be *accessible* from an s-arc S' if there exists a finite sequence $S_0, \ldots, S_n$ of s-arcs such that $S = S_0$, $S' = S_n$ and for all $0 \leqslant i \leqslant n-1$ the s-arc S_{i+1} is a predecessor or a successor of S_i.

Then we prove a preliminary lemma.

Lemma 10. *Let Γ be a connected graph such that each vertex has degree at least 2 and let S be an s-arc in Γ. Every vertex of Γ is contained in some s-arc that is accessible from S.*

Proof. Define $A \subseteq V\Gamma$ as the set of vertices that are contained in an s-arc accessible from S. Suppose α is a vertex in Γ that is adjacent to some vertex β in A. By assumption β

is contained in some s-arc S' that is accessible from S. We can assume without loss of generality that β is the initial vertex of S', since otherwise we can achieve this by taking predecessors of S' repeatedly (note that every s-arc accessible from S' is also accessible from S). If α is not in S' then α is clearly contained in an successor of S'. Thus α is in A. Therefore every neighbour of a vertex in A is in A and since Γ is a connected graph we conclude that every vertex of Γ is in A. □

And finally we prove the theorem.

Proof of Theorem 8. We start by showing that G acts transitively on the set of s-arcs for all $s \geqslant 1$. The assumptions say that G acts transitively on the set of 1-arcs. Suppose there is a number s such that G acts transitively on the set of s-arcs but does not act transitively on the set of $(s+1)$-arcs.

Let $S = (\alpha_0, \dots, \alpha_s)$ be an s-arc. Then there are exactly two $(s+1)$-arcs $S_1 = (\alpha_0, \dots, \alpha_s, \alpha_{s+1})$ and $S_2 = (\alpha_0, \dots, \alpha_s, \alpha'_{s+1})$ that contain both S and a predecessor of S. By assumption, if $(\beta_0, \dots, \beta_s, \beta_{s+1})$ is an $(s+1)$-arc, then there exists an element $g \in G$ with $(\beta_0, \dots, \beta_s)g = S$. Then g maps the $(s+1)$-arc $(\beta_0, \dots, \beta_s, \beta_{s+1})$ to either S_1 or S_2. Thus every $(s+1)$-arc lies in the orbit of one of S_1 and S_2. Since the action of G is not transitive on the set of $(s+1)$-arcs, we see that G has 2 orbits on the set of $(s+1)$-arcs and the stabiliser of the s-arc S must fix both S_1 and S_2, and thus fix both α_{s+1} and α'_{s+1}. Thus the stabiliser of S fixes both predecessors of S. In the same way we see that the stabiliser of S also fixes both successors of S.

Since S was arbitrary, we just showed that the stabiliser of an s-arc is contained in the stabilisers of its predecessors and successors. Iterating this argument, we get that the stabiliser of an s-arc is contained in the stabiliser of every s-arc that is accessible from it. Now Lemma 10 shows that the stabiliser of an s-arc fixes every vertex of Γ and is therefore trivial. Now recall that the orbit-stabiliser theorem says that $|G_{\alpha_0}/K| = |G_S/K| \cdot |S \cdot G_{\alpha_0}/K|$, where K is the kernel of the action of G on Γ. This shows that G_{α_0}/K has to be finite.

It is now easy to conclude that Γ is a tree. First observe that, since the valency of the graph is bigger than 2, every s-arc can be extended to an $(s+1)$-arc that is not a cycle. In particular, for every $s \geqslant 1$ there exists an s-arc that is not a cycle. Now by s-arc-transitivity, no s-arc can be a cycle.

From Lemma 3 it follows that if G is closed, then G acts transitively on the set of all 2-way ∞-arcs in Γ. That in turn implies that G acts 2-transitively on the set of ends of Γ. □

Remark 11. There are infinitely many simple groups having a vertex- and arc-transitive action on the cubic tree, see Remark A4 in the paper [4] by Caprace and Radu.

Next we consider Case **B**. We would like to point out that while usually in the present article it is an assumption that the action is non-discrete, here, it is a conclusion.

Theorem 12. *Suppose Γ is a connected cubic graph and G acts on Γ vertex- and edge-transitively, but not arc-transitively. Then Γ is a tree and the action of G on Γ is non-discrete. Moreover, G fixes an end ω of Γ and, if G is a closed subgroup of $\mathrm{Aut}(\Gamma)$, then G acts transitively on $\Omega\Gamma \setminus \{\omega\}$.*

In particular, let Γ_+ denote one of the two digraphs that has the same vertex set as Γ and the set of arcs is one of the arc-orbits of G on Γ. Then G acts highly arc-transitively on Γ_+.

Proof. Clearly G has two orbits on the arcs of Γ. Let Γ_+ be one of the two digraphs that has the same vertex set as Γ and has one of the arc-orbits as a set of arcs. Namely, we choose the orbit so that the in-degree is 1 and the out-degree is 2. Any cycle in Γ_+ would have to be a directed cycle, since otherwise we would have a vertex with in-degree 2. If α is a vertex in a directed cycle in Γ_+, then an automorphism fixing α and taking one of the outgoing arcs of α to the other will move our directed cycle to a different directed cycle that also includes α. This leads to a contradiction because the finite subdigraph consisting of these two cycles would have a vertex with in-degree 2. Thus Γ cannot contain a cycle and Γ is therefore a tree.

Suppose $(\alpha_0, \ldots, \alpha_s)$ and $(\beta_0, \ldots, \beta_s)$ are s-arcs in Γ_+ with $s \geqslant 2$. Since G acts arc-transitively on Γ_+ there is $g \in G$ such that $(\alpha_{s-1}, \alpha_s)g = (\beta_{s-1}, \beta_s)$. But the in-degree of Γ_+ is 1 and therefore $(\alpha_0, \ldots, \alpha_s)g = (\beta_0, \ldots, \beta_s)$. Hence G acts highly arc-transitively on Γ_+. In particular we see that the action of G on Γ is non-discrete.

From Lemma 3 it follows that G acts transitively on the set of 2-way ∞-arcs in the digraph Γ_+. Given a vertex α_0 there is a unique ∞-arc of the type $(\ldots, \alpha_{-1}, \alpha_0)$ and the end ω that contains the ray $\alpha_0, \alpha_{-1}, \ldots$ is fixed by the automorphism group. Thus G fixes an end ω and acts transitively on $\Omega\Gamma \setminus \{\omega\}$. □

The following corollary sums up Theorems 8 and 12.

Corollary 13. *Suppose Γ is a connected, cubic graph. Suppose there exists a group G acting vertex- and edge-transitively and non-discretely on Γ. Then Γ is the 3-regular tree.*

Remark 14. In his study of faithful, vertex-transitive actions with infinite vertex stabilisers on the regular cubic tree Nebbia gets the same conclusions about the action on the ends as in the above theorems, but he assumes from the start that the graph is a tree, see [22, Proposition 3.1].

A transitive action of a group G on a set Ω is said to be *primitive* if there is no non-trivial, proper equivalence relation on Ω that is preserved by G.

Corollary 15. *Let Γ be a connected, cubic graph. Let G be a group acting vertex-transitively and non-discretely on Γ. Then the action of G on the vertex set is not primitive.*

Proof. From Theorems 8 and 12 it follows that if the action of G on Γ satisfies the conditions in Cases **A** or **B**, then Γ is a tree and the natural bipartition of Γ gives an equivalence relation on the vertex set of Γ that is preserved by G. If the conditions in

Case **C** are satisfied, then we can define a G-congruence on the vertex set of Γ by saying that α and β are related if and only if $\{\alpha, \beta\}$ is a red edge. Hence the action is not primitive. □

5 The non-edge-transitive case

Now we turn our attention to Case **C**, i.e. Γ is a connected cubic graph and G acts on Γ vertex-transitively and non-discretely with two orbits on the edges. The arguments presented here are similar to arguments in the proof Theorem 8 and are based on Tutte's paper [28]. The assumptions about Γ and the action of G on Γ guarantee that we have an expedient colouring of the edges of Γ, see Definition 4. In particular, adjacent to a vertex $\alpha \in V\Gamma$ are two blue edges and one red edge. The stabiliser of the vertex α transposes the two blue edges but fixes the red edge.

Definition 16. Let $S = (\alpha_0, \ldots, \alpha_s)$ be an s-arc in Γ. We call S *alternating* if consecutive arcs are in different G-orbits, i.e. the arcs in S are alternatively red and blue. If $\{\alpha_0, \alpha_1\}$ is red and $\{\alpha_{s-1}, \alpha_s\}$ is blue, we call S an *rb-alternating s-arc*; *rr-alternating s-arcs*, *br-alternating s-arcs* and *bb-alternating s-arcs* are then defined in the obvious way.

Note that rr-alternating and bb-alternating arcs have odd length and rb-alternating and br-alternating arcs have even length. The partition of the set of alternating s-arcs into rr-, rb-, br- and bb-alternating s-arcs is invariant under elements in G.

The following definition takes after Definition 9, but here we restrict our attention to alternating arcs.

Definition 17. An alternating s-arc S' is an *alt-predecessor* of an alternating s-arc S if there exists an alternating $(s+1)$-arc $(\alpha_0, \ldots, \alpha_{s+1})$ with $S = (\alpha_0, \ldots, \alpha_s)$ and $S' = (\alpha_1, \ldots, \alpha_{s+1})$. We also say that S is an *alt-successor* of S'.

An alternating s-arc S is said to be *alt-accessible* from an alternating s-arc S' if there exists a finite sequence $S_0, \ldots, S_n$ of alternating s-arcs such that $S = S_0$, $S' = S_n$ and for all $0 \leqslant i \leqslant n-1$ the s-arc S_{i+1} is an alt-predecessor or an alt-successor of S_i.

Similarly, an alternating s-arc S' is a 2-*alt-predecessor* of an alternating s-arc S if there exits an alternating $(s+2)$-arc $(\alpha_0, \ldots, \alpha_{s+2})$ with $S = (\alpha_0, \ldots, \alpha_s)$ and $S' = (\alpha_2, \ldots, \alpha_{s+2})$. In this situation we also say that S is a 2-*alt-successor* of S'.

An alternating s-arc S is said to be 2-*alt-accessible* from an alternating s-arc S' if there exists a finite sequence $S_0, \ldots, S_n$ of alternating s-arcs such that $S = S_0$, $S' = S_n$ and for all $0 \leqslant i \leqslant n-1$ the s-arc S_{i+1} is a 2-alt-predecessor or a 2-alt-successor of S_i.

It is not difficult to see that alt-accessibility and 2-alt-accessibility are equivalence relations. Note that, by definition, every alternating predecessor of an alternating arc is an alt-predecessor as soon as $s \geqslant 2$, similarly for successors. Also, for alternating 2-alt-predecessors and 2-alt-successors of alternating arcs, the requirement that the $(s+2)-$arc is alternating is automatic unless $s \leqslant 2$ and S contains a blue arc. The following lemma is an analogue of Lemma 10.

Lemma 18. *Let Γ be a connected, cubic graph. Assume that the edges are coloured so that each vertex is incident with two blue edges and one red edge.*

1. *Let S be an alternating s-arc. Every vertex of Γ is contained in an alternating s-arc that is alt-accessible from S.*

2. *Let S be an rr-alternating s-arc with $s \geqslant 3$. Every vertex of Γ is contained in an rr-alternating s-arc that is 2-alt-accessible from S.*

Proof. 1. Define A as the set of all the vertices in Γ that are contained in some alternating s-arc that is alt-accessible from S.

Assume that α is a vertex in Γ that is adjacent to some vertex β in A. Since $\beta \in A$ we know that β is contained in some alternating s-arc S' that is alt-accessible from S and we can assume that β is the initial vertex of S' (if needed, we can repeatedly replace S' with an alt-predecessor). If α is in S' then $\alpha \in A$. So we assume that α is not in S'. Say the edge $\{\beta, \delta\}$ belongs to S'. If the edges $\{\alpha, \beta\}$ and $\{\beta, \delta\}$ have different colours, then the vertex α clearly belongs to an alt-successor of S' and is thus in A. Suppose now that the edges $\{\alpha, \beta\}$ and $\{\beta, \delta\}$ have the same colour, i.e. both are blue. Say $\{\beta, \gamma\}$ is the red edge incident with β. Then γ belongs to an alt-successor of S' and we can find an alternating s-arc S'' in W that has β as its terminal vertex and contains the vertex γ. Clearly α belongs to an alt-predecessor of S'' and hence $\alpha \in A$. Since the graph Γ is connected, every vertex in Γ belongs to A.

2. In Part 1 we saw that if α is a vertex in Γ then it is contained in some alternating s-arc S' that is alt-accessible from S. If S' is a bb-alternating s-arc then its alt-predecessor and its alt-successor are both rr-alternating s-arcs and we can be sure that at least one of them includes α and both are alt-accessible from S. Thus we may assume that S' is an rr-alternating s-arc.

Since S' is alt-accessible from S, there is a sequence of alternating s-arcs $S_0, \ldots, S_n$ such that $S_0 = S$ and $S_n = S'$ and for all $i = 0, \ldots, n-1$ either S_{i+1} is an alt-predecessor or it is an alt-successor of S_i. Note also that taking alt-successors and alt-predecessors change rr-alternating s-arcs into bb-alternating s-arcs and *vice versa*. Thus the number n is even. If it so happens that S_{2i+1} is an alt-predecessor of S_{2i} and S_{2i+2} is an alt-predecessor of S_{2i+1}, then S_{2i+2} is a 2-alt-predecessor of S_i and, similarly, if S_{2i+1} is an alt-successor of S_{2i} and S_{2i+2} is an alt-successor of S_{2i+1}, then S_{2i+2} is a 2-alt-successor of S_{2i}. In this case we can delete S_{2i+1} from the sequence. If S_{2i+1} is an alt-predecessor of S_{2i} and S_{2i+2} is an alt-successor of S_{2i+1}, then we let S'_{2i+1} be an alt-predecessor of S_{2i+1} and note that then S'_{2i+1} is a 2-alt-predecessor of S_{2i} and S_{2i+2} is a 2-alt-successor of S'_{2i+1}. In the case that S_{2i+1} is an alt-successor of S_{2i} and S_{2i+2} is an alt-predecessor of S_{2i+1} can be handled similarly. In these cases we replace S_{2i}, S_{2i+1} in our sequence with $S_{2i}, S'_{2i+1}, S_{2i+1}$. Thus we can construct a sequence of s-arcs starting with S and ending with S' such that each alternating s-arc in this sequence, except the first one, is the 2-alt-predecessor or 2-alt-successor of the previous one. Hence S' is 2-alt-accessible from S and every vertex in Γ is contained in an rr-alternating s-arc that is 2-alt-accessible from S. □

The following is now proved in the same way as Theorem 8.

Lemma 19. *Let Γ be a connected cubic graph. Assume that G acts on Γ such that the conditions in Case* **C** *are satisfied. Then G acts transitively on the set of rr-alternating s-arcs for every $s \geqslant 0$.*

Proof. Since G acts transitively on the set of blue arcs, we see that G acts transitively on the set of rr-alternating 3-arcs. Assume G acts transitively on the set of rr-alternating s-arcs, but not on the set of rr-alternating $(s+2)$-arcs. Let $S = (\alpha_0, \dots, \alpha_s)$ be an rr-alternating s-arc. Then there are exactly two rr-alternating $(s+2)$-arcs $S_1 = (\alpha_0, \dots, \alpha_s, \alpha_{s+1}, \alpha_{s+2})$ and $S_2 = (\alpha_0, \dots, \alpha_s, \alpha'_{s+1}, \alpha'_{s+2})$ that contain both S and the 2-alt-predecessors of S. For every rr-alternating $(s+2)$-arc $(\beta_0, \dots, \beta_s, \beta_{s+1}, \beta_{s+2})$ there exists an element $g \in G$ such that $(\beta_0, \dots, \beta_s)g = S$. Then g maps the $(s+2)$-arc $(\beta_0, \dots, \beta_s, \beta_{s+1}, \beta_{s+2})$ to either S_1 or S_2. Thus $(\beta_0, \dots, \beta_s, \beta_{s+1}, \beta_{s+2})$ lies in the orbit of one of S_1 and S_2. Since the action of G is not transitive on the set of rr-alternating $(s+2)$-arcs we see that G has 2 orbits on the set of rr-alternating $(s+2)$-arc and the stabiliser of the rr-alternating s-arc S must fix both 2-alt-predecessors of S. In the same way we see that the stabiliser of S also fixes both 2-alt-successors of S.

We have now shown that an element of G fixing an rr-alternating s-arc S has to fix all the vertices in its rr-alternating 2-alt-predecessors and 2-alt-successors. Using induction we see that the stabiliser of S has to fix all the vertices that are contained in any rr-alternating s-arc that is 2-alt-accessible from S. By Lemma 18 every vertex in Γ is contained in a s-arc that is 2-alt-accessible from S and thus the pointwise stabiliser of S fixes every vertex in the graph and is trivial. Hence the action of G is discrete. We have reached a contradiction and conclude that G acts transitively on the set of rr-alternating s-arcs for every $s \geqslant 0$. □

This paper is mainly about infinite cubic vertex-transitive graphs such that the action of the automorphism group is non-discrete, but the arguments used in the proof of Lemma 19 can be applied to cubic vertex-transitive graphs such that the automorphism group acts discretely.

Corollary 20. *Let Γ be a connected cubic graph. Let G act on Γ vertex-transitively, with two orbits on the edges and two orbits on the arcs. Assume that the edges in Γ are coloured red or blue according to an expedient colouring. Furthermore, assume the action of G on Γ is discrete. Then there is an odd number s such that G acts regularly on the sets of rr-alternating s-arcs, bb-alternating $(s-2)$-arcs, rb-alternating $(s-1)$-arcs and br-alternating $(s-1)$-arcs.*

Proof. Clearly G acts transitively on the set of rr-alternating 3-arcs. Since the action is discrete, there is an odd number s such that G acts transitively on the set of rr-alternating s-arcs, but does not act transitively on the set of rr-alternating $(s+2)$-arcs. Now the result follows from the proof of Lemma 19. □

Lemma 19 implies the following theorem.

Theorem 21. *Let Γ be a connected cubic graph. Suppose G acts vertex-transitively and non-discretely, but not edge-transitively, on Γ. Then, for every $s \geqslant 1$, the action of G on the set of all alternating s-arcs that start with an edge of a given colour is transitive.*

Proof. By Lemma 19 the group G acts transitively on the set of rr-alternating and the set of bb-alternating s-arcs for all odd $s \geqslant 1$. Then it also acts transitively on the set of all rb-alternating and the set of br-alternating arcs s-arcs for all even $s \geqslant 2$. □

We also want to consider the action of the group on infinite alternating arcs, but first we must show that such arcs actually exist. This is done in the next two corollaries.

Corollary 22. *Suppose Γ is a connected cubic graph and G acts on Γ vertex-transitively and non-discretely, but not edge-transitively. Let $(\alpha_0, \ldots, \alpha_s)$ be an alternating s-arc and $s \geqslant 2$. Then $\alpha_0 \neq \alpha_s$ and α_0 and α_s are not neighbours.*

Proof. Clearly $s \geqslant 3$.

Suppose first that $\alpha_0 = \alpha_s$. At least one of the edges $\{\alpha_0, \alpha_1\}$ and $\{\alpha_{s-1}, \alpha_s\}$ is blue, say it is the edge $\{\alpha_{s-1}, \alpha_s\}$. Let β be a vertex such that $\beta \neq \alpha_s$ and $\{\alpha_{s-1}, \beta\}$ is a blue edge. There is $g \in G$ such that g takes the alternating s-arc $(\alpha_0, \ldots, \alpha_{s-1}, \alpha_s)$ to the alternating s-arc $(\alpha_0, \ldots, \alpha_{s-1}, \beta)$, but this is clearly impossible, because $\alpha_0 = \alpha_s$. The case where $\{\alpha_0, \alpha_1\}$ is a blue edge is similar.

Suppose now that $(\alpha_0, \ldots, \alpha_s)$ is an alternating s-arc such that α_0 and α_s are neighbours. By the above the vertices $\alpha_0, \ldots, \alpha_s$ are all distinct. If the edge $\{\alpha_0, \alpha_s\}$ is red, then $(\alpha_s, \alpha_0, \ldots, \alpha_s)$ would be an alternating $(s+1)$-arc contradicting what is shown above. Thus the edge $\{\alpha_0, \alpha_s\}$ must be blue. If the edge $\{\alpha_0, \alpha_1\}$ is red, then $(\alpha_s, \alpha_0, \ldots, \alpha_s)$ would be an alternating $(s+1)$-arc and that is impossible, and if the edge $\{\alpha_{s-1}, \alpha_s\}$ is red, then $(\alpha_0, \ldots, \alpha_s, \alpha_0)$ would be an alternating $(s+1)$-arc. Hence we see that both the edges $\{\alpha_0, \alpha_1\}$ and $\{\alpha_{s-1}, \alpha_s\}$ must be blue. Let β be a vertex, distinct from α_s, such that $\{\alpha_{s-1}, \beta\}$ is a blue edge. Note that, by the above $\alpha_s \neq \alpha_1$ and $\beta \neq \alpha_1$. Let g be an element in G taking the alternating s-arc $(\alpha_0, \ldots, \alpha_{s-1}, \alpha_s)$ to the alternating s-arc $(\alpha_0, \ldots, \alpha_{s-1}, \beta)$. Then $\{\alpha_0, \alpha_s\}g = \{\alpha_0, \beta\}$ is a blue edge and α_0 is the end-vertex of 3 distinct blue edges $\{\alpha_0, \alpha_1\}, \{\alpha_0, \alpha_s\}$ and $\{\alpha_0, \beta\}$, which is impossible. We have reached a contradiction and our proof is complete. □

Corollary 23. *Suppose Γ is a connected cubic graph. Let G be a group acting on Γ vertex-transitively and non-discretely, but not edge-transitively. Then Γ contains an infinite alternating line and every alternating s-arc is a part of such a line.*

Proof. It is clear that every alternating s-arc can be extended to a 2-way infinite alternating arc. By Corollary 22 all the vertices in this infinite alternating arc must be distinct and thus we have an infinite alternating line. □

The argument used to prove Lemma 3 can be adapted to show the following.

Corollary 24. *Let Γ be a connected, cubic graph. Let G be a group acting on Γ vertex-transitively and non-discretely, but not edge-transitively. If $\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \ldots$ and $\ldots, \beta_{-1}, \beta_0, \beta_1, \beta_2, \ldots$ are two infinite alternating lines such that the edges $\{\alpha_0, \alpha_1\}$ and $\{\beta_0, \beta_1\}$ have the same colour, then there exists $g \in G$ such that $\alpha_i g = \beta_i$ for all i.*

6 Two-ended cubic graphs

Our next task is to classify connected, vertex-transitive, cubic graphs with two ends such that the automorphism group acts non-discretely. To do so we use the classification of highly arc-transitive digraphs with two ends and prime in- and out-degrees from [21, Corollary 16]. To describe the classification we need the concept of an *arc-digraph*.

Construction 25. (See [12, Section 4.2]) The *arc-digraph* of Γ, where Γ denotes an undirected graph or a digraph, is denoted with $\mathrm{Arc}(\Gamma)$. The set of vertices is the set of arcs in Γ; and if $(\alpha,\beta),(\gamma,\delta)$ are arcs in Γ, then $((\alpha,\beta),(\gamma,\delta))$ is an arc in $\mathrm{Arc}(\Gamma)$ if and only if $\beta=\gamma$ and $\alpha\neq\delta$ (i.e. (α,β,δ) is a 2-arc in Γ). The s-arc-digraph $\mathrm{Arc}^s(\Gamma)$ is defined such that the vertex set of $\mathrm{Arc}^s(\Gamma)$ is the set of s-arcs of Γ and the arcs in $\mathrm{Arc}^s(\Gamma)$ are pairs $((\alpha_0,\dots,\alpha_s),(\alpha_1,\dots,\alpha_{s+1}))$, where $(\alpha_0,\dots,\alpha_{s+1})$ is an $(s+1)$-arc in Γ. (Sometimes the digraph $\mathrm{Arc}(\Gamma)$ is called the *line graph* or the *partial line graph*. In our context the emphasis is on the fact that the vertex set of the arc-digraph is the set of arcs of Γ and the vertex set of the s-arc-digraph is the set of s-arcs in Γ. Thus the names *arc-digraph* and *s-arc-digraph* seem more appropriate.)

It is easy to see that the map

$$\mathrm{VArc}^s(\Gamma)\to\mathrm{VArc}(\mathrm{Arc}^{s-1}(\Gamma));(\alpha_0,\dots,\alpha_s)\mapsto((\alpha_0,\dots,\alpha_{s-1}),(\alpha_1,\dots,\alpha_s))$$

is a graph isomorphism (cf. [23, Lemma 3.2]). Thus the graph $\mathrm{Arc}^s(\Gamma)$ is isomorphic to $\mathrm{Arc}(\mathrm{Arc}^{s-1}(\Gamma))$. It is not hard to see that if Γ is a connected digraph such that if (α,β) is an arc then (β,α) is not an arc and in addition every vertex has non-zero in- and out-degree, then the same is true for $\mathrm{Arc}(\Gamma)$ and by induction one sees that $\mathrm{Arc}^s(\Gamma)$ is connected.

For a prime p, let Δ_p be the digraph with vertex set $\{1,\dots,p\}\times\mathbb{Z}$ and arc set the set of all pairs $((i,j),(i',j+1))$ with $i,i'\in\{1,\dots,p\}$ and $j\in\mathbb{Z}$. In [21, Corollary 16] the authors show that any highly arc-transitive digraph with two ends and in- and out-degree equal to p is isomorphic to Δ_p or one of its s-arc-digraphs $\mathrm{Arc}^s(\Delta_p)$ for some $s\geqslant 1$. These digraphs bear a close resemblance to the finite graphs defined by Praeger and Xu in [24], as remarked in [21]. (The authors thank the referees for pointing out that in [21] the arc-digraph is defined slightly differently as s-arcs in loc. cit. are allowed to backtrack. This does not make any difference here since there are no back-tracking arcs in Δ_p.)

Let Δ be a vertex- and arc-transitive digraph so that the in- and out-degrees are both equal to 2 and the degree of the underlying undirected graph is 4. Set $G=\mathrm{Aut}(\Delta)$ and let Γ denote the underlying undirected graph of Δ. The *reverse digraph* Δ^R has the same vertex set as Δ and (α,β) is an arc in Δ^R if and only if (β,α) is an arc in Δ. Suppose Δ is isomorphic to its reverse digraph Δ^{R} via some digraph isomorphism $f:\Delta\to\Delta^{\mathrm{R}}$. The group $\langle G,f\rangle$ acts vertex- and arc-transitively on Γ and acts locally as D_4, the dihedral group with 8 elements. The construction described in Part 6 of Example 6 now produces a cubic graph that $\langle G,f\rangle$ acts on such that the conditions in Case **C** are satisfied.

The digraphs Δ_2 and $\mathrm{Arc}^s(\Delta_2)$ have the property that they are isomorphic to their reverse digraph. Thus we can apply the construction described above and obtain from Δ_2

a cubic graph Θ_0 (depicted in Figure 2) and from $\mathrm{Arc}^s(\Delta_2)$ a cubic graph Θ_s satisfying the conditions in Case **C**. These graphs have precisely two ends.

Theorem 26. *Let* Γ *be a connected, cubic graph with two ends. Suppose* $\mathrm{Aut}(\Gamma)$ *acts on* Γ *vertex-transitively and non-discretely. Then* Γ *is isomorphic to* Θ_s *for some* $s \geqslant 0$*.*

Remark 27. In Section 8 we prove Theorem 3.1 from Trofimov's paper [25]. Trofimov remarks at the end of his proof that his argument contains a description of all connected, vertex-transitive, cubic graphs with two ends such that stabilisers in $\mathrm{Aut}(\Gamma)$ of vertices are infinite.

Before embarking on the task of proving Theorem 26 we have to prove a technical lemma. This lemma concerns ends of graphs and the action of the automorphism group on the ends (see the final paragraphs of Section 2.1 and the final paragraph of Section 2.2).

Lemma 28. *Let* Γ *be a connected, cubic graph with two ends. Suppose* $\mathrm{Aut}(\Gamma)$ *acts on* Γ *vertex-transitively and non-discretely.*

1. *The graph* Γ *satisfies the conditions in Case* **C**.
2. *Let* $\ldots, \alpha_{-1}, \beta_{-1}, \alpha_0, \beta_0, \alpha_1, \beta_1, \ldots$ *be an infinite alternating line such that the edges of type* $\{\alpha_i, \beta_i\}$ *are red and edges of the type* $\{\beta_i, \alpha_{i+1}\}$ *are blue. Then the rays* $\alpha_0, \beta_0, \alpha_1, \ldots$ *and* $\alpha_0, \beta_{-1}, \alpha_{-1}, \ldots$ *belong to different ends of* Γ*.*

Proof. The first part follows from Corollary 13 and the fact that the 3-regular tree has infinitely many ends.

Suppose that the rays $R_1 = \alpha_0, \beta_0, \alpha_1, \ldots$ and $R_2 = \alpha_0, \beta_{-1}, \alpha_{-1}, \ldots$ belong to the same end of Γ. Let Φ be a finite set of vertices such that the graph $\Gamma \setminus \Phi$ has precisely two infinite components B and B'. Because Γ is locally finite the graph $\Gamma \setminus \Phi$ has only finitely many components. By "adding" all the finite components to Φ we may assume that Φ is connected and that $V\Gamma = B \cup \Phi \cup B'$. Transitivity implies that we can assume that $\alpha_0 \in \Phi$. Suppose that B contains a subray of R_1 and as R_1 and R_2 are in the same end then B will also contain a subray of R_2. Let R be a ray in Γ that does not belong to the same end as R_1 and R_2. We can assume that the initial vertex in R is α_0 and that R contain no vertices from B. The set Φ separates the two ends of Γ and so will every translate of Φ. By Corollary 24 there exists an automorphisms g of Γ such that $\alpha_i g = \alpha_{i+1}$ and $\beta_i g = \beta_{i+1}$ for all i. Then $d_\Gamma(\alpha_i, \alpha_j)$ depends only on $|i-j|$ and since the graph Γ is locally finite we see that for every number c there is a number K such that if $|i-j| \geqslant K$, then $d_\Gamma(\alpha_i, \alpha_j) > c$. Since the set Φ is finite we see that there is a number j such that $\Phi g^j \subseteq B$ and Φg^j contains none of the vertices $\alpha_0, \beta_{-1}, \alpha_{-1}, \ldots$. Then the rays R and R_2 both belong to the same component of $\Gamma \setminus \Phi g^j$. The set Φg^j does thus not separate the two ends of Γ. Now we have reached a contradiction and conclude that the rays R_1 and R_2 can not belong to the same end. □

Proof of Theorem 26. Continue with the setup in the proof of the previous lemma. We aim to construct on the basis of Γ a connected highly arc-transitive digraph with two ends such that the in- and out-degrees of every vertex are both equal to 2 and the degree of the underlying undirected graph is 4.

From the above lemma we see that each end contains an alternating ray. Since $G = \mathrm{Aut}(\Gamma)$ acts transitively on the set of alternating rays that start with an edge of a given colour, we see that G acts transitively on the set of ends of Γ. Let G_0 denote the subgroup of G that fixes both ends. This subgroup G_0 has index 2 in G. Let $\{\alpha_0, \beta_0\}$ and $\{\alpha'_0, \beta'_0\}$ be two red edges in Γ. These two edges are parts of alternating lines $\ldots, \alpha_{-1}, \beta_{-1}, \alpha_0, \beta_0, \alpha_1, \beta_1, \ldots$ and $\ldots, \alpha'_{-1}, \beta'_{-1}, \alpha'_0, \beta'_0, \alpha'_1, \beta'_1, \ldots$, respectively. By renumbering if necessary we can assume that the alternating rays $R_1 = \alpha_0, \beta_0, \alpha_1, \beta_1, \ldots$ and $R_2 = \alpha'_0, \beta'_0, \alpha'_1, \beta'_1, \ldots$ belong to the same end of Γ. Corollary 24 says that $\mathrm{Aut}(\Gamma)$ contains an element g taking R_1 to R_2 and this element belongs to G_0 since it fixes the end that R_1 and R_2 belong to and $\{\alpha_0, \beta_0\}g = \{\alpha'_0, \beta'_0\}$. Thus G_0 acts transitively on the set of red edges and in the same way we see that G_0 acts transitively on the set of blue edges. But G_0 does not act transitively on the set of blue arcs. Suppose $g \in G_0$ reverses the the blue arc (β_0, α_1). Then $\ldots, \alpha_2 g, \beta_1 g, \beta_0, \alpha_1, \ldots$ is an alternating line. (The vertices are distinct by Corollary 22.) By Lemma 28 the rays $\alpha_1, \beta_1, \ldots$ and $\alpha_1, \beta_0, \beta_1 g, \ldots$ belong to different ends of Γ. But g maps the first one to the other contradicting the assumption that g fixes both ends of Γ. Similarly, we find that G_0 does not act transitively on the set of red arcs.

Let A be one of the G_0-orbits on the blue arcs in Γ and let A' be one of the G_0-orbits on the red arcs in Γ. Consider the digraph that has the same vertex set as Γ and $A \cup A'$ as arc set. Contract now all the red arcs in this digraph. We get a digraph Δ with two ends where the in- and out-degrees of every vertex are 2. By Theorem 21 this digraph is highly arc-transitive and thus isomorphic to Δ_2 or $\mathrm{Arc}^s(\Delta_2)$ for some $s \geqslant 1$. Then Γ is isomorphic to Θ_0 or Θ_s. □

More is to be said about these graphs since they all have the same (abstract) automorphism group, as we will show below.

We start by looking at the relationship between the automorphism group of a digraph and its arc-digraph and the relationship between the automorphism groups of the graphs in Part 6 in Example 6.

Lemma 29. 1. *(See [21, Lemma 15]) Let Γ be a digraph such that every vertex α is contained in an arc (α, β) and such that, for all arcs (α, β) and (α, β') of Γ, there exists $\gamma \in V\Gamma$ such that both (γ, α, β) and (γ, α, β') are 2-arcs. Then, the identity map $A\Gamma \to V\mathrm{Arc}(\Gamma)$ induces a group isomorphism $\mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\mathrm{Arc}(\Gamma))$.*

2. *Let Γ be a connected quartic graph such that $\mathrm{Aut}(\Gamma)$ acts vertex-transitively, non-discretely and locally like D_4. Then the graph Γ' constructed in Part 6 of Example 6 has the same automorphism group as Γ.*

Proof. 1. Clearly, if $g \in \mathrm{Aut}(\Gamma)$ and $(\alpha_0, \alpha_1, \alpha_2)$ is a 2−arc, then $(\alpha_0 g, \alpha_1 g, \alpha_2 g)$ is a 2-arc as well. Thus the action of $\mathrm{Aut}(\Gamma)$ on $A\Gamma = V\mathrm{Arc}(\Gamma)$ gives us an action of $\mathrm{Aut}(\Gamma)$ by automorphisms on $\mathrm{Arc}(\Gamma)$.

On the other hand, if $h \in \mathrm{Aut}(\mathrm{Arc}(\Gamma))$ and $\alpha \in \mathrm{V}\Gamma$ are such that (α, β) is an arc, we can simply define αh to be the vertex γ such that there is $\beta_1 \in \mathrm{V}\Gamma$ with $(\gamma, \beta_1) = (\alpha, \beta)h$. We have to check that this definition of αh is independent of β; it is then obvious that it is a graph automorphism and by the conditions on Γ, we have defined h on all of $\mathrm{V}\Gamma$.

Consider any vertex β' such that $(\alpha, \beta') \in \mathrm{A}\Gamma$ and $(\alpha, \beta')h = (\alpha_1, \beta_2)$. By assumption there is $\delta \in \mathrm{V}\Gamma$ such that (δ, α, β) and (δ, α, β') are 2-arcs. Then, viewed in $\mathrm{Arc}(\Gamma)$, by definition of a digraph morphism $(\delta, \alpha, \beta)h$ is an arc connecting $(\delta, \alpha)h$ and $(\alpha, \beta)h = (\gamma, \beta_1)$ and $(\delta, \alpha, \beta')h$ is an arc connecting $(\delta, \alpha)h$ and $(\alpha, \beta')h = (\alpha_1, \beta_2)$. This proves $\alpha_1 = \gamma$. Now we defined an action of $\mathrm{Aut}(\mathrm{Arc}(\Gamma))$ on Γ and thus a group homomorphism $\mathrm{Aut}(\mathrm{Arc}(\Gamma)) \to \mathrm{Aut}(\Gamma)$.

It is easy to check that the group homomorphisms $\mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\mathrm{Arc}(\Gamma)))$ and $\mathrm{Aut}(\mathrm{Arc}(\Gamma)) \to \mathrm{Aut}(\Gamma)$ are inverse to each other.

2. The automorphism group of Γ acts on the cubic graph Γ', which means that we have a homomorphism $\mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\Gamma')$. Since $\mathrm{Aut}(\Gamma)$ acts locally like D_4, the graph Γ cannot be the quartic tree and consequently the graph Γ' is not the cubic tree. As $\mathrm{Aut}(\Gamma')$ acts non-discretely and Γ' is not the cubic tree, it follows from Corollary 13 that Γ' and thus $\mathrm{Aut}(\Gamma)$ must satisfy the conditions in Case **C**. Now if we endow Γ' with an expedient edge colouring and then contract the red edges, we get the graph Γ. Therefore $\mathrm{Aut}(\Gamma')$ acts on Γ and we have a homomorphism $\mathrm{Aut}(\Gamma') \to \mathrm{Aut}(\Gamma)$. The homomorphisms in both directions between $\mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(\Gamma')$ are clearly the inverses of each other and are thus isomorphisms. □

We can determine the automorphism group of Δ_2 explicitly. Recall that $\mathrm{V}\Delta_2 = \{1,2\} \times \mathbb{Z}$. We identify the group C_2 with $\mathrm{Sym}(\{1,2\})$. Then the group $\prod_{\mathbb{Z}} C_2$ acts on Δ_2 via $(i,j)(f_k)_{k\in\mathbb{Z}} = (if_j, j)$. Also the group $\mathbb{Z}$ acts on Δ_2, namely via $(i,j)k = (i, j+k)$. One can say that the elements from $\prod_{\mathbb{Z}} C_2$ describe the action on the first coordinate of a vertex (i,j) and $\mathbb{Z}$ describes the action on the second coordinate. Thus every element of $\mathrm{Aut}(\Gamma)$ can be written as $(f_k)h$, where (f_k) comes from $\prod_{\mathbb{Z}} C_2$ and h comes from $\mathbb{Z}$. Multiplication is determined by the equation $h^{-1}(f_k)h = (f_{k+h})$. We have now described $\mathrm{Aut}(\Gamma)$ as a semidirect product $\prod_{\mathbb{Z}} C_2 \ltimes \mathbb{Z}$. This type of semidirect product is often called the (unrestricted permutation) wreath product of C_2 and $\mathbb{Z}$.

The map $f : \mathrm{V}\Delta_2 \to \mathrm{V}\Delta_2$ given by $(i,j)f = (i,-j)$ is an isomorphism from Δ_2 to its reverse digraph. The automorphism group of the underlying undirected graph of Δ_2 is thus $\prod_{\mathbb{Z}} C_2 \ltimes D_\infty$, where D_∞ denotes the infinite dihedral group.

From [21, Corollary 17] and Lemma 29(1) it follows that the groups $\mathrm{Aut}(\Delta_2)$ and $\mathrm{Aut}(\mathrm{Arc}^s(\Delta_2))$ for $s \geqslant 1$ are all isomorphic to the semidirect product $(\prod_{\mathbb{Z}} C_2) \ltimes \mathbb{Z}$. We obtain the graph Θ_s from $\mathrm{Arc}^s(\Delta_2)$ by first looking at the underlying undirected graph and then applying the construction from Part 6 in Example 6. The following corollary is then a consequence of Lemma 29.

Corollary 30. *Let Γ be a connected, cubic graph with two ends. Suppose $\mathrm{Aut}(\Gamma)$ acts on Γ vertex-transitively and non-discretely. Then $\mathrm{Aut}(\Gamma)$ is isomorphic to $(\prod_{\mathbb{Z}} C_2) \ltimes D_\infty$.*

7 Applications to totally disconnected, locally compact groups

A *topological group* is a group G with a topology on its underlying set such that the multiplication operation $G \times G \to G; (g_1, g_2) \mapsto g_1 g_2$ and the operation of inverting an element $G \to G; g \mapsto g^{-1}$ are continuous maps. The map $G \to G; x \mapsto xg$ given by right multiplication by a group element g is thus a homeomorphism. Often a separation condition, such as the topology being Hausdorff, is added to the definition. A topological group is said to be locally compact if the topology is locally compact, i.e. if every element in G has an open neighbourhood with compact closure. The group is *totally disconnected* if the topology is totally disconnected, i.e. the only connected subsets are singleton sets. Many of the topological groups occurring in other branches of mathematics are locally compact. The solution to Hilbert's fifth problem gives a way to use the theory of Lie groups in the study of connected topological groups. Let G be a locally compact group and define G_0 as the connected component containing the identity. Then G_0 is a normal subgroup and the quotient group G/G_0 is a totally disconnected, locally compact group. Thus one can say that the study of general locally compact groups can be divided into the study of connected, locally compact groups and the study of totally disconnected, locally compact groups. The study of totally disconnected, locally compact groups is also interesting in its own right and there are important examples of such groups coming from other branches of mathematics, such as matrix groups and Lie groups over the p-adic numbers and, as is explained below, automorphism groups of locally finite connected graphs.

The study of totally disconnected, locally compact groups has become an active field in recent years, largely due to the efforts of George Willis and his coworkers, see e.g. [32] and [5]. In this section we relate the work in the preceding section to the *scale function* introduced by Willis in [32].

Let G be a compactly generated, totally disconnected, locally compact group. If G acts vertex-transitively on a connected, locally finite graph Γ such that the stabilisers of vertices are compact, open subgroups of G, then we say that Γ is a *Cayley–Abels graph* for G. A Cayley–Abels graph for G can be constructed by starting with a compact generating set C and a compact open subgroup U of G (such a subgroup always exists by a theorem of van Dantzig, [30]). Then we form the Cayley graph of G with respect to C and define Γ as the quotient graph (see Section 2.2) with respect to the left action of U. Note that the vertex set of Γ is the set of right cosets of the subgroup U. This construction comes from Abels' paper [1, Beispiel 5.2], but for further information and another construction see the survey paper [20, Section 4] where the term *rough Cayley graph* is used instead of *Cayley–Abels graph*. For a survey from the perspective of geometric group theory, see [16]. Define $\mathrm{md}(G)$ as the minimal possible degree of a Cayley–Abels graph for G. This concept is the main topic of discussion in [2]. It follows easily from known results that $\mathrm{md}(G) = 2$ if and only if G has a Cayley–Abels graph with precisely two ends (and then every Cayley–Abels graph has precisely two ends), see [2, Theorem 4.1]. The results in this section give a special property for compactly generated, totally disconnected, locally compact groups such that $\mathrm{md}(G) = 3$.

The connection between totally disconnected, locally compact groups and group actions on graphs works in both directions. When G is a group acting on a set Ω, e.g. the automorphism group of a graph Γ acting on the vertex set $V\Gamma$, we can endow G with the *permutation topology*, see for instance [34] and [20]. One way to define the permutation topology is to say that a neighbourhood basis of the identity is formed by the family of all subgroups of the form $G_{(\Phi)}$, where Φ ranges over all finite subsets of Ω. In a topological group, a neighbourhood basis of the identity element completely determines the topology, since this neighbourhood basis can be translated to a neighbourhood basis of an element g by multiplying with g. If the group G already has a topology and the stabiliser G_α of a point $\alpha \in \Omega$ is open, then the permutation topology is a subset of the given topology on G. The permutation topology coincides with the *topology of pointwise convergence* and thus the convergence for sequences defined in Section 2.3 is the same as convergence in the permutation topology. Note that the permutation topology is Hausdorff if and only if the action is faithful. Using the permutation topology we find that a closed subgroup of the automorphism group of a locally finite graph Γ is a totally disconnected, locally compact group, see [34, Lemma 1] and, also, [20, Lemma 2.2].

In [32], Willis introduced the concepts of a *tidy subgroup* and the *scale function*. In this work we will only discuss the scale function and we use as definition a formulation from Willis' later paper [33]. The *scale function* on a totally disconnected, locally compact group G is the function $s : G \to \mathbb{Z}_+$ defined by the formula

$$s(g) = \min\{|U : U \cap g^{-1}Ug| \mid U \text{ a compact open subgroup of } G\}.$$

Note that $s(g)$ has to be finite; the reason is that $g^{-1}Ug \cap U$ is an open subgroup of G and its right U-cosets form a partition of U with cardinality $|U : U \cap g^{-1}Ug|$. Each element of the partition is open and by compactness of U, the partition has to be finite.

Let U be a compact, open subgroup of G and consider the action of G on the set of right cosets $\Omega = G/U$. Set $\alpha = U$ and think of α as a point in Ω. It is shown in [19, Corollary 7.8] that

$$s(g) = \lim_{n\to\infty} |(\alpha g^n)G_\alpha|^{1/n},$$

and, furthermore, $s(g) = 1$ if and only if there is a constant C such that $|(\alpha g^i)G_\alpha| \leqslant C$ for all $i = 0, 1, 2, \ldots$. A totally disconnected, locally compact group is said to be *uniscalar* if $s(g) = 1$ for all $g \in G$.

The arguments used in the proof of the following lemma are somewhat reminiscent of arguments found in [19] and the notation is chosen to reflect this similarity.

Lemma 31. *Suppose Γ is a connected, vertex-transitive, cubic graph. Assume there exists a closed subgroup $G \leqslant \mathrm{Aut}(\Gamma)$ such that the action of G on Γ satisfies the conditions in Case* **C**. *Consider an alternating line $\ldots, \alpha_{-1}, \beta_{-1}, \alpha_0, \beta_0, \alpha_1, \beta_1, \ldots$ in Γ such that the edges of type $\{\alpha_i, \beta_i\}$ are red and the edges of type $\{\beta_i, \alpha_{i+1}\}$ are blue. If there is a constant C such that $|\alpha_i G_{\alpha_0}| \leqslant C$ for all $i \geqslant 1$, then Γ has exactly two ends.*

Proof. By Corollary 24 there exists an element $g \in G$ such that $\alpha_i g = \alpha_{i+1}$ and $\beta_i g = \beta_{i+1}$ for all i. Set $U = G_{\alpha_0}$. Define $U_{-\infty,i}$ as the subgroup of G fixing pointwise the ray

$\ldots, \alpha_{i-1}, \beta_{i-1}, \alpha_i, \beta_i$. These subgroups are all conjugate via powers of g. Now define U_{++} as the subgroup $\bigcup_{i\in\mathbb{Z}} U_{-\infty,i}$ and $G_{++} = \langle U_{++}, g\rangle$. Note that $g^{-1}U_{++}g = U_{++}$. Clearly

$$G_{++} = \{h \in G \mid \text{there exist } m, n \in \mathbb{Z} \text{ such that } (\ldots, \alpha_m, \beta_m)h = (\ldots, \alpha_n, \beta_n)\}.$$

Let Γ_{++} denote the subgraph that has vertex set $\alpha_0 G_{++} \cup \beta_0 G_{++}$ and edge set $\{\alpha_0, \beta_0\}G_{++} \cup \{\beta_0, \alpha_1\}G_{++}$. Our aim is to show that Γ_{++} is equal to Γ. The graph Γ_{++} is connected. The group G_{++} has at most two orbits on the vertex set of Γ_{++} and also at most two orbits on the edge set. It follows from the transitivity on alternating lines in Γ (see Corollary 24 above) that all the vertices in the orbit $\beta_i G_{++}$ have degree 3 in the graph Γ_{++}.

Suppose n is a number such that $2^n > C$. There are 2^n alternating $2n$-arcs having α_0 as their initial vertex and starting with the red edge $\{\alpha_0, \beta_0\}$. From Corollary 24 we see that the group $U_{-\infty,0}$ acts transitively on the set of these arcs. But the orbit $\alpha_n U_{-\infty,0}$ has fewer than 2^n elements and thus there is some alternating $2n$-arc in Γ_{++} of the form $(\alpha_0, \beta_0, \alpha'_1 \ldots, \alpha'_{n-1}, \beta_{n-1}, \alpha_n)$ that is different from the $2n$-arc $(\alpha_0, \beta_0, \alpha_1 \ldots, \alpha_{n-1}, \beta_{n-1}, \alpha_n)$. Note that it is impossible that $\alpha_i = \alpha'_i$ for all i. Let i be the biggest number such that $\alpha_i \neq \alpha'_i$. Then $\beta_i \neq \beta'_i$ and $\beta'_{i+1} = \beta_{i+1}$. Hence the vertices β_i, β'_i and β_{i+1} are all distinct and all of them are neighbours of α_{i+1} (recall that the edges $\{\beta_i, \alpha_{i+1}\}$ and $\{\beta'_i, \alpha_{i+1}\}$ are both blue but the edge $\{\alpha_{i+1}, \beta_{i+1}\}$ is red). Thus the vertex α_{i+1} has degree 3 in Γ_{++}. Hence Γ_{++} is regular with degree 3. Since Γ is a connected, cubic graph, we see that $\Gamma_{++} = \Gamma$.

The orbits $\alpha_i U_{++}$ are all finite and each orbit has size at most C. The same holds true for the orbits $\beta_i U_{++}$. We also see that $(\alpha_i U_{++})g = \alpha_{i+1}U_{++}$ and similarly that $(\beta_i U_{++})g = \beta_{i+1}U_{++}$. Hence $\langle g\rangle$ has at most $2C$ orbits on Γ. A result of Jung and Watkins [14, Theorem 5.12] says that a connected, vertex-transitive graph that has an automorphism with only finitely many orbits has just two ends. (For the readers' convenience a direct proof of this fact is included in Appendix C.) □

Remark 32. From the argument above we see that it is enough to assume that there exists some positive integer n such that $|\alpha_n U_{-\infty,0}| < 2^n$ to get the conclusion that Γ has exactly two ends.

Lemma 33. *Let G be a totally disconnected, locally compact group that has a cubic Cayley–Abels graph Γ such that G has two orbits on the edges of Γ. Let K denote the kernel of the action of G on $V\Gamma$. Suppose the group G_α/K is infinite for one, and hence every, vertex α in Γ. If the group G is uniscalar, then Γ has two ends, and G has a compact, open, normal subgroup.*

Proof. The conditions in Case **C** are satisfied, except that we don't know if the action of G on Γ is faithful. Hence we have an expedient colouring of the edges of Γ. Let $\ldots, \alpha_{-1}, \beta_{-1}, \alpha_0, \beta_0, \alpha_1, \beta_1, \ldots$ be an alternating line in Γ such that the edges of type $\{\alpha_i, \beta_i\}$ are red and edges of the type $\{\beta_i, \alpha_{i+1}\}$ are blue. By Corollary 24, there exists $g \in G$ such that $\alpha_i g = \alpha_{i+1}$ and $\beta_i g = \beta_{i+1}$. As mentioned above, the assumption that

$s(g) = 1$ implies that there is a constant C such that $C \geqslant |(\alpha_0 g^n)G_{\alpha_0}| = |\alpha_n G_{\alpha_0}|$ for all n. From Lemma 31 we see that Γ has just two ends.

From Corollary 30 we see that $\mathrm{Aut}(\Gamma)$ is isomorphic to $\prod_{\mathbb{Z}} C_2 \rtimes D_\infty$. The subgroup $\prod_{\mathbb{Z}} C_2$ in $\mathrm{Aut}(\Gamma)$ is a compact, open normal subgroup in the permutation topology and thus there is a homomorphism $\mathrm{Aut}(\Gamma) \to D_\infty$ with a compact, open kernel. The image of this homomorphism is isomorphic to D_∞. Thus we have a homomorphism $G \to D_\infty$ and the kernel is a compact, open, normal subgroup of G. □

Theorem 34. *Suppose G is a compactly generated, totally disconnected, locally compact group. If G has a cubic Cayley–Abels graph, then either G has a compact, open, normal subgroup or G is not uniscalar.*

Proof. Let Γ be a cubic Cayley–Abels graph for G. Assume that G has no compact, open, normal subgroup. Denote the kernel of the action of G on Γ by K. Let α be a vertex in Γ. If G_α/K is a finite group, then there is a finite set of vertices in Γ such that the pointwise stabiliser of this set acts trivially on Γ, i.e. the kernel K is equal to the pointwise stabiliser of this finite set. Hence the kernel of the action of G on Γ is a compact, open, normal subgroup of G, contradicting our assumption. Whence G_α/K must be infinite. We consider separately what happens in Cases **A**, **B** and **C**.

Let us first look at Case **A**. Consider an infinite line $\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots$ in Γ. Let $g \in G$ be an element such that $\alpha_i g = \alpha_{i+1}$ for all i. Then $\alpha_0 g^n = \alpha_n$. By Theorem 8 we see that $|\alpha_n G_{\alpha_0}| = 3 \cdot 2^{n-1}$ and then

$$s(g) = \lim_{n\to\infty} |(\alpha g^n)G_\alpha|^{1/n} = \lim_{n\to\infty} \left(3 \cdot 2^{n-1}\right)^{1/n} = 2.$$

Hence G is not uniscalar.

In Case **B** we let Γ_+ be the digraph defined in the proof of Theorem 12. Suppose that $(\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots)$ is a 2-way ∞-arc in Γ_+ and $g \in G$ acts like a translation on this arc such that $\alpha_i g = \alpha_{i+1}$ for all i. The fact that G acts highly arc-transitively on Γ_+ implies that if $n \geqslant 0$, then $|\alpha_n G_{\alpha_0}| = 2^n$. Thus

$$s(g) = \lim_{n\to\infty} |(\alpha g^n)G_\alpha|^{1/n} = \lim_{n\to\infty} |\alpha_n G_\alpha|^{1/n} = \lim_{n\to\infty} \left(2^n\right)^{1/n} = 2.$$

Finally we consider Case **C**. If the action of G on Γ is uniscalar, then, by Lemma 33, we see that G must have a compact, open, normal subgroup. Since we are assuming that G has no compact, open, normal subgroup we conclude that G cannot be uniscalar. □

Let G be a totally disconnected, locally compact group and $g \in G$. From the definition of the scale function we see that $s(g) = 1$ if and only if g normalises some compact, open subgroup of G, and the group G is uniscalar if and only if for every element of G there is some compact, open subgroup normalised by g. If G has a compact, open, normal subgroup, then G is clearly uniscalar. Bhattacharjee and Macpherson [3, Section 3] (following up on work by Kepert and Willis, [15]), constructed an example of a compactly generated, totally disconnected, locally compact group that has no compact, open, normal subgroup, but every element normalises some compact open subgroup. On the other hand

Glöckner and Willis have shown in [11] that a compactly generated, uniscalar p-adic Lie group has a compact, open, normal subgroup.

Corollary 35. *Let G be a compactly generated, totally disconnected, locally compact group having a cubic Cayley–Abels graph. If every $g \in G$ normalises a compact open subgroup of G (i.e. G is uniscalar), then G has a compact, open, normal subgroup.*

8 Trofimov's result on cubic graphs

For a graph Γ we let Γ_n denote the the graph that has the same vertex set as Γ and two distinct vertices α and β are adjacent in Γ_n if and only if $d_\Gamma(\alpha, \beta) \leqslant n$.

Definition 36. Let Γ be a graph. We say that Γ *essentially includes a tree* if there exists a number n such that the graph Γ_n contains the cubic tree as a subgraph.

Remark 37. Trofimov uses the term *hyperbolic* for graphs having the property described above, but we follow Cornulier [6] in using the term *essentially includes a tree.*

An action of a group G on a set Ω is said to be *nearly discrete* if there is a G-congruence σ on Ω with finite equivalence classes such that if K is the kernel of the action of G on Ω/σ, then the action of G/K on Ω/σ is discrete.

In his paper from 1984, [25], Trofimov considers the following question:

Is it true that if Γ is a locally finite, connected graph and G is a subgroup of $\mathrm{Aut}(\Gamma)$ *acting transitively on the vertices, then the graph Γ essentially includes a tree or the action is nearly discrete?*

In [6] Cornulier constructs an example of a vertex-transitive, locally finite graph that does not essentially include a tree and the action of its automorphism group is not nearly discrete, thereby giving a negative answer to Trofimov's question. But, Trofimov showed in [25] that the answer is "yes" if it is assumed that the graph Γ has degree 3. Our methods yield a short proof of Trofimovs's result.

Theorem 38. ([25, Theorem 3.1]) *Let Γ be a vertex-transitive, cubic graph and $G =$* $\mathrm{Aut}(\Gamma)$*. Then, the action of G is nearly discrete or Γ_2 contains a subgraph isomorphic to the 3-regular tree.*

Proof. We may assume that the stabilisers in G of vertices in Γ are infinite. If G acts edge-transitively, then Corollary 13 says that Γ is a tree. Hence we may assume that G does not act transitively on the edges of Γ and that the conditions in Case **C** are satisfied. Once again we work with an expedient colouring. Let $\ldots, \alpha_{-1}, \beta_{-1}, \alpha_0, \beta_0, \alpha_1, \beta_1, \ldots$ be an alternating line in Γ such that the edges $\{\alpha_i, \beta_i\}$ are red and the edges $\{\beta_i, \alpha_{i+1}\}$ are blue. Let $U_{-\infty,0}$ denote the pointwise stabiliser of the ray $\ldots, \alpha_{-1}, \beta_{-1}, \alpha_0$. If there is a positive integer n such that $|\alpha_n U_{-\infty,0}| < 2^n$, then it follows from the remarks after Lemma 31 and Lemma 33 that Γ has exactly two ends and the action is nearly discrete.

Since $|\alpha_n U| \leqslant 2^n$, we are now left to consider the case where $|\alpha_n U| = 2^n$ for every positive integer n. First note that $\{\alpha_i, \alpha_{i+1}\}$ is an edge in the graph Γ_2. We consider the subgraph Δ of Γ_2 with vertex set $\bigcup_{i=0}^{\infty} \alpha_i U$ and edge set $\bigcup_{i=0}^{\infty}\{\alpha_i, \alpha_{i+1}\}U$. In this graph the vertex α_0 has degree 2 and every other vertex has degree 3. The set of vertices in Δ at distance n from α_0 is equal to $\alpha_n U$ and since $|\alpha_n U| = 2^n$ we conclude that Δ is the infinite rooted binary tree. When we apply the same argument to the ray $\alpha_{-1}, \alpha_{-2}, \ldots$ and its pointwise stabiliser we find another copy of the rooted binary tree inside Γ_2. This second tree has root α_{-1} and is disjoint from the first one. Since the two roots α_{-1} and α_0 are adjacent in Γ_2, these two rooted trees together with the edge $\{\alpha_{-1}, \alpha_0\}$ give a copy of the 3-regular tree. □

Combining Proposition 7 with Corollary 35 and Theorem 38 yields:

Proposition 39. ([26, Example 5.5]) *Let G be a group that acts vertex-transitively on a locally finite, connected graph Γ of degree d. Assume that G acts locally like the dihedral group with $2d$ elements in its usual action on a set with d elements. Then, either the action is nearly discrete or the graph Γ essentially includes a tree.*

Acknowledgements

The authors thank the anonymous referees for careful reading and for their suggestions, which helped improving the presentation of the article.

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Appendix A: Quartic graphs

The results involving Case **C** can be rephrased as results for a special class of group actions on quartic (4-regular) graphs. This connection with quartic graphs has already appeared in Parts 6 and 7 of Example 6 and the proof of Theorem 26. We will use the notation described in Example 6.

Let Γ be a connected quartic graph and G a group acting vertex-transitively on Γ such that the local action is D_4. For a vertex α there is a natural G_α-congruence σ_α on $\Gamma(\alpha)$ with two classes $\Sigma_{\alpha,1}$ and $\Sigma_{\alpha,2}$. Let Γ' denote the cubic graph constructed on the basis of Γ as in Example 6. An Σ-*alternating* s-*arc* in Γ is an s-arc $(\alpha_0, \ldots, \alpha_s)$ such that α_{i-1} and α_{i+1} belong to different σ_{α_i} classes. The Σ-alternating s-arc in Γ gives us an rr-alternating $2s$-arc in Γ' starting with either $\Sigma_{\alpha_0,1}$ or $\Sigma_{\alpha_0,2}$ and ending with either $\Sigma_{\alpha_s,1}$ or $\Sigma_{\alpha_s,2}$. If, on the other hand, we start with a cubic graph and a subgroup of the automorphism group so the conditions in Case **C** are satisfied, then an rr-alternating $2s$-arc gives an Σ-alternating s-arc in the quartic graph that we get upon contracting the red edges. Thus one can go back-and-forth between cubic graphs with a group action such that the conditions in Case **C** are satisfied and quartic graphs with a group acting vertex-transitively and locally like D_4. Applying Lemma 19 and Corollary 20 to Γ' we get:

Corollary A.1 *Let Γ be a connected quartic graph and G a group acting vertex-transitively on Γ such that the local action is D_4. If the action of G on Γ is non-discrete, then G acts transitively on the set of Σ-alternating s-arcs in Γ for every $s \geqslant 0$. If the action is discrete, then there is a number s such that G acts regularly on the set of Σ-alternating s-arcs.*

Remark 40. The part in the above corollary about groups with finite vertex-stabilisers is stated in [9, p. 25]. There the author, Djoković, attributes this observation to G. L. Miller and says that it can be proved by applying the same argument as in the proof of 7.72 in [29]. A result in similar vein as Corollary 8 is [31, Lemma 2.3].

Example A.2 Tutte showed in [27] and [28] that if Γ is a connected finite cubic graph and $G \leqslant \mathrm{Aut}(\Gamma)$ acts arc-transitively, then G acts regularly on the set of s-arcs for some $s \leqslant 5$. In Corollaries 20 and A.1, where the conclusion is that G acts regularly on a specific set of Σ-alternating s-arcs, we do not get a general bound on s as the following example shows.

Let Γ_s be a graph with vertex set $\{0, 1, \ldots, s\} \times \{1, 2\}$ for some $s \geqslant 2$. If $0 \leqslant r \leqslant s-1$, then $\{(r, i), (r+1, j)\}$ is an edge in Γ_s for all $i, j \in \{1, 2\}$ and in addition $\{(s, i), (0, j)\}$ is an edge in Γ_s for all $i, j \in \{1, 2\}$. This is clearly a connected quartic graph and its automorphism group acts vertex-transitively and locally like D_4. The automorphism group acts regularly on the Σ-alternating s-arcs.

Corollary A.3 below is an analogue of Theorem 26. It would also be possible to deduce this directly from [21, Corollary 16] without mentioning cubic graphs.

Corollary A.3 *Let* Γ *be a connected, vertex-transitive, quartic graph with two ends. Suppose* $\mathrm{Aut}(\Gamma)$ *acts non-discretely and locally like* D_4. *Then* Γ *is isomorphic to the underlying undirected graph of* Δ_2 *or the underlying undirected graph of* $\mathrm{Arc}^s(\Delta_2)$ *for some* $s \geqslant 1$.

Rephrasing Theorem 34 and Corollary 35 we get:

Corollary A.4 *Let* G *be a compactly generated, totally disconnected, locally compact group having a quartic Cayley–Abels graph such that the local action of* G *is* D_4. *Then either* G *has a compact, open, normal subgroup or* G *is not uniscalar. In particular, if every* $g \in G$ *normalises a compact open subgroup of* G *(i.e.* G *is uniscalar), then* G *has a compact, open, normal subgroup.*

Appendix B: Diestel-Leader graphs

The Diestel-Leader graphs were first defined by Diestel and Leader in [8]. Diestel and Leader start with a regular directed tree T with in-degree q and out-degree r (in their paper they have $q = 2$ and $r = 3$). Then they look at the s-arc-digraphs $\mathrm{Arc}^s(T)$ and construct the limit of this sequence of digraphs. The underlying undirected graph of this limit is the Diestel-Leader graph $\mathrm{DL}(q, r)$.

The following direct construction of the Diestel-Leader graphs comes from Woess, see [35, p. 131] and [36]: Start with a regular tree T with degree $q+1$. Select a fixed reference end ω of T. Let $\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots$ be a line in T such that the ray $\alpha_0, \alpha_{-1}, \ldots,$ belongs to ω. Set

$$H = \{g \in \mathrm{Aut}(T) \mid g \text{ fixes all the vertices } \alpha_N, \alpha_{N-1}, \ldots \text{ for some number } N\}.$$

Then H is a subgroup of $\mathrm{Aut}(T)$. The *horocycles* of T with respect to the end ω are defined as the orbits of H. The function $h : \mathrm{V}T \to \mathbb{Z}$ defined by setting $h(\alpha) = j$ if α is in the same horocycle as α_j is called a *Busemann function*.

Definition B.1 Let T be a regular tree with degree $q + 1$ and let T' be a regular tree with degree $r + 1$. Let $h : \mathrm{V}T \to \mathbb{Z}$ and $h' : \mathrm{V}T' \to \mathbb{Z}$ denote Busemann functions as above. The vertex set of the Diestel-Leader graph $\mathrm{DL}(q, r)$ is the set $\{(\alpha, \alpha') \in \mathrm{V}T \times \mathrm{V}T' \mid h(\alpha) + h(\alpha') = 0\}$. Two vertices (α, α') and (β, β') in $\mathrm{DL}(q, r)$ are adjacent if and only if α and β are adjacent in T, and α' and β' are adjacent in T'.

The graph $\mathrm{DL}(2, 2)$ is a quartic graph. It is vertex-transitive and the vertex-stabilisers in its automorphism group are infinite and act locally like D_4. Furthermore this graph has only one end. As explained in Part 7 of Example 6 it is thus possible to use $\mathrm{DL}(2, 2)$ to construct a one-ended cubic graph that satisfies the conditions in Case **C**.

An alternative description of the Diestel-Leader graphs is given in [18, Example 1]. This description is helpful if one wants to visualise these graphs. To simplify the exposition we only describe $\mathrm{DL}(2, 2)$. Start with a copy T of the cubic tree and let h denote a

Busemann function as above defined with respect to a fixed end ω. For a horocycle i (the horocycle containing the vertex α_i) take a copy T_i of T and let $\varphi_i : \mathrm{V}T \to \mathrm{V}T_i$ be an isomorphism. Now identify α and $\varphi(\alpha)$ for every vertex α in $\mathrm{V}T$ such that $h(\alpha) \geqslant i$. Do this for every i. When that is done, every vertex in T has degree 4. For each horocycle in T_i where the vertices still have degree 3 take a new copy of T and identify vertices as above. Continue like this *ad infinitum* until you have a graph with valency 4. This graph is the graph $\mathrm{DL}(2, 2)$.

Variants of the above description can be used to produce further examples of cubic graphs satisfying the conditions in Case **C**. Let T denote the cubic tree equipped with a Busemann function with respect to some end ω. We say a vertex β is a descendant of a vertex α if the unique ray in ω that has β as its initial vertex contains α. If A is a set of vertices in T, the descendant set of A is the union of the descendant sets of all the vertices in A. In the description of $\mathrm{DL}(2, 2)$ above we identify each vertex in T that is in the descendant set of horocycle i to some vertex that is in the descendant set in T_i of some particular horocycle.

Say, vertices β, β' are *siblings* if there is a vertex α such that both β and β' are adjacent to α and both are descendants of α. For each pair β, β' of siblings in T we get a new copy $T_{\beta,\beta'}$ of T and an isomorphism $\varphi : T \to T_{\beta,\beta'}$. For each vertex α in the descendant set of $\{\beta, \beta'\}$ identify α with $\varphi(\alpha)$. As above we continue this process until we end up with a quartic graph Γ. The stabilisers of vertices in the automorphism group of this graph are infinite and the automorphism group acts locally like D_4. The role of the pair of siblings in this construction could be taken over by a 4 element set $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ of *cousins* (meaning that there is a vertex α such that $\beta_1, \beta_2, \beta_3, \beta_4$ are all descendants of α and the $d_\Gamma(\alpha, \beta_i) = 2$ for all i). Instead of using 4 element sets one could use 8 element sets and so on.

Each set of cousins contains two pairs of siblings and we colour the vertices in one of these pairs white and the other black. For horocycle i in a cubic tree T we take two copies $T_{i,w}$ and $T_{i,b}$ of the cubic tree together with isomorphisms $\varphi_{i,w} : T \to T_{i,w}$ and $\varphi_{i,b} : T \to T_{i,b}$. For each vertex α that belongs to the descendant set of a white vertex in the i-th horocycle in T we identify α and $\varphi_{i,w}(\alpha)$ and similarly identify α and $\varphi_{i,b}(\alpha)$ if α belongs to the descendant set of a black vertex in the i-th horocycle in T. We continue this process until we get a quartic graph. The resulting graph has one end, is vertex-transitive, the automorphism group acts non-discretely and the local action of the automorphism group is D_4. Again, there are obvious variants of this construction.

Appendix C: Two-ended graphs

In the proof of Lemma 31 we make use of a result that is contained in the the paper [14, Section 5] by Jung and Watkins. The purpose of this appendix is to give a completely self-contained proof of this result.

Lemma C.1 *Let Γ be an infinite, connected, locally finite graph such that its automorphism group has only finitely many orbits on the vertex set. Then Γ has precisely two* *ends if and only if there exists an automorphism* $g \in \mathrm{Aut}(\Gamma)$ *such that* $\langle g \rangle$ *has only finitely many orbits on the vertex set of* Γ*.*

Proof. Recall that the graph Γ has precisely two ends if for every finite set Φ of vertices the graph $\Gamma \setminus \Phi$ has at most two infinite components and there exists a finite set Φ_0 of vertices such that the graph $\Gamma \setminus \Phi_0$ has two infinite connected components. We denote by $d_\Gamma(-,-)$ the graph theoretical distance on Γ.

We now assume that Γ has precisely two ends. Let Φ_0 be a set as described above and let B and B' denote the two infinite components of $\Gamma \setminus \Phi_0$. Since Γ is locally finite we can be assured that $\Gamma \setminus \Phi_0$ has at most finitely many components. Thus, we may assume that the set Φ_0 is connected and that $V\Gamma = B \cup \Phi_0 \cup B'$. Our first task is to find an automorphism g such that $(B \cup \Phi_0)g \subseteq B$ or such that $(B' \cup \Phi_0)g \subseteq B'$. The following argument comes from [13]. Since $\mathrm{Aut}(\Gamma)$ has only finitely many orbits on $V\Gamma$ and Γ is connected, we see that there is a constant c such that for every vertex α and every orbit O of $\mathrm{Aut}(\Gamma)$ there is some vertex α_O from O with $d_\Gamma(\alpha, \alpha_O) \leqslant c$. The graph Γ is assumed to be locally finite and thus the diameters of the infinite components B and B' are infinite. If d denotes the diameter of the finite set Φ_0 and α is a vertex in Φ_0 then there is a vertex β in B such that β is in the same $\mathrm{Aut}(\Gamma)$-orbit as α and the distance from β to any vertex in Φ_0 is at least d. Thus if h is an automorphism such that $\alpha h = \beta$ then $\Phi_0 h \subseteq B$. Similarly we can find an automorphism h' such that $\Phi_0 h' \subseteq B'$. (In this argument the assumption that the graph is locally finite is used in an essential way. If the assumption of local finiteness is dropped but it assumed that the graph is vertex-transitive then we get a similar result as we have just proved, see [7, Section 3].) If it so happens that $Bh \subseteq B$ or $B'h' \subseteq B'$, then we need look no further. So let us assume that Bh is not contained in B and that $B'h'$ is not contained in B'. The connected set Φ_0 is either contained in Bh or $B'h$. If $\Phi_0 \subseteq B'h$, then $Bh \subseteq B$. Since we are assuming that Bh is not contained in B we see that $\Phi_0 \subseteq Bh$ and then $(B' \cup \Phi_0)h \subseteq B$. Similarly $(B \cup \Phi_0)h' \subseteq B'$. Now set $g = h'h$. Then

$$(B \cup \Phi_0)g = (B \cup \Phi_0)h'h \subseteq B'h \subseteq B.$$

Now we have to show that g has only finitely many orbits on the vertex set of Γ. Since we assumed that Γ has precisely two ends, the graph $\Gamma \setminus (\Phi_0 \cup \Phi_0 g)$ has precisely two infinite components and they are B' and Bg. Thus the set $\Psi = V\Gamma \setminus (B' \cup Bg)$ is finite. Let α_0 be some vertex in Φ_0. If β is a vertex in Bg^j for some $j > 0$, then a path from α_0 to β has to contain vertices from all of the sets $\Phi_0 g, \ldots, \Phi_0 g^j$. As these sets are disjoint, we see that $d_\Gamma(\alpha_0, \beta) > j$. Thus $\bigcap_{j \geqslant 0} Bg^j = \emptyset$ and by symmetry $\bigcap_{j \leqslant 0} B'g^j = \emptyset$. From this we see that $V\Gamma = \bigcup_{j \in \mathbb{Z}} \Psi g^j$. Because Ψ is finite we conclude that g has only finitely many orbits on Γ.

Assume now that g is an automorphism of Γ such that $\langle g \rangle$ has only finitely many orbits on the vertex set of Γ. For a set $A \subseteq V\Gamma$ we define

$$N_r(A) = \{\beta \in V\Gamma \mid d_\Gamma(\beta, \gamma) \leqslant r \text{ for some } \gamma \in A\}.$$

Let α_0 be some vertex in Γ. Set $\alpha_i = \alpha_0 g^i$. Since $\langle g \rangle$ has only finitely many orbits on Γ, there is a finite set of vertices $F \subseteq V\Gamma$ that contains an vertex from each of the

$\langle g\rangle$ orbits and then $\bigcup_{i\in\mathbb{Z}} Fg^i = \mathrm{V}\Gamma$. But Γ is connected we see that there is a number c, which is at most the maximal distance between two vertices in F, such that for every vertex β in Γ there is some i such that $d_\Gamma(\beta,\alpha_i)\leqslant c$. The graph Γ is infinite and locally finite, implying that the orbit of α_0 under $\langle g\rangle$ is infinite. Since $\langle g\rangle$ acts by isometries, we see that $d_\Gamma(\alpha_i,\alpha_j)$ only depends on $|i-j|$ and thus there is a number K such that if $|i-j|\geqslant K$, then $d_\Gamma(\alpha_i,\alpha_j)\geqslant 2c+2$. Set $\Phi_0 = N_c(\{\alpha_1,\ldots,\alpha_K\})$. The set Φ_0 is finite. We want to show that the graph $\Gamma\setminus\Phi_0$ is not connected.

Let $i\leqslant 0$ and $j\geqslant K+1$ be numbers such that neither α_i nor α_j are in Φ_0. Suppose $\beta_0,\ldots,\beta_s$ is a path in $\Gamma\setminus\Phi_0$ such that $\beta_0=\alpha_i$ and $\beta_s=\alpha_j$. From the way K is defined it follows that $d_\Gamma(\alpha_i,\alpha_j)\geqslant 2c+2$. Let k be the largest number such that β_k is in distance at most c from one of the vertices $\alpha_0,\alpha_{-1},\ldots$. Note that k is well defined because $\beta_0=\alpha_i$ and $k<s$ because $\beta_s=\alpha_j$ and $d_\Gamma(\alpha_\ell,\alpha_j)\geqslant 2c+2$ for all $\ell=0,-1,\ldots$. Then there is a number $\ell\leqslant 0$ such that β_k is in distance at most c from α_ℓ and since $\beta_{k+1}\notin\Phi_0$ there must be a number $\ell'\geqslant K+1$ such that β_{k+1} is in distance at most c from $\alpha_{\ell'}$. This leads to a contradiction since

$$d_\Gamma(\alpha_\ell,\alpha_{\ell'})\leqslant d_\Gamma(\alpha_\ell,\beta_k)+d_\Gamma(\beta_k,\beta_{k+1})+d_\Gamma(\beta_{k+1},\alpha_{\ell'})\leqslant c+1+c=2c+1,$$

but $d_\Gamma(\alpha_\ell,\alpha_{\ell'})\geqslant 2c+2$. Hence it is impossible that such a path $\beta_0,\ldots,\beta_s$ exists. We have now shown that α_i and α_j cannot belong to the same component of $\Gamma\setminus\Phi_0$.

Note that $d_\Gamma(\alpha_{i-1},\alpha_i)$ is a constant independent of i. Thus we can find a number $M\leqslant 0$ such that if $i\leqslant M$, then there is a path from α_{i-1} to α_i that does not intersect Φ_0. From this we see that the vertices $\alpha_M,\alpha_{M-1},\ldots$ all belong to the same component of $\Gamma\setminus\Phi_0$. Similarly we can find a number $M'\geqslant K+1$ such that the vertices $\alpha_{M'},\alpha_{M'+1},\ldots$ all belong to the same component of $\Gamma\setminus\Phi_0$. As we saw above the vertices α_M and $\alpha_{M'}$ cannot belong to the same component of $\Gamma\setminus\Phi_0$ and thus $\Gamma\setminus\Phi_0$ has at least two infinite components.

The final step is to show that if Φ is a finite set of vertices, then $\Gamma\setminus\Phi$ has at most two infinite components. As above we can find numbers M and M' such that the vertices $\alpha_M,\alpha_{M-1},\ldots$ all belong to some component B of $\Gamma\setminus\Phi$ and all the vertices $\alpha_{M'},\alpha_{M'+1},\ldots$ all belong to some component B' of $\Gamma\setminus\Phi$. In addition we may assume that the numbers M and M' are chosen so that $N_c(\{\alpha_M,\alpha_{M-1},\ldots\})\subseteq B$ and $N_c(\{\alpha_{M'},\alpha_{M'+1},\ldots\})\subseteq B'$. But then

$$\begin{aligned}\Gamma\setminus(B\cup B') &\subseteq \Gamma\setminus(N_c(\{\alpha_M,\alpha_{M-1},\ldots\})\cup N_c(\{\alpha_{M'},\alpha_{M'+1},\ldots\}))\\ &\subseteq N_c(\{\alpha_{M+1},\ldots,\alpha_{M'-1}\})\end{aligned}$$

is finite. From this we see that if Φ is a finite set of vertices in Γ, then $\Gamma\setminus\Phi$ has at most two infinite components.

We have now shown that there exists a finite set Φ_0 of vertices in Γ such that $\Gamma\setminus\Phi_0$ has at least two infinite connected components and for every finite set Φ of vertices the graph $\Gamma\setminus\Phi$ can have at most have two infinite components. Hence Γ has precisely two ends. □